Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

Specimen paper

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Let A and B be events such that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.4$ and $\mathrm{P}(A \cup B) = 0.6$.
Find $\mathrm{P}(A \mid B)$.

2. [Maximum mark: 5]

(a) Show that $(2n-1)^2+(2n+1)^2=8n^2+2$, where $n \in \mathbb{Z}$. [2]

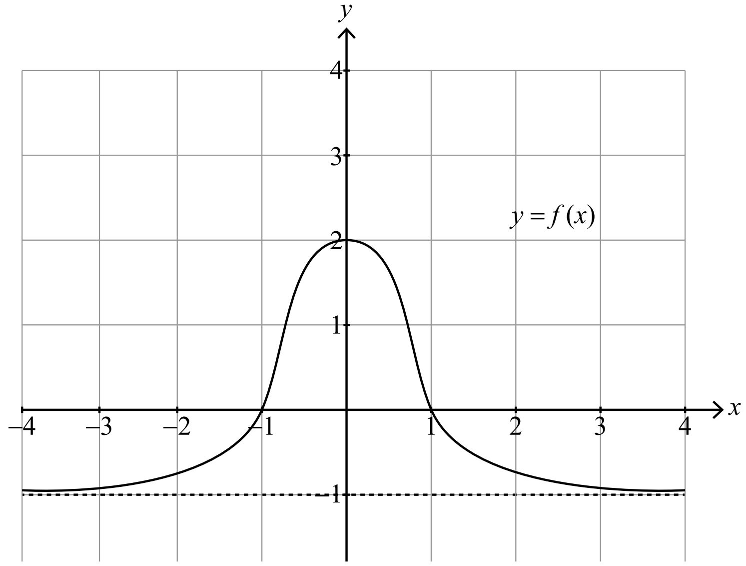
(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. [3]

3. [Maximum mark: 5]

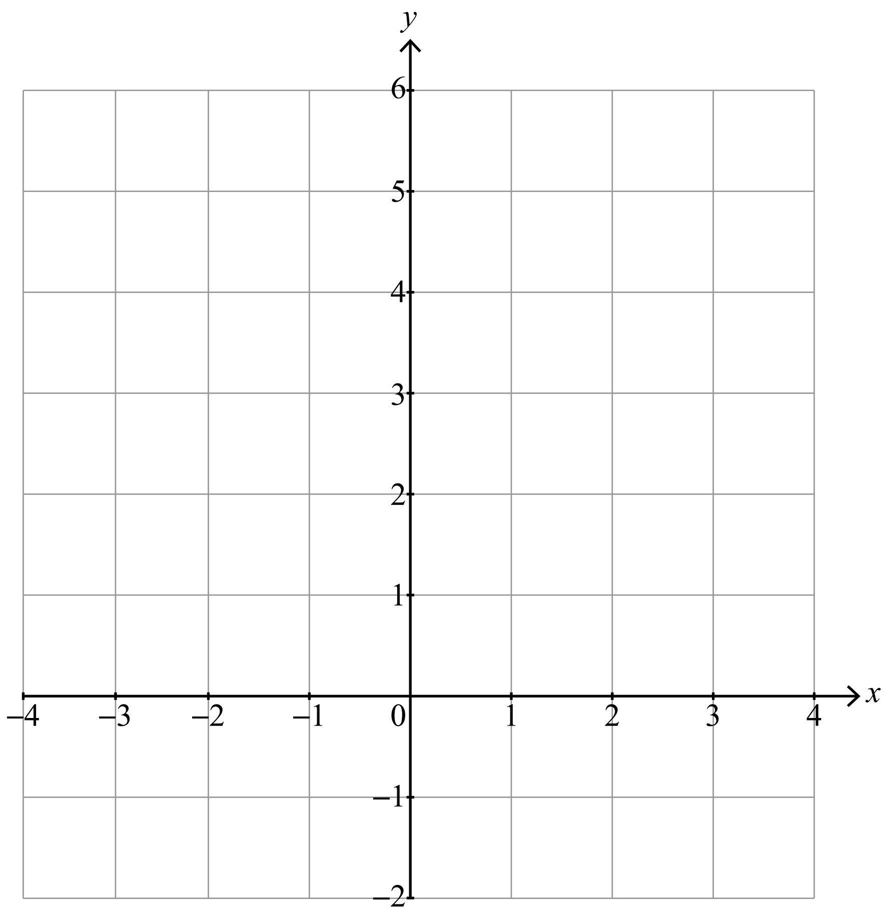
Let $f'(x) = \dfrac{8x}{\sqrt{2x^2 + 1}}$. Given that $f(0) = 5$, find $f(x)$.

4. [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$.



On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



5. [Maximum mark: 5]

The functions f and g are defined such that $f(x) = \frac{x+3}{4}$ and $g(x) = 8x + 5$.

(a) Show that $(g \circ f)(x) = 2x + 11$. [2]

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of a. [3]

6. [Maximum mark: 8]

(a) Show that $\log_9(\cos 2x + 2) = \log_3\sqrt{\cos 2x + 2}$. [3]

(b) Hence or otherwise solve $\log_3(2\sin x) = \log_9(\cos 2x + 2)$ for $0 < x < \frac{\pi}{2}$. [5]

. . . .

7. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \dfrac{\pi x}{36}\sin\left(\dfrac{\pi x}{6}\right), & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}.$$

Find $\mathrm{P}(0 \le X \le 3)$.

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8. [Maximum mark: 7]

The plane Π has the Cartesian equation $2x + y + 2z = 3$.

The line L has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line L and the plane Π is 30°.

Find the possible values of p.

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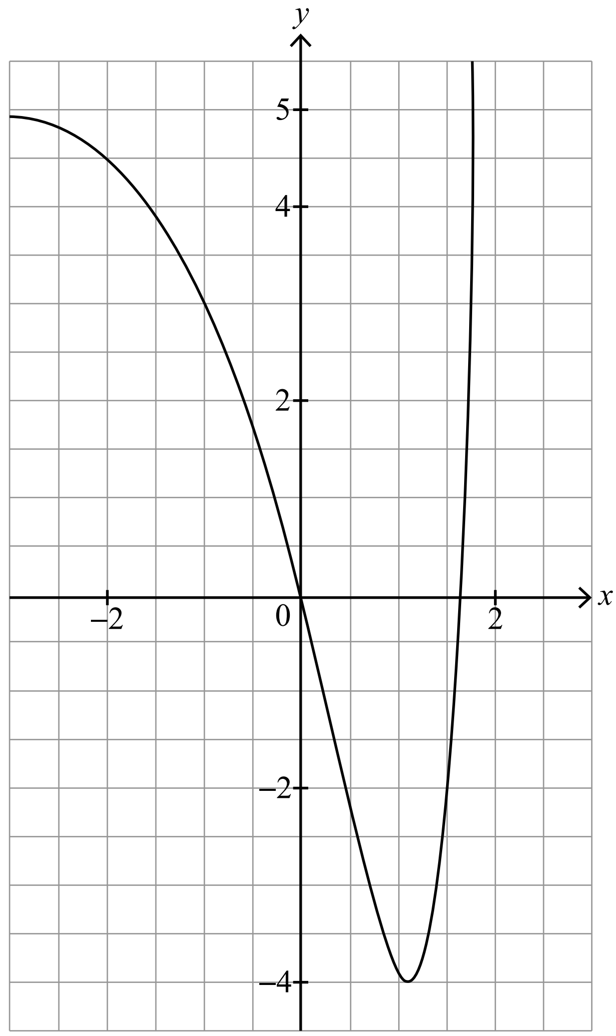
9. [Maximum mark: 8]

The function f is defined by $f(x) = e^{2x} - 6e^{x} + 5$, $x \in \mathbb{R}$, $x \leq a$. The graph of $y = f(x)$ is shown in the following diagram.



