15 Practice paper 1

Time allowed: 2 hours

- Answer all the questions
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Practice exam papers on CD: *IB examination papers include spaces for you to write your answers. There is a version of this practice paper with space for you to write your answers on the CD. You can also find an additional set of papers for further practice.*

Worked solutions on CD: *Detailed worked solutions for this practice paper are given as a PowerPoint presentation on the CD.*

SECTION A

1 Given that $4\ln 2 - 3\ln 4 + \ln k = 0$, find the value of k. *[4 marks]*

2 **a** Show that $p(x) = 2x^3 - 3x^2 + 8x + 5$ is divisible by $2x + 1$. *[4 marks]*

b Hence find all the zeros of $p(x)$. *[4 marks]*

3 The sum of the first two terms of a geometric sequence is $\frac{8}{9}$ and the sum of the first three terms is $\frac{26}{27}$.
Find possible values of the first term and the sum of all its terms. *[8 marks]*

4 Consider the events A and B such that $\text{P}(A) = 0.3$ and $\text{P}(B) = 0.2$.

Given that $\text{P}(A \cup B) = 3\text{P}(A \cap B)$, find $\text{P}(A|B)$ and $\text{P}(A \cap B')$. *[6 marks]*

5 Show that for any complex number z,

a $z + z^* = 2\text{Re}(z)$ **b** $z - z^* = 2\text{i}\,\text{Im}(z)$ **c** $\text{Re}(z) \le |z|$ *[6 marks]*

6 Find a vector equation of the line of intersection of the planes with equations $x + 2y - z = 5$ and $-3x - y + z = 1$. *[6 marks]*

7 A curve is defined by the equation $x^2 + 4y^2 - 2x + 16y + 13 = 0$.
Find the coordinates of the points on the curve where the tangent to the curve is parallel to the x-axis. *[6 marks]*

8 Use integration by parts to find the rational values of a and b such that

$$\int_1^9 \sqrt{x}\ln x \, dx = a\ln 3 + b.$$ *[5 marks]*

9 Consider the function f defined by $f(x) = \begin{cases} 0.1x & \text{if } 0 \le x \le 1 \\ 0.1(5x-4) & \text{if } 1 < x \le 2 \\ ax + b & \text{if } 2 < x \le c \\ 0 & \text{otherwise} \end{cases}$

a Given that f is a continuous pdf of a variable X, find the values of a, b and c.

b Hence state the value of the mode of X. *[7 marks]*

10 Consider the function defined by $f(x) = 12\sin x - 5\cos x$.
Find the range of f. *[4 marks]*

SECTION B

11 Consider the lines

$$L_1: \frac{x-2}{1} = \frac{y-1}{2} = \frac{z}{3} \text{ and } L_2: \frac{x-1}{4} = \frac{y-2}{1} = \frac{z-3}{-2}$$

a Show that the lines intersect and find their point of intersection. *[5 marks]*

b Hence find the equation of the plane that contains both lines. *[4 marks]*

c Show that the point A(1, −1, 0) does not lie on the plane π. *[2 marks]*

d Write down the equation of the line L_3 perpendicular to the plane π that contains the point A. *[1 mark]*

e Hence find the distance from A to the plane π. *[7 marks]*

12 a Prove by mathematical induction that

$$0^2 + 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}, \text{ for all } n \in \mathbb{N}.$$ *[7 marks]*

b Hence find an expression for $3^2 + 6^2 + \ldots + (3n)^2$. *[4 marks]*

c Given that $A_n = 1^2 + 4^2 + \ldots + (3n-2)^2$ and $B_n = 2^2 + 5^2 + 8^2 + \ldots + (3n-1)^2$, prove that $A_n + B_n = 6n^3 - n$ and $A_n - B_n = -3n^2$
Hence find A_n and B_n in terms of n. *[9 marks]*

13 Let $f : x \to e^{\cos x}$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$

a State with a reason whether or not the function f is even. *[2 marks]*

b Find $f'(x)$. *[2 marks]*

c Given that the graph of f has a maximum point, find its coordinates. *[5 marks]*

d Show there is a point of inflexion on the graph of f, for $0 < x < \frac{\pi}{2}$ and find its coordinates. *[6 marks]*

e Sketch the graph of f. *[1 mark]*

f A rectangle is drawn so that its lower vertices are on the x-axis and its upper vertices are on the curve $y = e^{\cos x}$ where $-\frac{\pi}{2} < x < \frac{\pi}{2}$

i Write down an expression for the area of the rectangle. *[1 mark]*

ii Show that there is a positive value $x = a$ for which the area of the rectangle reaches a maximum.

Hence show that its value is given by $2ae^{\frac{\sqrt{a^2-1}}{a}}$ *[4 marks]*

Use the mark scheme in the Answer section at the back of this book to mark your answers to this practice paper.

Practice paper 2

Time allowed: 2 hours

- Answer all the questions
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SECTION A

1 A rental agreement says that the yearly rent on an office shall increase by €600 each year. In the fifth year of the agreement the rent was €12200.

Find:

a the rent paid in the first year *[2 marks]*

b the total amount paid in the first 5 year. *[2 marks]*

c the first year that the annual rent will exceed €15000 *[3 marks]*

2 The complex numbers w and z are such that $w = \frac{az+b}{z+c}$, $a, b, c \in \mathbb{R}$.

a Given that $w = 3\text{i}$ when $z = -3\text{i}$ and $w = 1 - 4\text{i}$ when $z = 1 + 4\text{i}$, show that $b = 9$ and find the values of a and c. *[5 marks]*

b Hence, show that if $\text{Re } z = 4$ then $\text{Re } w = 4$. *[2 marks]*

3 A random variable X is normally distributed with mean and variance both equal to a. Given that $\text{P}(X < 2) = 0.3$, find the value of a. *[4 marks]*

4 Consider the function defined by $f(x) = \frac{1-x^2}{x}$

a Find the first and second derivatives and hence show that the graph of f has no maxima, no minima or points of inflexion. *[5 marks]*

b Hence sketch the graph of f, showing clearly the intercepts and any asymptotes. *[3 marks]*

5 Given the points A(1, −3, −1) and B(−5, 2, −4), find the coordinates of the point P that lies on the segment [AB] and is such that AP : PB = 1 : 2. *[6 marks]*

6 Given that $(1 + x)^5 (1 + ax)^6 \equiv 1 + bx + 10x^2 + \ldots + a^6x^{11}$, find the values of integers a and b. *[6 marks]*

7 Use substitution to find $\int \frac{e^{2x}}{4+e^{4x}}\,dx$ *[6 marks]*

8 The hands of a clock are 20 cm and 15 cm long. Let θ be the angle between the hands at any time t between 14:45 and 15:15.

a Express the distance d between the tips of the hands at the time t in terms of θ. *[2 marks]*

b Assuming that the movement of the hands of the clock is continuous, find the rate of change of θ in radians per minute. Hence find the rate of change of d at three o'clock, in cm per minute. *[6 marks]*

9 Consider the sequence (u_n) defined by $\begin{cases} u_1 = 2 \\ u_{n+1} = \frac{u_n + 1}{3}, \ n \in \mathbb{Z}^+ \end{cases}$

Investigate the numerical behavior of the terms of the sequence and deduce that $u_n = \frac{3^{2-n}+1}{2}$ *[8 marks]*

SECTION B

10 A box contains a very large number of ribbons of which 25% are red, 30% are white and the rest are blue. Twelve ribbons are selected at random from the box.

