# Exam-style questions

**11 P1:** Consider the points given by the coordinates A(1, 0, 1), B(3, 0, 0), C(4, 2, 3).

**a** Find the vector $\overrightarrow{AB} \times \overrightarrow{AC}$ (4 marks)

**b** Find an exact value for the area of triangle ABC. (3 marks)

**c** Show that the cartesian equation of the plane $\Pi_1$ which containing the triangle ABC is $2x-7y+4z=6$ (3 marks)

**d** A second plane is given by the equation $\Pi_2: 3x-5y+z=1$. Find the line of intersection of the planes $\Pi_1$ and $\Pi_2$ (5 marks)

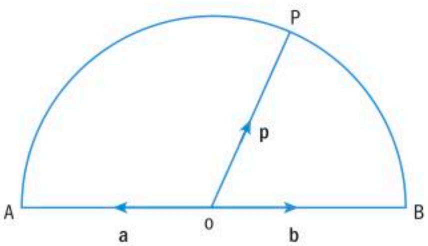
**12 P1:** A tetrahedron has vertices at the points A(1, 0, 1), B(−2, 2, 3), C(0, 4,2), D(3, 1, 3) relative to a fixed point O.

Find the volume of the tetrahedron. (6 marks)

**13 P1:** In the given semicircle, AB is the diameter and P is a general point on the arc. O is the centre of AB.

It is also given that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OP} = \mathbf{p}$.

By using the properties of the scalar product, prove that $A\hat{P}B = 90°$. (8 marks)




**14 P1:** Consider a point P(1, 0, 2) and the plane $\pi$: $4x - 3y + z = 19$. Point Q is such that P and Q are equidistant from $\pi$, and the line PQ is perpendicular to $\pi$.

**a** Determine the coordinates of Q. (6 marks)

**b** Find the exact value of the distance PQ. (3 marks)

**15 P1:** The plane $\pi$ has equation $4x + 3y - z = 14$ and the line $L$ has equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix}.$$

**a** Given that L meets $\pi$ at the point P, find the coordinates of P. (4 marks)

**b** Find the shortest distance from the origin O to $\pi$. (4 marks)

**16 P2:** Points A,B,C,D have coordinates given by A(8,2,0), B(2,0,6), C(4,4,4) and D(12, 3, 0).

**a** Find a vector equation of the line AB. (3 marks)

**b** Find a vector equation of the line CD. (3 marks)

**c** Hence, or otherwise, find the shortest distance between the lines AB and CD. (7 marks)

**17 P1:** A plane $\Pi$ contains the point (5,8,0) and the line $\mathbf{r} = \begin{pmatrix} 10 \\ -4 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Find the equation of $\Pi$ in the form $ax + by + cz = 1$, where $a$, $b$ and $c$ are constants. (12 marks)

**18 P2:** A line is given by the equation $\frac{x-1}{2} = \frac{y}{5} = \frac{z-5}{p}$ and a plane is given by the equation $5x + py + pz = 8$, where $p$ is a constant.

Determine the value of $p$ that maximises the angle between the line and the plane, and hence find the maximum acute angle between the line and the plane. Give your answer in degrees, correct to 1 decimal place. (10 marks)

**19 P1:** Determine whether or not the lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \text{ and}$$

$$L_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ are skew.}$$ (9 marks)

precisely to the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. As we saw in (a), these can be written as the cosines of the angles. Hence

$$\boldsymbol{n} = \cos\alpha\,\boldsymbol{i} + \cos\beta\,\boldsymbol{j} + \cos\gamma\,k$$

Then the equation of the plane can be written as

$$x\cos\alpha + y\cos\beta + z\cos\gamma = 0$$

**10 a** We calculate the vectors

$$AP = \begin{pmatrix}1\\2\\4\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\2\\4\end{pmatrix}$$

and

$$AQ = \begin{pmatrix}0\\1\\4\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\1\\4\end{pmatrix}$$

These will be the two vectors on the plane equation. Additionally we take a point, choosing for simplicity A = (2, 0, 0). Then the plane equation in vector form is

$$p = \begin{pmatrix}2\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\4\end{pmatrix} + \mu\begin{pmatrix}-2\\1\\4\end{pmatrix}$$

