International Baccalaureate
Baccalauréat International
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Mathematics: analysis and approaches

Practice paper 1 HL markscheme — Total 110

Section A [54 marks]

1.

METHOD 1

$2\ln x - \ln 9 = 4$

uses $m\ln x = \ln x^m$ **(M1)**

$\ln x^2 - \ln 9 = 4$

uses $\ln a - \ln b = \ln\frac{a}{b}$ **(M1)**

$\ln\frac{x^2}{9} = 4$

$\frac{x^2}{9} = e^4$ **A1**

$x^2 = 9e^4 \Rightarrow x = \sqrt{9e^4}$ $(x > 0)$ **A1**

$x = 3e^2$ $(p = 3, q = 2)$ **A1**

METHOD 2

expresses 4 as $4\ln e$ and uses $\ln x^m = m\ln x$ **(M1)**

$2\ln x = 2\ln 3 + 4\ln e$ $\left(\ln x = \ln 3 + 2\ln e\right)$ **A1**

uses $2\ln e = \ln e^2$ and $\ln a + \ln b = \ln ab$ **(M1)**

$\ln x = \ln\left(3e^2\right)$ **A1**

$x = 3e^2$ $(p = 3, q = 2)$ **A1**

METHOD 3

expresses 4 as $4\ln e$ and uses $m\ln x = \ln x^m$ **(M1)**

$\ln x^2 = \ln 3^2 + \ln e^4$ **A1**

uses $\ln a + \ln b = \ln ab$ **(M1)**

$\ln x^2 = \ln\left(3^2 e^4\right)$

$x^2 = 3^2 e^4 \Rightarrow x = \sqrt{3^2 e^4}$ ($x > 0$) **A1**

so $x = 3e^2$ $\left(x > 0\right)$ ($p = 3, q = 2$) **A1**

Total [5 marks]

2.

uses $\sum \mathrm{P}\left(X = x\right)\left(=1\right)$ **(M1)**

$k^2 + \left(7k+2\right) + \left(-2k\right) + \left(3k^2\right)\left(=1\right)$

$4k^2 + 5k + 1\left(=0\right)$ **A1**

EITHER

attempts to factorize their quadratic **M1**

$\left(k+1\right)\left(4k+1\right) = 0$

OR

attempts use of the quadratic formula on their equation **M1**

$k = \frac{-5 \pm \sqrt{5^2 - 4\left(4\right)\left(1\right)}}{8}$ $\left(= \frac{-5 \pm 3}{8}\right)$

THEN

$k = -1, -\frac{1}{4}$ **A1**

rejects $k = -1$ as this value leads to invalid probabilities, for example, $\mathrm{P}\left(X = 2\right) = -5 < 0$ **R1**

so $k = -\frac{1}{4}$ **A1**

Note: Award **R0A1** if $k = -\frac{1}{4}$ is stated without a valid reason given for rejecting $k = -1$.

Total [6 marks]

3.

(a) **EITHER**

uses $u_2 - u_1 = u_3 - u_2$ **(M1)**

$6u_1 = 24$ **A1**

OR

uses $u_2 = \frac{u_1 + u_3}{2}$ **(M1)**

$5u_1 - 8 = \frac{u_1 + (3u_1 + 8)}{2}$

$3u_1 = 12$ **A1**

THEN

so $u_1 = 4$ **AG**

[2 marks]

(b) $d = 8$ **(A1)**

uses $S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$ **M1**

$S_n = \frac{n}{2}\left(8 + 8(n-1)\right)$ **A1**

$= 4n^2$

$= (2n)^2$ **A1**

Note: The final **A1** can be awarded for clearly explaining that $4n^2$ is a square number.

so sum of the first n terms is a square number **AG**

[4 marks]

Total [6 marks]

4.

$(f \circ g)(x) = ax + b - 2$ **(M1)**

$(f \circ g)(2) = -3 \Rightarrow 2a + b - 2 = -3 \ (2a + b = -1)$ **A1**

$(g \circ f)(x) = a(x-2) + b$ **(M1)**

$(g \circ f)(1) = 5 \Rightarrow -a + b = 5$ **A1**

a valid attempt to solve their two linear equations for a and b **M1**

so $a = -2$ and $b = 3$ **A1**

Total [6 marks]

5.

attempts either product rule or quotient rule differentiation **M1**

EITHER

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x^2 + bx}{(x+2)^2} + \frac{6x+b}{x+2}$$ **A1**

OR

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+2)(6x+b) - (3x^2 + bx)}{(x+2)^2}$$ **A1**

Note: Award **A0** if the denominator is incorrect. Subsequent marks can be awarded.