a Find the expected number of red ribbons selected. *[1 mark]*

b Find the probability that exactly six of these ribbons are blue. *[3 marks]*

c Find the probability that at least two of these ribbons are blue. *[2 marks]*

d Find the most likely number of white ribbons selected. State any assumptions you have made about the probability of selecting a white ribbon. *[4 marks]*

There are two other boxes with large number of colored ribbons: one where 25% of the ribbons are blue, 25% white and the others red, and another box where 50% of the ribbons are white, 20% blue and the rest red.

e Kathy picks a box at random and, without looking, takes a ribbon out. If she takes a white ribbon out, what is the probability that this ribbon was taken from the first box? State any assumptions made. *[4 marks]*

11 The points A(1, 2, −3), B(2, −1, 0) and C(−1, 0, 3) are given.

a Find the vector equation of a line (AB) that passes through the points A and B. *[3 marks]*

b Find the midpoint M of the line segment [AB]. Hence show that the equation of the plane α perpendicular to [AB] that bisects the line segment [AB] is

$$2x - 6y + 6z = -9$$ *[3 marks]*

c Show that the equation of the plane β perpendicular to [AC] that bisects the line segment [AC] is

$$x + y - 3z = 1.$$ *[3 marks]*

d Find the angle between the planes α and β.
The plane γ perpendicular to [BC] that bisects the line segment [BC] has an equation

$$6x - 2y - 6z = -5.$$ *[4 marks]*

e Show that the planes α, β and γ intersect and find the vector equation of the line of intersection. *[4 marks]*

f Consider the plane π defined by the equation $x + y + z = 0$
Find the coordinates of the point P on the plane π that is at the same distance from the points A, B and C. *[6 marks]*

12 Let $f(x) = \cos(2x) + 1$ and $g(x) = \frac{e^x + e^{-x}}{2}$

a Show that both functions are even. *[3 marks]*

b Find the derivatives $f'(x)$ and $g'(x)$. *[4 marks]*

c Show that both derivative functions are odd. *[2 marks]*

d Sketch the curves $y = f(x)$ and $y = g(x)$ and find their points of intersection. *[4 marks]*

e Show that the tangents to the curves $y = f(x)$ and $y = g(x)$ at the point of intersection in the first quadrant have equations $y = -1.95x + 2.53$ and $y = 0.719x + 0.751$ respectively. *[3 marks]*

f Find the area of the region enclosed by all four tangents to the curves $y = f(x)$ and $y = g(x)$ at the points of intersection. *[3 marks]*

g The region enclosed by the curves $y = f(x)$ and $y = g(x)$ is rotated by 2π about the x-axis. Find the volume of revolution generated. *[4 marks]*

Accuracy of answers

Since 2011, the accuracy penalty per paper no longer applies. Read the instruction on the cover page of the exam carefully: you may be told to use a minimum number of significant figures in questions where the rounding criterion is not given. In any case, you should also look carefully and see if your answer makes sense in the context of the question.

Practice paper 1

SECTION A

1 $4\ln 2 - 3\ln 4 + \ln k = 0 \Rightarrow \ln k = -4\ln 2 + 3\ln 4$ M1

$\ln k = \ln 2^{-4} + \ln 4^3$ A1

$\ln k = \ln 2^{-4} \times 4^3$ (A1)

$k = 2^{-4} \times 4^3 = 4$ A1 [4 marks]

2 a

$$\begin{array}{r} x^2 - 2x + 5 \\ 2x+1 \overline{)\, 2x^3 - 3x^2 + 8x + 5} \\ \underline{-2x^3 - x^2 } \\ -4x^2 + 8x + 5 \\ \underline{4x^2 + 2x } \\ 10x + 5 \\ \underline{-10x - 5} \\ 0 \end{array}$$

M1A2

as the remainder is 0, $p(x)$ is divisible by $2x + 1$.

Alternative method:

$\left(-\frac{1}{2}\right)$	2	−3	8	5
		+	+	+
		−1	2	−5
	2	−4	10	0

Use synthetic division for the value of $-\frac{1}{2}$.

R1

b $p(x) = (2x+1)(x^2 - 2x + 5)$ (A1)

$p(x) = 0 \Rightarrow 2x + 1 = 0$ or $x^2 - 2x + 5 = 0$ (M1)

$x = -\frac{1}{2}$ or $x = 1 \pm 2\mathrm{i}$ A1A1 [8 marks]

3 $a_1 + a_2 = \frac{8}{9} \Rightarrow a + ar = \frac{8}{9} \Rightarrow a(1+r) = \frac{8}{9}$ M1

$\Rightarrow a = \frac{8}{9(r+1)}$ A1

$\underbrace{a_1 + a_2}_{\frac{8}{9}} + a_3 = \frac{26}{27} \Rightarrow \frac{8}{9} + ar^2 = \frac{26}{27}$ M1

$\Rightarrow \frac{8}{9} + \frac{8}{9(r+1)} r^2 = \frac{26}{27} \Rightarrow 12r^2 = r + 1$ M1

$\Rightarrow r = \frac{1}{3}$ or $r = -\frac{1}{4}$ A1

$a = \frac{2}{3}$ or $a = \frac{32}{27}$ A1

$S = \frac{\frac{2}{3}}{1-\frac{1}{3}} = 1$ or $S = \frac{\frac{32}{27}}{1+\frac{1}{4}} = \frac{128}{135}$ A1A1 [8 marks]

4 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and

$3P(A \cap B) = 0.3 + 0.2 - P(A \cap B)$ M1

$P(A \cap B) = \frac{1}{8}$ (0.125) $\left(\text{accept } \frac{5}{40}\right)$ A1

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{8}}{0.2} = \frac{5}{8}$ M1A1

$P(A \cap B') = P(A) - P(A \cap B) = 0.3 - 0.125 = 0.175$ M1A1 [6 marks]

5 Let $z = a + ib$ where $a = \text{Re}(z)$ and $b = \text{Im}(z)$,

a $z + z^* = (a + ib) + (a - ib) = 2a$ M1A1

$= 2\,\text{Re}(z)$ AG

b $z - z^* = (a + ib) - (a - ib) = 2bi$ M1A1

$= 2i\,\text{Im}(z)$ AG

c $\text{Re}(z) = a \le |a| = \sqrt{a^2}$ R1

$\le \sqrt{a^2 + b^2} = |z$ A1

Therefore $\text{Re}(z) \le |z|$ AG [6 marks]

6 $\begin{cases} x + 2y - z = 5 \\ -3x - y + z = 1 \end{cases}$ eliminate z first, M1A1

$\Rightarrow -2x + y = 6 \Rightarrow y = 2x + 6$, express all in terms of x. A1

$z = 1 + 3x + y \Rightarrow z = 5x - 5$ A1

$r = \begin{pmatrix} 0 \\ -6 \\ -5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, t \in \mathbb{Z}$ M1A1

7 $x^2 + 4y^2 - 2x + 16y + 13 = 0 \Rightarrow 2x + 8y\frac{dy}{dx} - 2 + 16\frac{dy}{dx} = 0$ M1A1

$\Rightarrow \frac{dy}{dx} = \frac{2 - 2x}{8y + 16}$ $\left(= \frac{1 - x}{4y + 8}\right)$ A1

$\frac{dy}{dx} = 0 \Rightarrow x = 1$

$1^2 + 4y^2 - 2 + 16y + 13 = 0 \Rightarrow y^2 + 4y + 3 = 0$ M1

$\Rightarrow y = -3$ or $y = -1$ M1

So the points are $(1, -3)$ and $(1, -1)$ A1 [6 marks]

