Paper 1

Time allowed: 2 hours

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

You are not allowed to use a calculator for this paper.

Short questions

1 **a** Find $\int \sin^2 x \, dx$. (3 marks)

b Hence, evaluate $\int_0^{\pi} \sin^2 x \, dx$. (2 marks)

[Total 5 marks]

2 Let $\log x = p$ and $\log y = q$. Find

a $\log(xy)$ (2 marks)

b $\log\left(\frac{x^2}{y}\right)$ (2 marks)

c $\log\sqrt{x}$ (2 marks)

d $\log(100y)$. (2 marks)

[Total 8 marks]

3 A piece of wood is 290 cm long. It is going to be cut into 10 pieces. The length of each piece should be 2 cm longer than the piece cut off before it, and all the wood will be used up.

Find

a the length of the first piece of wood that is cut (3 marks)

b the length of the last piece. (2 marks)

[Total 5 marks]

4 Find $\int_0^4 \frac{1}{2x+1} dx$.

Give your answer in the form $\ln k$, where $k \in \mathbb{N}$. (5 marks)

[Total 5 marks]

5 Let **a, b** and **c** be three, 3-dimensional vectors.

State whether each expression below is a meaningful vector expression, or is meaningless.

Briefly, justify your answer if you claim an expression is meaningless.

- **i** $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$
- **ii** $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$
- **iii** $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$
- **iv** $\mathbf{a}\times(\mathbf{b}\cdot\mathbf{c})$
- **v** $(\mathbf{b}\times\mathbf{c})\mathbf{a}$
- **vi** $(\mathbf{b}\cdot\mathbf{c})\mathbf{a}$ (6 marks)

[Total 6 marks]

6 By repeated use of L'Hôpital's Rule, find $\lim_{x\to 0}\frac{\tan x - x}{x^3}$. At each stage, you should justify that any indeterminant form meets the criteria to use L'Hopial's Rule. (7 marks)

[Total 7 marks]

7 Differentiate the function $f(x)=x^4$ from first principles

(ie using $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$). (5 marks)

[Total 5 marks]

8 When the polynomial $p(x)=x^3-x^2+ax+b$ is divided by x the remainder is 9, and when $p(x)$ is divided by $(x+1)$ the remainder is 16.

a Find the values of a and b. (4 marks)

b Factorize $p(x)$ into linear factors. (4 marks)

[Total 8 marks]

9 A polynomial is defined by $p(x)=x^4+10x^3+35x^2+50x+24$.

a For $p(x)=0$, write down (i) the sum of the roots, (ii) the product of the roots. (2 marks)

Another polynomial is defined by $q(x)=p(x-1)$.

b For $q(x)=0$, find (i) the sum of the roots, (ii) the product of the roots. (5 marks)

[Total 7 marks]

Long questions

10 Let $f(x)=x^2+6x+(8+k)$.

a Find the discriminant, D, of this quadratic, in terms of k. (2 marks)

b Given that $f(x)>0$ for all values of x find the set of values that
i D **ii** k can take. In each case, you must justify your answer. (5 marks)

c Write $f(x)$ in the form $f(x)=(x+p)^2+q$, where $p\in\mathbb{Z}$ and q is an expression involving k. (2 marks)

d Explain how expression you obtained for $f(x)$ in part **c** confirms your findings in part **bii** for the range of k. (2 marks)

e If $k = 4$, write down the minimum point of the quadratic. (2 marks)

[Total 13 marks]

11 a Prove by induction that $\sum_{i=1}^{n} i \times i! = (n+1)! - 1$ for $i, n \in \mathbb{Z}^+$. (9 marks)

b By writing $i \times i!$ as $(i+1-1) \times i!$ construct a direct proof (that is, a proof not using induction) for the result given in part **a**. (4 marks)

[Total 13 marks]