THEN

sets their $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ **M1**

$(x+2)(6x+b) - (3x^2 + bx) = 0$

$3x^2 + 12x + 2b = 0$ **A1**

(exactly one point of zero gradient requires) $12^2 - (4)(3)(2b) = 0$ **M1**

$b = 6$ **A1**

Total [6 marks]

6.

attempts to apply l'Hôpital's rule on $\lim_{x \to 0}\left(\frac{2x\cos\left(x^2\right)}{5\tan x}\right)$ **M1**

$= \lim_{x \to 0}\left(\frac{2\cos\left(x^2\right) - 4x^2\sin\left(x^2\right)}{5\sec^2 x}\right)$ **M1A1A1**

Note: Award **M1** for attempting to use product and chain rule differentiation on the numerator, **A1** for a correct numerator and **A1** for a correct denominator. The awarding of **A1** for the denominator is independent of the **M1**.

$= \frac{2}{5}$ **A1**

Total [5 marks]

7.

METHOD 1

from vertex P, draws a line parallel to $[\text{QR}]$ that meets $[\text{SR}]$ at a point X **(M1)**

uses the sine rule in ΔPSX **M1**

$\frac{\text{PS}}{\sin\beta} = \frac{y - x}{\sin\left(180° - \alpha - \beta\right)}$ **A1**

$\sin\left(180° - \alpha - \beta\right) = \sin\left(\alpha + \beta\right)$ **(A1)**

$\text{PS} = \frac{\left(y - x\right)\sin\beta}{\sin\left(\alpha + \beta\right)}$ **A1**

METHOD 2

let the height of quadrilateral PQRS be h

$h = \text{PS}\sin\alpha$ **A1**

attempts to find a second expression for h **M1**

$h = \left(y - x - \text{PS}\cos\alpha\right)\tan\beta$

$\text{PS}\sin\alpha = \left(y - x - \text{PS}\cos\alpha\right)\tan\beta$

writes $\tan\beta$ as $\frac{\sin\beta}{\cos\beta}$, multiplies through by $\cos\beta$ and expands the RHS **M1**

$$\text{PS}\sin\alpha\cos\beta = (y-x)\sin\beta - \text{PS}\cos\alpha\sin\beta$$

$$\text{PS} = \frac{(y-x)\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$$ **A1**

$$\text{PS} = \frac{(y-x)\sin\beta}{\sin(\alpha+\beta)}$$ **A1**

Total [5 marks]

8.

(a) attempts to calculate $\begin{pmatrix}2\\1\\m\end{pmatrix}\cdot\begin{pmatrix}2\\-5\\-m\end{pmatrix}$ **(M1)**

$= -1 - m^2$ **A1**

since $m^2 \geq 0$, $-1-m^2 < 0$ for $m \in \mathbb{R}$ **R1**

so l_1 and l_2 are never perpendicular to each other **AG**

[3 marks]

(b) (i) (since l_1 is parallel to Π, l_1 is perpendicular to the normal of Π and so)

$$\begin{pmatrix}2\\1\\m\end{pmatrix}\cdot\begin{pmatrix}1\\4\\-1\end{pmatrix} = 0$$ **R1**

$$2 + 4 - m = 0$$

$$m = 6$$ **A1**

(ii) since there are no points in common, $(3,-2,0)$ does not lie in Π

EITHER

substitutes $(3,-2,0)$ into $x+4y-z(\neq p)$ **(M1)**

OR

$$\begin{pmatrix}3\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\4\\-1\end{pmatrix}(\neq p)$$ **(M1)**

THEN

$p\neq -5$ **A1**

[4 marks]

Total [7 marks]

9.

let $\mathrm{P}(n)$ be the proposition that $\sum_{r=1}^{n}\cos(2r-1)\theta=\frac{\sin 2n\theta}{2\sin\theta}$ for $n\in\mathbb{Z}^{+}$

considering $\mathrm{P}(1)$:

$\mathrm{LHS}=\cos(1)\theta=\cos\theta$ and $\mathrm{RHS}=\frac{\sin 2\theta}{2\sin\theta}=\frac{2\sin\theta\cos\theta}{2\sin\theta}=\cos\theta=\mathrm{LHS}$

so $\mathrm{P}(1)$ is true **R1**

assume $\mathrm{P}(k)$ is true, i.e. $\sum_{r=1}^{k}\cos(2r-1)\theta=\frac{\sin 2k\theta}{2\sin\theta}$ $\left(k\in\mathbb{Z}^{+}\right)$ **M1**

Note: Award **M0** for statements such as "let $n=k$".