To write it in Cartesian form, we write out the system of equations

$x = 2 - \lambda - 2\mu$

$y = 2\lambda + \mu$

$z = 4\lambda + 4\mu$

We subtract the third one from twice the second one, to get

$z - 2y = 2\mu$

so $\mu = \dfrac{z-2y}{2}$ and we add the second one to twice the first one, to get

$y + 2x = 4 + 2\lambda - 2\lambda + \mu - 4\mu$

or equivalently

$y + 2x = 4 - 3\mu$

Then we substitute with our value for $\mu$ to get

$$y + 2x = 4 - 3\left(\frac{z-2y}{2}\right)$$

This simplifies to

$4x - 4y + 3z = 8$

**b** using the equation of the plane written in a. $BG$ gives the direction vector of the line.

$$BG = \begin{pmatrix}0\\0\\4\end{pmatrix} - \begin{pmatrix}2\\2\\0\end{pmatrix} = \begin{pmatrix}-2\\-2\\4\end{pmatrix}$$

Then the equation of the line is written as

$$\boldsymbol{p} = \begin{pmatrix}0\\0\\4\end{pmatrix} - \lambda\begin{pmatrix}-2\\-2\\4\end{pmatrix}$$

**c** Angle between plane $4x - 4y + 3z = 8$ and line

$$\boldsymbol{p} = \begin{pmatrix}0\\0\\4\end{pmatrix} - \lambda\begin{pmatrix}-2\\-2\\4\end{pmatrix}$$

We have that

$$\sin\theta = \frac{|\boldsymbol{d}\cdot\boldsymbol{n}|}{|\boldsymbol{d}||\boldsymbol{n}|}$$

$$= \frac{|(-2)(4)+(-2)(-4)+(4)(3)|}{\sqrt{2^2+2^2+4^2}\sqrt{4^2+4^2+3^2}}$$

$$= \frac{12}{2\sqrt{246}} = \frac{6}{\sqrt{246}}$$

Then

$\theta = 22.5°$

## Exam-style questions

**11 a** $\overrightarrow{AB} = \begin{pmatrix}2\\0\\-1\end{pmatrix}$ (1 mark)

$\overrightarrow{AC} = \begin{pmatrix}3\\2\\2\end{pmatrix}$ (1 mark)

$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{pmatrix}2\\0\\-1\end{pmatrix}\times\begin{pmatrix}3\\2\\2\end{pmatrix}$

(1 mark)

$= \begin{pmatrix}2\\-7\\4\end{pmatrix}$ (1 mark)

**b** $\dfrac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \dfrac{1}{2}\sqrt{2^2+(-7)^2+4^2}$

(2 marks)

$= \dfrac{\sqrt{69}}{2}$ (1 mark)

**c** $\mathbf{r}.\begin{pmatrix}2\\-7\\4\end{pmatrix} = \begin{pmatrix}3\\0\\0\end{pmatrix}.\begin{pmatrix}2\\-7\\4\end{pmatrix}$

(2 marks)

$\mathbf{r}.\begin{pmatrix}2\\-7\\4\end{pmatrix} = 6$ (1 mark)

$2x - 7y + 4z = 6$

**d** $\begin{pmatrix}3\\-5\\1\end{pmatrix}\times\begin{pmatrix}2\\-7\\4\end{pmatrix} = \begin{pmatrix}-13\\-10\\-11\end{pmatrix}$

(2 marks)

$\mathbf{n} = \begin{pmatrix}13\\10\\11\end{pmatrix}$

$y = 0 \Rightarrow x = -\dfrac{1}{5}, z = \dfrac{8}{5}$

(or equivalent) (2 marks)

$\mathbf{r} = \begin{pmatrix}-\frac{1}{5}\\0\\\frac{8}{5}\end{pmatrix} + \lambda\begin{pmatrix}13\\10\\11\end{pmatrix}$

(or equivalent) (1 mark)

**12**

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}-2\\2\\3\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}-3\\2\\2\end{pmatrix}$

(1 mark)

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}0\\4\\2\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\4\\1\end{pmatrix}$

(1 mark)

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix}3\\1\\3\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\2\end{pmatrix}$

(1 mark)

Volume

$= \dfrac{1}{6}\left|\overrightarrow{AB}.\overrightarrow{AC}\times\overrightarrow{AD}\right| = \dfrac{1}{6}\left|\begin{pmatrix}-3\\2\\2\end{pmatrix}.\begin{pmatrix}7\\4\\-9\end{pmatrix}\right|$

