Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

November 2020

Mathematics

Higher level

Paper 1

30 pages

International Baccalaureate
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Bachillerato Internacional

Section A

1. $\mathrm{E}(X)=(0\times p)+\left(1\times\frac{1}{4}\right)+\left(2\times\frac{1}{6}\right)+3q\left(=\frac{19}{12}\right)$ ***(M1)***

$\left(\Rightarrow \frac{1}{4}+\frac{1}{3}+3q=\frac{19}{12}\right)$

$q=\frac{1}{3}$ ***A1***

$p+\frac{1}{4}+\frac{1}{6}+q=1$ ***(M1)***

$\left(\Rightarrow p+q=\frac{7}{12}\right)$

$p=\frac{1}{4}$ ***A1***

[4 marks]

2. $(x=0\Rightarrow)\ y=1$ ***(A1)***

appreciate the need to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***(M1)***

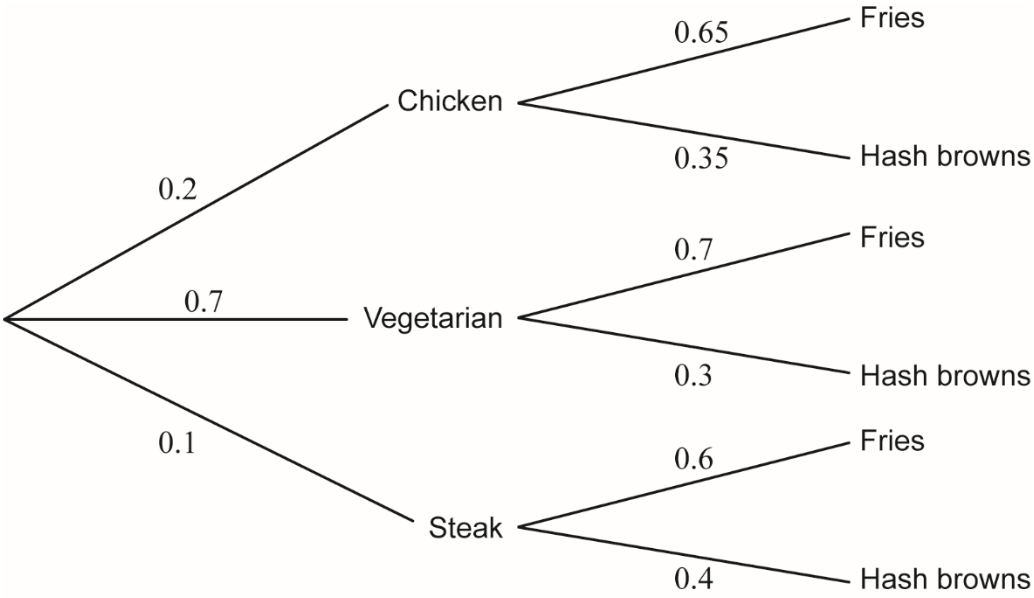
$\left(\frac{\mathrm{d}y}{\mathrm{d}x}=\right)2\mathrm{e}^{2x}-3$ ***A1***

$(x=0\Rightarrow)\frac{\mathrm{d}y}{\mathrm{d}x}=-1$ ***A1***

$\frac{y-1}{x-0}=-1\ \ (y=1-x)$ ***A1***

[5 marks]

3. (a)



A1A1

> **Note:** Award ***A1*** for probabilities for type of omelette and ***A1*** for probabilities for fries / hash browns.

