International Baccalaureate® Baccalauréat International Bachillerato Internacional

Mathematics: analysis and approaches

Practice paper 1 HL

Total 110

Section A [54 marks]

1. [Maximum mark: 5]

Solve the equation $2\ln x = \ln 9 + 4$. Give your answer in the form $x = pe^{q}$ where $p, q \in \mathbb{Z}^{+}$.

2. [Maximum mark: 6]

The following table shows the probability distribution of a discrete random variable X where $x = 1, 2, 3, 4$.

x	1	2	3	4
$\mathrm{P}(X = x)$	k^2	$7k + 2$	$-2k$	$3k^2$

Find the value of k, justifying your answer.

3. [Maximum mark: 6]

The first three terms of an arithmetic sequence are $u_1, 5u_1 - 8$ and $3u_1 + 8$.

(a) Show that $u_1 = 4$. [2]

(b) Prove that the sum of the first n terms of this arithmetic sequence is a square number. [4]

4. [Maximum mark: 6]

The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = x - 2$ and $g(x) = ax + b$, where $a, b \in \mathbb{R}$.

Given that $(f \circ g)(2) = -3$ and $(g \circ f)(1) = 5$, find the value of a and the value of b.

5. [Maximum mark: 6]

Consider the function $f(x) = \frac{3x^2 + bx}{x + 2}$ where $x \neq -2$ and $b \in \mathbb{R}$.

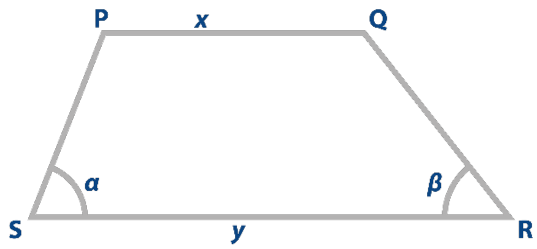
Find the value of b for which the graph of f has exactly one point of zero gradient.

6. [Maximum mark: 5]

Use l'Hôpital's rule to determine the value of $\lim_{x \to 0} \left(\frac{2x\cos(x^2)}{5\tan x} \right)$.

7. [Maximum mark: 5]

Consider quadrilateral PQRS where $[\text{PQ}]$ is parallel to $[\text{SR}]$.



In PQRS, $\text{PQ} = x$, $\text{SR} = y$, $\text{R}\hat{\text{S}}\text{P} = \alpha$ and $\text{Q}\hat{\text{R}}\text{S} = \beta$.

Find an expression for PS in terms of $x, y, \sin\beta$ and $\sin(\alpha + \beta)$.

8. [Maximum mark: 7]

The lines l_1 and l_2 have the following vector equations where $\lambda, \mu \in \mathbb{R}$ and $m \in \mathbb{R}$.

$$l_1 : \boldsymbol{r}_1 = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ m \end{pmatrix} \quad l_2 : \boldsymbol{r}_2 = \begin{pmatrix} -1 \\ -4 \\ -2m \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ -m \end{pmatrix}$$

(a) Show that l_1 and l_2 are never perpendicular to each other. [3]

The plane Π has Cartesian equation $x + 4y - z = p$ where $p \in \mathbb{R}$.

(b) Given that l_1 and Π have no points in common, find

(i) the value of m

(ii) and the condition on the value of p. [4]

9. [Maximum mark: 8]