(a) Find the largest value of a such that f has an inverse function. [3]

(b) For this value of a, find an expression for $f^{-1}(x)$, stating its domain. [5]

(This question continues on the following page)

(Question 9 continued)

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Let $f(x) = \dfrac{\ln 5x}{kx}$ where $x > 0$, $k \in \mathbb{R}^{+}$.

(a) Show that $f'(x) = \dfrac{1 - \ln 5x}{kx^2}$. [3]

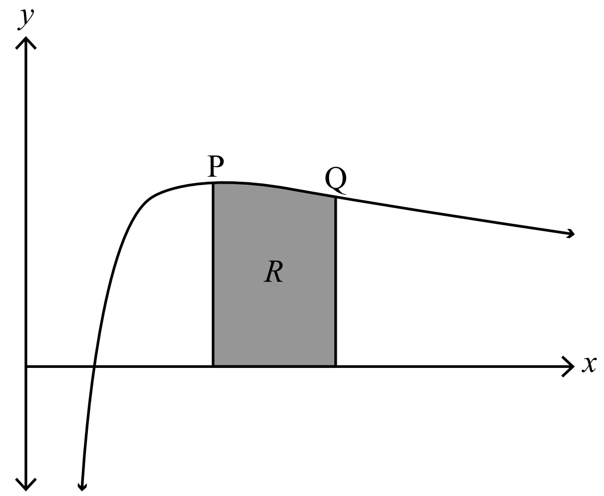
The graph of f has exactly one maximum point P.

(b) Find the x-coordinate of P. [3]

The second derivative of f is given by $f''(x) = \dfrac{2\ln 5x - 3}{kx^3}$. The graph of f has exactly one point of inflexion Q.

(c) Show that the x-coordinate of Q is $\dfrac{1}{5}\mathrm{e}^{\frac{3}{2}}$. [3]

The region R is enclosed by the graph of f, the x-axis, and the vertical lines through the maximum point P and the point of inflexion Q.



(d) Given that the area of R is 3, find the value of k. [7]

Do **not** write solutions on this page.

11. [Maximum mark: 18]

(a) Express $-3+\sqrt{3}\mathrm{i}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [5]

Let the roots of the equation $z^3=-3+\sqrt{3}\mathrm{i}$ be u, v and w.

(b) Find u, v and w expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\leq\pi$. [5]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(c) Find the area of triangle UVW. [4]

(d) By considering the sum of the roots u, v and w, show that $\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0$. [4]

12. [Maximum mark: 21]

The function f is defined by $f(x)=\mathrm{e}^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the x^2 term. [8]

(b) Show that the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $\mathrm{e}^{3x}-1$, find the Maclaurin series for $\arctan\left(\mathrm{e}^{3x}-1\right)$ up to and including the x^3 term. [6]

(d) Hence, or otherwise, find $\lim_{x\to 0}\frac{f(x)-1}{\arctan\left(\mathrm{e}^{3x}-1\right)}$. [3]

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will not be marked.

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Markscheme

Specimen paper

Mathematics: analysis and approaches

Higher level

Paper 1

16 pages

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme eg ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***M2***, ***N3***, *etc*., do **not** split the marks, unless there is a note.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	**Further working seen**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

4 Follow through marks (only applied after an error is made)

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) or subpart(s). Usually, to award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if the only marks awarded in a subpart are for the answer (i.e. there is no working expected), then **FT** marks should be awarded if appropriate.*

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, use of $r > 1$ for the sum of an infinite GP, $\sin\theta = 1.5$, non integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- Exceptions to this rule will be explicitly noted on the markscheme.
- If a candidate makes an error in one part, but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the question says hence. It is often possible to use a different approach in subsequent parts that does not depend on the answer to previous parts.

5 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- The ***MR*** penalty can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

8 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. There are two types of accuracy errors, and the final answer mark should not be awarded if these errors occur.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

9 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

Section A

1. attempt to substitute into $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***(M1)***

Note: Accept use of Venn diagram or other valid method.

$0.6 = 0.5 + 0.4 - \mathrm{P}(A \cap B)$ ***(A1)***

$\mathrm{P}(A \cap B) = 0.3$ (seen anywhere) ***A1***

attempt to substitute into $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ ***(M1)***

$= \dfrac{0.3}{0.4}$

$\mathrm{P}(A \mid B) = 0.75 \left(= \dfrac{3}{4}\right)$ ***A1***

Total [5 marks]

2. (a) attempting to expand the LHS ***(M1)***

$\mathrm{LHS} = \left(4n^2 - 4n + 1\right) + \left(4n^2 + 4n + 1\right)$ ***A1***

$= 8n^2 + 2 \left(= \mathrm{RHS}\right)$ ***AG***

[2 marks]

(b) ***METHOD 1***

recognition that $2n-1$ and $2n+1$ represent two consecutive odd integers (for $n \in \mathbb{Z}$) ***R1***

$8n^2 + 2 = 2\left(4n^2 + 1\right)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

METHOD 2

recognition, *eg* that n and $n+2$ represent two consecutive odd integers (for $n \in \mathbb{Z}$) ***R1***

$n^2 + (n+2)^2 = 2\left(n^2 + 2n + 2\right)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

Total [5 marks]

3. attempt to integrate ***(M1)***

$u = 2x^2 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 4x$

$\int \frac{8x}{\sqrt{2x^2+1}}\,\mathrm{d}x = \int \frac{2}{\sqrt{u}}\,\mathrm{d}u$ ***(A1)***

EITHER

$= 4\sqrt{u}\,(+C)$ ***A1***

OR

$= 4\sqrt{2x^2+1}\,(+C)$ ***A1***

THEN

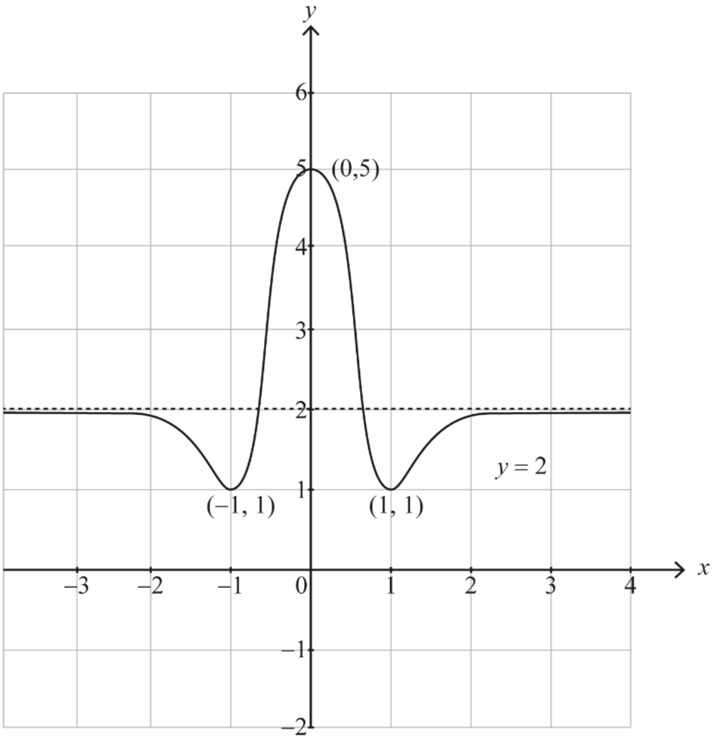
correct substitution into **their** integrated function (must have C) ***(M1)***