[2 marks]

(b) $(0.2\times0.65)+(0.7\times0.7)+(0.1\times0.6)$ ***(M1)***

$=0.68\left(=\frac{17}{25}\right)$ ***A1***

[2 marks]

(c) $\frac{\mathrm{P}(\text{ordered fries and did not order chicken omelette})}{\mathrm{P}(\text{ did not order chicken omelette})}$ ***(M1)***

$\frac{0.7\times0.7+0.1\times0.6}{0.7+0.1}\left(=\frac{0.49+0.06}{0.8}=\frac{0.55}{0.8}\right)$ ***(A1)***

$=\frac{55}{80}\left(=\frac{11}{16}\right)$ ***A1***

[3 marks]
Total [7 marks]

4. substituting $z = x + \mathrm{i}y$ **and** $z^* = x - \mathrm{i}y$ ***M1***

$$\frac{2\left(x+\mathrm{i}y\right)}{3-\left(x-\mathrm{i}y\right)} = \mathrm{i}$$

$$2x + 2\mathrm{i}y = -y + \mathrm{i}\left(3-x\right)$$

equate real and imaginary: ***M1***

$y = -2x$ AND $2y = 3 - x$ ***A1***

Note: If they multiply top and bottom by the conjugate, the equations $6x - 2x^2 + 2y^2 = 0$ and $6y - 4xy = (3-x)^2 + y^2$ may be seen. Allow for ***A1***.

solving simultaneously:

$x = -1, y = 2$ $\left(z = -1 + 2\mathrm{i}\right)$ ***A1A1***

[5 marks]

5. $u_5 = 4 + 4d = \log_2 625$ ***(A1)***

$4d = \log_2 625 - 4$

attempt to write an integer (*eg* 4 or 1) in terms of $\log_2$ ***M1***

$4d = \log_2 625 - \log_2 16$

attempt to combine two logs into one ***M1***

$4d = \log_2\left(\frac{625}{16}\right)$

$d = \frac{1}{4}\log_2\left(\frac{625}{16}\right)$

attempt to use power rule for logs ***M1***

$d = \log_2\left(\frac{625}{16}\right)^{\frac{1}{4}}$

$d = \log_2\left(\frac{5}{2}\right)$ ***A1***

[5 marks]

Note: Award method marks in any order.

6. **METHOD 1**

$\sin\theta\cos\theta = \frac{c}{a}$ and $\sin\theta + \cos\theta = -\frac{b}{a}$ ***A1***

attempt to square $\sin\theta + \cos\theta$ ***M1***

$\left(\frac{b^2}{a^2} = \right)(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta$ ***A1***

$\frac{b^2}{a^2}(= 1 + 2\sin\theta\cos\theta) = 1 + \frac{2c}{a}$ ***A1***

$b^2 = a^2 + 2ac$ ***AG***

[4 marks]

METHOD 2

$a\sin^2\theta + b\sin\theta + c = 0$ and $a\cos^2\theta + b\cos\theta + c = 0$ ***A1***

adding the two equations ***M1***

$a + b(\sin\theta + \cos\theta) + 2c = 0$ ***A1***

$a + b \times -\frac{b}{a} + 2c = 0$ ***A1***

$a^2 - b^2 + 2ac = 0$

$b^2 = a^2 + 2ac$ ***AG***

[4 marks]

7. (a) (i) $\frac{z_1}{z_2} = \cos\left(\frac{11\pi}{12} - \frac{\pi}{6}\right) + i\sin\left(\frac{11\pi}{12} - \frac{\pi}{6}\right)$ (***M1***)

$= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$ ***A1***

(ii) $\frac{z_2}{z_1} = \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}$ ***A1***

Note: Allow equivalent forms in part (a), *e.g.* $\text{cis}\left(-\frac{3\pi}{4}\right)$.

Note: Ignore subsequent work once correct answer(s) are seen.

[3 marks]

(b) valid attempt to calculate area of their triangle ***(M1)***

(angle between OA and OB is $\frac{\pi}{2}$) $\Rightarrow$ area $\left(= \frac{1}{2} \times 1 \times 1\right) = \frac{1}{2}$ ***A1***

[2 marks]

8. (a) **METHOD 1**

attempt to replace $\tan x = \dfrac{\sin x}{\cos x}$ ***M1***

$$\frac{\sin x \tan x}{1-\cos x} \equiv \frac{\sin^2 x}{\cos x(1-\cos x)}$$

attempt to use $\sin^2 x + \cos^2 x = 1$ ***M1***

$$= \frac{1-\cos^2 x}{\cos x(1-\cos x)}$$

$$= \frac{(1+\cos x)(1-\cos x)}{\cos x(1-\cos x)}$$ ***A1***

$$= \frac{(1+\cos x)}{\cos x}$$

$$= 1 + \frac{1}{\cos x}$$ ***AG***

Note: Award marks in reverse for working from RHS to LHS.

