Mark scheme

Practice paper 1

1 $\boldsymbol{n}_1 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \boldsymbol{n}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \cos\theta = \frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}$ (A1) (A1) (M1)

$= \frac{2+2-1}{\sqrt{6}\cdot\sqrt{6}} = \frac{1}{2} \Rightarrow \theta = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$ (A1) (A1) [5 marks]

2 a $f(-x) = (-x)\sin^3(-x) = (-x)(-\sin^3 x) = x\sin^3 x = f(x)$ (M1) (A1) (A1) even

b $g(-x) = \sqrt[3]{\frac{e^{(-x)^2}+1}{-x}} = \sqrt[3]{\frac{e^{x^2}+1}{-x}} = -\sqrt[3]{\frac{e^{x^2}+1}{x}} = -g(x)$ (M1) (A1) (A1) odd [6 marks]

3 a $\lim_{x\to 2}\frac{2x^2-5x+2}{3x^2-4x-4} = \lim_{x\to 2}\frac{(2x-1)(x-2)}{(3x+2)(x-2)}$ (M1) (A1)

$= \lim_{x\to 2}\frac{2x-1}{3x+2} = \frac{2\cdot 2-1}{3\cdot 2+2} = \frac{3}{8}$ (A1)

b $\lim_{x\to 0}\frac{\sin 3x}{2x} = \lim_{x\to 0}\frac{3}{2}\cdot\frac{\sin 3x}{3x}$ (M1) (A1)

$= \frac{3}{2}\lim_{3x\to 0}\frac{\sin 3x}{3x} = \frac{3}{2}\cdot 1 = \frac{3}{2}$ (A1) [6 marks]

4 a $(a+2d)^2 = a(a+5d)$ (M1) (A1)

$a^2+4ad+4d^2 = a^2+5ad \Rightarrow ad = 4d^2 \Rightarrow a = 4d$ (A1) (A1)

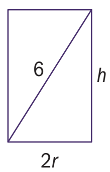
b $(a+4d)^2 = a(a+12d) \Rightarrow (8d)^2 = 4d(16d)$ (M1) (A1)

$\Rightarrow 64d^2 = 64d^2$ (A1) [7 marks]

5 a $\frac{1}{2}(8-1)! = 2520$ (M1) (A1)

b $P = \frac{\frac{1}{2}\cdot 2\cdot(7-1)!}{\frac{1}{2}(8-1)!} = \frac{2\cdot 6!}{7!} = \frac{2}{7}$ (M1) (M1) (A1) [5 marks]

6



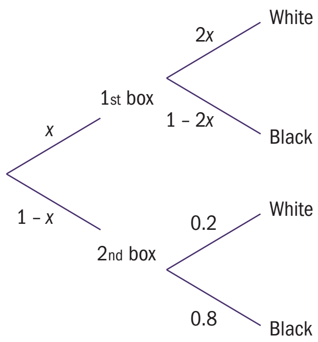
$h^2 + 4r^2 = 36 \Rightarrow r^2 = 9 - \frac{1}{4}h^2$ (A1)

$V = r^2\pi h = \left(9-\frac{1}{4}h^2\right)\pi h = \left(9h-\frac{1}{4}h^3\right)\pi$ (M1) (A1)

$\frac{dV}{dh} = \left(9-\frac{3}{4}h^2\right)\pi = 0 \Rightarrow h = \sqrt{12}$ (M1) (A1)

$r = \sqrt{9-\frac{1}{4}\cdot 12} = \sqrt{6}$ cm (A1) [6 marks]

7 **a**



b $\mathrm{P}(W) = x \cdot 2x + (1-x)\cdot 0.2 = 0.44$ (M1) (A1)

$2x^2 - 0.2x - 0.24 = 0 \Rightarrow (2x+0.6)(x-0.4) = 0$ (M1) (A1)

~~$x_1 = -0.3$~~, $x_2 = 0.4$ (A1) [7 marks]

8 $2 \times 25^x = 3 \times 10^x + 5 \times 4^x \Rightarrow 2 \times \frac{25^x}{4^x} = 3 \times \frac{10^x}{4^x} + 5 \times \frac{4^x}{4^x}$ (M1)

$\Rightarrow 2 \times \left(\frac{5}{2}\right)^{2x} - 3 \times \left(\frac{5}{2}\right)^x - 5 = 0$ (A1)

Let's use a substitution $\left(\frac{5}{2}\right)^x = t \Rightarrow 2t^2 - 3t - 5 = 0$ (M1)

$(2t-5)(t+1) = 0 \Rightarrow t_1 = \frac{5}{2}$, ~~$t_2 = -1$~~ (A1) (A1)

$\left(\frac{5}{2}\right)^x = \frac{5}{2} \Rightarrow x = 1$ (A1) [6 marks]

9 **a** $\log_x 10 = \log_x 5 + \log_x 2$ (M1)

$= \frac{1}{\log_5 x} + \frac{1}{\log_2 x} = \frac{1}{p} + \frac{1}{q}$ (A1) (A1)

b $\log x = \frac{1}{\log_x 10}$ (M1)