8 $\int_1^9 \sqrt{x} \ln x \, dx = \left[\frac{2}{3}x\sqrt{x}\ln x\right]_1^9 - \frac{2}{3}\int_1^9 \sqrt{x}\,dx$ M1A1

$= \left[\frac{2}{3}x\sqrt{x}\ln x\right]_1^9 - \frac{2}{3}\left[\frac{2}{3}x\sqrt{x}\right]_1^9 = 36\ln 3 - \frac{104}{9}$ A1

$\int_1^9 \sqrt{x} \ln x \, dx = a\ln 3 + b \Rightarrow a = 36$ and $b = -11\frac{5}{9}$ A1A1 [5 marks]

9 a f is a continuous PDF of X $\Rightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$ M1

$\Rightarrow 0.6 = 2a + b$ A1

$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) \Rightarrow ac + b = 0$ A1

The area under the graph is 1 M1

$0.05 + \frac{0.1+0.6}{2} \times 1 + \frac{(2a+b)+(ac+b)}{2} \times (c-2) = 1$

$\Rightarrow c = 4$ A1

Solve simultaneously to obtain

$a = -0.3$ and $b = 1.2$ A1

b 2 A1 [7 marks]

10 $f(x) = 12\sin x - 5\cos x \Rightarrow f(x) = 13\left(\underbrace{\frac{12}{13}}_{\cos\alpha}\sin x - \underbrace{\frac{5}{13}}_{\sin\alpha}\cos x\right)$ M1A1

$\Rightarrow f(x) = 13\sin(x - \alpha)$ where $\alpha = \arctan\frac{5}{12}$ (A1)

The range of f is [–13, 13] A1 [4 marks]

SECTION B

11 a $x - 2 = \frac{y-1}{2} \Rightarrow 2x - y = 3$ and $\frac{x-1}{4} = y - 2 \Rightarrow x - 4y = -7$ M1

$\Rightarrow x = \frac{19}{7}$ and $y = \frac{17}{7}$ A1A1

$z = \frac{15}{7}$ A1

Therefore, as the system of equations has unique solution R1

The lines meet at $\left(\frac{19}{7}, \frac{17}{7}, \frac{15}{7}\right)$

b $\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4\\1\\-2\end{pmatrix} = \begin{pmatrix}-7\\14\\-7\end{pmatrix}$ or $7\begin{pmatrix}-1\\2\\-1\end{pmatrix}$ M1A1

$\begin{pmatrix}-1\\2\\-1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\2\\-1\end{pmatrix} \cdot \begin{pmatrix}\frac{19}{7}\\\frac{17}{7}\\\frac{15}{7}\end{pmatrix} \Rightarrow -x + 2y - z = 0$ (or $x - 2y + z = 0$) M1A1

c As $-1\times 1 + 2\times(-1) - 1\times 0 \neq 0$ M1R1

A(1, –1, 0) does not lie on the plane π AG

d $\frac{x-1}{-1} = \frac{y+1}{2} = \frac{z}{-1}$ (or equivalent) A1

e $z=t \Rightarrow x=t+1$ and $y=-2t-1$ M1A1

$-x+2y-z=0 \Rightarrow -(t+1)+2(-2t-1)-t=0$ M1A1

$\Rightarrow t=-\frac{1}{2}$ A1

The intersection point of L_3 with the plane is $\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$

So the distance from A to the plane is

$\sqrt{\left(1-\frac{1}{2}\right)^2+(-1-0)^2+\left(0+\frac{1}{2}\right)^2}=\frac{\sqrt{6}}{2}$ M1A1 [19 marks]

12 a Let $P(n): 0^2+1^2+2^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$ where $n \in \mathbb{N}$.

Verify that $P(0)$ is a true statement:

$P(0): 0^2=\frac{0(0+1)(2\times 0+1)}{6}$ (verified true) A1

Next, assume the truth of the proposition for a particular value of n, say k:

$P(k): 0^2+1^2+2^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}$ (assumed true) M1

Consider the proposition for the next value of n, ie, $n = k+1$:

$P(k+1): 0^2+1^2+2^2+\ldots+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$

(under consideration) M1

As $0^2+1^2+2^2+\ldots+k^2+(k+1)^2$

$=\frac{k(k+1)(2k+1)}{6}+(k+1)^2$ (using the induction hypothesis) A1

$=(k+1)\left(\frac{k(2k+1)}{6}+(k+1)\right)$ M1

$=(k+1)\left(\frac{2k^2+7k+6}{6}\right)$ A1

$=\frac{(k+1)(k+2)(2k+3)}{6}$ (QED)

We have shown that $P(k)$ true $\Rightarrow P(k+1)$ true and, as we had established that $P(0)$ is true, by the principle of mathematical induction, we can conclude that $P(n)$ is true for any $n \in \mathbb{N}$. R1

b $3^2+6^2+\ldots+(3n)^2=9\times 1^2+9\times 2^2+\ldots+9\times n^2$ M1

$=9\times\left(1^2+2^2+\ldots+n^2\right)$ (A1)

$=9\left(\frac{n(n+1)(2n+1)}{6}\right)$ M1

$=\frac{3n(n+1)(2n+1)}{2}$ A1

c $A_n + B_n = \left(1^2 + 4^2 + \ldots + (3n-2)^2\right) + \left(2^2 + 5^2 + 8^2 + \ldots + (3n-1)^2\right)$

$= \left(1^2 + 2^2 + 4^2 + 5^2 + \ldots + (3n-2)^2 + (3n-1)^2\right)$ M1

$= \left(0^2 + 1^2 + 2^2 + 3^2 \ldots + (3n)^2\right) - \left(3^2 + 6^2 + \ldots + (3n)^2\right)$ A1

$= \frac{3n(3n+1)(6n+1)}{6} - \frac{3n(n+1)(2n+1)}{2}$ M1

$= \frac{3n(18n^2 + 9n + 1 - 3n^2 - 9n - 3)}{6} = 6n^3 - n$ A1AG

$A_n - B_n = \left(1^2 + 4^2 + \ldots + (3n-2)^2\right) - \left(2^2 + 5^2 + 8^2 + \ldots + (3n-1)^2\right)$

$= (1^2 - 2^2) + (4^2 - 5^2) + \ldots + \left((3n-2)^2 - (3n-1)^2\right)$ M1

$= -3 - 9 - \ldots - (6n-3)$

$= \frac{-3 + (-6n+3)}{2} \times n$ M1A1

$= -3n^2$ AG

Solve simultaneously $A_n + B_n = 6n^3 - n$ and $A_n - B_n = -3n^2$ M1

$A_n = \frac{6n^3 - 3n^2 - n}{2}$ and $B_n = \frac{6n^3 + 3n^2 - n}{2}$ A1 [20 marks]