[3 marks]

METHOD 2

attempt to replace $\tan x = \dfrac{\sin x}{\cos x}$ ***M1***

$$\frac{\sin x \tan x}{1-\cos x} \equiv \frac{\sin^2 x}{\cos x(1-\cos x)}$$

$$\equiv \frac{\sin^2 x(1+\cos x)}{\cos x(1-\cos x)(1+\cos x)} \equiv \frac{\sin^2 x(1+\cos x)}{\cos x(1-\cos^2 x)}$$

attempt to use $\sin^2 x + \cos^2 x = 1$ ***M1***

$$\equiv \frac{\sin^2 x(1+\cos x)}{\cos x \sin^2 x}$$ ***A1***

$$= \frac{(1+\cos x)}{\cos x}$$

$$= 1 + \frac{1}{\cos x}$$ ***AG***

Note: Award marks in reverse for working from RHS to LHS.

[3 marks]

(b) **METHOD 1**

consider $1+\dfrac{1}{\cos x}=k$, leading to $\cos x=\dfrac{1}{k-1}$ (***M1)***

consider graph of $y=\dfrac{1}{x-1}$ or range of solutions for $y=\cos x$ (***M1)***

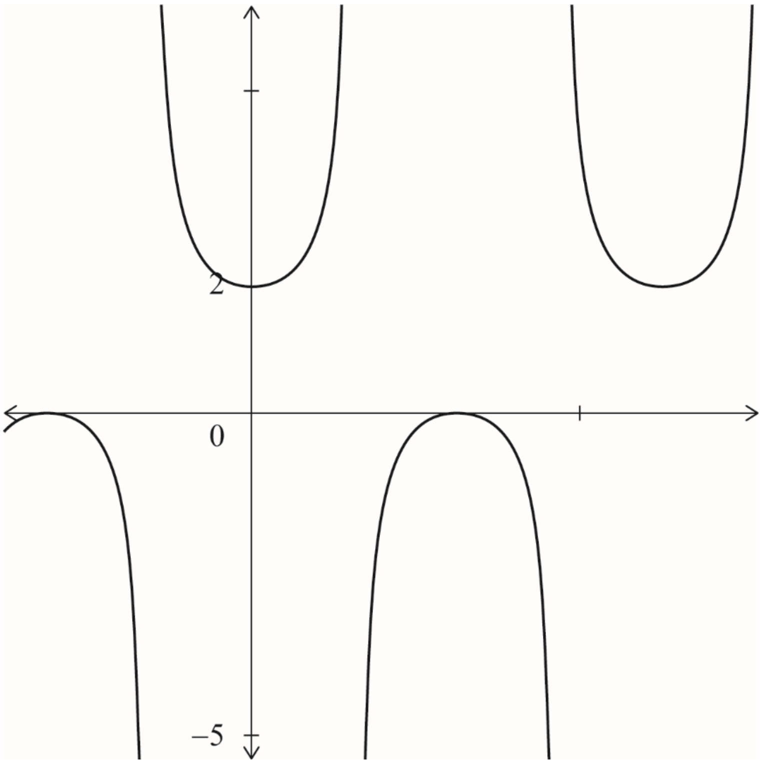
(no solutions if $y<-1$ or $y>1 \Rightarrow$) $0<k<2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities. These may also be expressed as '$k>0$ and $k<2$'.

[4 marks]

METHOD 2

consider graph of $y = 1 + \sec x$ ***M1***



A1

no real solutions if $0 < k < 2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities. These may also be expressed as '$k > 0$ and $k < 2$'.