$= \frac{1}{\frac{1}{p}+\frac{1}{q}} = \frac{pq}{p+q}$ (A1) (A1)

[6 marks]

10 Let's assume that $\cos\alpha \neq 0$ and $\sin\alpha \neq 0$.

$\begin{cases} \sin\alpha \cdot x - \cos\alpha \cdot y = 1 \\ \cos\alpha \cdot x + \sin\alpha \cdot y = 1 \end{cases} \Rightarrow \left\{ \begin{array}{l} \sin^2\alpha \cdot x - \sin\alpha \cdot \cos\alpha \cdot y = \sin\alpha \\ \cos^2\alpha \cdot x + \cos\alpha \cdot \sin\alpha \cdot y = \cos\alpha \end{array} \right) +$ (M1)

$\underbrace{\left(\sin^2\alpha + \cos^2\alpha\right)}_{1} x = \sin\alpha + \cos\alpha \Rightarrow x = \sin\alpha + \cos\alpha$ (A1)

To find y we use the second equation

$\cos\alpha(\sin\alpha + \cos\alpha) + \sin\alpha \cdot y = 1$

$\Rightarrow \sin\alpha \cdot y = 1 - \cos^2\alpha - \sin\alpha \cdot \cos\alpha$ (M1)

$\Rightarrow \sin\alpha \cdot y = \sin^2\alpha - \sin\alpha \cdot \cos\alpha, \quad \sin\alpha \neq 0$ (A1)

$\Rightarrow y = \sin\alpha - \cos\alpha$ (A1)

If $\begin{cases} \sin\alpha = 0 \Rightarrow \cos\alpha \neq 0 \Rightarrow y = -\sec\alpha, x = \sec\alpha \\ \cos\alpha = 0 \Rightarrow \sin\alpha \neq 0 \Rightarrow y = \csc\alpha, x = \csc\alpha \end{cases}$

so we always have a unique solution. (R1) (R1) [6 marks]

11 a $x^2 - x^3 \neq 0 \Rightarrow x^2(1-x) \neq 0$ (M1)

b $D(f) = \{x \in \mathbb{R} : x \neq 0, 1\}$ (A1)

$$= \lim_{x\to\pm\infty} \frac{1-\frac{2}{x}+\frac{5}{x^3}}{\frac{1}{x}-1} = -1$$ (M1) (A1) (A1)

c $\frac{x^3-2x^2+5}{x^2-x^3} = 1$ (M1)

$x^3-2x^2+5 = x^2-x^3 \Rightarrow 2x^3-3x^2+5=0$ (A1)

On inspection, by using synthetic division we obtain possible x-values.

	2	–3	0	5
·(–1)		+ –2	+ 5	+ –5
	2	–5	5	0

$2x^3 - 3x^2 + 5 = 0 \Rightarrow (x+1)(2x^2-5x+5) = 0$ (M1) (A1)

$x_1 = -1, x_{2,3} \notin \mathbb{R} \Rightarrow (-1,1)$ (A1) [10 marks]

12 a $\int \sin^n(x)\,dx \int \sin^{n-1}(x)\cdot \sin x\,dx$ (M1)

Let $\begin{cases} \sin^{n-1}(x) = u \Rightarrow (n-1)\sin^{n-2}(x)\cdot\cos x\,dx = du \\ \sin x\,dx = dv \Rightarrow -\cos x = v \end{cases}$ (A1) (A1)

$= \sin^{n-1}(x)\cdot(-\cos x) - \int(-\cos x)\cdot(n-1)\sin^{n-2}(x)\cdot -\cos x\,dx$ (A1)

$= -\sin^{n-1}(x)\cos x + (n-1)\int \sin^{n-2}(x)\cdot\cos^2 x\,dx$

$= -\sin^{n-1}(x)\cos x + (n-1)\int \sin^{n-2}(x)\cdot(1-\sin^2 x)\,dx$ (A1)

$= -\sin^{n-1}(x)\cos x + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)\int \sin^n(x)\,dx$ (A1)

$n\int \sin^n(x)\,dx = -\sin^{n-1}(x)\cos x + (n-1)\int \sin^{n-2}(x)\,dx$

$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos x + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$ (A1)

$\int_0^{\frac{\pi}{2}} \sin^n(x)\,dx = \left[-\frac{1}{n}\sin^{n-1}(x)\cos x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\,dx$ (A1)

$= -\frac{1}{n}\left(\left(\sin^{n-1}\left(\frac{\pi}{2}\right)\underbrace{\cos\frac{\pi}{2}}_{0}\right) - \left(\underbrace{\sin^{n-1}(0)}_{0}\cos 0\right)\right) + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\,dx$ (A1)

$= \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}(x)\,dx$ (AG)

b $\int_0^{\frac{\pi}{2}} \sin^4(x)\,dx = \frac{3}{4}\int_0^{\frac{\pi}{2}} \sin^2(x)\,dx$ (A1)

$= \frac{3}{4}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} dx\right)$ (A1)

$= \frac{3}{8}\left(\frac{\pi}{2} - 0\right) = \frac{3\pi}{16}$ (A1) [12 marks]

13 a $\sum_{r=1}^{n}(3r-1) = 2 + 5 + 8 + \ldots + (3n-1)$

$n=1 \Rightarrow 2 = \frac{1}{2}\cdot 1\cdot(3\cdot 1+1)$ (M1)

$2 = 2$, true for $n = 1$ (A1)

Assume that the formula works for $n = k$.