(2 marks)

$= \dfrac{1}{6}|(-21+8-18)|$

$= \dfrac{31}{6}$ units². (1 mark)

**13** $\overrightarrow{AP} = \mathbf{p} - \mathbf{a}$ (1 mark)

$\overrightarrow{BP} = \mathbf{p} - \mathbf{b}$ (1 mark)

$\overrightarrow{AP}.\overrightarrow{BP} = (\mathbf{p} - \mathbf{a}).(\mathbf{p} - \mathbf{b})$ (1 mark)

$= (\mathbf{p} - \mathbf{a}) . (\mathbf{p} + \mathbf{a})$ (1 mark)

$= \mathbf{p} . \mathbf{p} - \mathbf{a} . \mathbf{p} + \mathbf{a} . \mathbf{p} - \mathbf{a} . \mathbf{a}$ (1 mark)

$= \mathbf{p} . \mathbf{p} - \mathbf{a} . \mathbf{a}$ (1 mark)

$= |\mathbf{p}|^2 - |\mathbf{a}|^2$ (1 mark)

$= 0$ since $|\mathbf{p}| = |\mathbf{a}|$ (1 mark)

Therefore $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{BP}$ and $\angle APB = 90°$

**14 a** Equation of line perpendicular to $\Pi$ and passing through $P$ is

$$\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}4\\-3\\1\end{pmatrix}$$ (2 marks)

Attempting to solve $P$ and $\Pi$ simultaneously: (1 mark)

$4(1 + 4\lambda) - 3(-3\lambda) + (2 + \lambda) = 19$

$4 + 16\lambda + 9\lambda + 2 + \lambda = 19$

$26\lambda + 6 = 19$

$\lambda = \frac{1}{2}$ (1 mark)

Therefore

$$\overrightarrow{OQ} = \begin{pmatrix}1\\0\\2\end{pmatrix} + 2\times\frac{1}{2}\times\begin{pmatrix}4\\-3\\1\end{pmatrix}$$ (1 mark)

$$= \begin{pmatrix}5\\-3\\3\end{pmatrix}$$ (1 mark)

**b** Distance between $P(1, 0, 2)$ and $Q(5, -3, 3)$ is given by

$$\sqrt{(5-1)^2 + (-3-0)^2 + (3-2)^2}$$ (2 marks)

$= \sqrt{16+9+1}$

$= \sqrt{26}$ (1 mark)

**15 a** $4(1 + 6\lambda) + 3(5 - 2\lambda) - (-3 + 2\lambda) = 14$ (1 mark)

$4 + 24\lambda + 15 - 6\lambda + 3 - 2\lambda = 14$

$22 + 16\lambda = 14$

$\lambda = -\frac{1}{2}$ (1 mark)

$$\mathbf{r} = \begin{pmatrix}1\\5\\-3\end{pmatrix} - \frac{1}{2}\begin{pmatrix}6\\-2\\2\end{pmatrix} = \begin{pmatrix}-2\\6\\-4\end{pmatrix}$$ (2 marks)

So $P(-2, 6, 4)$.

**b** $\begin{pmatrix}-2\\6\\-4\end{pmatrix}$ lies on the plane and

$$\mathbf{n} = \begin{pmatrix}4\\3\\-1\end{pmatrix}$$ (2 marks)

So distance

$$= \frac{\begin{pmatrix}-2\\6\\-4\end{pmatrix}.\begin{pmatrix}4\\3\\-1\end{pmatrix}}{\sqrt{4^2+3^2+(-1)^2}}$$ (1 mark)

$$= \frac{-8+18+4}{\sqrt{26}}$$

$$= \frac{14}{\sqrt{26}}\left(= \frac{14\sqrt{26}}{26} = \frac{7\sqrt{26}}{13}\right)$$ (1 mark)

**16 a**

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}2\\0\\6\end{pmatrix} - \begin{pmatrix}8\\2\\0\end{pmatrix} = \begin{pmatrix}-6\\-2\\6\end{pmatrix}$$ (1 mark)

$$\mathbf{r} = \begin{pmatrix}8\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}-6\\-2\\6\end{pmatrix}$$ (2 marks)

