International Baccalaureate®
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Mathematics: analysis and approaches

Practice paper 2 HL markscheme — Total 110

Section A [56 marks]

1.

$$\frac{\sum_{i=1}^{16} x_i}{16} = 14.5$$ **(M1)**

Note: Award **M1** for use of $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$.

$$\Rightarrow \sum_{i=1}^{16} x_i = 232$$ **(A1)**

new $\bar{x} = \frac{232-9}{15}$ **(A1)**

$$= 14.9 \left(= 14.8\overline{6}, = \frac{223}{15} \right)$$ **A1**

Note: Do not accept 15.

Total [4 marks]

2.

(a) attempts either graphical or symbolic means to find the value of t when $\frac{\mathrm{d}v}{\mathrm{d}t} = 0$ **(M1)**

$T = 0.465$ (s) **A1**

[2 marks]

(b) attempts to find the value of either $s_1 = \int_0^{0.46494...} v\ \mathrm{d}t$ or $s_2 = \int_{0.46494...}^{1} v\ \mathrm{d}t$ **(M1)**

$s_1 = 3.02758...$ and $s_2 = 3.47892...$ **A1A1**

Note: Award as above for obtaining, for example, $s_2 - s_1 = 0.45133...$ or $\frac{s_2}{s_1} = 1.14907....$

Note: Award a maximum of **M1A1A0FT** for use of an incorrect value of T from part (a).

so $s_2 > s_1$ **AG**

[3 marks]

Total [5 marks]

3.

(a) (i) $r = 0.946$ **A2**

(ii) the value of r shows a (very) strong positive correlation between age and (systolic) blood pressure **A1**

[3 marks]

(b) $p = 1.05t + 69.3$ **A1A1**

Note: Only award marks for an equation. Award **A1** for $a = 1.05$ and **A1** for $b = 69.3$. Award **A1A0** for $y = 1.05x + 69.3$.

[2 marks]

(c) 122 (mmHg) **(M1)A1**

[2 marks]

(d) the regression equation should not be used because it involves extrapolation **A1**

[1 mark]

Total [8 marks]

4.

attempts to find an expression for the discriminant, Δ, in terms of k **(M1)**

$\Delta = 4 - 4(k-1)(2k-3) \quad (= -8k^2 + 20k - 8)$ **(A1)**

Note: Award **M1A1** for finding $x = \dfrac{-2 \pm \sqrt{4 - 4(k-1)(2k-3)}}{2(k-1)}$.

attempts to solve $\Delta > 0$ for k **(M1)**

Note: Award **M1** for attempting to solve $\Delta = 0$ for k.

$\dfrac{1}{2} < k < 2$ **A1A1**

Note: Award **A1** for obtaining critical values $k = \dfrac{1}{2}, 2$ and **A1** for correct inequality signs.

Total [5 marks]

5.

(a) attempts to solve $x^2 \sin x = -1 - \sqrt{1 + 4(x+2)^2}$ **(M1)**

$x = -2.76, -1.54$ **A1A1**

Note: Award **A1A0** if additional solutions outside the domain are given.

[3 marks]

(b) $A=\int_{-2.762...}^{-1.537...}\left(-1-\sqrt{1+4(x+2)^2}-x^2\sin x\right)\mathrm{d}x$ (or equivalent) **(M1)(A1)**

Note: Award **M1** for attempting to form an integrand involving "top curve" – "bottom curve".

so $A=1.47$ **A2**

[4 marks]

Total [7 marks]

6.

(a) attempts implicit differentiation on both sides of the equation **M1**

$2\mathrm{e}^{2y}\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2+\frac{\mathrm{d}y}{\mathrm{d}x}$ **A1**

$\left(2\mathrm{e}^{2y}-1\right)\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2$ **A1**

so $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x^2}{2\mathrm{e}^{2y}-1}$ **AG**

[3 marks]

(b) attempts to solve $2\mathrm{e}^{2y}-1=0$ for y **(M1)**

$y=-0.346...\left(=\frac{1}{2}\ln\frac{1}{2}\right)$ **A1**

attempts to solve $\mathrm{e}^{2y}=x^3+y$ for x given their value of y **(M1)**

$x=0.946\left(=\left(\frac{1}{2}\left(1-\ln\frac{1}{2}\right)\right)^{\frac{1}{3}}\right)$ **A1**

[4 marks]

Total [7 marks]

7.

