Name:

1. (11 points)

   Consider a continuous random variable X with probability density function:

$$f(x) = \begin{cases} \frac{2}{3} - \frac{2}{3}x, & 0 \leqslant x \leqslant 1 \\ \sqrt{x-1}, & 1 < x \leqslant a \end{cases}$$

   (a) Find the value of a. [3]

   (b) Find the probability that X is greater than $\frac{1}{2}$. [2]

   (c) Find:

   (i) the mean, [2]

   (ii) the mode, [1]

   (iii) the median [3]

   of X.

2. (11 points)

The speeds of the cars passing the school at Myśliwiecka are normally distributed. The speed limit is 30 kmh^{-1}. 37% of the cars exceed the speed limit and 7% of the cars exceed this limit by more than 10 kmh^{-1}.

(a) Find the mean and the standard deviation of the speeds of the cars passing the school. [5]

(b) Find the probability that a randomly chosen car exceeds the limit by more than 20 kmh^{-1}. [1]

30 cars passed the school between 8:00 and 8:05.

(c) Find the probability that at least half of them exceeded the speed limit. [2]

(d) At least half of the cars exceeded the speed limit. Find the probability that no car exceeded the speed limit by more than 10 kmh^{-1}. [3]

3. (18 points)

Consider the function $f(x) = (\arcsin x)^2$, with $-1 \leqslant x \leqslant 1$.

(a) Show that $f'(0) = 0$. [2]

(b) Calculate $f''(x)$ and hence show that the function satisfies the equation: [6]

$$(1 - x^2)f''(x) - xf'(x) - 2 = 0$$

(c) By differentiating the above equation, show that the function satisfies the following equations: [5]

$$(1 - x^2)f^{(3)} - 3xf''(x) - f'(x) = 0$$

and

$$(1 - x^2)f^{(4)}(x) - 5xf^{(3)}(x) - 4f''(x) = 0$$

where $f^{(3)}$ and $f^{(4)}$ denote the 3rd and 4th derivative of $f(x)$ respectively.

(d) By substituting $x = 0$ into the above equations find the Maclaurin series for $f(x)$ up to and including the x^4 term. [3]

(e) Use this Maclaurin series approximation for $f(x)$ with $x = \dfrac{1}{2}$ to find an approximate value of π^2. [2]