13 Let $f: x \to e^{\cos x}$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$

a As $\cos(-x) = \cos x$ R1

$f(-x) = e^{\cos(-x)} = e^{\cos x} = f(x)$ A1

Therefore f is even. AG

b $f'(x) = -\sin x\ e^{\cos x}$ M1A1

c $f'(x) = 0 \Rightarrow \sin x = 0$ M1A1

$\Rightarrow x = 0$ (as $-\frac{\pi}{2} < x < \frac{\pi}{2}$) A1

$f'(0^-) > 0$ and $f'(0^+) < 0$ R1

f has a maximum at (0, e) A1

d $f''(x) = -\cos x\ e^{\cos x} + \sin^2 x\ e^{\cos x}$ M1A1

$= \left(-\cos x + \sin^2 x\right) e^{\cos x}$

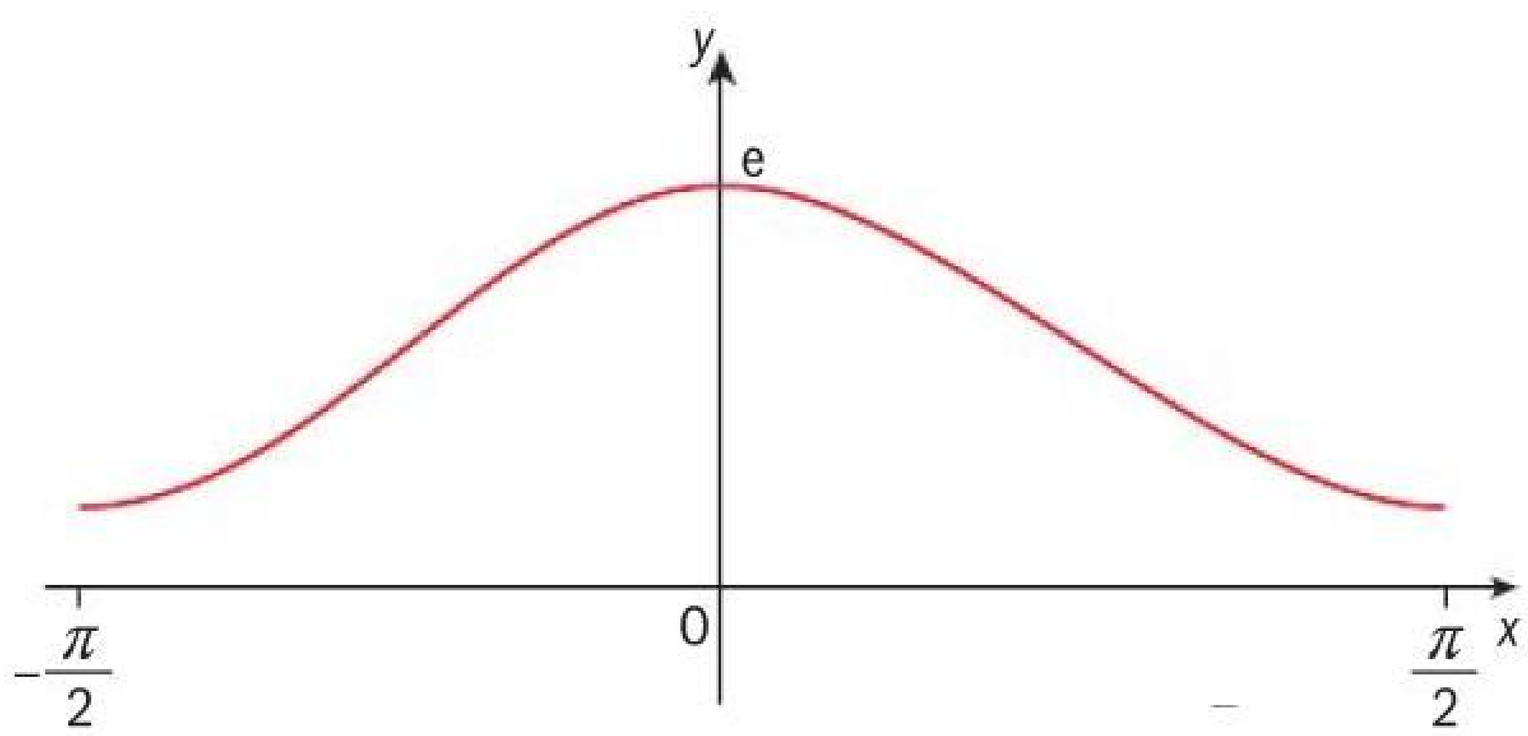
$f''(x) = 0 \Rightarrow -\cos x + \sin^2 x = 0$ M1

$\Rightarrow -\cos x + 1 - \cos^2 x = 0$ (or $\cos^2 x + \cos x - 1 = 0$) M1

$\Rightarrow \cos x = -\frac{1+\sqrt{5}}{2}$ (as $\cos x > 0$) A1

The inflexion point is arccos $\left(\frac{-1+\sqrt{5}}{2}, e^{\frac{-1+\sqrt{5}}{2}}\right)$ A1

e A1 (1)



f i $A(x)=2x\ \mathrm{e}^{\cos x}$ A1

ii $A'(x)=2\ \mathrm{e}^{\cos x}-2x\sin x\ \mathrm{e}^{\cos x}$ M1A1 (1)

$=2(1-x\sin x)\ \mathrm{e}^{\cos x}$

$A'(x)=0 \Leftrightarrow 1-x\sin x=0$ and this equation must have a zero in $\left[0,\frac{\pi}{2}\right]$ because $g(x)=1-x\sin x$ is continuous and changes sign in this interval R1

$A'(a)=0 \Rightarrow \sin a=\frac{1}{a}$ and $\cos a=\sqrt{1-\frac{1}{a^2}}=\frac{\sqrt{a^2-1}}{a}$ M1

Therefore $A(a)=2a\ \mathrm{e}^{\frac{\sqrt{a^2-1}}{a}}$ AG [21 marks]

Mark scheme

Practice paper 2

SECTION A

1 a $12200=a+4\times600 \Rightarrow a=9800$ M1A1

b $S_5=\frac{9800+12200}{2}\times5=55000$ M1A1

c $9800+600(n+1)>15000 \Rightarrow n>7\frac{2}{3}$ M1A1

So on the 8th year. A1 [7 marks]

2 a $3\mathrm{i}=\frac{a(-3\mathrm{i})+b}{-3\mathrm{i}+c} \Rightarrow 9+3c\mathrm{i}=b-3a\mathrm{i}$ M1

$\Rightarrow b=9$ and $c=-a$ A1

$1-4\mathrm{i}=\frac{a(1+4\mathrm{i})+b}{1+4\mathrm{i}+c} \Rightarrow (1+4\mathrm{i})(1+c-4\mathrm{i})=a+b+4a\mathrm{i}$ M1