(a) $2+7x \equiv A(1-x)+B(1+2x)$

EITHER

substitutes $x=1$ and attempts to solve for B and substitutes $x=-\frac{1}{2}$ and attempts to solve for A **(M1)**

$9=3B \Rightarrow B=3$; $\frac{3A}{2}=-\frac{3}{2} \Rightarrow A=-1$

OR

forms $A+B=2$ and $-A+2B=7$ and attempts to solve for A and B **(M1)**

THEN

$A=-1$ and $B=3$ **A1A1**

[3 marks]

(b) uses the binomial expansion on either $3(1-x)^{-1}$ or $(1+2x)^{-1}$ **M1**

$3(1-x)^{-1}=3(1+x+x^2+...)$ **A1**

$(1+2x)^{-1}=\left(1-2x+\frac{(-1)(-2)}{2!}(2x)^2+...\right)(=1-2x+4x^2+...)$ **A1**

$3+3x+3x^2-(1-2x+4x^2)$

so the expansion is $2+5x-x^2$ (in ascending powers of x) **A1**

[4 marks]

(c) $(1+2x)^{-1}$ (is convergent) requires $|x|<\frac{1}{2}$ and $x=\frac{3}{4}$ is outside this so the expansion is not valid **R1**

[1 mark]

Total [8 marks]

8.

assume there exist $p, q \in \mathbb{N}$ where $q \geq 1$ such that $\log_2 5 = \frac{p}{q}$ **M1A1**

Note: Award **M1** for attempting to write the negation of the statement as an assumption. Award **A1** for a correctly stated assumption.

$\log_2 5 = \frac{p}{q} \Rightarrow 5 = 2^{\frac{p}{q}}$ **A1**

$5^q = 2^p$ **A1**

EITHER

5 is a factor of 5^q but not a factor of 2^p **R1**

OR

2 is a factor of 2^p but not a factor of 5^q **R1**

OR

5^q is odd and 2^p is even **R1**

THEN

no $p, q \in \mathbb{N}$ (where $q \geq 1$) satisfy the equation $5^q = 2^p$ and this is a contradiction **R1**

so $\log_2 5$ is an irrational number **AG**

Total [6 marks]

9.

METHOD 1

E_n is the event "the first tail occurs on the $2\text{nd}, 4\text{th}, 6\text{th}, \ldots, 2n\text{th}$ toss"

$\mathrm{P}(E) = \sum_{n=1}^{\infty} \mathrm{P}(E_n)$ **(A1)**

Note: Award **A1** for deducing that either 1 head before a tail or 3 heads before a tail or 5 heads before a tail etc. is required. In other words, deduces $(2n-1)$ heads before a tail.

$\mathrm{P}(E) = 0.4 \times 0.6 + (0.4)^3 \times 0.6 + (0.4)^5 \times 0.6 + \ldots$ **M1A1**

Note: Award **M1** for attempting to form an infinite geometric series.

Note: Award **A1** for $\mathrm{P}(E) = \sum_{n=1}^{\infty} (0.4)^{2n-1}(0.6)$.

uses $S_\infty = \frac{u_1}{1-r}$ with $u_1 = 0.6 \times 0.4$ and $r = (0.4)^2$ **(M1)**

Note: Award **M1** for using $S_\infty = \frac{u_1}{1-r}$ with $u_1 = 0.4$ and $r = (0.4)^2$.

$= \frac{0.6 \times 0.4}{1-(0.4)^2}$ **A1**

$= 0.286 \left(= \frac{2}{7}\right)$ **A1**

METHOD 2

let T_1 be the event "tail occurs on the first toss"

uses $\mathrm{P}(E) = \mathrm{P}(E \mid T_1)\mathrm{P}(T_1) + \mathrm{P}(E \mid T_1')\mathrm{P}(T_1')$ **M1**

concludes that $\mathrm{P}(E \mid T_1) = 0$ and so $\mathrm{P}(E) = \mathrm{P}(E \mid T_1')\mathrm{P}(T_1')$ **R1**

$\mathrm{P}(E \mid T_1') = \mathrm{P}(E')(= 1 - \mathrm{P}(E))$ **A1**

Note: Award **A1** for concluding: given that a tail is not obtained on the first toss, then $\mathrm{P}(E \mid T_1')$ is the probability that the first tail is obtained after a further odd number of tosses, $\mathrm{P}(E')$.