[4 marks]

METHOD 3

consider $-1 \le \cos x \le 1$, ***(M1)***

$$\frac{1}{\cos x} \le -1 \text{ or } \frac{1}{\cos x} \ge 1$$ ***(M1)***

$$1 + \frac{1}{\cos x} \le 0 \text{ or } 1 + \frac{1}{\cos x} \ge 2$$

no solutions if $0 < k < 2$ ***A1A1***

Note: Award ***A1*** for 0 and 2 seen as critical values, ***A1*** for correct inequalities.
These may also be expressed as '$k > 0$ and $k < 2$'.

[4 marks]
Total [7 marks]

9. $x=\tan u \Rightarrow \dfrac{dx}{du}=\sec^2 u$ OR $u=\arctan x \Rightarrow \dfrac{du}{dx}=\dfrac{1}{1+x^2}$ ***A1***

attempt to write the integral in terms of u ***M1***

$$\int_0^{\frac{\pi}{4}} \frac{\tan^2 u \sec^2 u\ du}{\left(1+\tan^2 u\right)^3}$$

$$\int_0^{\frac{\pi}{4}} \frac{\tan^2 u \sec^2 u\ du}{\left(\sec^2 u\right)^3}$$ ***(A1)***

$$=\int_0^{\frac{\pi}{4}} \sin^2 u \cos^2 u\ du$$ ***A1***

$$=\frac{1}{4}\int_0^{\frac{\pi}{4}} \sin^2 2u\ du$$ ***M1***

$$=\frac{1}{8}\int_0^{\frac{\pi}{4}} (1-\cos 4u)\ du$$ ***M1***

$$=\frac{1}{8}\left[u-\frac{\sin 4u}{4}\right]_0^{\frac{\pi}{4}}$$ ***A1***

$$=\frac{1}{8}\left[\frac{\pi}{4}-\frac{\sin \pi}{4}-0-0\right]$$

$$=\frac{\pi}{32}$$ ***A1***

Total [8 marks]

Section B

10. (a) (i) $f'(x) = 3ax^2 + 2bx + c$ ***A1***

(ii) since f^{-1} does not exist, there must be two turning points ***R1***

($\Rightarrow f'(x) = 0$ has more than one solution)

using the discriminant $\Delta > 0$ ***M1***

$4b^2 - 12ac > 0$ ***A1***

$b^2 - 3ac > 0$ ***AG***

[4 marks]

(b) (i) **METHOD 1**

$b^2 - 3ac = (-3)^2 - 3 \times \frac{1}{2} \times 6$ ***M1***

$= 9 - 9$

$= 0$ ***A1***

hence g^{-1} exists ***AG***

METHOD 2

$g'(x)=\frac{3}{2}x^2-6x+6$ ***M1***

$\Delta=(-6)^2-4\times\frac{3}{2}\times 6$

$\Delta=36-36=0\Rightarrow$ there is (only) one point with gradient of 0
and this must be a point of inflexion (since $g(x)$ is a cubic.) ***R1***

hence g^{-1} exists ***AG***

(ii) $p=\frac{1}{2}$ ***A1***

$(x-2)^3=x^3-6x^2+12x-8$ ***(M1)***

$\frac{1}{2}(x^3-6x^2+12x-8)=\frac{1}{2}x^3-3x^2+6x-4$

$g(x)=\frac{1}{2}(x-2)^3-4\Rightarrow q=-4$ ***A1***

(iii) $x=\frac{1}{2}(y-2)^3-4$ ***(M1)***

Note: Interchanging x and y can be done at any stage.

$2(x+4)=(y-2)^3$ ***(M1)***

$\sqrt[3]{2(x+4)}=y-2$

$y=\sqrt[3]{2(x+4)}+2$

$g^{-1}(x)=\sqrt[3]{2(x+4)}+2$ ***A1***

Note: $g^{-1}(x)=\ldots$ must be seen for the final ***A*** mark.