12 a Solve $z^4 - 1 = 0$ for $z \in \mathbb{C}$ by factorizing the left-hand side as far as possible. Give your answers in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$ (4 marks)

b Solve $z^8 = 1$ for $z \in \mathbb{C}$ using De Moivre's theorem. Give your answers in the form $r\text{cis}\theta$ where $r \in \mathbb{R}^+, -\pi < \theta \le \pi$. (6 marks)

c Express the solution to $z^8 = 1$ which lies in the first quadrant of the Argand diagram (with $0 < \theta < \frac{\pi}{2}$) in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$ (2 marks)

d Hence, write down one of the values of $\sqrt{\text{i}}$ in the form $a + bi$, where $a, b \in \mathbb{R}$ (1 mark)

[Total 13 marks]

13 A rumour about Katie is spreading in a college with a large number of students. Let x be the proportion of the college students who have heard the rumour and let t be the time in hours, after 9.00 a.m. This situation is modelled by the differential equation $\frac{\text{d}x}{\text{d}t} = kx(1-x)$, where k is a constant.

a Use partial fractions to solve this differential equation and hence show that $\frac{x}{1-x} = A\text{e}^{kt}$, where A is a constant. (7 marks)

b At 9.00 a.m., one third of the students know about the rumour. Find the value of A. (2 marks)

c At 10.00 a.m., half of the students knew about the rumour. Find the value of e^k. (2 marks)

d Hence, find the proportion of the students who knew about the rumour at 11.00 a.m. (4 marks)

[Total 15 marks]

Paper 2

Time allowed: 2 hours

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

You need a graphic display calculator for this paper.

Short questions

1 The random variable X is normally distributed, with mean equal to 8. Given that $P(X > 7) = 0.69146$, find the value of the standard deviation of X. (6 marks)

[Total 6 marks]

2 Maria sees a huge monster. The angle of elevation from Maria's feet to the top of the monster's head is 40°. Maria runs 5 m further away from the monster, along horizontal ground. The new angle of elevation from Maria's feet to the top of the monster's head is 35°.

a Sketch a diagram to represent the information given above. (1 mark)

b Find the height of the monster. (6 marks)

[Total 7 marks]

3 An object is initially at the origin. It moves in a straight line with velocity v m/s given by $v = 5\sin\left(t^2\right)$, where t s (measures in radians) is the time which has elapsed.

a Write down the initial velocity of the object. (1 mark)

b Find the total distance that the object travels in the first 3 seconds. (3 marks)

c Find the displacement of the object from the origin when $t = 3$. (3 marks)

[Total 7 marks]

4 A very old aircraft has four engines, two on each wing. On a particular flight the probability that any engine breaks down is 0.1. All engine breakdowns are independent of each other. The aircraft will be able to complete the flight as long as there is at least one engine operating on each wing. Otherwise it will have to perform an emergency landing. Find the probability that it does not have to perform an emergency landing, giving your answer to 4 decimal places. (4 marks)

[Total 4 marks]

5 A function is given by $f(x) = e^{x^2}$.

a Calculate the area of the region bounded by $y = f(x)$, $y = 0$, $x = -1$ and $x = 1$. (3 marks)

b This region is rotated through 2π about the x-axis. Find the volume of the solid of revolution that is generated. (3 marks)

[Total 6 marks]

6 A triangle ABC has $B\hat{A}C = 40^\circ$, $AB = 5$, $BC = 4$. Find the value of AC given that area $ABC < 5$. (7 marks)

[Total 7 marks]

7 The discrete random variable X satisfies the $B\left(2n, \frac{1}{2}\right)$ distribution.

a By considering symmetry, write down what the mode is. (1 mark)

b Given that $P(X = \text{the mode}) = \frac{35}{128}$, find the value of n. (2 marks)

[Total 3 marks]

8 A forest fire is spreading out in the shape of a circle. The radius is increasing at a rate of 0.1 km per hour.

a Find the rate at which the burned area is increasing when the radius is 5 km. (4 marks)

b Find the radius when the burned area is increasing at a rate of 6 km^2 per hour. (2 marks)

[Total 6 marks]

9 Consider the system of equations

$$x + 2y + 3z = 7$$
$$4x - y - z = 6$$
$$5x + y + kz = 12\cdot$$

a Find the value of k for which there are no solutions. (4 marks)

b For $k = 1$ find the solution. (3 marks)

