Markscheme

May 2019

Mathematics

Higher level

Paper 2

18 pages

Section A

1. attempt to apply cosine rule ***M1***

$$\cos A = \frac{5^2 + 11^2 - 14^2}{2 \times 5 \times 11} = -0.4545...$$

$\Rightarrow A = 117.03569...^\circ$

$\Rightarrow A = 117.0^\circ$ ***A1***

attempt to apply sine rule or cosine rule: ***M1***

$$\frac{\sin 117.03569...^\circ}{14} = \frac{\sin B}{11}$$

$\Rightarrow B = 44.4153...^\circ$

$\Rightarrow B = 44.4^\circ$ ***A1***

$C = 180^\circ - A - B$

$C = 18.5^\circ$ ***A1***

Note: Candidates may attempt to find angles in any order of their choosing.

[5 marks]

2. (a) $X \sim N\left(820, 230^2\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to use normal distribution. Accept labelled normal graph.

$\Rightarrow P\left(X > 1000\right) = 0.217$ ***A1***

[2 marks]

(b) $Y \sim B\left(24, 0.217...\right)$ ***(M1)***

Note: Award ***M1*** for recognition of binomial distribution with parameters.

$P\left(Y \le 10\right) - P\left(Y \le 4\right)$ ***(M1)***

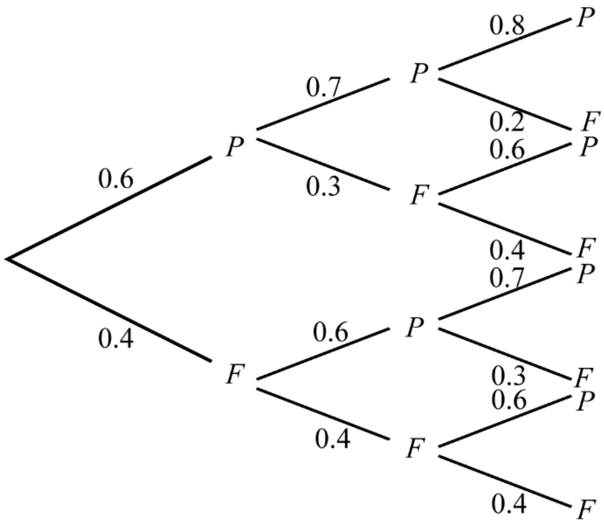
Note: Award ***M1*** for an attempt to find $P(5 \le Y \le 10)$ or $P\left(Y \le 10\right) - P\left(Y \le 4\right)$.

$= 0.613$ ***A1***

[3 marks]

Total [5 marks]

3. (a)



A1A1A1

Note: Award ***A1*** for each correct column of probabilities.

[3 marks]

(b) probability (at least twice) =

EITHER

$$(0.6\times0.7\times0.8)+(0.6\times0.7\times0.2)+(0.6\times0.3\times0.6)+(0.4\times0.6\times0.7)$$ ***(M1)***

OR

$$(0.6\times0.7) + (0.6\times0.3\times0.6) + (0.4\times0.6\times0.7)$$ ***(M1)***

Note: Award ***M1*** for summing all required probabilities.

THEN

$=0.696$ ***A1***

[2 marks]

(c) P(passes third paper given only one paper passed before)

$$=\frac{\text{P (passes third AND only one paper passed before)}}{\text{P (passes once in first two papers)}}$$ ***(M1)***