$5 = 4 + C \Rightarrow C = 1$

$f(x) = 4\sqrt{2x^2+1} + 1$ ***A1***

Total [5 marks]

4.



no y values below 1 — ***A1***

horizontal asymptote at $y = 2$ with curve approaching from below as $x \to \pm\infty$ — ***A1***

$(\pm 1, 1)$ local minima — ***A1***

$(0, 5)$ local maximum — ***A1***

smooth curve and smooth stationary points — ***A1***

Total [5 marks]

5. (a) attempt to form composition — ***M1***

correct substitution $g\left(\frac{x+3}{4}\right) = 8\left(\frac{x+3}{4}\right) + 5$ — ***A1***

$(g \circ f)(x) = 2x + 11$ — ***AG***

[2 marks]

(b) attempt to substitute 4 (seen anywhere) — ***(M1)***

correct equation $a = 2 \times 4 + 11$ — ***(A1)***

$a = 19$ — ***A1***

[3 marks]

Total [5 marks]

6. (a) attempting to use the change of base rule ***M1***

$\log_9(\cos 2x+2)=\dfrac{\log_3(\cos 2x+2)}{\log_3 9}$ ***A1***

$=\dfrac{1}{2}\log_3(\cos 2x+2)$ ***A1***

$=\log_3\sqrt{\cos 2x+2}$ ***AG***

[3 marks]

(b) $\log_3(2\sin x)=\log_3\sqrt{\cos 2x+2}$

$2\sin x=\sqrt{\cos 2x+2}$ ***M1***

$4\sin^2 x=\cos 2x+2$ (or equivalent) ***A1***

use of $\cos 2x=1-2\sin^2 x$ ***(M1)***

$6\sin^2 x=3$

$\sin x=(\pm)\dfrac{1}{\sqrt{2}}$ ***A1***

$x=\dfrac{\pi}{4}$ ***A1***

Note: Award ***A0*** if solutions other than $x=\dfrac{\pi}{4}$ are included.

[5 marks]

Total [8 marks]

7. attempting integration by parts, *eg*

$u = \frac{\pi x}{36}, \mathrm{d}u = \frac{\pi}{36}\mathrm{d}x, \mathrm{d}v = \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x, v = -\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)$ ***(M1)***

$\mathrm{P}(0 \le X \le 3) = \frac{\pi}{36}\left(\left[-\frac{6x}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^3 + \frac{6}{\pi}\int_0^3 \cos\left(\frac{\pi x}{6}\right)\mathrm{d}x\right)$ (or equivalent) ***A1A1***

Note: Award ***A1*** for a correct uv and ***A1*** for a correct $\int v\ \mathrm{d}u$.

attempting to substitute limits ***M1***

$\frac{\pi}{36}\left[-\frac{6x}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^3 = 0$ ***(A1)***

so $\mathrm{P}(0 \le X \le 3) = \frac{1}{\pi}\left[\sin\left(\frac{\pi x}{6}\right)\right]_0^3$ (or equivalent) ***A1***

$= \frac{1}{\pi}$ ***A1***

Total [7 marks]

8. recognition that the angle between the normal and the line is $60°$ (seen anywhere) ***R1***
attempt to use the formula for the scalar product ***M1***

$\cos 60° = \frac{\left|\begin{pmatrix}2\\1\\2\end{pmatrix} \bullet \begin{pmatrix}1\\-2\\p\end{pmatrix}\right|}{\sqrt{9} \times \sqrt{1+4+p^2}}$ ***A1***

$\frac{1}{2} = \frac{|2p|}{3\sqrt{5+p^2}}$ ***A1***

$3\sqrt{5+p^2} = 4|p|$

attempt to square both sides ***M1***

$9(5+p^2) = 16p^2 \Rightarrow 7p^2 = 45$

$p = \pm 3\sqrt{\frac{5}{7}}$ (or equivalent) ***A1A1***

Total [7 marks]

9. (a) attempt to differentiate and set equal to zero ***M1***

$f'(x) = 2e^{2x} - 6e^{x} = 2e^{x}(e^{x} - 3) = 0$ ***A1***

minimum at $x = \ln 3$

$a = \ln 3$ ***A1***

[3 marks]

(b) **Note:** Interchanging x and y can be done at any stage.

$y = \left(e^{x} - 3\right)^{2} - 4$ ***(M1)***

$e^{x} - 3 = \pm\sqrt{y+4}$ ***A1***

as $x \le \ln 3$, $x = \ln\left(3 - \sqrt{y+4}\right)$ ***R1***

so $f^{-1}(x) = \ln\left(3 - \sqrt{x+4}\right)$ ***A1***

domain of f^{-1} is $x \in \mathbb{R}, -4 \le x < 5$ ***A1***

[5 marks]

Total [8 marks]

Section B

10. (a) attempt to use quotient rule *(M1)*
correct substitution into quotient rule

$$f'(x) = \frac{5kx\left(\frac{1}{5x}\right) - k\ln 5x}{(kx)^2} \text{ (or equivalent)}$$ ***A1***

$$= \frac{k - k\ln 5x}{k^2x^2}, \left(k \in \mathbb{R}^+\right)$$ ***A1***

$$= \frac{1 - \ln 5x}{kx^2}$$ ***AG***

[3 marks]

(b) $f'(x) = 0$ ***M1***

$$\frac{1 - \ln 5x}{kx^2} = 0$$

$\ln 5x = 1$ ***(A1)***

$x = \frac{e}{5}$ ***A1***

[3 marks]

(c) $f''(x) = 0$ ***M1***

$$\frac{2\ln 5x - 3}{kx^3} = 0$$

$\ln 5x = \frac{3}{2}$ ***A1***

$5x = e^{\frac{3}{2}}$ ***A1***

so the point of inflexion occurs at $x = \frac{1}{5}e^{\frac{3}{2}}$ ***AG***

[3 marks]

continued…

Question 10 continued

(d) attempt to integrate ***(M1)***

$u = \ln 5x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\int \frac{\ln 5x}{kx} \mathrm{d}x = \frac{1}{k}\int u \ \mathrm{d}u$ ***(A1)***

EITHER

$= \frac{u^2}{2k}$ ***A1***

so $\frac{1}{k}\int_1^{\frac{3}{2}} u \ \mathrm{d}u = \left[\frac{u^2}{2k}\right]_1^{\frac{3}{2}}$ ***A1***

OR

$= \frac{(\ln 5x)^2}{2k}$ ***A1***

so $\int_{\frac{e}{5}}^{\frac{1}{5}e^{\frac{3}{2}}} \frac{\ln 5x}{kx} \mathrm{d}x = \left[\frac{(\ln 5x)^2}{2k}\right]_{\frac{e}{5}}^{\frac{1}{5}e^{\frac{3}{2}}}$ ***A1***

THEN

$= \frac{1}{2k}\left(\frac{9}{4} - 1\right)$

$= \frac{5}{8k}$ ***A1***

setting **their** expression for area equal to 3 ***M1***

$\frac{5}{8k} = 3$

$k = \frac{5}{24}$ ***A1***

[7 marks]

Total [16 marks]

11. (a) attempt to find modulus ***(M1)***

$r = 2\sqrt{3}\left(=\sqrt{12}\right)$ ***A1***

attempt to find argument in the correct quadrant ***(M1)***

$\theta = \pi + \arctan\left(-\frac{\sqrt{3}}{3}\right)$ ***A1***

$= \frac{5\pi}{6}$ ***A1***

$-3+\sqrt{3}\mathrm{i} = \sqrt{12}\mathrm{e}^{\frac{5\pi \mathrm{i}}{6}}\left(=2\sqrt{3}\mathrm{e}^{\frac{5\pi \mathrm{i}}{6}}\right)$

[5 marks]

(b) attempt to find a root using de Moivre's theorem ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{5\pi \mathrm{i}}{18}}$ ***A1***

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{-\frac{7\pi \mathrm{i}}{18}}$ ***A1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{17\pi \mathrm{i}}{18}}$ ***A1***

Note: Ignore labels for u, v and w at this stage.