$\sum_{r=1}^{k}(3r-1) = \frac{1}{2}k(3k+1)$ (M1)

$n = k+1 \Rightarrow \sum_{r=1}^{k+1}(3r-1) = \underbrace{2 + 5 + 8 + \ldots + (3k-1)}_{\text{Assumption}} + (3k+2)$ (M1)

$= \frac{1}{2}k(3k+1) + (3k+2) = \frac{1}{2}\left(3k^2 + k + 6k + 4\right)$ (A1)

$= \frac{1}{2}\left(3k^2 + 7k + 4\right) = \frac{1}{2}(k+1)(3k+4)$ (A1) (A1)

The formula works for $n = 1$ and, from the assumption that it works for $n = k$, we obtained that it works for $n = k + 1$, therefore it works for all $n \in \mathbb{Z}^+$. (R1)

b $1 - 2 + 3 + 4 - 5 + \ldots + (3n-2) - (3n-1) + 3n$ (M1) (A2)

$= \frac{3n\cdot(3n+1)}{2} - 2\cdot\frac{1}{2}n(3n+1)$ (A1) (A1)

$= \frac{n\cdot(3n+1)}{2}$ (A1) [14 marks]

14 a $z^2 + z + 1 = 0 \Rightarrow z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$ (M1) (A1)

$\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\omega^* = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ (A1) (A1)

b i $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \omega^*$ (A1)

ii $\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^2 = \cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$ (A1)

iii $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3 = \cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3} = \cos 2\pi + i\sin 2\pi = 1$ (A1)

$\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^3 = \cos\frac{12\pi}{3} + i\sin\frac{12\pi}{3} = \cos 4\pi + i\sin 4\pi = 1$ (A1)

c i $(1+\omega)^{3n} = 1 + \binom{3n}{1}\omega + \binom{3n}{2}\omega^2 + \binom{3n}{3}\omega^3 + \ldots$ (M1)

$= 1 + \binom{3n}{1}\omega + \binom{3n}{2}\omega^* + \binom{3n}{3} + \ldots$ (A1)

ii $(1+\omega^*)^{3n} = 1 + \binom{3n}{1}\omega^* + \binom{3n}{2}(\omega^*)^2 + \binom{3n}{3}(\omega^*)^3 + \ldots$ (M1)

$= 1 + \binom{3n}{1}\omega^* + \binom{3n}{2}\omega + \binom{3n}{3} + \ldots$ (A1)

d i $1 + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ (M1) (A1)

ii $1 + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right) = \frac{1}{2} - \frac{\sqrt{3}}{2} = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$ (A1)

e $\binom{3n}{0}+\binom{3n}{1}+\binom{3n}{2}+\ldots+\binom{3n}{3n}=(1+1)^{3n}=2^{3n}$ (M1) (A1)

f

$$\left.\begin{aligned}2^{3n}&=\binom{3n}{0}+\binom{3n}{1}+\binom{3n}{2}+\binom{3n}{3}+\ldots+\binom{3n}{3n}\\(1+\omega)^{3n}&=\binom{3n}{0}+\binom{3n}{1}\omega+\binom{3n}{2}\omega^*+\binom{3n}{3}+\ldots+\binom{3n}{3n}\\(1+\omega^*)^{3n}&=\binom{3n}{0}+\binom{3n}{1}\omega^*+\binom{3n}{2}\omega+\binom{3n}{3}+\ldots+\binom{3n}{3n}\end{aligned}\right\}+$$ (M1) (A1)

$$2^{3n}+\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)^{3n}+\left(\cos\frac{5\pi}{3}+\mathrm{i}\sin\frac{5\pi}{3}\right)^{3}$$

$$=3\cdot\left(\binom{3n}{0}+\binom{3n}{3}+\binom{3n}{6}+\ldots+\binom{3n}{3n}\right)$$ (A1) (A1)

$$2^{3n}+\left(\cos n\pi+\mathrm{i}\underbrace{\sin n\pi}_{0}\right)+\left(\cos 5n\pi+\mathrm{i}\underbrace{\sin 5n\pi}_{0}\right)$$

$$=3\cdot\left(\binom{3n}{0}+\binom{3n}{3}+\binom{3n}{6}+\ldots+\binom{3n}{3n}\right)$$ (A1) (A1)

Since $\cos(n\pi)=\cos(5n\pi)=\begin{cases}1, n \text{ is even}\\-1, n \text{ is odd}\end{cases}$ we can write (R1)

$$\binom{3n}{0}+\binom{3n}{3}+\binom{3n}{6}+\ldots+\binom{3n}{3n}=\frac{2^{3n}+2\cos(n\pi)}{3}.$$ (AG) [24 marks]