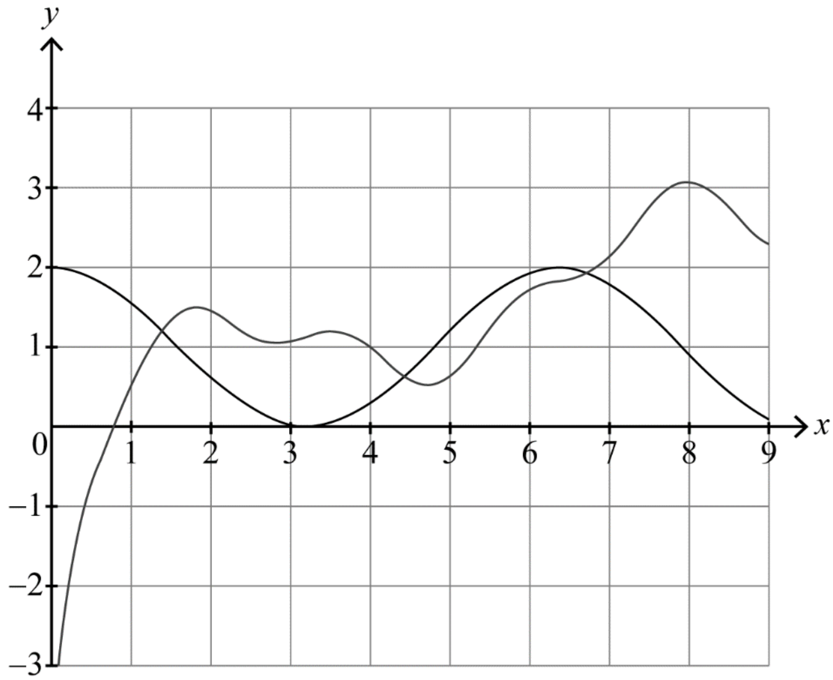
$$=\frac{(0.6\times0.3\times0.6)+(0.4\times0.6\times0.7)}{(0.6\times0.3)+(0.4\times0.6)}$$ ***A1***

$=0.657$ ***A1***

[3 marks]

Total [8 marks]

4. (a)



A1A1

Note: Award ***A1*** for each correct curve, showing all local max & mins.

Note: Award ***A0A0*** for the curves drawn in degrees.

[2 marks]

(b) $x = 1.35, 4.35, 6.64$ ***(M1)***

Note: Award ***M1*** for attempt to find points of intersections between two curves.

$0 < x < 1.35$ ***A1***

Note: Accept $x < 1.35$.

$4.35 < x < 6.64$ ***A1A1***

Note: Award ***A1*** for correct endpoints, ***A1*** for correct inequalities.

Note: Award ***M1FTA1FTA0FTA0FT*** for $0 < x < 7.31$.

Note: Accept $x < 7.31$.

[4 marks]

Total [6 marks]

5. (a) **METHOD 1**

$\text{LHS} = \dfrac{1+\sin 2x}{\cos 2x} = \dfrac{1+2\sin x\cos x}{\cos^2 x - \sin^2 x}$ ***M1***

$= \dfrac{\left(\cos^2 x + \sin^2 x\right) + 2\sin x\cos x}{\cos^2 x - \sin^2 x}$ ***M1***

$= \dfrac{\left(\cos x + \sin x\right)^2}{\left(\cos x + \sin x\right)\left(\cos x - \sin x\right)}$ ***A1***

$= \dfrac{\cos x + \sin x}{\cos x - \sin x}$

$= \dfrac{\dfrac{\cos x}{\cos x} + \dfrac{\sin x}{\cos x}}{\dfrac{\cos x}{\cos x} - \dfrac{\sin x}{\cos x}}$ ***A1***

$= \dfrac{1+\tan x}{1-\tan x}$ ***AG***

Note: Candidates may start with RHS, apply MS in reverse.

[4 marks]

METHOD 2

$\text{LHS} = \dfrac{1+\sin 2x}{\cos 2x} = \dfrac{1+2\sin x\cos x}{\cos^2 x - \sin^2 x}$ ***M1***

dividing numerator and denominator by $\cos^2 x$ ***M1***

$= \dfrac{\sec^2 x + 2\tan x}{1-\tan^2 x}$

$= \dfrac{1+\tan^2 x + 2\tan x}{1-\tan^2 x}$ ***A1***

$= \dfrac{(\tan x + 1)^2}{(1-\tan x)(1+\tan x)}$ ***A1***

$= \dfrac{1+\tan x}{1-\tan x}$ ***AG***

Note: Candidates may start with RHS; apply MS in reverse.