[8 marks]

(c) translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, ***A1***

Note: This can be seen anywhere.

EITHER

a stretch scale factor $\frac{1}{2}$ parallel to the y-axis then a translation through $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ ***A2***

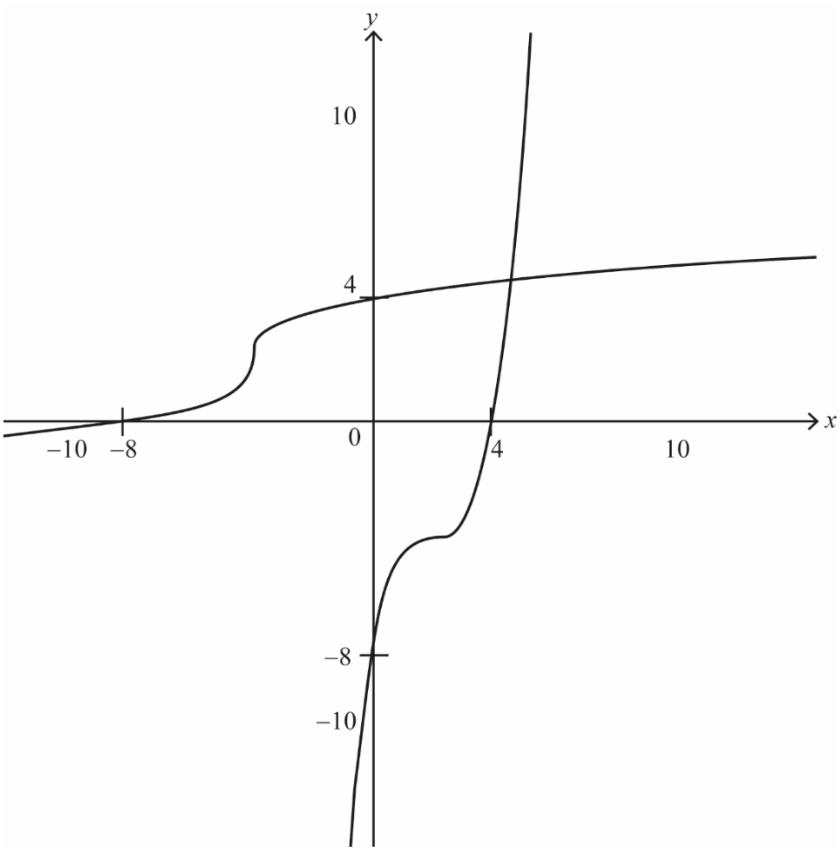
OR

a translation through $\begin{pmatrix} 0 \\ -8 \end{pmatrix}$ then a stretch scale factor $\frac{1}{2}$ parallel to the y-axis ***A2***

Note: Accept 'shift' for translation, but do not accept 'move'. Accept 'scaling' for 'stretch'.

[3 marks]

(d)



A1A1A1
M1A1

Note: Award ***A1*** for correct 'shape' of g (allow non-stationary point of inflexion)
Award ***A1*** for each correct intercept of g
Award ***M1*** for attempt to reflect their graph in $y = x$, ***A1*** for completely correct g^{-1} including intercepts

[5 marks]
Total [20 marks]

11. (a) attempt at implicit differentiation ***M1***

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(xy)\left[x\frac{\mathrm{d}y}{\mathrm{d}x} + y\right]$$ ***A1M1A1***

Note: Award ***A1*** for LHS, ***M1*** for attempt at chain rule, ***A1*** for RHS.