**b**

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \begin{pmatrix}12\\3\\0\end{pmatrix} - \begin{pmatrix}4\\4\\4\end{pmatrix} = \begin{pmatrix}8\\-1\\-4\end{pmatrix}$$ (1 mark)

$$\mathbf{r} = \begin{pmatrix}4\\4\\4\end{pmatrix} + \mu\begin{pmatrix}8\\-1\\4\end{pmatrix}$$ (2 marks)

**c** Direction vectors are $\begin{pmatrix}-6\\-2\\6\end{pmatrix}$ and $\begin{pmatrix}8\\-1\\4\end{pmatrix}$

$$\begin{pmatrix}-6\\-2\\6\end{pmatrix} \times \begin{pmatrix}8\\-1\\4\end{pmatrix} = \begin{pmatrix}-2\\72\\22\end{pmatrix}$$ (2 marks)

$(8, 2, 0)$ lies on AB and $(4, 4, 4)$ lies on CD

$$\overrightarrow{AC} = \begin{pmatrix}-4\\2\\4\end{pmatrix}$$ (1 mark)

Projection of $\overrightarrow{AC}$ to the vector $\begin{pmatrix}-2\\72\\22\end{pmatrix}$ is

$$\frac{\left|\begin{pmatrix}-4\\2\\4\end{pmatrix}.\begin{pmatrix}-2\\72\\22\end{pmatrix}\right|}{\sqrt{(-2)^2+72^2+22^2}}$$ (2 marks)

$$= \frac{8+144+88}{\sqrt{(-2)^2+72^2+22^2}}$$ (1 mark)

$$= \frac{240}{\sqrt{5672}}$$ (1 mark)

$$\left(= \frac{240\sqrt{5672}}{5672} = \frac{480\sqrt{1418}}{5672} = \frac{60\sqrt{1418}}{709} (= 3.19)\right)$$

**17** Choosing $\lambda = 1$ (say), gives

$$\mathbf{r} = \begin{pmatrix}10\\-4\\4\end{pmatrix} + \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}11\\-2\\5\end{pmatrix}$$ (1 mark)

Therefore A(5, 8, 0), B(10, −4, 4) and C(11, −2, 5) lie on $\Pi$ (2 marks)

$$\overrightarrow{AB}=\begin{pmatrix}5\\-12\\4\end{pmatrix} \text{ and } \overrightarrow{AC}=\begin{pmatrix}6\\-10\\5\end{pmatrix}$$
(2 marks)

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}5\\-12\\4\end{pmatrix}\times\begin{pmatrix}6\\-10\\5\end{pmatrix}=\begin{pmatrix}-20\\-1\\22\end{pmatrix}$$
(2 marks)

So equation of plane is

$$\mathbf{r}.\begin{pmatrix}-20\\-1\\22\end{pmatrix}=\begin{pmatrix}5\\8\\0\end{pmatrix}.\begin{pmatrix}-20\\-1\\22\end{pmatrix}$$
(2 marks)

$$\mathbf{r}.\begin{pmatrix}-20\\-1\\22\end{pmatrix}=-108$$ (1 mark)

$-20x-y+22z=-108$
(1 mark)

$$\frac{20}{108}x+\frac{1}{108}y-\frac{22}{108}z=1$$
(1 mark)

$$\left(\frac{5}{27}x+\frac{1}{108}y-\frac{11}{54}z=1\right)$$

**18** Direction vector of line is $\begin{pmatrix}2\\5\\p\end{pmatrix}$
(1 mark)

Direction normal to plane is $\begin{pmatrix}5\\p\\p\end{pmatrix}$
(1 mark)

If the angle between the line and the plane is $\theta$, then

$$\sin\theta=\frac{\begin{pmatrix}2\\5\\p\end{pmatrix}.\begin{pmatrix}5\\p\\p\end{pmatrix}}{\sqrt{2^2+5^2+p^2}\sqrt{5^2+p^2+p^2}}$$
(3 marks)

$$=\frac{10+5p+p^2}{\sqrt{2^2+5^2+p^2}\sqrt{5^2+p^2+p^2}}$$
(1 mark)

$\theta$ is maximum when $\sin\theta$ is maximum. (1 mark)

By GDC, maximum occurs when $p=6.797$ (1 mark)

So maximum value of $\sin\theta$ is 0.96 (1 mark)