[4 marks]

(b) valid attempt to solve $\dfrac{1+\tan x}{1-\tan x} = \sqrt{3}$ ***(M1)***

$\tan x = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$x = 0.262\left(=\dfrac{\pi}{12}\right),\ x = 3.40\left(=\dfrac{13\pi}{12}\right)$ ***A1***

Note: Award ***M1A0*** if only one correct solution is given.

[2 marks]

Total [6 marks]

6. attempt to integrate a to find v ***M1***

$v = \int a \ \mathrm{d}t = \int (2t-1) \ \mathrm{d}t$

$= t^2 - t + c$ ***A1***

$s = \int v \ \mathrm{d}t = \int (t^2 - t + c) \ \mathrm{d}t$

$= \frac{t^3}{3} - \frac{t^2}{2} + ct + d$ ***A1***

attempt at substitution of given values ***(M1)***

at $t = 6$, $18.25 = 72 - 18 + 6c + d$

at $t = 15$, $922.75 = 1125 - 112.5 + 15c + d$

solve simultaneously: ***(M1)***

$c = -6$; $d = 0.25$ ***A1***

$\Rightarrow s = \frac{t^3}{3} - \frac{t^2}{2} - 6t + \frac{1}{4}$

[6 marks]

7. $n = 1 \Rightarrow S_1 = u_1$, so true for $n = 1$ ***R1***

assume true for $n = k$, ie. $S_k = \frac{u_1(1-r^k)}{1-r}$ ***M1***

Note: Award ***M0*** for statements such as "let $n = k$".

Note: Subsequent marks after the first ***M1*** are independent of this mark and can be awarded.

$S_{k+1} = S_k + u_1 r^k$ ***M1***

$S_{k+1} = \frac{u_1(1-r^k)}{1-r} + u_1 r^k$ ***A1***

$S_{k+1} = \frac{u_1(1-r^k)}{1-r} + \frac{u_1 r^k (1-r)}{1-r}$

$S_{k+1} = \frac{u_1 - u_1 r^k + u_1 r^k - r u_1 r^k}{1-r}$ ***A1***

$S_{k+1} = \frac{u_1(1-r^{k+1})}{1-r}$ ***A1***

true for $n = 1$ and if true for $n = k$ then true for $n = k + 1$,
the statement is true for any positive integer (or equivalent). ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks are gained.

[7 marks]

8. (a) **METHOD 1**

$w^3 = 8\mathrm{i}$

writing $8\mathrm{i} = 8\left(\cos\left(\frac{\pi}{2} + 2\pi k\right) + \mathrm{i}\sin\left(\frac{\pi}{2} + 2\pi k\right)\right)$ ***(M1)***

Note: Award ***M1*** for an attempt to find cube roots of w using modulus-argument form.

cube roots $w = 2\left(\cos\left(\frac{\frac{\pi}{2} + 2\pi k}{3}\right) + \mathrm{i}\sin\left(\frac{\frac{\pi}{2} + 2\pi k}{3}\right)\right)$ ***(M1)***

ie. $w = \sqrt{3} + \mathrm{i}, -\sqrt{3} + \mathrm{i}, -2\mathrm{i}$ ***A2***

Note: Award ***A2*** for all 3 correct, ***A1*** for 2 correct.

Note: Accept $w = 1.73 + \mathrm{i}$ and $w = -1.73 + \mathrm{i}$.

[4 marks]

METHOD 2

$w^3 + (2\mathrm{i})^3 = 0$

$(w + 2\mathrm{i})\left(w^2 - 2w\mathrm{i} - 4\right) = 0$ ***M1***

$w = \frac{2\mathrm{i} \pm \sqrt{12}}{2}$ ***M1***

$w = \sqrt{3} + \mathrm{i}, -\sqrt{3} + \mathrm{i}, -2\mathrm{i}$ ***A2***

Note: Award ***A2*** for all 3 correct, ***A1*** for 2 correct.

