Markscheme

November 2019

Mathematics

Higher level

Paper 1

18 pages

Section A

1. (a) $p = 1 - \frac{1}{2} - \frac{1}{5} - \frac{1}{5}$ ***(M1)***

$= \frac{1}{10}$ ***A1***

[2 marks]

(b) attempt to find $\mathrm{E}(X)$ ***(M1)***

$\frac{1}{2} + 1 + 2 + \frac{N}{10} = 10$ ***A1***

$\Rightarrow N = 65$ ***A1***

Note: Do not allow FT in part (b) if their p is outside the range $0 < p < 1$.

[3 marks]

Total [5 marks]

2. $\frac{1}{2}\mathrm{e}^{2x}$ seen ***(A1)***

attempt at using limits in an integrated expression $\left(\left[\frac{1}{2}\mathrm{e}^{2x}\right]_0^{\ln k} = \frac{1}{2}\mathrm{e}^{2\ln k} - \frac{1}{2}\mathrm{e}^{0}\right)$ ***(M1)***

$= \frac{1}{2}\mathrm{e}^{\ln k^2} - \frac{1}{2}\mathrm{e}^{0}$ ***(A1)***

Setting their equation $= 12$ ***M1***

Note: their equation must be an integrated expression with limits substituted.

$\frac{1}{2}k^2 - \frac{1}{2} = 12$ ***A1***

$\left(k^2 = 25 \Rightarrow\right) k = 5$ ***A1***

Note: Do not award final ***A1*** for $k = \pm 5$.

[6 marks]

3. attempt to eliminate a variable (or attempt to find det A) ***M1***

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & 3 & -1 & 4 \\ 3 & -5 & a & b \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 7 & -3 & 3 \\ 0 & -14 & a+3 & b-12 \end{array}\right)$$ (or $\det A = 14(a-3)$)

(or two correct equations in two variables) ***A1***

$$\to \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 0 & 7 & -3 & 3 \\ 0 & 0 & a-3 & b-6 \end{array}\right)$$ (or solving $\det A = 0$)

(or attempting to reduce to one variable, e.g. $(a-3)z = b-6$) ***M1***

$a = 3,\ b \neq 6$ ***A1A1***

[5 marks]

4. attempt to use $\cos(2A+B) = \cos 2A \cos B - \sin 2A \sin B$ (may be seen later) ***M1***

attempt to use any double angle formulae (seen anywhere) ***M1***

attempt to find either $\sin A$ or $\cos B$ (seen anywhere) ***M1***

$\cos A = \frac{2}{3} \Rightarrow \sin A \left(= \sqrt{1 - \frac{4}{9}} \right) = \frac{\sqrt{5}}{3}$ ***(A1)***

$\sin B = \frac{1}{3} \Rightarrow \cos B \left(= \sqrt{1 - \frac{1}{9}} = \frac{\sqrt{8}}{3} \right) = \frac{2\sqrt{2}}{3}$ ***A1***

$\cos 2A \left(= 2\cos^2 A - 1 \right) = -\frac{1}{9}$ ***A1***

$\sin 2A \left(= 2 \sin A \cos A \right) = \frac{4\sqrt{5}}{9}$ ***A1***

So $\cos(2A+B) = \left(-\frac{1}{9}\right)\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{4\sqrt{5}}{9}\right)\left(\frac{1}{3}\right)$

$= -\frac{2\sqrt{2}}{27} - \frac{4\sqrt{5}}{27}$ ***AG***

[7 marks]

5. (a) **METHOD 1**

$|z| = \sqrt[4]{4} \left(= \sqrt{2}\right)$ ***(A1)***

$\arg(z_1) = \frac{\pi}{4}$ ***(A1)***

first solution is $1 + \mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/– their components) ***(M1)***

other solutions are $-1 + \mathrm{i}$, $-1 - \mathrm{i}$, $1 - \mathrm{i}$ ***A1***

[5 marks]

METHOD 2

$z^4 = -4$

$(a + \mathrm{i}b)^4 = -4$

attempt to expand and equate **both** reals and imaginaries. ***(M1)***

$a^4 + 4a^3b\mathrm{i} - 6a^2b^2 - 4ab^3\mathrm{i} + b^4 = -4$

$\left(a^4 - 6a^4 + a^4 = -4 \Rightarrow\right) a = \pm 1$ ***and*** $\left(4a^3b - 4ab^3 = 0 \Rightarrow\right) a = \pm b$ ***(A1)***

first solution is $1 + \mathrm{i}$ ***A1***

valid attempt to find all roots (De Moivre or +/- their components) ***(M1)***

other solutions are $-1 + \mathrm{i}$, $-1 - \mathrm{i}$, $1 - \mathrm{i}$ ***A1***

[5 marks]

(b) complete method to find area of 'rectangle' ***(M1)***

$= 4$ ***A1***

[2 marks]

Total [7 marks]

6. $f'(x) = e^{2x} + 2xe^{2x}$ ***A1***

> **Note:** This must be obtained from the candidate differentiating $f(x)$.