$\Rightarrow c+17-4c\mathrm{i}=a+b+4a\mathrm{i}$ A1

As $b=9$ and $c=-a$,

$-a+17+4a\mathrm{i}=a+9+4a\mathrm{i} \Rightarrow a=4$ and $c=-4$ A1

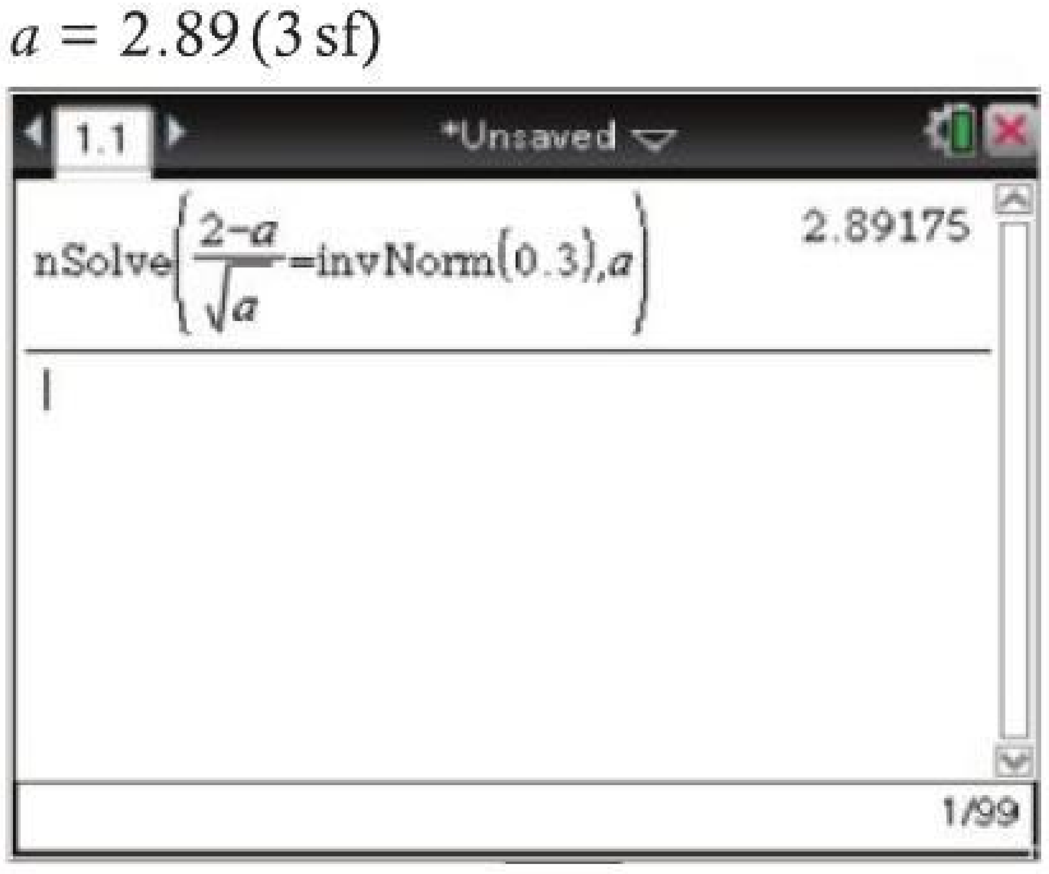
b $w=\frac{4z+9}{z-4} \Rightarrow w=\frac{4(4+y\mathrm{i})+9}{4+y\mathrm{i}-4}$ M1

$w=\frac{25+4y\mathrm{i}}{y\mathrm{i}} \Rightarrow w=4-\frac{25}{y}\mathrm{i}$ A1

So, $\mathrm{Re}\,w=4$ AG [7 marks]

3 $\mathrm{P}(X<2)=0.3 \Rightarrow \frac{2-a}{\sqrt{a}}=\underbrace{\phi^{-1}(0.3)}_{-0.5244\ldots}$ M1A1

$a=2.89$ (3 sf)



(Can be solved on a GDC)

(M1)A1 [4 marks]

4 a **Method 1:**

$f'(x)=\dfrac{-2x^2-1+x^2}{x^2}$ M1A1

$\Rightarrow f'(x)=-\dfrac{x^2+1}{x^2}\neq 0$ in the domain of f R1

So no maxima/minima points AG

$f''(x)=-\dfrac{2x^3-2x\left(x^2+1\right)}{x^4}$ A1

$\Rightarrow f''(x)=\dfrac{2}{x^3}\neq 0$ in the domain of f R1

Method 2:

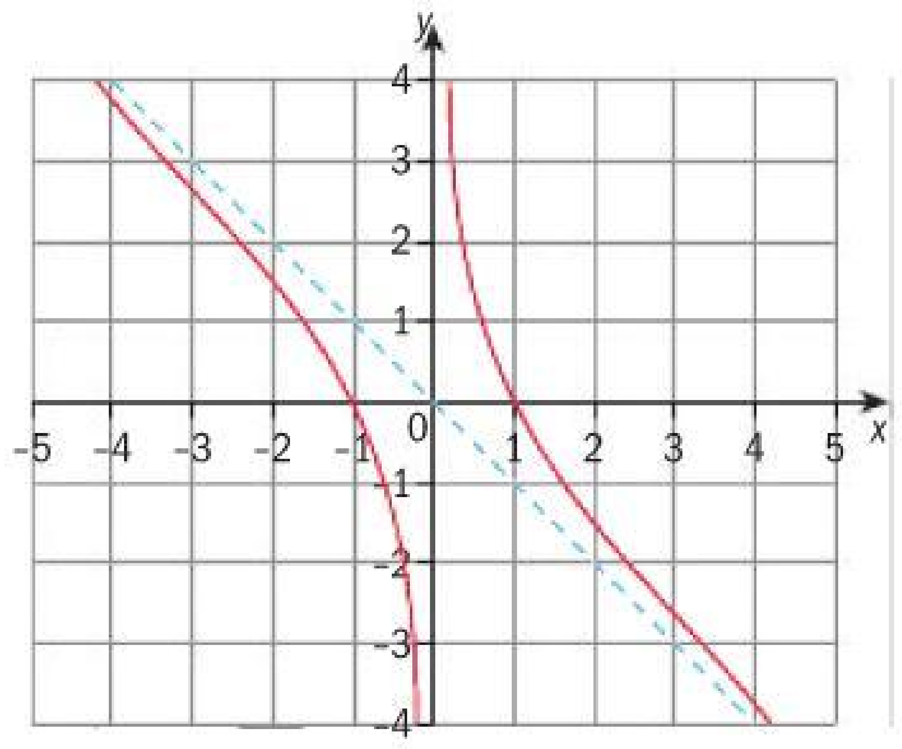
$f(x)=\dfrac{1-x^2}{x}=\dfrac{1}{x}-x$ M1

$f'(x)=-\dfrac{1}{x^2}-1\neq 0$ A1R1

So no maxima/minima points AG

$f''(x)=\dfrac{2}{x^3}\neq 0$ A1R1

b



Note: A1 for shape, A1 for zeros, A1 for asymptotes
(no equations required) A3 [8 marks]

5 $\overrightarrow{AB}=\begin{pmatrix}-6\\5\\-3\end{pmatrix}$ A1

either

$AP:PB=1:2\Rightarrow \overrightarrow{AP}=\dfrac{1}{3}\overrightarrow{AB}$ M1

$\overrightarrow{AP}=\begin{pmatrix}-2\\\frac{5}{3}\\-1\end{pmatrix}$ A1

$\overrightarrow{OP}=\overrightarrow{OA}+\overrightarrow{AP}=\begin{pmatrix}1\\-3\\-1\end{pmatrix}+\begin{pmatrix}-2\\\frac{5}{3}\\-1\end{pmatrix}=\begin{pmatrix}-1\\-\frac{4}{3}\\-2\end{pmatrix}$ M1A1

or

$AP:PB=1:2\Rightarrow PB=2PA$ M1

$\sqrt{(6-6t)^2+(5t-4)^2+(3-3t)^2}=2\sqrt{(-6t)^2+(5t)^2+(-3t)^2}$ M1A1

$(6-6t)^2+(5t-4)^2+(3-3t)^2=4\left((-6t)^2+(5t)^2+(-3t)^2\right)$

$\Rightarrow t=\dfrac{1}{3}\quad (t>0)$ A1

Therefore

$P\left(-1,-\dfrac{4}{3},-2\right)$ A1 [6 marks]

6 $(1+x)^5(1+ax)^6=(1+5x+10x^2+\ldots+x^5)(1+6ax+10x^2+\ldots+a^6x^{11})$ M1

$=1+(6a+5)x+(15a^2+30a+10)x^2+\ldots$ A1A1

$(1+x)^5(1+ax)^6\equiv 1+bx+10x^2+\ldots+a^6x^{11}$

Note: award A1 for first two terms, A1 for third term

$\Rightarrow 6a+5=b$ and $15a^2+30a+10=10$ M1

$\Rightarrow a=0$ and $b=5$ A1

$a=-2$ and $b=-7$ A1 [6 marks]