[Total 7 marks]

Long questions

10 a Find the acute angle, in degrees, between the vectors $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. (5 marks)

b Find the acute angle, in degrees, between the planes $\Pi_1 : x + 2y + 3z + 4 = 0$ and $\Pi_2 : 2x - y + 2z - 11 = 0$. (3 marks)

c Find the acute angle, in degrees, between the plane Π_1 and the line $L_1 : \frac{x-5}{2} = \frac{y+6}{-1} = \frac{z-5}{2}$. (3 marks)

d Find the point of intersection between the plane Π_1 and the line L_1. (5 marks)

[Total marks 16]

11 A function is defined by $f(x)=\dfrac{6x+3}{2x-10}, x \in \mathbb{R}, x \neq 5$.

a **i** Write down the equation of the vertical asymptote.

ii Write down the equation of the horizontal asymptote. (2 marks)

b **i** Write down the intercept on the y-axis.

ii Write down the intercept on the x-axis. (2 marks)

c **i** Find $f'(x)$.

ii State what your answer to part **i** tells you about the graph of $f(x)$. (4 marks)

d Sketch the graph of $f(x)$, showing the information found in **a**, **b**, and **c**. (3 marks)

e Write down the equation of the tangent to the curve at $x = 2$. (2 marks)

[Total 13 marks]

12 A very bouncy ball is dropped from a height of 2 m. After each bounce on the floor it rebounds to four-fifths of its previous height.

a Find the height that the ball bounces to after the 5th bounce. (2 marks)

b Find the least number of bounces after which the ball bounces to height less than 0.25 m. (3 marks)

c Calculate the total distance the ball travels before it comes to rest on the floor. (6 marks)

The time t that it takes for the ball to drop a distance of s on each bounce is given by $s = 5t^2$, where t is measured in seconds. For each bounce, the ball takes the same time going up as it does coming down.

d Calculate the total time between the ball being dropped and it coming to rest on the floor. (7 marks)

[Total 18 marks]

13 A continuous random variable has a probability density function defined by

$$f(x)=\begin{cases} 0 & \text{if } x<-\dfrac{1}{2} \\ a\cos(\pi x) & \text{if } -\dfrac{1}{2}\le x\le\dfrac{1}{2} \\ 0 & \text{if } x>\dfrac{1}{2}. \end{cases}$$

x is measured in radians.

a Find the exact value of a. (3 marks)

b For this distribution find (i) the mean, (ii) the variance. (4 marks)

c Given that $x > 0$ find the probability that $x < \dfrac{1}{4}$. (3 marks)

[Total 10 marks]

Paper 3

Time allowed: 1 hour

Answer all the questions.

All numerical answers must be given exactly or correct to three significant figures, unless otherwise stated in the question.

Answers should be supported by working and/or explanations. Where an answer is incorrect, some marks may be awarded for a correct method, provided this is shown clearly.

You need a graphic display calculator for this paper.

1 Consider a sequence, with first term of u_1, defined by a recurrence relation of the form $u_{n+1} = au_n + b$, where a and b are constants.

For each of the given the criteria in parts **a** and **b**,

i state the type of sequence defined by the recurrence relation

ii write down an expression for u_n in terms of u_1 and b

iii write down an expression for S_n (the sum of the first n terms) in terms of u_1 and b.

a $a = 1$ (3 marks)

b $b = 0, a \neq 1, a \neq 0$ (3 marks)

c The *Towers of Hanoi* is a famous example of a sequence defined by a recurrence relation of the form $u_{n+1} = au_n + b$. In this sequence, $u_1 = 1$ and $u_{n+1} = 2u_n + 1$.

i Find u_2, u_3, u_4, and u_5.

ii Using your answers to part **i**, suggest an explicit expression for u_n in terms of n.

iii Prove your expression from part **ii** using mathematical induction. (9 marks)

d When $a \neq 1$, the recurrence relation $u_{n+1} = au_n + b$ can be rewritten in the form $u_{n+1} + c = a(u_n + c)$ for some constant c.

i Find the constant c in terms of a and b.