Note: Subsequent marks after this **M1** are independent of this mark and can be awarded.

considering $\mathrm{P}(k+1)$:

$$\sum_{r=1}^{k+1}\cos(2r-1)\theta=\sum_{r=1}^{k}\cos(2r-1)\theta+\cos(2(k+1)-1)\theta$$ **M1**

$$=\frac{\sin 2k\theta}{2\sin\theta}+\cos(2(k+1)-1)\theta$$ **A1**

$$=\frac{\sin 2k\theta+2\cos((2k+1)\theta)\sin\theta}{2\sin\theta}$$

$$=\frac{\sin 2k\theta+\sin((2k+1)\theta+\theta)-\sin((2k+1)\theta-\theta)}{2\sin\theta}$$ **M1**

Note: Award **M1** for use of $2\cos A\sin B=\sin(A+B)-\sin(A-B)$ with $A=(2k+1)\theta$ and $B=\theta$.

$$=\frac{\sin 2k\theta+\sin(2k+2)\theta-\sin 2k\theta}{2\sin\theta}$$ **A1**

$$=\frac{\sin 2(k+1)\theta}{2\sin\theta}$$ **A1**

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true, $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n\in\mathbb{Z}^+$ **R1**

Note: Award the final **R1** mark provided at least five of the previous marks have been awarded.

Total [8 marks]

Section B [56 marks]

10.

(a) attempts to find $h(0)$ **(M1)**

$h(0)=0.4\cos(0)+1.8(=2.2)$

2.2 (m) (above the ground) **A1**

[2 marks]

(b) **EITHER**

uses the minimum value of $\cos(\pi t)$ which is -1 **M1**

$0.4(-1)+1.8$ (m)

OR

the amplitude of motion is 0.4 (m) and the mean position is 1.8 (m) **M1**

OR

finds $h'(t)=-0.4\pi\sin(\pi t)$, attempts to solve $h'(t)=0$ for t and determines that the minimum height above the ground occurs at $t=1,3,...$ **M1**

$0.4(-1)+1.8$ (m)

THEN

1.4 (m) (above the ground) **A1**

[2 marks]

(c) **EITHER**

the ball is released from its maximum height and returns there a period later **R1**

the period is $\frac{2\pi}{\pi}(=2)$ (s) **A1**

OR

attempts to solve $h(t)=2.2$ for t **M1**

$\cos(\pi t)=1$

$t=0,2,...$ **A1**

THEN

so it takes 2 seconds for the ball to return to its initial position for the first time **AG**

[2 marks]

(d) $0.4\cos(\pi t)+1.8=1.8+0.2\sqrt{2}$ **(M1)**

$0.4\cos(\pi t)=0.2\sqrt{2}$

$\cos(\pi t)=\frac{\sqrt{2}}{2}$ **A1**

$\pi t=\frac{\pi}{4},\frac{7\pi}{4}$ **(A1)**

Note: Accept extra correct positive solutions for πt .

$t=\frac{1}{4},\frac{7}{4}\ (0\le t\le 2)$ **A1**

Note: Do not award **A1** if solutions outside $0\le t\le 2$ are also stated.

the ball is less than $1.8+0.2\sqrt{2}$ metres above the ground for $\frac{7}{4}-\frac{1}{4}$(s)

1.5 (s) **A1**

[5 marks]

(e) **EITHER**

attempts to find $h'(t)$ **(M1)**

OR

recognizes that $h'(t)$ is required **(M1)**

THEN

$h'(t) = -0.4\pi \sin(\pi t)$ **A1**

attempts to evaluate their $h'\left(\frac{1}{3}\right)$ **(M1)**

$h'\left(\frac{1}{3}\right) = -0.4\pi \sin\frac{\pi}{3}$

$= -0.2\pi\sqrt{3}\ \left(\text{ms}^{-1}\right)$ **A1**

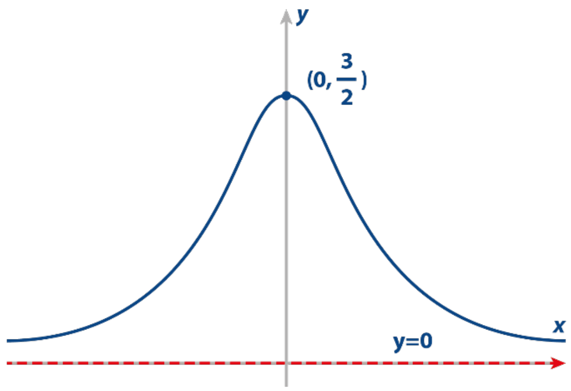
Note: Accept equivalent correct answer forms where $p \in \mathbb{Q}$. For example, $-\frac{1}{5}\pi\sqrt{3}$.