7 $\int\frac{e^{2x}}{4+e^{4x}}dx=\int\frac{e^{2x}}{4+(e^{2x})^2}dx$ M1

$u=e^{2x}\Rightarrow du=2e^{2x}\ dx$ (A1)

$\int\frac{e^{2x}}{4+e^{4x}}dx=\frac{1}{2}\int\frac{1}{4+u^2}du$ A1

$=\frac{1}{2}\int\frac{\frac{1}{2}}{1+\left(\frac{u}{2}\right)^2}du$ (A1)

$=\frac{1}{4}\arctan\frac{u}{2}+C$ (A1)

$=\frac{1}{4}\arctan\frac{e^{2x}}{2}+C$ A1 [6 marks]

8 **a** Attempt to apply cosine rule M1

$d=(20^2+15^2-2\times20\times15\cos\theta)^{\frac{1}{2}}$ (or equivalent) A1

b Minute hand moves $\frac{2\pi}{60}$ $\left(=\frac{\pi}{30}\right)$ A1

Hour hand $\frac{2\pi}{12\times60}$ $\left(=\frac{\pi}{360}\right)$ radians per minute A1

$\frac{d\theta}{dt}=\frac{\pi}{360}-\frac{\pi}{30}=-\frac{11\pi}{360}$ A1

$d'=\frac{1}{2}(20^2+15^2-2\times20\times15\cos\theta)^{-\frac{1}{2}}600\sin\theta\frac{d\theta}{dt}$ M1A1

$=-\frac{55\pi}{6}(20^2+15^2-2\times20\times15\cos\theta)^{-\frac{1}{2}}\sin\theta$ (or equivalent)

At 3 o'clock, $\theta=\frac{\pi}{2}\Rightarrow\ d'=-\frac{55\pi}{6}(20^2+15^2)^{-\frac{1}{2}}=1.15$ (3 sf) A1 [6 marks]

9 For investigating the pattern or deducing that $\left(u_n-\frac{1}{2}\right)$ is a GP with M3

$r=\frac{1}{3}$ and $a=\frac{3}{2}$ A1A1

$u_n-\frac{1}{2}=\frac{3}{2}\left(\frac{1}{3}\right)^{n-1}$ M1A1

$u_n=\frac{1}{2}+\frac{3\times3^{1-n}}{2}=\frac{3^{2-n}+1}{2}$ A1AG [8 marks]

SECTION B

10 a $12 \times 0.25 = 3$ — A1

b Let B be the number of blue ribbons taken from the box.

$B \sim \mathrm{B}(12, 0.45)$ — A1

$\mathrm{P}(B=6) = \binom{12}{6}(0.45)^6(0.55)^6 = 0.212$ (3 sf) — (M1)A1

c $\mathrm{P}(B \geq 2) = 1 - \mathrm{P}(B \leq 1) = 0.992$ (3 sf) — M1A1

d Let W be the number of white ribbons taken from the box.

$W \sim \mathrm{B}(12, 0.3)$ — (A1)

The value of W for which the pdf takes the maximum value is 3 — (M1)A1

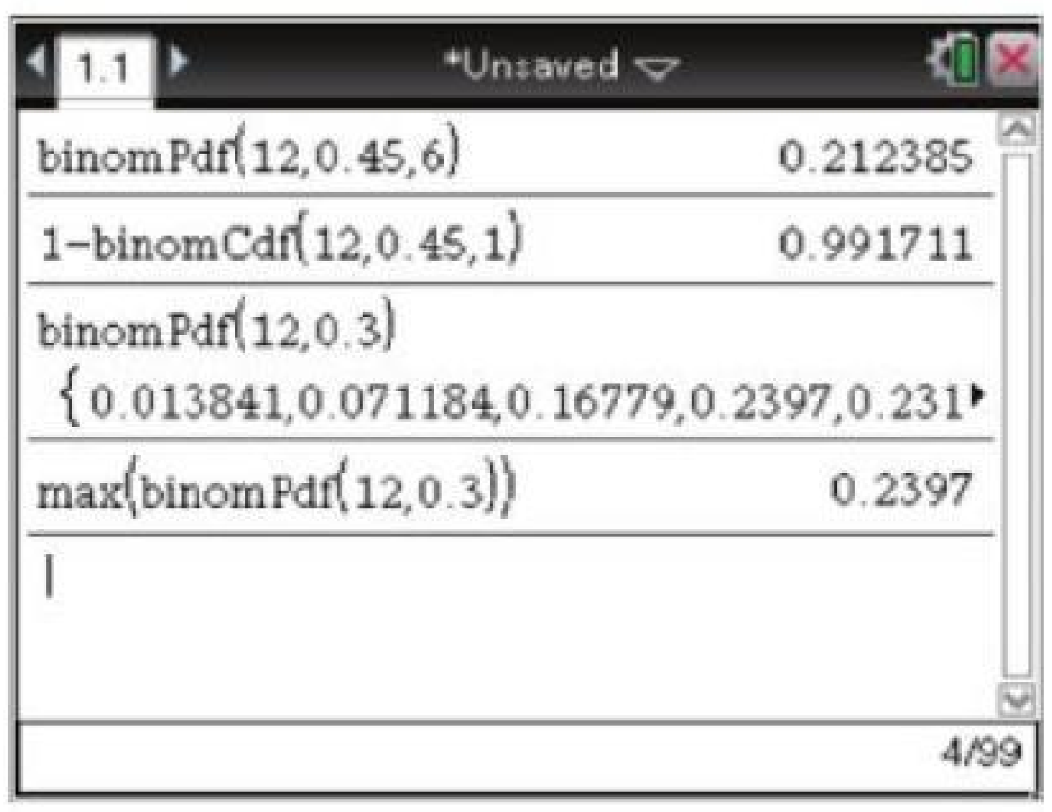
Assumptions: independence of events and — A1

probability of success constant due to big number of ribbons in the box — A1

e Attempt to use Bayes' theorem or correct tree diagram — M1

$\mathrm{P}(box\ 1|W) = \dfrac{0.3}{0.3 + 0.25 + 0.5}$ — A1

$= 0.286$ (3sf) — A1 [14 marks]



(GDC may be used)

11 a $\mathbf{d} = \overrightarrow{\mathrm{AB}} = \begin{pmatrix} 2-1 \\ -1-2 \\ 0+3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$ — A1

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d} \Rightarrow \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$ — M1A1

b $M\left(\dfrac{1+2}{2}, \dfrac{2-1}{2}, \dfrac{-3+0}{2}\right) = \left(\dfrac{3}{2}, \dfrac{1}{2}, -\dfrac{3}{2}\right)$ — A1

$\mathbf{n} = \mathbf{d} \Rightarrow \mathbf{r} \cdot \mathbf{n} = \mathbf{m} \cdot \mathbf{n} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -\frac{3}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 3 \end{pmatrix}$ — M1A1

$x - 3y + 3z = \dfrac{3}{2} - \dfrac{3}{2} - \dfrac{9}{2} \Rightarrow 2x - 6y + 6z = -9$ — AG

c Let Q be the midpoint of [AC].