$= \left(2^1 x + 1 \times 2^{1-1}\right) e^{2x}$ ***A1***

(hence true for $n = 1$)

assume true for $n = k$: ***M1***

$f^{(k)}(x) = \left(2^k x + k2^{k-1}\right)e^{2x}$

> **Note:** Award ***M1*** if truth is assumed. Do not allow "let $n = k$".

consider $n = k + 1$:

$f^{(k+1)}(x) = \dfrac{d}{dx}\left(\left(2^k x + k2^{k-1}\right)e^{2x}\right)$

attempt to differentiate $f^{(k)}(x)$ ***M1***

$f^{(k+1)}(x) = 2^k e^{2x} + 2\left(2^k x + k2^{k-1}\right)e^{2x}$ ***A1***

$f^{(k+1)}(x) = \left(2^k + 2^{k+1}x + k2^k\right)e^{2x}$

$f^{(k+1)}(x) = \left(2^{k+1}x + (k+1)2^k\right)e^{2x}$ ***A1***

$= \left(2^{k+1}x + (k+1)2^{(k+1)-1}\right)e^{2x}$

True for $n = 1$ and $n = k$ true implies true for $n = k + 1$.

Therefore the statement is true for all $n\left(\in \mathbb{Z}^+\right)$ ***R1***

> **Note:** Do not award final ***R1*** if the two previous ***M1s*** are not awarded.
> Allow full marks for candidates who use the base case $n = 0$.

[7 marks]

7. (a) attempt to complete the square or multiplication and equating coefficients ***(M1)***

$2x - x^2 = -(x-1)^2 + 1$ ***A1***

$a = -1,\ h = 1,\ k = 1$

[2 marks]

(b) use of their identity from part (a) $\left(\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{\sqrt{1-(x-1)^2}}\,\mathrm{d}x\right)$ ***(M1)***

$= \left[\arcsin(x-1)\right]_{\frac{1}{2}}^{\frac{3}{2}}$ or $\left[\arcsin(u)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ ***A1***

Note: Condone lack of, or incorrect limits up to this point.

$= \arcsin\left(\frac{1}{2}\right) - \arcsin\left(-\frac{1}{2}\right)$ ***(M1)***

$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)$ ***(A1)***

$= \frac{\pi}{3}$ ***A1***

[5 marks]

Total [7 marks]

8. a vector normal to Π_p is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ ***(A1)***

> **Note:** Allow any scalar multiple of $\begin{pmatrix}1\\0\\0\end{pmatrix}$, including $\begin{pmatrix}p\\0\\0\end{pmatrix}$

attempt to find scalar product (or vector product) of direction vector of line with any scalar multiple of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ ***M1***

$\begin{pmatrix}1\\0\\0\end{pmatrix} \bullet \begin{pmatrix}5\\\sin\theta\\\cos\theta\end{pmatrix} = 5$ (or $\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}5\\\sin\theta\\\cos\theta\end{pmatrix} = \begin{pmatrix}0\\-\cos\theta\\\sin\theta\end{pmatrix}$) ***A1***

(if α is the angle between the line and the normal to the plane)

$\cos\alpha = \frac{5}{1 \times \sqrt{25 + \sin^2\theta + \cos^2\theta}}$ (or $\sin\alpha = \frac{1}{1 \times \sqrt{25 + \sin^2\theta + \cos^2\theta}}$) ***A1***

$\Rightarrow \cos\alpha = \frac{5}{\sqrt{26}}$ or $\sin\alpha = \frac{1}{\sqrt{26}}$ ***A1***

this is independent of p and θ, hence the angle between the line and the plane, $(90 - \alpha)$, is also independent of p and θ ***R1***

> **Note:** The final ***R*** mark is independent, but is conditional on the candidate obtaining a value independent of p and θ.

[6 marks]

Section B

9. (a) $\cos 105^\circ = \cos(180^\circ - 75^\circ) = -\cos 75^\circ$ ***R1***

$= -q$ ***AG***

Note: Accept arguments using the unit circle or graphical/diagrammatical considerations.