A new sequence is defined by $v_n = u_n + c$.

ii Write down the recurrence relation for the sequence v_n, and state what type of sequence v_n is.

iii Hence, write down an explicit formula for v_n in terms of v_1, a and n.

iv Hence, write down an explicit formula for u_n in terms of u_1, a, b, and n. (8 marks)

e Verify that the expression you obtained for u_n in part **cii** for the Towers of Hanoi is consistent with the formula you derived for u_n in part **div**. (2 marks)

[Total 25 marks]

2 In this question you will find the value of the indefinite integral $I = \int \frac{1}{\sqrt{1+x^2}}\,dx$ using different methods.

Suppose that, first, you are asked to find the value of a definite integral. You can do this using technology.

a Write down the value of $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$. (2 marks)

Another integral, which looks similar to I, can be found using a trigonometric substitution.

b Write down $\int \frac{1}{\sqrt{1-x^2}}\,dx$. (1 mark)

c Show how the answer in (b) can be proved by applying the substitution $x = \sin\theta \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right)$, to $\int \frac{1}{\sqrt{1-x^2}}\,dx$. (4 marks)

To find the integral I by substitution, we are going to first define two new functions.

$$\sinh x = \frac{e^x - e^{-x}}{2},\ \cosh x = \frac{e^x + e^{-x}}{2}.$$

d Show that $(\cosh x)^2 - (\sinh x)^2 = 1$. (2 marks)

e Find the derivative of **i** $\sinh x$ **ii** $\cosh x$. give your answers in terms of $\sinh x$ and $\cosh x$. (4 marks)

The inverse function $\sinh^{-1} x$ is denoted by arsinh x.

f By solving a quadratic in e^y, show and justify that $\text{arsinh}\, x = \ln\left(x + \sqrt{x^2+1}\right)$. (5 marks)

g Find the indefinite integral I by applying the substitution $x = \sinh u$. (4 marks)

h Hence, find the exact value of $\int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$ and verify that it agrees with the answer from part (a). (2 marks)

We will now attempt to see if we can find the integral I just by using the usual trigonometrical functions.

i Apply the substitution $x = \tan\theta$, $\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$, to show that I can be changed to $\int \sec\theta\,d\theta$. (2 marks)

j Multiply the sec θ by 1, written as $\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$, to obtain the answer to $\int \sec\theta\,d\theta$ in terms of θ. (2 marks)

k Now express the answer found in part **j** in terms of x and verify this is consistent with the expression you obtained for I in part **g**. (2 marks)

You will now attempt to find the integral I without using a substitution.

Let $f(x) = x + \sqrt{1+x^2}$.

l Find $f'(x)$. (2 marks)

m Multiply $\frac{1}{\sqrt{1+x^2}}$ by 1, written as $\frac{f'(x)}{f'(x)}$ using your expression for $f(x)$ from part **l**. Hence simplify the expression for I into an expression involving only $f(x)$ and $f'(x)$. In this way, find the integral I directly. (3 marks)

[Total 35 marks]

d $\frac{12}{37}\int_0^m (10x^2 - x^3)\,dx = \frac{1}{2}$ (1 mark)

$\frac{12}{37}\left[\frac{10x^3}{3} - \frac{x^4}{4}\right]_0^m = \frac{1}{2}$ (1 mark)

$\frac{12}{37}\left(\frac{10m^3}{3} - \frac{m^4}{4}\right) = \frac{1}{2}$ (1 mark)

$\frac{10m^3}{3} - \frac{m^4}{4} = \frac{37}{24}$

$80m^3 - 6m^4 = 37$

GDC $\Rightarrow m = 0.789$ (2 marks)

18 a $P\left(Z < \frac{110-\mu}{\sigma}\right) = 0.10$

$\Rightarrow \frac{110-\mu}{\sigma} = -1.282$ (2 marks)