$Q\left(\frac{1-1}{2},\frac{2-0}{2},\frac{-3+3}{2}\right)=(0,1,0)$ A1

$\overrightarrow{AC}=\begin{pmatrix}-1-1\\0-2\\3+3\end{pmatrix}=\begin{pmatrix}-2\\-2\\6\end{pmatrix}=-2\times\begin{pmatrix}1\\1\\-3\end{pmatrix}\Rightarrow \mathbf{n}_2=\begin{pmatrix}1\\1\\-3\end{pmatrix}$ A1

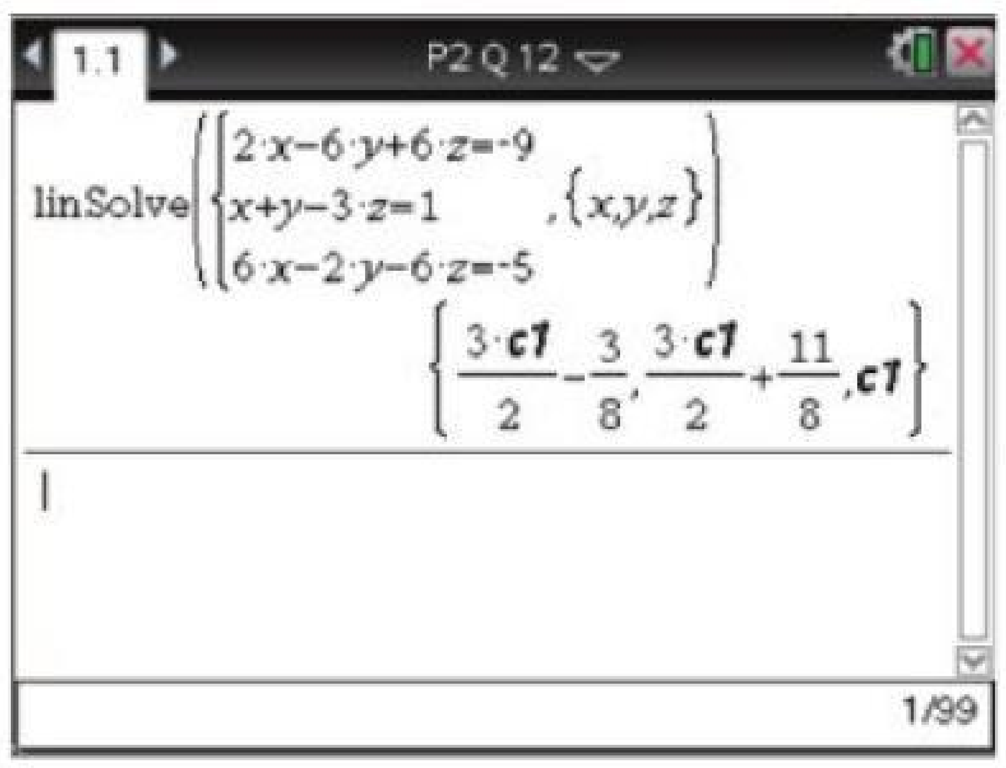
$\mathbf{r}\cdot\mathbf{n}_2=\mathbf{q}\cdot\mathbf{n}_2\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}$ A1

$x+y-3z=1$ AG

d $\cos\theta=\dfrac{\begin{pmatrix}1\\-3\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\3\end{pmatrix}}{\sqrt{1^2+(-3)^2+3^2}\sqrt{1^2+1^2+3^2}}=\dfrac{7}{\sqrt{19}\sqrt{11}}$ M1A1A1

$\theta=1.07(61.0°)$ A1

e Attempt to solve the system of three simultaneous equations: (M2)



(Can be solved with GDC)

$\begin{cases}x=\frac{3}{2}\lambda-\frac{3}{8}\\ y=\frac{3}{2}\lambda+\frac{11}{8},\lambda\in\mathbb{R}\\ z=\lambda\end{cases}$ A2

f Since the point P is equally distant to the points A, B and C therefore it lies in the bisecting planes of the line segments [AB], [AC] and [BC]. R1

Attempt to find the point of intersection between the line in part e) and the plane $x+y+z=0$.

$\frac{3}{2}\lambda-\frac{3}{8}+\frac{3}{2}\lambda+\frac{11}{8}+\lambda=0$ M1 A1

$4\lambda=-1\Rightarrow\lambda=-\frac{1}{4}$ A1

$\begin{cases}x=\frac{3}{2}\left(-\frac{1}{4}\right)-\frac{3}{8}\\ y=\frac{3}{2}\left(-\frac{1}{4}\right)+\frac{11}{8}\Rightarrow P\left(-\frac{3}{4},1,-\frac{1}{4}\right)\\ z=-\frac{1}{4}\end{cases}$ M1A1 [23 marks]

12 a $f(-x) = \cos(-2x) + 1 = \cos(2x) + 1 = f(x)$ M1A1

$g(-x) = \dfrac{e^{-x} + e^{x}}{2} = g(x)$ A1

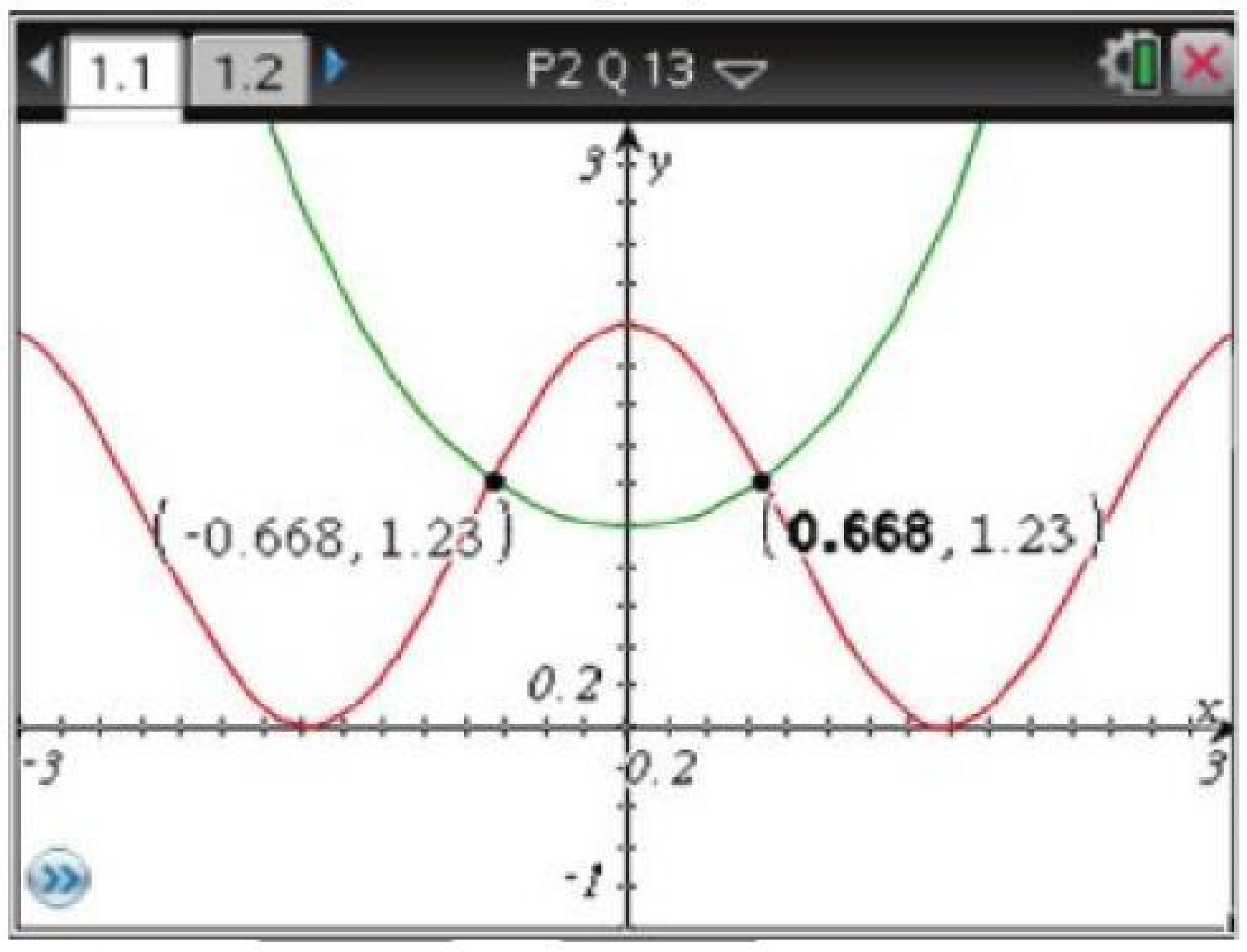
b $f'(x) = -\sin(2x) \cdot 2 = -2\sin(2x)$ M1A1