[5 marks]

continued…

Question 11 continued

(c) ***METHOD 1***

attempting to find the total area of (congruent) triangles UOV, VOW and UOW ***M1***

$\text{Area} = 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$.

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

METHOD 2

$\text{UV}^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$ (or equivalent) ***A1***

$\text{UV} = \sqrt{3}\left(12^{\frac{1}{6}}\right)$ (or equivalent) ***A1***

attempting to find the area of UVW using $\text{Area} = \frac{1}{2}\times\text{UV}\times\text{VW}\times\sin\alpha$ for example ***M1***

$\text{Area} = \frac{1}{2}\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

(d) $u + v + w = 0$ ***R1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \text{i}\sin\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \text{i}\sin\frac{5\pi}{18} + \cos\frac{17\pi}{18} + \text{i}\sin\frac{17\pi}{18}\right) = 0$ ***A1***

consideration of real parts ***M1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right) + \cos\frac{5\pi}{18} + \cos\frac{17\pi}{18}\right) = 0$

$\cos\left(-\frac{7\pi}{18}\right) = \cos\frac{7\pi}{18}$ explicitly stated ***A1***

$\cos\frac{5\pi}{18} + \cos\frac{7\pi}{18} + \cos\frac{17\pi}{18} = 0$ ***AG***

[4 marks]

Total [18 marks]

12. (a) attempting to use the chain rule to find the first derivative ***M1***

$f'(x)=(\cos x)e^{\sin x}$ ***A1***

attempting to use the product rule to find the second derivative ***M1***

$f''(x)=e^{\sin x}(\cos^2 x-\sin x)$ (or equivalent) ***A1***

attempting to find $f(0)$, $f'(0)$ and $f''(0)$ ***M1***

$f(0)=1$; $f'(0)=(\cos 0)e^{\sin 0}=1$; $f''(0)=e^{\sin 0}(\cos^2 0-\sin 0)=1$ ***A1***

substitution into the Maclaurin formula $f(x)=f(0)+xf'(0)+\dfrac{x^2}{2!}f''(0)+...$ ***M1***

so the Maclaurin series for $f(x)$ up to and including the x^2 term is $1+x+\dfrac{x^2}{2}$ ***A1***

[8 marks]

(b) ***METHOD 1***

attempting to differentiate $f''(x)$ ***M1***

$f'''(x)=(\cos x)e^{\sin x}(\cos^2 x-\sin x)-(\cos x)e^{\sin x}(2\sin x+1)$ (or equivalent) ***A2***

substituting $x=0$ into **their** $f'''(x)$ ***M1***

$f'''(0)=1(1-0)-1(0+1)=0$

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

METHOD 2

substituting $\sin x$ into the Maclaurin series for e^x ***(M1)***

$e^{\sin x}=1+\sin x+\dfrac{\sin^2 x}{2!}+\dfrac{\sin^3 x}{3!}+...$

substituting Maclaurin series for $\sin x$ ***M1***

$$e^{\sin x}=1+\left(x-\frac{x^3}{3!}+...\right)+\frac{\left(x-\frac{x^3}{3!}+...\right)^2}{2!}+\frac{\left(x-\frac{x^3}{3!}+...\right)^3}{3!}+...$$ ***A1***

coefficient of x^3 is $-\dfrac{1}{3!}+\dfrac{1}{3!}=0$ ***A1***

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

[4 marks]

continued…

Question 12 continued

(c) substituting $3x$ into the Maclaurin series for e^x ***M1***

$$e^{3x} = 1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + ...$$ ***A1***

substituting $\left(e^{3x} - 1\right)$ into the Maclaurin series for $\arctan x$ ***M1***

$$\arctan\left(e^{3x} - 1\right) = \left(e^{3x} - 1\right) - \frac{\left(e^{3x} - 1\right)^3}{3} + \frac{\left(e^{3x} - 1\right)^5}{5} - ...$$

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots\right) - \frac{\left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \dots\right)^3}{3} + ...$$ ***A1***

selecting correct terms from above ***M1***

$$= \left(3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!}\right) - \frac{(3x)^3}{3}$$

$$= 3x + \frac{9x^2}{2} - \frac{9x^3}{2}$$ ***A1***

[6 marks]

(d) ***METHOD 1***

substitution of **their** series ***M1***

$$\lim_{x \to 0} \frac{x + \frac{x^2}{2} + ...}{3x + \frac{9x^2}{2} + ...}$$ ***A1***

$$= \lim_{x \to 0} \frac{1 + \frac{x}{2} + ...}{3 + \frac{9x}{2} + ...}$$

$$= \frac{1}{3}$$ ***A1***

METHOD 2

use of l'Hôpital's rule ***M1***

$$\lim_{x \to 0} \frac{(\cos x)e^{\sin x}}{\frac{3e^{3x}}{1 + \left(e^{3x} - 1\right)^2}}$$ (or equivalent) ***A1***

$$= \frac{1}{3}$$ ***A1***

[3 marks]

Total [21 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 2

Specimen

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

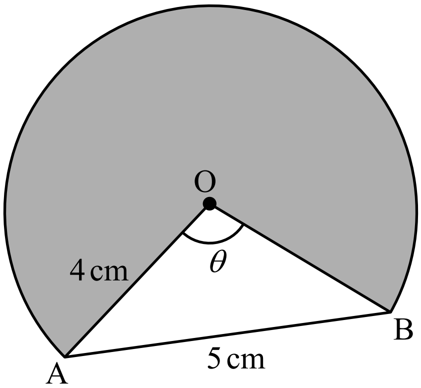
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows part of a circle with centre O and radius $4\,\text{cm}$.



Chord AB has a length of $5\,\text{cm}$ and $\text{A}\hat{\text{O}}\text{B} = \theta$.

(a) Find the value of θ, giving your answer in radians. [3]

(b) Find the area of the shaded region. [3]

2. [Maximum mark: 6]

On 1st January 2020, Laurie invests $\$P$ in an account that pays a nominal annual interest rate of 5.5%, compounded **quarterly**.

The amount of money in Laurie's account **at the end of each year** follows a geometric sequence with common ratio, r.

(a) Find the value of r, giving your answer to four significant figures. [3]

Laurie makes no further deposits to or withdrawals from the account.

(b) Find the year in which the amount of money in Laurie's account will become double the amount she invested. [3]

3. [Maximum mark: 6]

A six-sided biased die is weighted in such a way that the probability of obtaining a "six" is $\frac{7}{10}$.

The die is tossed five times. Find the probability of obtaining

(a) at most three "sixes". [3]

(b) the third "six" on the fifth toss. [3]

4. [Maximum mark: 7]

The following table below shows the marks scored by seven students on two different mathematics tests.

Test 1 (x)	15	23	25	30	34	34	40
Test 2 (y)	20	26	27	32	35	37	35

Let L_1 be the regression line of x on y. The equation of the line L_1 can be written in the form $x = ay + b$.