$P\left(Z > \frac{130-\mu}{\sigma}\right) = 0.45$

$\Rightarrow \frac{130-\mu}{\sigma} = 0.126$ (2 marks)

Attempt to solve simultaneously: (1 mark)

$\mu = 128$ (1 mark)

$\sigma = 14.2$ (1 mark)

b $P(|X-\mu|) < 0.22$

$0.5 - \frac{0.22}{2} = 0.39$ (1 mark)

$P(X < a) = 0.39 \Rightarrow a = 124.2$ (1 mark)

$P(X > b) = 0.39 \Rightarrow b = 132.2$ (1 mark)

So $124.2 < X < 132.2$ (1 mark)

19 a $k\int_0^{\frac{\pi}{2}} \cos x\,dx = 1$ (1 mark)

$k[\sin x]_0^{\frac{\pi}{2}} = 1$ (1 mark)

$k\left(\sin\frac{\pi}{2} - \sin 0\right) = 1$ (1 mark)

$k(1 - 0) = 1$ (1 mark)

$k = 1$

b

$\int_0^{\frac{\pi}{2}} x\cos x\,dx = [x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,dx$ (2 marks)

$\int_0^{\frac{\pi}{2}} x\cos x\,dx = [x\sin x]_0^{\frac{\pi}{2}} + [\cos x]_0^{\frac{\pi}{2}}$ (1 mark)

$= \left(\frac{\pi}{2} - 0\right) + (0 - 1)$ (1 mark)

$= \frac{\pi}{2} - 1$ (1 mark)

$\int_0^{\frac{\pi}{2}} x^2\cos x\,dx = [x^2\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 2x\sin x\,dx$ (2 marks)

$= [x^2\sin x]_0^{\frac{\pi}{2}} - 2[-x\cos x + \sin x]_0^{\frac{\pi}{2}}$ (1 mark)

$= [x^2\sin x - 2\sin x + 2x\cos x]_0^{\frac{\pi}{2}}$

$= \frac{\pi^2}{4} - 2$ (1 mark)

$\text{Var}(X) = \int_0^{\frac{\pi}{2}} x^2\cos x\,dx - \left(\int_0^{\frac{\pi}{2}} x\cos x\,dx\right)^2$ (1 mark)

$= \frac{\pi^2}{4} - 2 - \left(\frac{\pi}{2} - 1\right)^2$ (1 mark)

$= \frac{\pi^2}{4} - 2 - \left(\frac{\pi^2}{4} - \pi + 1\right)$ (1 mark)

$= \pi - 3$

Paper 1

1 a $\frac{x}{2} - \frac{\sin(2x)}{4} + c$ **b** $\frac{\pi}{2}$

2 a $p + q$ **b** $2p - q$

c $\frac{1}{2}p$ **d** $2 + q$

3 a 20 cm **b** 38 cm

4 ln 3

5 i Meaningful

ii Meaningless; $(\mathbf{b}\cdot\mathbf{c})$ is not a vector, so you cannot take the scalar product of **a** with it.

iii Meaningful

iv Meaningless; $(\mathbf{b}\cdot\mathbf{c})$ is not a vector, so you cannot take vector product of **a** with it.

v Meaningless; both **a** and $(\mathbf{b}\cdot\mathbf{c})$ are vectors, so you cannot perform scalar multiplication.

vi Meaningful

6 $\frac{1}{3}$

7 $4x^3$

8 a $a = -9$, $b = 9$

b $(x - 1)(x - 3)(x + 3)$

9 a i -10 **ii** 24

b i New sum -6 **ii** 0

10 a $D = 36 - 4(8 + k) = 4 - 4k$

b i Concave-up quadratic, always positive, so no roots so $D < 0$

ii $D = 4 - 4k < 0 \Rightarrow k > 1$

c $f(x) = (x + 3)^2 + (k - 1)$

d $f(x) > 0 \Rightarrow (x + 3)^2 + (k - 1) > 0 \Rightarrow k - 1 > 0$ since $(x + 3)^2 \geq 0$ for all x. Hence $k > 1$ as found in **b** part **ii**

e $(-3,3)$

11 a

Let $P(n)$ be the statement

$\sum_{i=1}^{n} i \times i! = (n+1)! - 1$

LHS of $P(1)$ is 1. RHS of $P(1)$ is $2! - 1 = 1$. So $P(1)$ is true.