$\mathrm{P}(T_1') = 0.4$

$\mathrm{P}(E) = 0.4(1 - \mathrm{P}(E))$ **A1**

attempts to solve for $\mathrm{P}(E)$ **(M1)**

$= 0.286 \left(= \frac{2}{7}\right)$ **A1**

Total [6 marks]

Section B [54 marks]

10.

(a) $T \sim \mathrm{N}\left(\mu, 8.6^2\right)$

$\mathrm{P}(T \le 36.8) = 0.7$ **(A1)**

states a correct equation, for example, $\dfrac{36.8-\mu}{8.6} = 0.5244...$ **A1**

attempts to solve their equation **(M1)**

$\mu = 36.8 - (0.5244...)(8.6) \quad (=32.2902...)$ **A1**

the solution to the equation is $\mu = 32.29$, correct to two decimal places **AG**

[4 marks]

(b) let $t_{0.86}$ be the 86th percentile

attempts to use the inverse normal feature of a GDC to find $t_{0.86}$ **(M1)**

$t_{0.86} = 41.6$ (mins) **A1**

[2 marks]

(c) evidence of identifying the correct area under the normal curve **(M1)**

Note: Award **M1** for a clearly labelled sketch.

$\mathrm{P}(T > 30) = 0.605$ **A1**

[2 marks]

(d) let X represent the number of people out of the six who take more than 30 minutes to complete the jigsaw puzzle

$X \sim \mathrm{B}(6, 0.6049...)$ **(M1)**

for example, $\mathrm{P}(X=5)+\mathrm{P}(X=6)$ or $1-\mathrm{P}(X \le 4)$ **(A1)**

$\mathrm{P}(X \ge 5) = 0.241$ **A1**

[3 marks]

(e) recognizes that $\mathrm{P}(T > 30 \mid T \geq 25)$ is required **(M1)**

Note: Award **M1** for recognizing conditional probability.

$$= \frac{\mathrm{P}(T > 30 \cap T \geq 25)}{\mathrm{P}(T \geq 25)}$$ **(A1)**

$$= \frac{\mathrm{P}(T > 30)}{\mathrm{P}(T \geq 25)} = \frac{0.6049...}{0.8016...}$$ **M1**

$$= 0.755$$ **A1**

[4 marks]

Total [15 marks]

11.

(a) $\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix} \left(= 6\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right)$ **A1**

$\overrightarrow{\mathrm{AC}} = \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix} \left(= 6\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \right)$ **A1**

[2 marks]

(b) attempts to use $\cos \mathrm{B\hat{A}C} = \dfrac{\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{AC}}}{\left|\overrightarrow{\mathrm{AB}}\right|\left|\overrightarrow{\mathrm{AC}}\right|}$ **(M1)**

$$= \frac{\begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 0 \\ -6 \end{pmatrix}}{\sqrt{72} \times \sqrt{72}}$$ **A1**

$$= \frac{1}{2}$$ **A1**

so $\mathrm{B\hat{A}C} = 60°$ **AG**

[3 marks]

(c) attempts to find a vector normal to Π **M1**

for example, $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \begin{pmatrix} -36 \\ 36 \\ 36 \end{pmatrix} \left(= 36 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right)$ leading to **A1**

a vector normal to Π is $\boldsymbol{n} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

EITHER

substitutes $(5, -2, 5)$ (or $(5, 4, -1)$ or $(-1, -2, -1)$) into $-x + y + z = d$ and attempts to find the value of d **M1**

for example, $d = -5 - 2 + 5 \ (= -2)$

OR

attempts to use $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$ **M1**

for example, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

THEN

leading to the Cartesian equation of Π as $-x + y + z = -2$ **AG**

[3 marks]

(d) (i) $\boldsymbol{r}=\begin{pmatrix}7\\-4\\-3\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$ $(\lambda \in \mathbb{R})$ **A1**

(ii) substitutes $x=7-\lambda, y=-4+\lambda, z=-3+\lambda$ into $-x+y+z=-2$ **(M1)**