[1 mark]

(b) $\mathrm{AD} = \mathrm{CD} \Rightarrow \mathrm{C\hat{A}D} = 45^\circ$ ***A1***

valid method to find $\mathrm{B\hat{A}C}$ ***(M1)***

for example: $\mathrm{BC} = r \Rightarrow \mathrm{B\hat{C}A} = 60^\circ$

$\Rightarrow \mathrm{B\hat{A}C} = 30^\circ$ ***A1***

hence $\mathrm{B\hat{A}D} = 45^\circ + 30^\circ = 75^\circ$ ***AG***

[3 marks]

(c) (i) $\mathrm{AB} = r\sqrt{3}$, $\mathrm{AD}(=\mathrm{CD}) = r\sqrt{2}$ ***A1A1***

applying cosine rule ***(M1)***

$\mathrm{BD}^2 = (r\sqrt{3})^2 + (r\sqrt{2})^2 - 2(r\sqrt{3})(r\sqrt{2})\cos 75^\circ$ ***A1***

$= 3r^2 + 2r^2 - 2r^2\sqrt{6}\cos 75^\circ$

$= 5r^2 - 2r^2 q\sqrt{6}$ ***AG***

(ii) $\mathrm{B\hat{C}D} = 105^\circ$ ***(A1)***

attempt to use cosine rule on $\Delta\mathrm{BCD}$ ***(M1)***

$\mathrm{BD}^2 = r^2 + (r\sqrt{2})^2 - 2r(r\sqrt{2})\cos 105^\circ$

$= 3r^2 + 2r^2 q\sqrt{2}$ ***A1***

[7 marks]

(d) $5r^2 - 2r^2 q\sqrt{6} = 3r^2 + 2r^2 q\sqrt{2}$ ***(M1)(A1)***

$2r^2 = 2r^2 q(\sqrt{6} + \sqrt{2})$ ***A1***

Note: Award ***A1*** for any correct intermediate step seen using only two terms.

$q = \dfrac{1}{\sqrt{6} + \sqrt{2}}$ ***AG***

Note: Do not award the final ***A1*** if follow through is being applied.

[3 marks]

Total [14 marks]

10. (a) (i) attempt to use quotient rule (or equivalent) ***(M1)***

$$f'(x) = \frac{(x^2-1)(2)-(2x-4)(2x)}{(x^2-1)^2}$$ ***A1***

$$= \frac{-2x^2+8x-2}{(x^2-1)^2}$$

(ii) $f'(x) = 0$

simplifying numerator (may be seen in part (i)) ***(M1)***

$\Rightarrow x^2 - 4x + 1 = 0$ or equivalent quadratic equation ***A1***

EITHER

use of quadratic formula

$$\Rightarrow x = \frac{4 \pm \sqrt{12}}{2}$$ ***A1***

OR

use of completing the square

$(x-2)^2 = 3$ ***A1***

THEN

$x = 2 - \sqrt{3}$ (since $2 + \sqrt{3}$ is outside the domain) ***AG***

Note: Do not condone verification that $x = 2 - \sqrt{3} \Rightarrow f'(x) = 0$. Do not award the final ***A1*** as follow through from part (i).

[5 marks]

(b) (i) $(0, 4)$ ***A1***

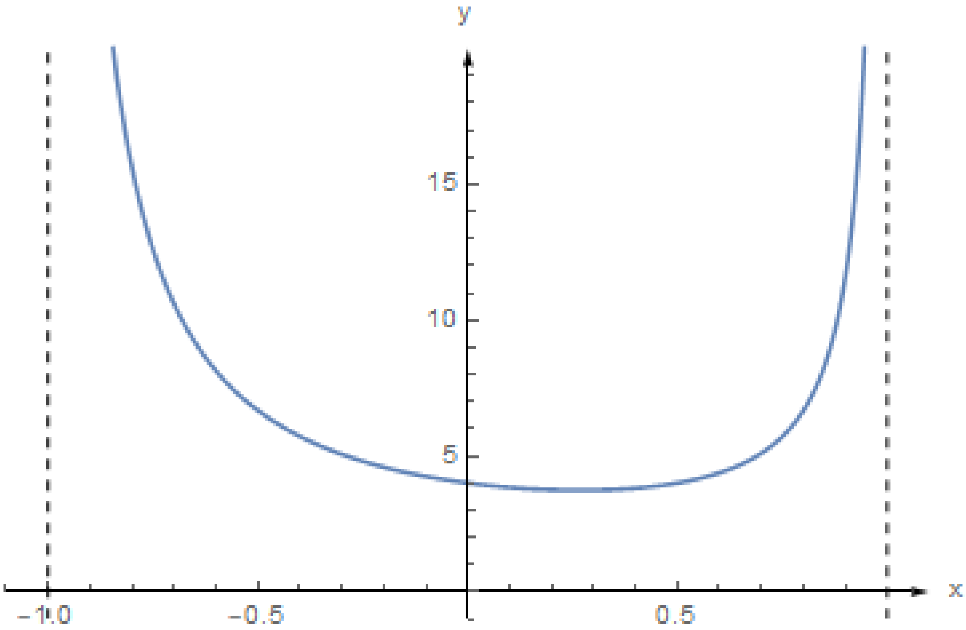
(ii) $2x - 4 = 0 \Rightarrow x = 2$ ***A1***

outside the domain ***R1***

continued…

Question 10 continued

(iii)