(a) Find the value of a and the value of b. [2]

Let L_2 be the regression line of y on x. The lines L_1 and L_2 pass through the same point with coordinates (p, q).

(b) Find the value of p and the value of q. [3]

(c) Jennifer was absent for the first test but scored 29 marks on the second test. Use an appropriate regression equation to estimate Jennifer's mark on the first test. [2]

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5. [Maximum mark: 7]

The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = t^2 \cos t + 2t \sin t$, $0 \le t \le 5$.

(a) Find the maximum distance of the particle from O. [3]

(b) Find the acceleration of the particle at the instant it first changes direction. [4]

6. [Maximum mark: 6]

In a city, the number of passengers, X, who ride in a taxi has the following probability distribution.

x	1	2	3	4	5
$\mathrm{P}(X=x)$	0.60	0.30	0.03	0.05	0.02

After the opening of a new highway that charges a toll, a taxi company introduces a charge for passengers who use the highway. The charge is $\$2.40$ per taxi plus $\$1.20$ per passenger. Let T represent the amount, in dollars, that is charged by the taxi company per ride.

(a) Find $\mathrm{E}(T)$. [4]

(b) Given that $\mathrm{Var}(X)=0.8419$, find $\mathrm{Var}(T)$. [2]

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7. [Maximum mark: 5]

Two ships, A and B, are observed from an origin O. Relative to O, their position vectors at time t hours after midday are given by

$$\boldsymbol{r}_{\mathrm{A}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{B}} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

where distances are measured in kilometres.

Find the minimum distance between the two ships.

8. [Maximum mark: 7]

The complex numbers w and z satisfy the equations

$$\frac{w}{z} = 2\mathrm{i}$$
$$z^* - 3w = 5 + 5\mathrm{i}.$$

Find w and z in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{Z}$.

Turn over

9. [Maximum mark: 5]

Consider the graphs of $y = \dfrac{x^2}{x-3}$ and $y = m(x+3)$, $m \in \mathbb{R}$.

Find the set of values for m such that the two graphs have no intersection points.

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The length, X mm, of a certain species of seashell is normally distributed with mean 25 and variance, σ^2.

The probability that X is less than 24.15 is 0.1446.

(a) Find $\mathrm{P}(24.15 < X < 25)$. [2]

(b) (i) Find σ, the standard deviation of X.

(ii) Hence, find the probability that a seashell selected at random has a length greater than 26 mm. [5]

A random sample of 10 seashells is collected on a beach. Let Y represent the number of seashells with lengths greater than 26 mm.

(c) Find $\mathrm{E}(Y)$. [3]

(d) Find the probability that exactly three of these seashells have a length greater than 26 mm. [2]

A seashell selected at random has a length less than 26 mm.

(e) Find the probability that its length is between 24.15 mm and 25 mm. [3]

Do **not** write solutions on this page.

11. [Maximum mark: 21]

A large tank initially contains pure water. Water containing salt begins to flow into the tank The solution is kept uniform by stirring and leaves the tank through an outlet at its base. Let x grams represent the amount of salt in the tank and let t minutes represent the time since the salt water began flowing into the tank.

The rate of change of the amount of salt in the tank, $\frac{\mathrm{d}x}{\mathrm{d}t}$, is described by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = 10e^{-\frac{t}{4}} - \frac{x}{t+1}$.

(a) Show that $t+1$ is an integrating factor for this differential equation. [2]

(b) Hence, by solving this differential equation, show that $x(t) = \frac{200 - 40e^{-\frac{t}{4}}(t+5)}{t+1}$. [8]

(c) Sketch the graph of x versus t for $0 \le t \le 60$ and hence find the maximum amount of salt in the tank and the value of t at which this occurs. [5]

(d) Find the value of t at which the amount of salt in the tank is decreasing most rapidly. [2]

The rate of change of the amount of salt leaving the tank is equal to $\frac{x}{t+1}$.

(e) Find the amount of salt that left the tank during the first 60 minutes. [4]

12. [Maximum mark: 19]

(a) Show that $\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$. [1]

(b) Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)x - 1 = 0$. [7]

(c) Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. [5]

(d) Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$.
Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]

Markscheme

Specimen paper

Mathematics: analysis and approaches

Higher level

Paper 2

16 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

*Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***M2***, ***N3***, *etc.*, do **not** split the marks, unless there is a note.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

4 Follow through marks (only applied after an error is made)

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s) or subpart(s). Usually, to award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part. However, if the only marks awarded in a subpart are for the answer (i.e. there is no working expected), then* ***FT*** *marks should be awarded if appropriate.*

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, use of $r > 1$ for the sum of an infinite GP, $\sin\theta = 1.5$, non integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- Exceptions to this rule will be explicitly noted on the markscheme.
- If a candidate makes an error in one part, but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the question says hence. It is often possible to use a different approach in subsequent parts that does not depend on the answer to previous parts.

5 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- The ***MR*** penalty can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

8 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. There are two types of accuracy errors, and the final answer mark should not be awarded if these errors occur.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

9 Calculators

A GDC is required for paper 2, but calculators with symbolic manipulation features/ CAS functionality are not allowed.

Calculator notation
The subject guide says:
Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

Section A

1. (a) ***METHOD 1***

attempt to use the cosine rule ***(M1)***

$\cos\theta = \dfrac{4^2 + 4^2 - 5^2}{2 \times 4 \times 4}$ (or equivalent) ***A1***

$\theta = 1.35$ ***A1***

[3 marks]

METHOD 2

attempt to split triangle AOB into two congruent right triangles ***(M1)***

$\sin\left(\dfrac{\theta}{2}\right) = \dfrac{2.5}{4}$ ***A1***

$\theta = 1.35$ ***A1***

[3 marks]

(b) attempt to find the area of the shaded region ***(M1)***

$\dfrac{1}{2} \times 4 \times 4 \times (2\pi - 1.35...)$ ***A1***

$= 39.5\ (\text{cm}^2)$ ***A1***

[3 marks]

Total [6 marks]

2. (a) $\left(1 + \dfrac{5.5}{4 \times 100}\right)^4$ ***(M1)(A1)***

1.056 ***A1***

[3 marks]

continued…

Question 2 continued

(b) ***EITHER***

$2P = P\times\left(1+\frac{5.5}{100\times 4}\right)^{4n}$ **OR** $2P = P\times\left(\text{their } (a)\right)^{m}$ ***(M1)(A1)***

Note: Award ***(M1)*** for substitution into loan payment formula. Award ***(A1)*** for correct substitution.

OR

$\text{PV} = \pm 1$
$\text{FV} = \mp 2$
$\text{I\%} = 5.5$
$\text{P/Y} = 4$
$\text{C/Y} = 4$
$n = 50.756\ldots$ ***(M1)(A1)***

OR

$\text{PV} = \pm 1$
$\text{FV} = \mp 2$
$\text{I\%} = 100(\text{their } (a) - 1)$
$\text{P/Y} = 1$
$\text{C/Y} = 1$ ***(M1)(A1)***

THEN

$\Rightarrow 12.7$ years
Laurie will have double the amount she invested during 2032 ***A1***

[3 marks]

Total [6 marks]

3. (a) recognition of binomial ***(M1)***

$X \sim \text{B}(5,\ 0.7)$

attempt to find $\text{P}(X \le 3)$ ***M1***

$= 0.472 (= 0.47178)$ ***A1***

[3 marks]

(b) recognition of 2 sixes in 4 tosses ***(M1)***

$\text{P}(\text{3rd six on the 5th toss}) = \left[\binom{4}{2}\times(0.7)^2\times(0.3)^2\right]\times 0.7 (= 0.2646\times 0.7)$ ***A1***

$= 0.185 (= 0.18522)$ ***A1***

[3 marks]

Total [6 marks]

4. (a) $a = 1.29$ and $b = -10.4$ ***A1A1***

[2 marks]

(b) recognising both lines pass through the mean point ***(M1)***

$p = 28.7$, $q = 30.3$ ***A2***

[3 marks]

(c) substitution into **their** x on y equation ***(M1)***

$x = 1.29082(29) - 10.3793$

$x = 27.1$ ***A1***

Note: Accept 27.