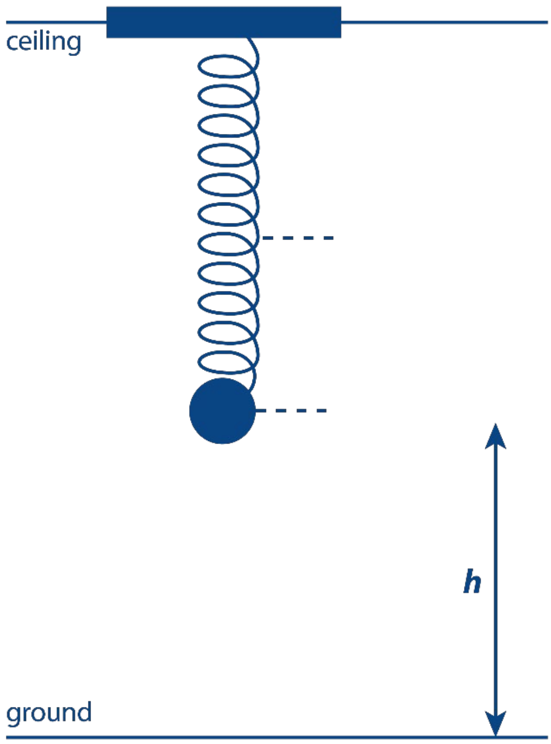
It is given that $2\cos A \sin B \equiv \sin(A+B) - \sin(A-B)$. (Do **not** prove this identity.)

Using mathematical induction and the above identity, prove that $\sum_{r=1}^{n} \cos(2r-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}$ for $n \in \mathbb{Z}^+$.

Section B [56 marks]

10. [Maximum mark: 15]

The following diagram shows a ball attached to the end of a spring, which is suspended from a ceiling.



The height, h metres, of the ball above the ground at time t seconds after being released can be modelled by the function $h(t) = 0.4\cos(\pi t) + 1.8$ where $t \geq 0$.

(a) Find the height of the ball above the ground when it is released. [2]

(b) Find the minimum height of the ball above the ground. [2]

(c) Show that the ball takes 2 seconds to return to its initial height above the ground for the first time. [2]

(d) For the first 2 seconds of its motion, determine the amount of time that the ball is less than $1.8 + 0.2\sqrt{2}$ metres above the ground. [5]

(e) Find the rate of change of the ball's height above the ground when $t = \frac{1}{3}$. Give your answer in the form $p\pi\sqrt{q}$ ms^{-1} where $p \in \mathbb{Q}$ and $q \in \mathbb{Z}^{+}$. [4]

11. [Maximum mark: 21]

A function f is defined by $f(x) = \frac{3}{x^2+2}$, $x \in \mathbb{R}$.

(a) Sketch the curve $y = f(x)$, clearly indicating any asymptotes with their equations and stating the coordinates of any points of intersection with the axes. [4]

The region R is bounded by the curve $y = f(x)$, the x- axis and the lines $x = 0$ and $x = \sqrt{6}$. Let A be the area of R.

(b) Show that $A = \frac{\sqrt{2}\pi}{2}$. [4]

The line $x = k$ divides R into two regions of equal area.

(c) Find the value of k. [4]

Let m be the gradient of a tangent to the curve $y = f(x)$.

(d) Show that $m = -\frac{6x}{\left(x^2+2\right)^2}$. [2]

(e) Show that the maximum value of m is $\frac{27}{32}\sqrt{\frac{2}{3}}$. [7]

12. [Maximum mark: 20]

(a) Use the binomial theorem to expand $(\cos\theta + \mathrm{i}\sin\theta)^4$. Give your answer in the form $a + b\mathrm{i}$ where a and b are expressed in terms of $\sin\theta$ and $\cos\theta$. [3]

(b) Use de Moivre's theorem and the result from part (a) to show that $\cot 4\theta = \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$. [5]

(c) Use the identity from part (b) to show that the quadratic equation $x^2 - 6x + 1 = 0$ has roots $\cot^2\dfrac{\pi}{8}$ and $\cot^2\dfrac{3\pi}{8}$. [5]

(d) Hence find the exact value of $\cot^2\dfrac{3\pi}{8}$. [4]

(e) Deduce a quadratic equation with integer coefficients, having roots $\operatorname{cosec}^2\dfrac{\pi}{8}$ and $\operatorname{cosec}^2\dfrac{3\pi}{8}$. [3]