$-(7-\lambda)+(-4+\lambda)+(-3+\lambda)=-2 \ (3\lambda=12)$

$\lambda=4$ **A1**

shows a correct calculation for finding $d_{\min}$, for example, attempts to find

$\left|4\begin{pmatrix}-1\\1\\1\end{pmatrix}\right|$ **M1**

$d_{\min}=4\sqrt{3} \ (=6.93)$ **A1**

[5 marks]

(e) let the area of triangle ABC be A

EITHER

attempts to find $A=\frac{1}{2}\left|\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AC}}\right|$, for example **M1**

$A=\frac{1}{2}\left|\begin{pmatrix}-36\\36\\36\end{pmatrix}\right|$

OR

attempts to find $\frac{1}{2}\left|\overrightarrow{\mathrm{AB}}\right|\left|\overrightarrow{\mathrm{AC}}\right|\sin\theta$, for example **M1**

$A=\frac{1}{2}\times 6\sqrt{2}\times 6\sqrt{2}\times\frac{\sqrt{3}}{2}$ (where $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$)

THEN

$A = 18\sqrt{3}\ (= 31.2)$ **A1**

uses $V = \frac{1}{3}Ah$ where A is the area of triangle ABC and $h = d_{\min}$ **M1**

$V = \frac{1}{3} \times 18\sqrt{3} \times 4\sqrt{3}$

$= 72$ **A1**

[4 marks]

Total [17 marks]

12.

(a) $y = vx \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$ **M1**

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = f(v)$ **A1**

$\int \frac{\mathrm{d}v}{f(v) - v} = \int \frac{\mathrm{d}x}{x}$ **A1**

integrating the RHS, $\int \frac{\mathrm{d}v}{f(v) - v} = \ln x + C$ **AG**

[3 marks]

(b) **EITHER**

attempts to find $f(v)$ **M1**

$f(v) = v^2 + 3v + 2$ **(A1)**

substitutes their $f(v)$ into $\int \frac{\mathrm{d}v}{f(v) - v}$ **M1**

$\int \frac{\mathrm{d}v}{f(v) - v} = \int \frac{\mathrm{d}v}{v^2 + 2v + 2}$

attempts to complete the square **(M1)**

$= \int \frac{\mathrm{d}v}{(v+1)^2 + 1}$ **A1**

$\arctan(v+1) (= \ln x + C)$ **A1**

OR

attempts to find $f(v)$ **M1**

$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = v^2 + 3v + 2$ **A1**

$\int \frac{\mathrm{d}v}{v^2 + 2v + 2} = \int \frac{\mathrm{d}x}{x}$ **M1**

attempts to complete the square **(M1)**

$\int \frac{\mathrm{d}v}{(v+1)^2 + 1} \left(= \int \frac{\mathrm{d}x}{x} \right)$ **A1**

$\arctan(v+1) = \ln x (+C)$ **A1**

THEN

when $x = 1$, $v = -1$ (or $y = -1$) and so $C = 0$ **M1**

substitutes for v into their expression **M1**

$$\arctan\left(\frac{y}{x}+1\right) = \ln x$$

$$\frac{y}{x}+1 = \tan(\ln x)$$ **A1**

so $y = x(\tan(\ln x) - 1)$ **AG**

[9 marks]

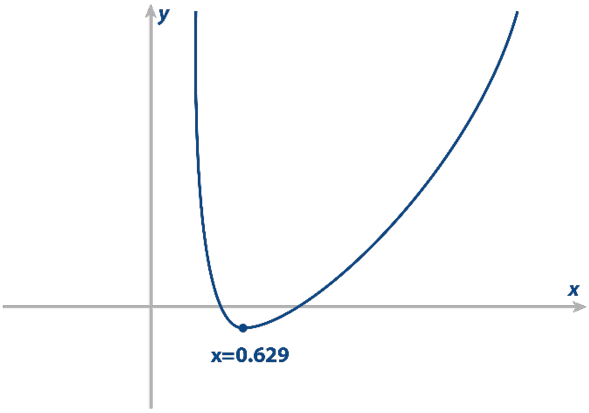
(c) **METHOD 1**

EITHER

a correct graph of $y = f'(x)$ (for approximately $e^{-\frac{\pi}{2}} < x < e^{\frac{\pi}{2}}$) with a local minimum point below the x- axis **A2**