[2 marks]

Total [7 marks]

5. (a) use of a graph to find the coordinates of the local minimum ***(M1)***

$s = -16.513...$ ***(A1)***

maximum distance is $16.5\,\text{cm}$ (to the left of O) ***A1***

[3 marks]

(b) attempt to find time when particle changes direction *eg* considering the first maximum on the graph of s or the first t – intercept on the graph of s'. ***(M1)***

$t = 1.51986...$ ***(A1)***

attempt to find the gradient of s' for **their** value of *t*, $s''(1.51986...)$ ***(M1)***

$= -8.92\ (\text{cm/s}^2)$ ***A1***

[4 marks]

Total [7 marks]

6. (a) ***METHOD 1***

attempting to use the expected value formula ***(M1)***

$E(X)=(1\times 0.60)+(2\times 0.30)+(3\times 0.03)+(4\times 0.05)+(5\times 0.02)$

$E(X)=1.59(\$)$ ***(A1)***

use of $E(1.20X+2.40)=1.20E(X)+2.40$ ***(M1)***

$E(T)=1.20(1.59)+2.40$

$=4.31(\$)$ ***A1***

METHOD 2

attempting to find the probability distribution for T ***(M1)***

t	3.60	4.80	6.00	7.20	8.40
$P(T=t)$	0.60	0.30	0.03	0.05	0.02

(A1)

attempting to use the expected value formula ***(M1)***

$E(T)=(3.60\times 0.60)+(4.80\times 0.30)+(6.00\times 0.03)+(7.20\times 0.05)+(8.40\times 0.02)$

$=4.31(\$)$ ***A1***

[4 marks]

(b) ***METHOD 1***

using $\text{Var}(1.20X+2.40)=(1.20)^2\text{Var}(X)$ with $\text{Var}(X)=0.8419$ ***(M1)***

$\text{Var}(T)=1.21$ ***A1***

METHOD 2

finding the standard deviation for **their** probability distribution found in part (a) ***(M1)***

$\text{Var}(T)=(1.101...)^2$

$=1.21$ ***A1***

Note: Award ***M1A1*** for $\text{Var}(T)=(1.093...)^2=1.20$.

[2 marks]

Total [6 marks]

7. attempting to find $\boldsymbol{r}_B - \boldsymbol{r}_A$ for example *(M1)*

$\boldsymbol{r}_B - \boldsymbol{r}_A = \begin{pmatrix} 3 \\ -6 \end{pmatrix} + t\begin{pmatrix} -5 \\ 4 \end{pmatrix}$

attempting to find $|\boldsymbol{r}_B - \boldsymbol{r}_A|$ ***M1***

distance $d(t) = \sqrt{(3-5t)^2 + (4t-6)^2} \left(= \sqrt{41t^2 - 78t + 45}\right)$ ***A1***

using a graph to find the d – coordinate of the local minimum ***M1***

the minimum distance between the ships is $2.81 \text{ (km)} \left(= \frac{11\sqrt{41}}{41} \text{ (km)}\right)$ ***A1***

Total [5 marks]

8. substituting $w = 2\text{i}z$ into $z^* - 3w = 5 + 5\text{i}$ ***M1***

$z^* - 6\text{i}z = 5 + 5\text{i}$ ***A1***

let $z = x + y\text{i}$

comparing real and imaginary parts of $(x - y\text{i}) - 6\text{i}(x + y\text{i}) = 5 + 5\text{i}$ ***M1***

to obtain $x + 6y = 5$ and $-6x - y = 5$ ***A1***

attempting to solve for x and y ***M1***

$x = -1$ and $y = 1$ and so $z = -1 + \text{i}$ ***A1***

hence $w = -2 - 2\text{i}$ ***A1***

9. ***METHOD 1***

sketching the graph of $y=\frac{x^2}{x-3}$ ($y=x+3+\frac{9}{x-3}$) ***M1***

the (oblique) asymptote has a gradient equal to 1

and so the maximum value of m is 1 ***R1***

consideration of a straight line steeper than the horizontal line joining $(-3,0)$ and $(0,0)$ ***M1***

so $m>0$ ***R1***

hence $0<m\le 1$ ***A1***

METHOD 2

attempting to eliminate y to form a quadratic equation in x ***M1***

$x^2=m\left(x^2-9\right)$

$\Rightarrow (m-1)x^2-9m=0$ ***A1***

EITHER

attempting to solve $-4(m-1)(-9m)<0$ for m ***M1***

OR

attempting to solve $x^2<0$ *ie* $\frac{9m}{m-1}<0$ $(m\ne 1)$ for m ***M1***

THEN

$\Rightarrow 0<m<1$ ***A1***

a valid reason to explain why $m=1$ gives no solutions *eg* if $m=1$, $(m-1)x^2-9m=0\Rightarrow -9=0$ and so $0<m\le 1$ ***R1***

Total [5 marks]

Section B

10. (a) attempt to use the symmetry of the normal curve ***(M1)***
eg diagram, $0.5-0.1446$

$\mathrm{P}(24.15<X<25)=0.3554$ ***A1***

[2 marks]

(b) (i) use of inverse normal to find z score ***(M1)***
$z=-1.0598$

correct substitution $\frac{24.15-25}{\sigma}=-1.0598$ ***(A1)***

$\sigma=0.802$ ***A1***

(ii) $\mathrm{P}(X>26)=0.106$ ***(M1)A1***

[5 marks]

(c) recognizing binomial probability ***(M1)***
$\mathrm{E}(Y)=10\times0.10621$ ***(A1)***
$=1.06$ ***A1***

[3 marks]

(d) $\mathrm{P}(Y=3)$ ***(M1)***
$=0.0655$ ***A1***

[2 marks]

(e) recognizing conditional probability ***(M1)***
correct substitution ***A1***

$\frac{0.3554}{1-0.10621}$

$=0.398$ ***A1***

[3 marks]

Total [15 marks]

11. (a) ***METHOD 1***

using $I(t) = e^{\int P(t)\mathrm{d}t}$ ***M1***

$e^{\int \frac{1}{t+1}\mathrm{d}t}$

$= e^{\ln(t+1)}$ ***A1***

$= t + 1$ ***AG***

METHOD 2

attempting product rule differentiation on $\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t+1)\right)$ ***M1***

$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t+1)\right) = \frac{\mathrm{d}x}{\mathrm{d}t}(t+1) + x$

$= (t+1)\left(\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{t+1}\right)$ ***A1***

so $t+1$ is an integrating factor for this differential equation ***AG***

[2 marks]

continued…

Question 11 continued

(b) attempting to multiply through by $(t+1)$ and rearrange to give ***(M1)***

$(t+1)\frac{dx}{dt}+x=10(t+1)e^{-\frac{t}{4}}$ ***A1***

$\frac{d}{dt}\left(x(t+1)\right)=10(t+1)e^{-\frac{t}{4}}$

$x(t+1)=\int 10(t+1)e^{-\frac{t}{4}}\,dt$ ***A1***

attempting to integrate the RHS by parts ***M1***

$=-40(t+1)e^{-\frac{t}{4}}+40\int e^{-\frac{t}{4}}\,dt$

$=-40(t+1)e^{-\frac{t}{4}}-160e^{-\frac{t}{4}}+C$ ***A1***

Note: Condone the absence of C .