Assume $P(k)$ is true and attempt to prove $P(k + 1)$

$\sum_{i=1}^{k+1} i \times i! = \sum_{i=1}^{k} i \times i! + (k+1)\times(k+1)!$

$= (k+1)! - 1 + (k+1)\times(k+1)!$

$= (k + 2)\times(k + 1)! - 1$

$= (k + 2)! - 1$ as required

Since $P(1)$ is true and $P(k)$ true implies $P(k+1)$ true, by the principle of mathematical induction the statement has been proved for all $n \in \mathbb{Z}^+$.

b

$$\sum_{i=1}^{n}(i+1-1)\times i! = \sum_{i=1}^{n}(i+1)\times i! - i!$$

$$= \sum_{i=1}^{n}(i+1)! - \sum_{i=1}^{n} i! = (n+1)! - 1$$

as almost all the terms cancel.

12 **a** $z = 1, -1, \text{i}, \text{ or } -\boldsymbol{i}$

b $z = 1\text{cis}0,\ 1\text{cis}\dfrac{\pi}{4},\ 1\text{cis}\dfrac{\pi}{2},$

$1\text{cis}\dfrac{3\pi}{4},\ 1\text{cis}\pi,\ 1\text{cis}-\dfrac{\pi}{4},$

$1\text{cis}-\dfrac{\pi}{2},\ 1\text{cis}-\dfrac{3\pi}{4}$

c $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\text{i}$ **d** $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}\text{i}$

13 **a** $\displaystyle\int \frac{1}{x(1-x)}\,\text{d}x = \int k\,\text{d}t$

$\displaystyle\int \frac{1}{x} + \frac{1}{1-x}\,\text{d}x = kt + c$

$\ln x - \ln(1-x) = kt + c$

$\ln\left(\dfrac{x}{1-x}\right) = kt + c$

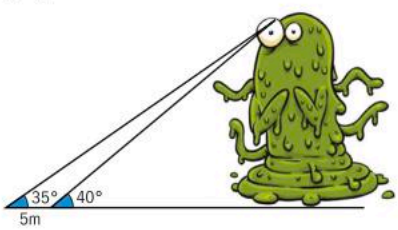
$\dfrac{x}{1-x} = \text{e}^{kt+c} = \text{e}^{c}\text{e}^{kt} = A\text{e}^{kt}$

b $\dfrac{1}{2}$ **c** 2 **d** $\dfrac{2}{3}$

Paper 2

1 **a** 2

2 **a**



b 21.2 m

3 **a** 0 **b** 8.51 m

c +3.87 m (3 s.f.)

4 0.9801

5 **a** 2.93 (3 s.f.) **b** 14.9 (3 s.f.)

6 1.45 (3 s.f.)

7 **a** $X = n$ **b** 4

8 **a** 3.14 km^2/h

b 9.55 m (3 s.f.)

9 **a** 2 **b** $x = 2, y = 1, z = 1$

10 **a** 57.7° (3s.f.)

b 57.7° (3s.f.)

c 32.3° (3s.f.)

d $(1, -4, 1)$

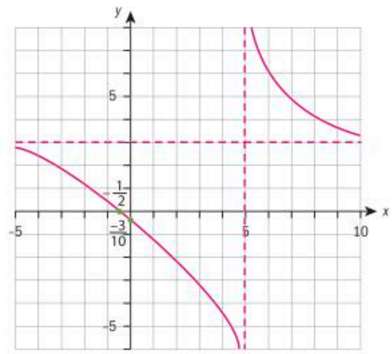
11 **a** **i** $x = 5$ **ii** $y = 3$

b **i** $\left(0, \frac{-3}{10}\right)$ **ii** $\left(-\frac{1}{2}, 0\right)$

c **i** $\dfrac{-66}{(2x-10)^2}$

ii Graph is always decreasing

d



e $y = -1.83x + 1.17$ (3 s.f.)