A1A1

award ***A1*** for concave up curve over correct domain with one minimum point in the first quadrant
award ***A1*** for approaching $x = \pm 1$ asymptotically

[5 marks]

(c) valid attempt to combine fractions (using common denominator) ***M1***

$$\frac{3(x-1)-(x+1)}{(x+1)(x-1)}$$ ***A1***

$$= \frac{3x-3-x-1}{x^2-1}$$

$$= \frac{2x-4}{x^2-1}$$ ***AG***

[2 marks]

continued…

Question 10 continued

(d)

$f(x)=4 \Rightarrow 2x-4=4x^2-4$ ***M1***

($x=0$ or) $x=\frac{1}{2}$ ***A1***

area under the curve is $\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ ***M1***

$=\int_0^{\frac{1}{2}} \frac{3}{x+1}-\frac{1}{x-1}\,\mathrm{d}x$

Note: Ignore absence of, or incorrect limits up to this point.

$=\left[3\ln|x+1|-\ln|x-1|\right]_0^{\frac{1}{2}}$ ***A1***

$=3\ln\frac{3}{2}-\ln\frac{1}{2}(-0)$

$=\ln\frac{27}{4}$ ***A1***

area is $2-\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ or $\int_0^{\frac{1}{2}} 4\,\mathrm{d}x-\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x$ ***M1***

$=2-\ln\frac{27}{4}$

$=\ln\frac{4\mathrm{e}^2}{27}$ ***A1***

$\left(\Rightarrow v=\frac{4\mathrm{e}^2}{27}\right)$

[7 marks]

Total [19 marks]

11. (a) (i) $\overrightarrow{AV} = \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix}$ ***A1***

$$\overrightarrow{AB} \times \overrightarrow{AV} = \begin{pmatrix} 0 \\ 10 \\ -10 \end{pmatrix} \times \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix} = \begin{pmatrix} 10(p-10)+10p \\ -10p \\ -10p \end{pmatrix}$$ ***A1***

$$= \begin{pmatrix} 20p-100 \\ -10p \\ -10p \end{pmatrix} = -10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix}$$ ***AG***

$$\overrightarrow{AC} \times \overrightarrow{AV} = \begin{pmatrix} 10 \\ 0 \\ -10 \end{pmatrix} \times \begin{pmatrix} p \\ p \\ p-10 \end{pmatrix} = \begin{pmatrix} 10p \\ 100-20p \\ 10p \end{pmatrix}\left(= 10\begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix}\right)$$ ***A1***

(ii) attempt to find a scalar product ***M1***

$$-10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix} \bullet 10\begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} = 100\left(3p^2-20p\right)$$

OR $-\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix} \bullet \begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix} = 3p^2-20p$ ***A1***

attempt to find magnitude of either $\overrightarrow{AB} \times \overrightarrow{AV}$ or $\overrightarrow{AC} \times \overrightarrow{AV}$ ***M1***

$$\left|-10\begin{pmatrix} 10-2p \\ p \\ p \end{pmatrix}\right| = \left|10\begin{pmatrix} p \\ 10-2p \\ p \end{pmatrix}\right| = 10\sqrt{(10-2p)^2+2p^2}$$ ***A1***

$$100\left(3p^2-20p\right) = 100\left(\sqrt{(10-2p)^2+2p^2}\right)^2 \cos\theta$$

$$\cos\theta = \frac{3p^2-20p}{(10-2p)^2+2p^2}$$ ***A1***

Note: Award ***A1*** for any intermediate step leading to the correct answer.

$$= \frac{p(3p-20)}{6p^2-40p+100}$$ ***AG***

Note: Do not allow FT marks from part (a)(i).

[8 marks]

continued…

Question 11 continued

(b) (i) $p(3p-20)=0 \Rightarrow p=0$ or $p=\frac{20}{3}$ ***M1A1***

coordinates are $(0,0,0)$ and $\left(\frac{20}{3},\frac{20}{3},\frac{20}{3}\right)$ ***A1***

Note: Do not allow column vectors for the final ***A*** mark.

(ii) two points are mirror images in the plane
or opposite sides of the plane
or equidistant from the plane
or the line connecting the two Vs is perpendicular to the plane ***R1***

[4 marks]

(c) (i) geometrical consideration or attempt to solve $-1=\frac{p(3p-20)}{6p^2-40p+100}$ ***(M1)***

$p=\frac{10}{3}$, $\theta=\pi$ or $\theta=180^\circ$ ***A1A1***

(ii) $p \to \infty \Rightarrow \cos\theta \to \frac{1}{2}$ ***M1***

hence the asymptote has equation $\theta=\frac{\pi}{3}$ ***A1***

[5 marks]

Total [17 marks]