EITHER

substituting $t=0, x=0 \Rightarrow C=200$ ***M1***

$x=\frac{-40(t+1)e^{-\frac{t}{4}}-160e^{-\frac{t}{4}}+200}{t+1}$ ***A1***

using $-40e^{-\frac{t}{4}}$ as the highest common factor of $-40(t+1)e^{-\frac{t}{4}}$ and $-160e^{-\frac{t}{4}}$ ***M1***

OR

using $-40e^{-\frac{t}{4}}$ as the highest common factor of $-40(t+1)e^{-\frac{t}{4}}$ and $-160e^{-\frac{t}{4}}$ giving

$x(t+1)=-40e^{-\frac{t}{4}}(t+5)+C$ (or equivalent) ***M1A1***

substituting $t=0$, $x=0 \Rightarrow C=200$ ***M1***

THEN

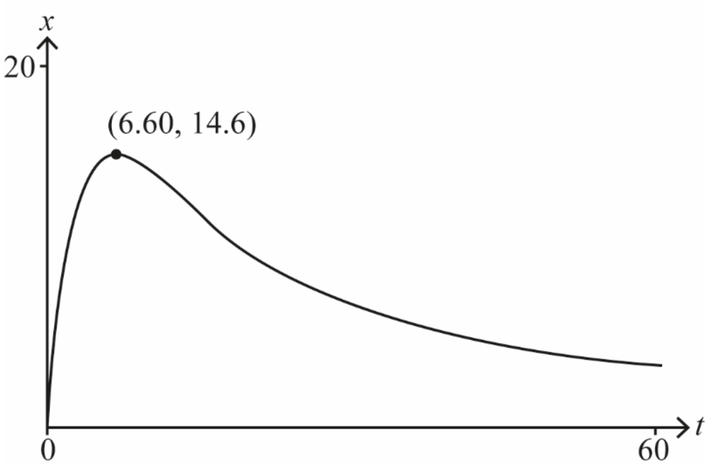
$x(t)=\frac{200-40e^{-\frac{t}{4}}(t+5)}{t+1}$ ***AG***

[8 marks]

continued…

Question 11 continued

(c)



graph starts at the origin and has a local maximum (coordinates not required) ***A1***
sketched for $0 \le t \le 60$ ***A1***
correct concavity for $0 \le t \le 60$ ***A1***
maximum amount of salt is 14.6 (grams) at $t = 6.60$ (minutes) ***A1A1***

[5 marks]

(d) using an appropriate graph or equation (first or second derivative) ***M1***
amount of salt is decreasing most rapidly at $t = 12.9$ (minutes) ***A1***

[2 marks]

(e) **EITHER**

attempting to form an integral representing the amount of salt that left the tank ***M1***

$$\int_0^{60} \frac{x(t)}{t+1}\,dt$$

$$\int_0^{60} \frac{200 - 40e^{-\frac{t}{4}}(t+5)}{(t+1)^2}\,dt$$ ***A1***

OR

attempting to form an integral representing the amount of salt that entered the tank minus the amount of salt in the tank at $t = 60$ (minutes) ***M1***

amount of salt that left the tank is $\int_0^{60} 10e^{-\frac{t}{4}}\,dt - x(60)$ ***A1***

THEN

$= 36.7$ (grams) ***A2***

[4 marks]

Total [21 marks]

12. (a) stating the relationship between cot and tan and stating the identity for $\tan 2\theta$ ***M1***

$\cot 2\theta = \dfrac{1}{\tan 2\theta}$ and $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta}$ ***AG***

[1 mark]

(b) ***METHOD 1***

attempting to substitute $\tan\theta$ for x and using the result from (a) ***M1***

$\text{LHS} = \tan^2\theta + 2\tan\theta\left(\dfrac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$\tan^2\theta + 1 - \tan^2\theta - 1 = 0\,(= \text{RHS})$ ***A1***

so $x = \tan\theta$ satisfies the equation ***AG***

attempting to substitute $-\cot\theta$ for x and using the result from (a) ***M1***

$\text{LHS} = \cot^2\theta - 2\cot\theta\left(\dfrac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$= \dfrac{1}{\tan^2\theta} - \left(\dfrac{1-\tan^2\theta}{\tan^2\theta}\right) - 1$ ***A1***

$\dfrac{1}{\tan^2\theta} - \dfrac{1}{\tan^2\theta} + 1 - 1 = 0\,(= \text{RHS})$ ***A1***

so $x = -\cot\theta$ satisfies the equation ***AG***

METHOD 2

let $\alpha = \tan\theta$ and $\beta = -\cot\theta$

attempting to find the sum of roots ***M1***

$\alpha + \beta = \tan\theta - \dfrac{1}{\tan\theta}$

$= \dfrac{\tan^2\theta - 1}{\tan\theta}$ ***A1***

$= -2\cot 2\theta$ (from part (a)) ***A1***

attempting to find the product of roots ***M1***

$\alpha\beta = \tan\theta \times (-\cot\theta)$ ***A1***

$= -1$ ***A1***

the coefficient of x and the constant term in the quadratic are $2\cot 2\theta$ and -1 respectively ***R1***

hence the two roots are $\alpha = \tan\theta$ and $\beta = -\cot\theta$ ***AG***

[7 marks]

continued…

Question 12 continued

(c) ***METHOD 1***

$x = \tan\frac{\pi}{12}$ and $x = -\cot\frac{\pi}{12}$ are roots of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$ ***R1***

> **Note:** Award ***R1*** if only $x = \tan\frac{\pi}{12}$ is stated as a root of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$.

$x^2 + 2\sqrt{3}x - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$x = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ $\left(-\cot\frac{\pi}{12} < 0\right)$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

METHOD 2

attempting to substitute $\theta = \frac{\pi}{12}$ into the identity for $\tan 2\theta$ ***M1***

$$\tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1 - \tan^2\frac{\pi}{12}}$$

$\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$\tan\frac{\pi}{12} = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

[5 marks]

(d) $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$ is the sum of the roots of $x^2 + \left(2\cot\frac{\pi}{12}\right)x - 1 = 0$ ***R1***

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24} = -2\cot\frac{\pi}{12}$ ***A1***

$= \frac{-2}{2 - \sqrt{3}}$ ***A1***

attempting to rationalise **their** denominator ***(M1)***

$= -4 - 2\sqrt{3}$ ***A1A1***

[6 marks]

Total [19 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

Specimen

1 hour

Instructions to candidates

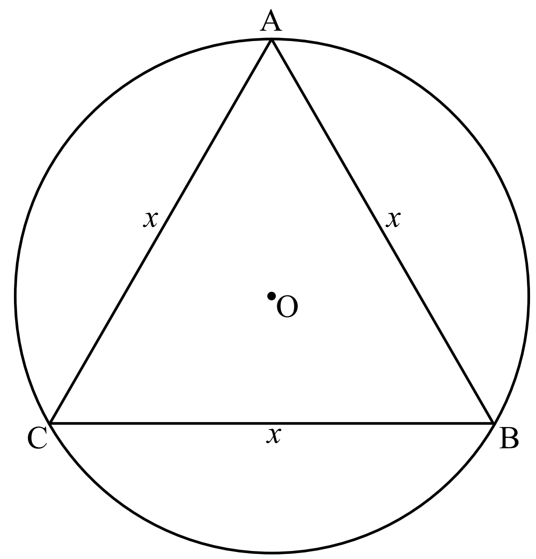
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

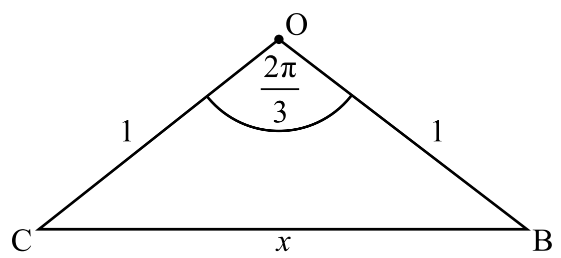
1. [Maximum mark: 30]

This question asks you to investigate regular n-sided polygons inscribed and circumscribed in a circle, and the perimeter of these as n tends to infinity, to make an approximation for π.