12 **a** 0.655 m (3 s.f.) **b** 10

c 18 m **d** 11.3 s

13 **a** $\dfrac{\pi}{2}$

b **i** 0 **ii** 0.0474 (3s.f.)

c 0.707 (3 s.f.)

Paper 3

1 **a** **i** Arithmetic progression

ii $u_n = u_1 + (n-1)b$

iii $S_n = \dfrac{n}{2}\left(2u_1 + (n-1)b\right)$

b **i** Geometric progression

ii $u_n = u_1 a^{n-1}$

iii $S_n = u_1\dfrac{(a^n - 1)}{(a-1)}$

c **i** $u_2 = 3,\ u_3 = 7,\ u_4 = 15,\ u_5 = 31$

ii $u_n = 2^n - 1$

iii Let $\dfrac{\pi}{3}(n)$ be the statement $u_n = 2^n - 1$

$\text{P}(1) = 2^1 - 1 = 1$ which is true, since the question gives $u_1 = 1$

Assume the result for $\text{P}(k)$ and attempt to prove for $\text{P}(k+1)$

By the recurrence relation, $u_{k+1} = 2u_k + 1$ $= 2(2^k - 1) + 1 = 2^{k+1} - 1$ as required

Since P(1) is true and P(k) true implies P($k+1$) is also true. Hence, by the principle of mathematical induction, P(n) is true for all $n \in \mathbb{Z}^+$.

d **i** $c = \dfrac{b}{a-1}$

ii $v_{n+1} = av_n$, geometric progression

iii $v_n = a^{n-1}v_1$

iv $u_n = 2^n - 1$

2 **a** 0.881 (3s.f.)

b $\arcsin x + c$

c Let $x = \sin\theta$, integral becomes $\dfrac{1}{\sqrt{1-\sin^2\theta}}\cos\theta\,\text{d}\theta$

$= \displaystyle\int 1\,\text{d}\theta = \theta + c = \arcsin x + c$

d $(\cosh x)^2 - (\sinh x)^2$

$= \dfrac{\text{e}^{2x} + 2 + \text{e}^{-2x}}{4} - \dfrac{\text{e}^{2x} - 2 + \text{e}^{-2x}}{4}$

$= \dfrac{4}{4} = 1$

e **i** $\dfrac{\text{d}\sinh x}{\text{d}x} = \dfrac{\text{e}^{x} + \text{e}^{-x}}{2} = \cosh x$

ii $\dfrac{\text{d}\cosh x}{\text{d}x} = \dfrac{\text{e}^{x} - \text{e}^{-x}}{2} = \sinh x$

f Let $y = \sinh x = \frac{e^x - e^{-x}}{2}$.

Then inverse function is given by making y the subject of $x = \frac{e^y - e^{-y}}{2}$:

$2x = e_y - e^{-y} \Rightarrow 2xe_y$

$= (e_y)^2 - 1 \Rightarrow (e_y)^2 - 2x(e_y)$

$- 1 = 0$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

g $\ln\left(x + \sqrt{x^2+1}\right)$

h $\ln\left(1+\sqrt{2}\right)$

i $I = \int \frac{1}{\sqrt{1+\tan^2\theta}} \sec^2\theta \, d\theta$

$= \int \sec\theta \, d\theta$

j $\ln(\tan\theta + \sec\theta) + c$

k $\tan\theta = x, \quad 1 + \tan^2\theta = \sec^2\theta,$

$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+x^2}$

So $\ln(\tan\theta + \sec\theta) + c$

$= \ln\left(x + \sqrt{1+x^2}\right) + c$

as before.

l $1 + x\left(1+x^2\right)^{-\frac{1}{2}}$

m $\int \frac{f'(x)}{\sqrt{1+x^2}+x} dx, \int \frac{f'(x)}{f(x)} dx,$

$\ln\left(x + \sqrt{1+x^2}\right) + c$