$\Rightarrow \theta_{MAX}=73.7°$ (1 mark)

**19**

$$\begin{pmatrix}3\\-1\\1\end{pmatrix}\neq k\begin{pmatrix}1\\-1\\1\end{pmatrix}$$, so $L_1$ and $L_2$ are not parallel. (2 marks)

Consider **i** and **j** components:
(1 mark)

$1+3\lambda=2+\mu$ and $-\lambda=1-\mu$
(1 mark)

Solving simultaneously:
(1 mark)

$\lambda=1$ , $\mu=2$ (1 mark)

Substitute into **k** component:
(1 mark)

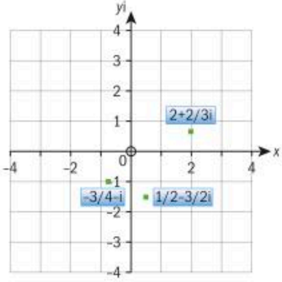
$2+\lambda=1+\mu$, $2+1=1+2$
(so equations are consistent).
(1 mark)

Therefore $L_1$ and $L_2$ intersect at the point where $\lambda=1$ and $\mu=2$, so are not skew.
(1 mark)

# Chapter 10

## Skills check

**1**




**2** $\text{Re}(z_1)=2, \text{Im}(z_1)=\frac{2}{3}$,

$\text{Re}(z_2)=-\frac{3}{4}, \text{Im}(z_2)=-1$,

$\text{Re}(z_3)=\frac{1}{2}, \text{Im}(z_3)=-\frac{3}{2}$.

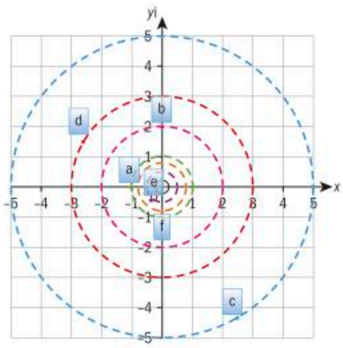
**3** **a** $1-13i$ **b** $-\frac{17}{4}-\frac{7}{4}\text{i}$

**4** **a** $z^*=2+3\text{i}, -z=-2+3\text{i}$,
$\frac{1}{z}=\frac{2}{13}+\frac{3}{13}\text{i}, |z|=\sqrt{13}$

**b** $z^*=\frac{4}{5}-\frac{3}{5}\text{i}, -z=-\frac{4}{5}-\frac{3}{5}\text{i}$,
$\frac{1}{z}=\frac{4}{5}-\frac{3}{5}\text{i}, |z|=1$

## Exercise 10A

**1**




**2** **a** $2\text{cis}\left(\frac{\pi}{4}\right)$ **b** $\frac{3}{2}\text{cis}\left(\frac{\pi}{2}\right)$

**c** $5\text{cis}\left(-\pi+\arctan\left(\frac{3}{4}\right)\right)$

**d** $29\text{cis}\left(-\arctan\left(\frac{20}{21}\right)\right)$

**e** $2\text{cis}\left(\frac{2\pi}{3}\right)$ **f** $\frac{4}{3}\text{cis}\left(\frac{3\pi}{2}\right)$

**g** $\frac{5\sqrt{2}}{12}\text{cis}\left(-\arctan\left(\frac{3}{4}\right)\right)$

**4** **a** $\frac{7}{12}\text{cis}\frac{\pi}{9}$ **b** $\frac{7}{12}\text{cis}\left(-\frac{\pi}{9}\right)$

**c** $-\frac{7}{12}\text{cis}\left(-\frac{\pi}{9}\right)$

## Exercise 10B

**1** **a** $8\text{e}^{\text{i}\frac{7\pi}{12}}$ **b** $30\text{cis}(135°)$

**c** $\frac{5}{9}\text{e}^{\text{i}\left(\frac{45}{14}\pi\right)}$ **d** $\text{cis}(135°)$

**2** **a** $-\frac{1}{\sqrt{2}}+\text{i}\frac{1}{\sqrt{2}}$ **b** $\text{e}^{\text{i}\frac{2\pi}{3}}$

**c** $\frac{-\sqrt{6}+\sqrt{2}}{4}-\frac{\sqrt{6}+\sqrt{2}}{4}i$

**d** See answer to part c.

**e** $2+\sqrt{3}$