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x}\cos(xy) + y\cos(xy)$$

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} - x\frac{\mathrm{d}y}{\mathrm{d}x}\cos(xy) = y\cos(xy)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(2y - x\cos(xy)\right) = y\cos(xy)$$ ***M1***

Note: Award ***M1*** for collecting derivatives and factorising.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y\cos(xy)}{2y - x\cos(xy)}$$ ***AG***

[5 marks]

(b) setting $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$y\cos(xy)=0$ ***(M1)***

$(y\neq 0)\Rightarrow \cos(xy)=0$ ***A1***

$\Rightarrow \sin(xy)\left(=\pm\sqrt{1-\cos^2(xy)}=\pm\sqrt{1-0}\right)=\pm 1$ ***OR*** $xy=(2n+1)\frac{\pi}{2}\ (n\in\mathbb{Z})$

OR $xy=\frac{\pi}{2},\frac{3\pi}{2},\ldots$ ***A1***

Note: If they offer values for xy , award ***A1*** for at least two correct values in two different ‘quadrants’ and no incorrect values.

$y^2\left(=\sin(xy)\right)>0$ ***R1***

$\Rightarrow y^2=1$ ***A1***

$\Rightarrow y=\pm 1$ ***AG***

[5 marks]

(c) $y = \pm 1 \Rightarrow 1 = \sin(\pm x) \Rightarrow \sin x = \pm 1$ OR $y = \pm 1 \Rightarrow 0 = \cos(\pm x) \Rightarrow \cos x = 0$ ***(M1)***

$(\sin x = 1 \Rightarrow)\left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{2}, 1\right)$ ***A1A1***

$(\sin x = -1 \Rightarrow)\left(\frac{3\pi}{2}, -1\right), \left(\frac{7\pi}{2}, -1\right)$ ***A1A1***

Note: Allow 'coordinates' expressed as $x = \frac{\pi}{2}, y = 1$ for example.

Note: Each of the ***A*** marks may be awarded independently and are not dependent on ***(M1)*** being awarded.

Note: Mark only the candidate's first two attempts for each case of $\sin x$.

[5 marks]

Total [15 marks]

12. (a) $x = k$ ***A1***

[1 mark]

(b) $y = k$ ***A1***

[1 mark]

(c) **METHOD 1**

$$(f \circ f)(x) = \frac{k\left(\frac{kx-5}{x-k}\right)-5}{\left(\frac{kx-5}{x-k}\right)-k}$$ ***M1***

$$= \frac{k(kx-5)-5(x-k)}{kx-5-k(x-k)}$$ ***A1***

$$= \frac{k^2x-5k-5x+5k}{kx-5-kx+k^2}$$

$$= \frac{k^2x-5x}{k^2-5}$$ ***A1***

$$= \frac{x(k^2-5)}{k^2-5}$$

$$= x$$

$(f \circ f)(x) = x$, (hence f is self-inverse) ***R1***

Note: The statement $f(f(x)) = x$ could be seen anywhere in the candidate's working to award ***R1***.

[4 marks]

METHOD 2

$f(x) = \frac{kx-5}{x-k}$

$x = \frac{ky-5}{y-k}$ ***M1***

Note: Interchanging x and y can be done at any stage.