[4 marks]

Total [15 marks]

11.

(a)



a curve symmetrical about the y- axis with correct concavity that has a local maximum point on the positive y- axis **A1**

a curve clearly showing that $y \to 0$ as $x \to \pm\infty$ **A1**

$\left(0, \frac{3}{2}\right)$ **A1**

horizontal asymptote $y = 0$ (x- axis) **A1**

[4 marks]

(b) attempts to find $\int \frac{3}{x^2+2}\mathrm{d}x$ **(M1)**

$= \left[\frac{3}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}\right]$ **A1**

Note: Award **M1A0** for obtaining $\left[k\arctan\frac{x}{\sqrt{2}}\right]$ where $k \neq \frac{3}{\sqrt{2}}$.

Note: Condone the absence of or use of incorrect limits to this stage.

$= \frac{3}{\sqrt{2}}\left(\arctan\sqrt{3} - \arctan 0\right)$ **(M1)**

$= \frac{3}{\sqrt{2}}\times\frac{\pi}{3}\left(= \frac{\pi}{\sqrt{2}}\right)$ **A1**

$A = \frac{\sqrt{2}\pi}{2}$ **AG**

[4 marks]

(c) **METHOD 1**

EITHER

$\int_0^k \frac{3}{x^2+2}\mathrm{d}x = \frac{\sqrt{2}\pi}{4}$

$\frac{3}{\sqrt{2}}\arctan\frac{k}{\sqrt{2}} = \frac{\sqrt{2}\pi}{4}$ **(M1)**

OR

$\int_k^{\sqrt{6}} \frac{3}{x^2+2}\mathrm{d}x = \frac{\sqrt{2}\pi}{4}$

$\frac{3}{\sqrt{2}}\left(\arctan\sqrt{3} - \arctan\frac{k}{\sqrt{2}}\right) = \frac{\sqrt{2}\pi}{4}$ **(M1)**

$\arctan\sqrt{3} - \arctan\frac{k}{\sqrt{2}} = \frac{\pi}{6}$

THEN

$\arctan \frac{k}{\sqrt{2}} = \frac{\pi}{6}$ **A1**

$\frac{k}{\sqrt{2}} = \tan \frac{\pi}{6} \left(= \frac{1}{\sqrt{3}} \right)$ **A1**

$k = \frac{\sqrt{6}}{3} \left(= \sqrt{\frac{2}{3}} \right)$ **A1**

METHOD 2

$\int_0^k \frac{3}{x^2+2} \mathrm{d}x = \int_k^{\sqrt{6}} \frac{3}{x^2+2} \mathrm{d}x$

$\frac{3}{\sqrt{2}} \arctan \frac{k}{\sqrt{2}} = \frac{3}{\sqrt{2}} \left(\arctan \sqrt{3} - \arctan \frac{k}{\sqrt{2}} \right)$ **(M1)**

$\arctan \frac{k}{\sqrt{2}} = \frac{\pi}{6}$ **A1**

$\frac{k}{\sqrt{2}} = \tan \frac{\pi}{6} \left(= \frac{1}{\sqrt{3}} \right)$ **A1**

$k = \frac{\sqrt{6}}{3} \left(= \sqrt{\frac{2}{3}} \right)$ **A1**

[4 marks]

(d) attempts to find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{x^2+2} \right)$ **(M1)**

$= (3)(-1)(2x)(x^2+2)^{-2}$ **A1**

so $m = -\frac{6x}{(x^2+2)^2}$ **AG**

[2 marks]

(e) attempts product rule or quotient rule differentiation **M1**

EITHER

$$\frac{dm}{dx} = (-6x)(-2)(2x)(x^2+2)^{-3} + (x^2+2)^{-2}(-6)$$ **A1**

OR

$$\frac{dm}{dx} = \frac{(x^2+2)^2(-6) - (-6x)(2)(2x)(x^2+2)}{(x^2+2)^4}$$ **A1**

Note: Award **A0** if the denominator is incorrect. Subsequent marks can be awarded.