Note: Accept $w = 1.73 + \mathrm{i}$ and $w = -1.73 + \mathrm{i}$.

[4 marks]

(b) $w_1 = -2\mathrm{i}$

$\frac{z}{z - \mathrm{i}} = -2\mathrm{i}$ ***M1***

$z = -2\mathrm{i}(z - \mathrm{i})$

$z(1 + 2\mathrm{i}) = -2$

$z = \frac{-2}{1 + 2\mathrm{i}}$ ***A1***

$z = -\frac{2}{5} + \frac{4}{5}\mathrm{i}$ ***A1***

Note: Accept $a = -\frac{2}{5}, b = \frac{4}{5}$.

[3 marks]

Total [7 marks]

Section B

9. (a) **METHOD 1**

attempt to find roots or factors ***(M1)***

roots are $-3,\ 1,\ (4+\mathrm{i}),(4-\mathrm{i})$ ***A1A1***

Note: Award ***A1*** for each pair of roots or factors, real and complex.

attempt to form quadratic ***M1***

$(z-1)(z+3)=z^2+2z-3$ ***A1***

$(z-(4+\mathrm{i}))(z-(4-\mathrm{i}))$

$=z^2-(4-\mathrm{i})z-(4+\mathrm{i})z+17$ ***(A1)***

$=z^2-8z+17$ ***A1***

$z^4-6z^3-2z^2+58z-51=(z^2-8z+17)(z^2+2z-3)$

[7 marks]

METHOD 2

attempt to find roots or factors ***(M1)***

real roots are -3, 1 (or real factors $(z+3),(z-1)$) ***A1***

attempt to form quadratic ***M1***

$(z-1)(z+3)=z^2+2z-3$ ***A1***

$z^4-6z^3-2z^2+58z-51=[z^2+2z-3][z^2+kz+17]$

equate coefficients of z^2 ***M1***

$-2=2k-3+17$ ***A1***

solve to give $k=-8$ ***A1***

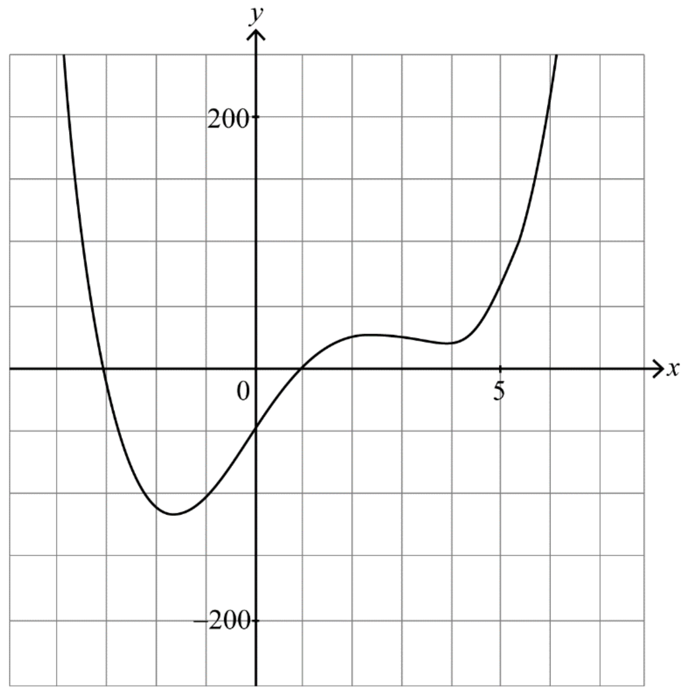
$z^4-6z^3-2z^2+58z-51=(z^2-8z+17)(z^2+2z-3)$

[7 marks]

continued…

Question 9 continued

(b)



shape **A1**

x-axis intercepts at $(-3, 0)$, $(1, 0)$ and y-axis intercept at $(0, -51)$ **A1A1**

minimum points at $(-1.62, -118)$ and $(3.72, 19.7)$ **A1A1**

maximum point at $(2.40, 26.9)$ **A1**

Note: Coordinates may be seen on the graph or elsewhere.