Note: Award **M1A1** for $\frac{dy}{dx} = \tan(\ln x) + \sec^2(\ln x) - 1$.

attempts to find the x- coordinate of the local minimum point on the graph of $y = f'(x)$ **(M1)**

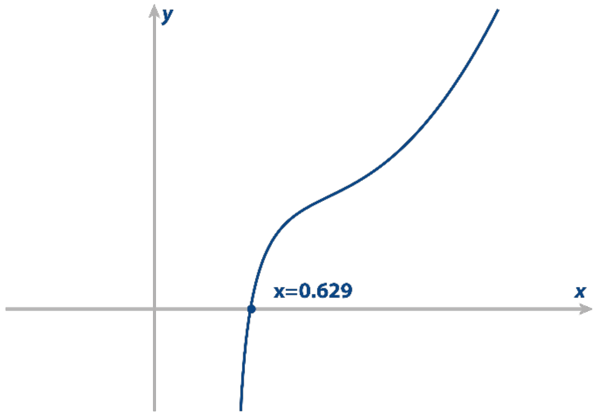


OR

a correct graph of $y = f''(x)$ (for approximately $e^{-\frac{\pi}{2}} < x < e^{\frac{\pi}{2}}$) showing the location of the x- intercept **A2**

Note: Award **M1A1** for $\frac{d^2y}{dx^2} = \frac{\sec^2(\ln x)}{x} + \frac{2\sec^2(\ln x)\tan(\ln x)}{x}$.

attempts to find the x- intercept **(M1)**



THEN

$x = 0.629 \left(= e^{-\arctan\frac{1}{2}}\right)$ **A1**

attempts to find $f(0.629...)$ $\left(f\left(e^{-\arctan\frac{1}{2}}\right)\right)$ **(M1)**

the coordinates are $(0.629, -0.943)$ $\left(e^{-\arctan\frac{1}{2}}, -\frac{3}{2}e^{-\arctan\frac{1}{2}}\right)$ **A1**

METHOD 2

attempts implicit differentiation on $\frac{\mathrm{d}y}{\mathrm{d}x}$ to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ **M1**

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{(2y+3x)\left(x\frac{\mathrm{d}y}{\mathrm{d}x} - y\right)}{x^3}$ (or equivalent)

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \Rightarrow y = -\frac{3x}{2} \left(\frac{\mathrm{d}y}{\mathrm{d}x} \neq \frac{y}{x}\right)$ **A1**

attempts to solve $-\frac{3x}{2} = x(\tan(\ln x) - 1)$ for x where $\mathrm{e}^{-\frac{\pi}{2}} < x < \mathrm{e}^{\frac{\pi}{2}}$ **M1**

$x = 0.629 \left(= \mathrm{e}^{-\arctan\frac{1}{2}}\right)$ **A1**

attempts to find $f(0.629...)$ $\left(f\left(\mathrm{e}^{-\arctan\frac{1}{2}}\right)\right)$ **(M1)**

the coordinates are $(0.629, -0.943)$ $\left(\mathrm{e}^{-\arctan\frac{1}{2}}, -\frac{3}{2}\mathrm{e}^{-\arctan\frac{1}{2}}\right)$ **A1**

[6 marks]

(d) $\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow y^2 + 3xy + 2x^2 = 0$ **M1**

attempts to solve $y^2 + 3xy + 2x^2 = 0$ for y **M1**

$(y+2x)(y+x) = 0$ or $y = \frac{-3x \pm \sqrt{(3x)^2 - 4(2x^2)}}{2}$ $\left(= \frac{-3x \pm x}{2}, (x > 0)\right)$ **A1**

$y = -2x$ and $y = -x$ $(m = -2, -1)$ **A1**

Note: Award **M1** for stating $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, **M1** for substituting $y = mx$ into $\frac{\mathrm{d}y}{\mathrm{d}x}(=0)$, **A1** for $(m+2)(m+1) = 0$ and ***A1*** for $m = -2, -1 \Rightarrow y = -2x$ and $y = -x$.

[4 marks]

Total [22 marks]