THEN

attempts to express their $\frac{dm}{dx}$ as a rational fraction with a factorized numerator **M1**

$$\frac{dm}{dx} = \frac{6(x^2+2)(3x^2-2)}{(x^2+2)^4} \left(= \frac{6(3x^2-2)}{(x^2+2)^3} \right)$$

attempts to solve their $\frac{dm}{dx} = 0$ for x **M1**

$$x = \pm\sqrt{\frac{2}{3}}$$ **A1**

from the curve, the maximum value of m occurs at $x = -\sqrt{\frac{2}{3}}$ **R1**

(the minimum value of m occurs at $x = \sqrt{\frac{2}{3}}$)

Note: Award **R1** for any equivalent valid reasoning.

maximum value of m is $-\frac{6\left(-\sqrt{\frac{2}{3}}\right)}{\left(\left(-\sqrt{\frac{2}{3}}\right)^2 + 2\right)^2}$ **A1**

leading to a maximum value of $\frac{27}{32}\sqrt{\frac{2}{3}}$ **AG**

[7 marks]

Total [21 marks]

12.

(a) uses the binomial theorem on $(\cos\theta + \mathrm{i}\sin\theta)^4$ **M1**

$$= {}^4C_0\cos^4\theta + {}^4C_1\cos^3\theta(\mathrm{i}\sin\theta) + {}^4C_2\cos^2\theta(\mathrm{i}^2\sin^2\theta) + {}^4C_3\cos\theta(\mathrm{i}^3\sin^3\theta) + {}^4C_4(\mathrm{i}^4\sin^4\theta)$$ **A1**

$$= (\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + \mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$$ **A1**

[3 marks]

(b) (using de Moivre's theorem with $n = 4$ gives) $\cos 4\theta + \mathrm{i}\sin 4\theta$ **(A1)**

equates both the real and imaginary parts of $\cos 4\theta + \mathrm{i}\sin 4\theta$ and

$(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta) + \mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$ **M1**

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ and $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

recognizes that $\cot 4\theta = \dfrac{\cos 4\theta}{\sin 4\theta}$ **(A1)**

substitutes for $\sin 4\theta$ and $\cos 4\theta$ into $\dfrac{\cos 4\theta}{\sin 4\theta}$ **M1**

$$\cot 4\theta = \frac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$$

divides the numerator and denominator by $\sin^4\theta$ to obtain

$$\cot 4\theta = \frac{\dfrac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{\sin^4\theta}}{\dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\sin^4\theta}}$$ **A1**

$$\cot 4\theta = \frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$$ **AG**

[5 marks]

(c) setting $\cot 4\theta = 0$ and putting $x = \cot^2 \theta$ in the numerator of

$\cot 4\theta = \dfrac{\cot^4 \theta - 6\cot^2 \theta + 1}{4\cot^3 \theta - 4\cot \theta}$ gives $x^2 - 6x + 1 = 0$ **M1**

attempts to solve $\cot 4\theta = 0$ for θ **M1**

$4\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \ldots \left(4\theta = \dfrac{1}{2}(2n+1)\pi, n = 0, 1, \ldots\right)$ **(A1)**

$\theta = \dfrac{\pi}{8}, \dfrac{3\pi}{8}$ **A1**

Note: Do not award the final **A1** if solutions other than $\theta = \dfrac{\pi}{8}, \dfrac{3\pi}{8}$ are listed.

finding the roots of $\cot 4\theta = 0$ $\left(\theta = \dfrac{\pi}{8}, \dfrac{3\pi}{8}\right)$ corresponds to finding the roots of

$x^2 - 6x + 1 = 0$ where $x = \cot^2 \theta$ **R1**

so the equation $x^2 - 6x + 1 = 0$ has roots $\cot^2 \dfrac{\pi}{8}$ and $\cot^2 \dfrac{3\pi}{8}$ **AG**

[5 marks]

(d) attempts to solve $x^2 - 6x + 1 = 0$ for x **M1**

$x = 3 \pm 2\sqrt{2}$ **A1**

since $\cot^2 \dfrac{\pi}{8} > \cot^2 \dfrac{3\pi}{8}$, $\cot^2 \dfrac{3\pi}{8}$ has the smaller value of the two roots **R1**

Note: Award **R1** for an alternative convincing valid reason.

so $\cot^2 \dfrac{3\pi}{8} = 3 - 2\sqrt{2}$ **A1**

[4 marks]

(e) let $y = \operatorname{cosec}^2\theta$

uses $\cot^2 \theta = \operatorname{cosec}^2\theta - 1$ where $x = \cot^2 \theta$ **(M1)**

$x^2 - 6x + 1 = 0 \Rightarrow (y-1)^2 - 6(y-1) + 1 = 0$ **M1**

$y^2 - 8y + 8 = 0$ **A1**

[3 marks]

Total [20 marks]