Note: Accept -3, 1 and -51 marked on the axes.

[6 marks]

(c) from graph, $19.7 \le k \le 26.9$ **A1A1**

Note: Award ***A1*** for correct endpoints and ***A1*** for correct inequalities.

[2 marks]

Total [15 marks]

10. (a) $X \sim \text{Po}(2.1)$

$\text{P}(X = 0) = 0.122 \left(= \text{e}^{-2.1}\right)$ ***(M1)A1***

[2 marks]

continued…

Question 10 continued

(b)

y	0	1	2	3	4
$\mathrm{P}(Y=y)$	$0.122...$ $\left(=\mathrm{e}^{-2.1}\right)$	$0.257...$ $\left(=\mathrm{e}^{-2.1}2.1\right)$	$0.270...$ $\left(=\frac{\mathrm{e}^{-2.1}2.1^2}{2!}\right)$	$0.189...$ $\left(=\frac{\mathrm{e}^{-2.1}2.1^3}{3!}\right)$	$0.161...$

A1A1A1A1

Note: Award ***A1*** for each correct probability for $Y=1, 2, 3, 4$. Accept 0.162 for $\mathrm{P}(Y=4)$.

[4 marks]

(c) $\mathrm{E}(Y)=\sum y\mathrm{P}(Y=y)$ ***(M1)***

$=1\times0.257...+2\times0.270...+3\times0.189...+4\times0.161...$ ***(A1)***

$=2.01$ ***A1***

[3 marks]

(d) let T be the no of days per year that Steffi does not visit

$T\sim\mathrm{B}(365, 0.122...)$ ***(M1)***

require $0.45\le\mathrm{P}(T\le n)<0.55$ ***(M1)***

$\mathrm{P}(T\le44)=0.51$

$n=44$ ***A1***

[3 marks]

(e) **METHOD 1**

let V be the discrete random variable "number of times Steffi is not fed per day"

$\mathrm{E}(V)=1\times\mathrm{P}(X=5)+2\times\mathrm{P}(X=6)+3\times\mathrm{P}(X=7)+\cdots$ ***M1***

$=1\times0.0416...+2\times0.0145...+3\times0.00437...+\cdots$ ***A1***

$=0.083979...$ ***A1***

expected no of occasions per year $>0.083979...\times365=30.7$ ***A1***

hence Steffi can expect not to be fed on at least 30 occasions ***AG***

Note: Candidates may consider summing more than three terms in their calculation for $\mathrm{E}(V)$.

[4 marks]

METHOD 2

$\mathrm{E}(X)-\mathrm{E}(Y)=0.0903...$ ***M1A1***

$0.0903...\times365$ ***M1***

$=33.0>30$ ***A1AG***

[4 marks]

Total [16 marks]

11. (a) **METHOD 1**

for example

$\vec{PQ} = \begin{pmatrix} -1 \\ -5 \\ 8 \end{pmatrix}$, $\vec{PR} = \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix}$ ***A1A1***

$\vec{PQ} \times \vec{PR} = 33\boldsymbol{i} + 11\boldsymbol{j} + 11\boldsymbol{k}$ ***(M1)A1***

$\boldsymbol{r.n} = \boldsymbol{a.n}$

$33x + 11y + 11z = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 33 \\ 11 \\ 11 \end{pmatrix} = 22$ ***(M1)***

$\Rightarrow 3x + y + z = 2$ or equivalent ***A1***

[6 marks]

METHOD 2

assume plane can be written as $ax + by + cz = 1$ ***M1***

substituting each set of coordinates gives the system of equations:

$a + 6b - 7c = 1$

$0a + b + c = 1$

$2a + 0b - 4c = 1$ ***A1***

solving by GDC ***(M1)***

$a = \frac{3}{2}$, $b = \frac{1}{2}$, $c = \frac{1}{2}$ ***A1A1A1***

$\Rightarrow \frac{3}{2}x + \frac{1}{2}y + \frac{1}{2}z = 1$ or equivalent

[6 marks]

(b) **METHOD 1**

substitution of equation of line into both equations of planes ***M1***

$3\left(\frac{5}{4} + \frac{\lambda}{2}\right) + \lambda + \left(-\frac{7}{4} - \frac{5\lambda}{2}\right) = 2$ ***A1***

$\left(\frac{5}{4} + \frac{\lambda}{2}\right) - 3\lambda - \left(-\frac{7}{4} - \frac{5\lambda}{2}\right) = 3$ ***A1***

[3 marks]

continued…

Question 11 continued

METHOD 2

adding Π_1 and Π_2 gives $4x - 2y = 5$ ***M1***

given $y = \lambda \Rightarrow x = \frac{5}{4} + \frac{\lambda}{2}$ ***A1***

$z = 2 - y - 3x = -\frac{7}{4} - \frac{5\lambda}{2}$ ***A1***

$$\Rightarrow \boldsymbol{r} = \begin{pmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \end{pmatrix} + \lambda \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{5}{2} \end{pmatrix}$$ ***AG***

[3 marks]

METHOD 3

$$\boldsymbol{n}_1 \times \boldsymbol{n}_2 = \begin{pmatrix} 2 \\ 4 \\ -10 \end{pmatrix}$$ ***A1***

$$= 4 \begin{pmatrix} \frac{1}{2} \\ 1 \\ -\frac{5}{2} \end{pmatrix}$$ ***R1***

common point $\frac{5}{4} - 3(0) - \left(-\frac{7}{4}\right) = 3$ and $-3\left(\frac{5}{4}\right) - 0 - \left(-\frac{7}{4}\right) = -2$ ***A1***

[3 marks]

(c) normal to Π_3 is perpendicular to direction of L

$$\Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix} = 0$$ ***A1***

$\Rightarrow a + 2b - 5c = 0$ ***AG***

[1 mark]

continued…

Question 11 continued

(d) (i) substituting $\begin{pmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \end{pmatrix}$ into Π_3: ***M1***

$\frac{5a}{4} - \frac{7c}{4} = 1$ ***A1***

$5a - 7c = 4$ ***AG***

(ii) attempt to find scalar products for Π_1 and Π_3, Π_2 and Π_3 and equating ***M1***

$\frac{3a+b+c}{\sqrt{11}\sqrt{a^2+b^2+c^2}} = \frac{a-3b-c}{\sqrt{11}\sqrt{a^2+b^2+c^2}}$ ***M1***

Note: Accept $3a+b+c = a-3b-c$.

$\Rightarrow a+2b+c=0$ ***A1***

attempt to solve $a+2b+c=0$, $a+2b-5c=0$, $5a-7c=4$ ***M1***

$\Rightarrow a = \frac{4}{5}, b = -\frac{2}{5}, c = 0$ ***A1***

hence equation is $\frac{4x}{5} - \frac{2y}{5} = 1$

for second equation:

$\frac{3a+b+c}{\sqrt{11}\sqrt{a^2+b^2+c^2}} = -\frac{a-3b-c}{\sqrt{11}\sqrt{a^2+b^2+c^2}}$ ***(M1)***

$\Rightarrow 2a-b=0$

attempt to solve $2a-b=0$, $a+2b-5c=0$, $5a-7c=4$

$\Rightarrow a=-2$, $b=-4$, $c=-2$ ***A1***

hence equation is $-2x-4y-2z=1$

[9 marks]

Total [19 marks]