$g'(x) = \dfrac{1}{2}\left(e^{x} + (-1) \times e^{-x}\right) = \dfrac{e^{x} - e^{-x}}{2}$ M1A1

c $f'(-x) = -2\sin(-2x) = 2\sin(2x) = -f'(x)$ A1

$g'(-x) = \dfrac{e^{-x} - e^{x}}{2} = -\dfrac{e^{x} - e^{-x}}{2} = -g'(x)$ A1

d Correct shape of the graphs A1A1



$(-0.668, 1.23)$ A1

$(0.668, 1.23)$ A1

Notice that we stored the *x*-value of the point in the first quadrant to a variable called *a*, that will be seen in the following parts

e Attempt to use the formula for the tangent $y = f'(x_1)(x - x_1) + y_1$ M1

$y = f'(0.668)(x - 0.668) + 1.23$ A1

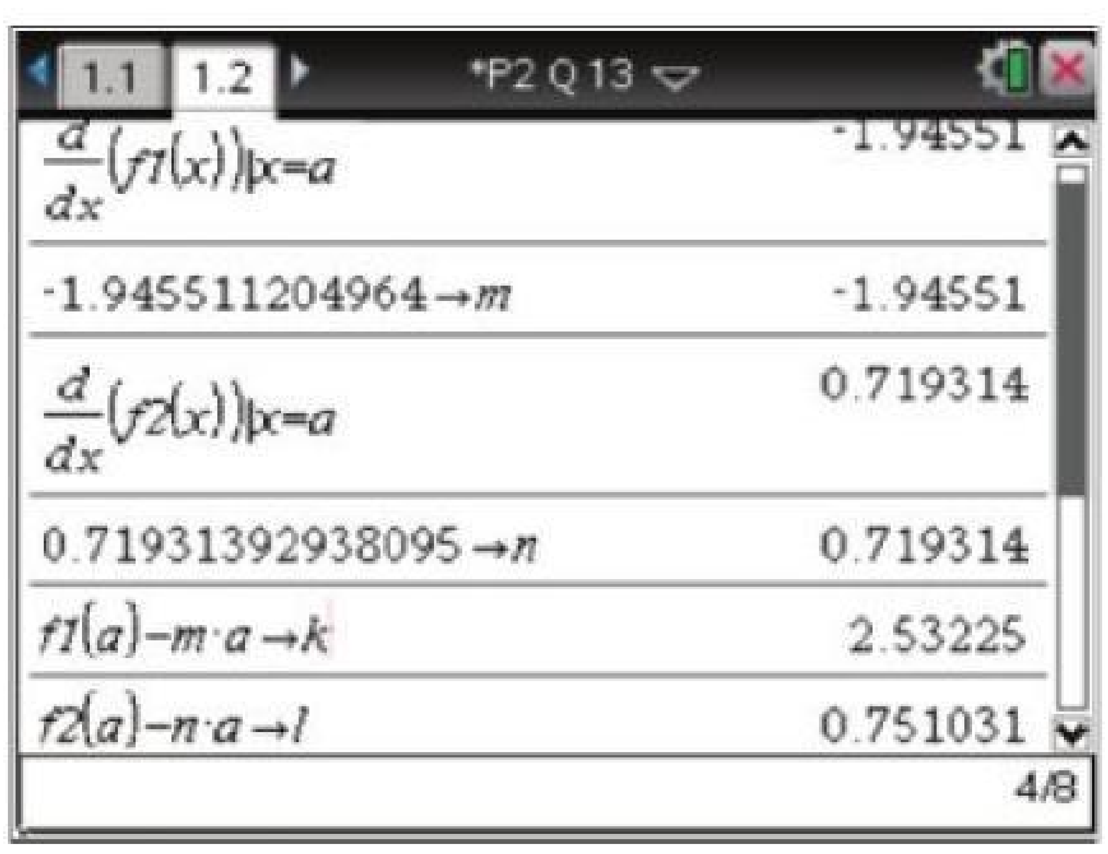
$y = -1.95(x - 0.668) + 1.23$

$y = -1.95x + 2.53$ AG

$y = g'(0.668)(x - 0.668) + 1.23$

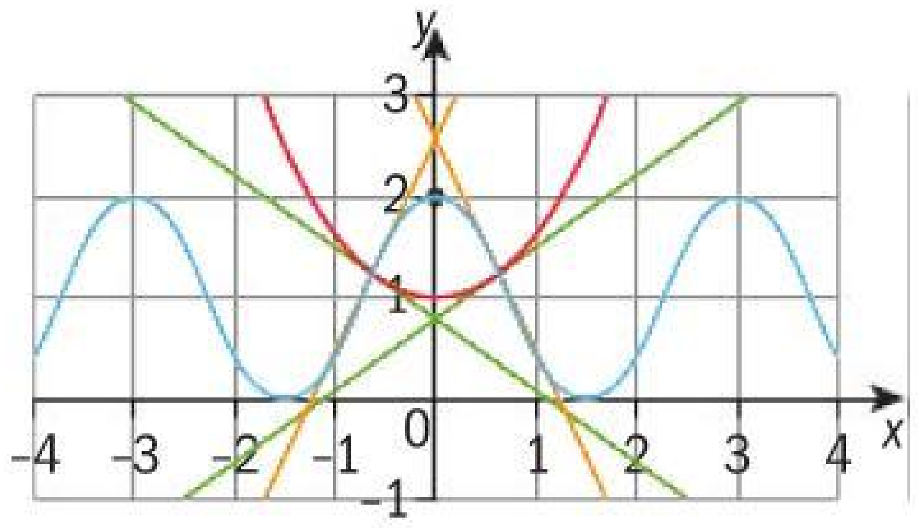
$y = 0.719(x - 0.668) + 1.23$ A1

$y = 0.719x + 0.751$ AG



Store all the values found. It saves you time and reduces the risk of errors due to randing and mistakes.

f Since the functions are even, their graphs and the respective tangents are symmetrical with respect to the y-axis. They form a kite so the area is half the product of the diagonals. R1



$$A = \frac{e \cdot f}{2} \Rightarrow A = \frac{(2.53 - 0.751) \cdot (2 \cdot 0.668)}{2} = 1.19$$ M1A1

g Either $V = \pi \int_{-0.688}^{0.668} (f^2(x) - g^2(x))\,dx = 7.88$ M1A1A2

or

$V = 2\pi \int_{0}^{0.668} (f^2(x) - g^2(x))\,dx = 7.88$ M1A1A2 [23 marks]

Note: M1 for correct formula, A1 for correct limits, A2 for correct final answer