$x(y-k) = ky-5$ ***A1***

$xy - xk = ky - 5$

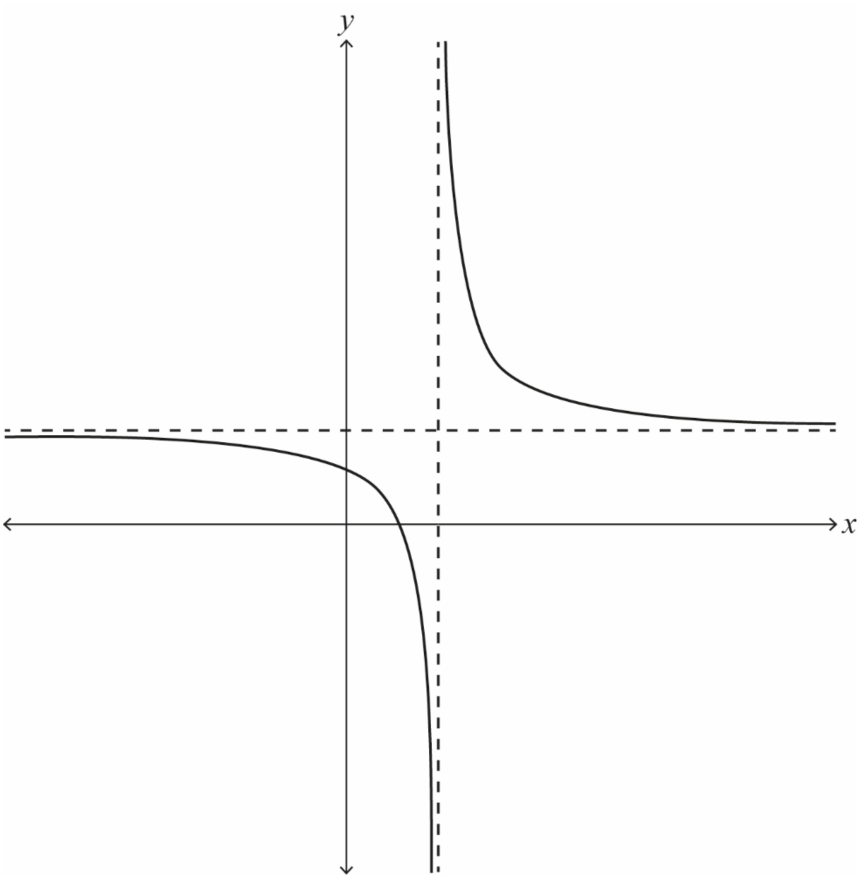
$xy - ky = xk - 5$

$y(x-k) = kx-5$ ***A1***

$y = f^{-1}(x) = \frac{kx-5}{x-k}$ (hence f is self-inverse.) ***R1***

[4 marks]

(d)



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attempt to draw both branches of a rectangular hyperbola ***M1***

$x = 3$ and $y = 3$ ***A1***

$\left(0, \frac{5}{3}\right)$ and $\left(\frac{5}{3}, 0\right)$ ***A1***

[3 marks]

(e) **METHOD 1**

$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$ ***(M1)***

EITHER

attempt to express $\frac{3x-5}{x-3}$ in the form $p+\frac{q}{x-3}$ ***M1***

$\frac{3x-5}{x-3} = 3+\frac{4}{x-3}$ ***A1***

OR

attempt to expand $\left(\frac{3x-5}{x-3}\right)^2$ or $(3x-5)^2$ and divide out ***M1***

$\left(\frac{3x-5}{x-3}\right)^2 = 9+\frac{24x-56}{(x-3)^2}$ ***A1***

THEN

$\left(\frac{3x-5}{x-3}\right)^2 = 9+\frac{24}{x-3}+\frac{16}{(x-3)^2}$ ***A1***

volume $= \pi\int_5^7 \left(9+\frac{24}{x-3}+\frac{16}{(x-3)^2}\right) \mathrm{d}x$

$= \pi\left[9x+24\ln(x-3)-\frac{16}{x-3}\right]_5^7$ ***A1***

$= \pi\left[(63+24\ln 4-4)-(45+24\ln 2-8)\right]$

$= \pi(22+24\ln 2)$ ***A1***

[6 marks]

METHOD 2

$$\text{volume} = \pi\int_5^7 \left(\frac{3x-5}{x-3}\right)^2 \mathrm{d}x$$ ***(M1)***

substituting $u = x - 3 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ ***A1***

$$3x - 5 = 3(u+3) - 5 = 3u + 4$$

$$\text{volume} = \pi\int_2^4 \left(\frac{3u+4}{u}\right)^2 \mathrm{d}u$$ ***M1***

$$= \pi\int_2^4 9 + \frac{16}{u^2} + \frac{24}{u}\,\mathrm{d}u$$ ***A1***

$$= \pi\left[9u - \frac{16}{u} + 24\ln u\right]_2^4$$ ***A1***

Note: Ignore absence of or incorrect limits seen up to this point.

$$= \pi\left(22 + 24\ln 2\right)$$ ***A1***

[6 marks]
Total [15 marks]