(a) Consider an equilateral triangle ABC of side length, x units, inscribed in a circle of radius 1 unit and centre O as shown in the following diagram.



The equilateral triangle ABC can be divided into three smaller isosceles triangles, each subtending an angle of $\frac{2\pi}{3}$ at O, as shown in the following diagram.



Using right-angled trigonometry or otherwise, show that the perimeter of the equilateral triangle ABC is equal to $3\sqrt{3}$ units. [3]

(b) Consider a square of side length, x units, inscribed in a circle of radius 1 unit. By dividing the inscribed square into four isosceles triangles, find the exact perimeter of the inscribed square. [3]

(This question continues on the following page)

(Question 1 continued)

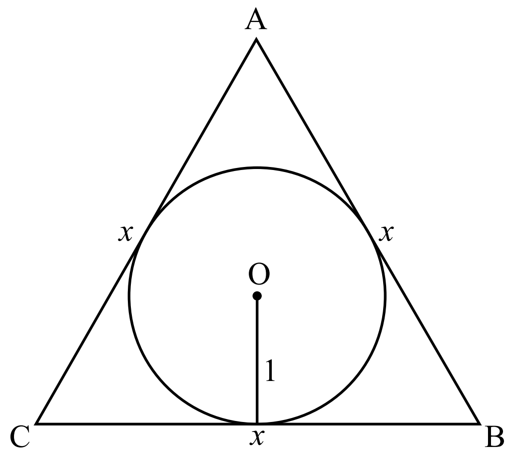
(c) Find the perimeter of a regular hexagon, of side length, x units, inscribed in a circle of radius 1 unit. [2]

Let $P_i(n)$ represent the perimeter of any n-sided regular polygon inscribed in a circle of radius 1 unit.

(d) Show that $P_i(n) = 2n\sin\left(\frac{\pi}{n}\right)$. [3]

(e) Use an appropriate Maclaurin series expansion to find $\lim_{n\to\infty} P_i(n)$ and interpret this result geometrically. [5]

Consider an equilateral triangle ABC of side length, x units, circumscribed about a circle of radius 1 unit and centre O as shown in the following diagram.



Let $P_c(n)$ represent the perimeter of any n-sided regular polygon circumscribed about a circle of radius 1 unit.

(f) Show that $P_c(n) = 2n\tan\left(\frac{\pi}{n}\right)$. [4]

(g) By writing $P_c(n)$ in the form $\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}$, find $\lim_{n\to\infty} P_c(n)$. [5]

(h) Use the results from part (d) and part (f) to determine an inequality for the value of π in terms of n. [2]

The inequality found in part (h) can be used to determine lower and upper bound approximations for the value of π.

(i) Determine the least value for n such that the lower bound and upper bound approximations are both within 0.005 of π. [3]

2. [Maximum mark: 25]

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$ and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

(a) On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \le x \le 1$. [2]

(b) For odd values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

(i) local maximum points;

(ii) local minimum points. [4]

(c) On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \le x \le 1$. [2]

(d) For even values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of n describing, in terms of n, the number of

(i) local maximum points;

(ii) local minimum points. [4]

(e) Solve the equation $f_n'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and $0 < k < n$. [4]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

(f) Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. [2]

Consider $f_{n+1}(x) = \cos((n+1)\arccos x)$.

(g) Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$. [2]

(h) Hence

(i) show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n(x)$, $n \in \mathbb{Z}^+$;

(ii) express $f_3(x)$ as a cubic polynomial. [5]

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

Specimen paper

Mathematics: analysis and approaches

Higher level

Paper 3

9 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

*Award marks using the annotations as noted in the markscheme eg **M1**, **A2**.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***M2***, ***N3***, *etc.*, do **not** split the marks, unless there is a note.
- Once a correct answer to a question or part-question is seen, ignore further correct working. However, if further working indicates a lack of mathematical understanding do not award the final ***A1***. An exception to this may be in numerical answers, where a correct exact value is followed by an incorrect decimal. However, if the incorrect decimal is carried through to a subsequent part, and correct ***FT*** working shown, award ***FT*** marks as appropriate but do not award the final ***A1*** in that part.

Examples

	Correct answer seen	Further working seen	Action
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	Award the final ***A1*** *(ignore the further working)*
2.	$\frac{1}{4}\sin 4x$	$\sin x$	Do not award the final ***A1***
3.	$\log a - \log b$	$\log(a-b)$	Do not award the final ***A1***

3 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

4 Follow through marks (only applied after an error is made)

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s) or subpart(s). Usually, to award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part. However, if the only marks awarded in a subpart are for the answer (i.e. there is no working expected), then* ***FT*** *marks should be awarded if appropriate.*

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g*. probability greater than 1, use of $r > 1$ for the sum of an infinite GP, $\sin\theta = 1.5$, non integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- Exceptions to this rule will be explicitly noted on the markscheme.
- If a candidate makes an error in one part, but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the question says hence. It is often possible to use a different approach in subsequent parts that does not depend on the answer to previous parts.

5 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates’ own work does **not** constitute a misread, it is an error.
- The ***MR*** penalty can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

8 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. There are two types of accuracy errors, and the final answer mark should not be awarded if these errors occur.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

9 Calculators

A GDC is required for paper 3, but calculators with symbolic manipulation features/ CAS functionality are not allowed.

Calculator notation
The subject guide says:
Students must always use correct mathematical notation, not calculator notation.

Do **not** accept final answers written using calculator notation. However, do not penalize the use of calculator notation in the working.

1. (a) ***METHOD 1***

consider right-angled triangle OCX where $\text{CX} = \frac{x}{2}$

$\sin\frac{\pi}{3} = \frac{\frac{x}{2}}{1}$ ***M1A1***

$\Rightarrow \frac{x}{2} = \frac{\sqrt{3}}{2} \Rightarrow x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

METHOD 2

eg use of the cosine rule $x^2 = 1^2 + 1^2 - 2(1)(1)\cos\frac{2\pi}{3}$ ***M1A1***

$x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

Note: Accept use of sine rule.

[3 marks]

(b) $\sin\frac{\pi}{4} = \frac{1}{x}$ where $x =$ side of square ***M1***

$x = \sqrt{2}$ ***A1***

$P_i = 4\sqrt{2}$ ***A1***

[3 marks]

(c) 6 equilateral triangles $\Rightarrow x = 1$ ***A1***

$P_i = 6$ ***A1***

[2 marks]

(d) in right-angled triangle $\sin\left(\frac{\pi}{n}\right) = \frac{\frac{x}{2}}{1}$ ***M1***

$\Rightarrow x = 2\sin\left(\frac{\pi}{n}\right)$ ***A1***

$P_i = n \times x$

$P_i = n \times 2\sin\left(\frac{\pi}{n}\right)$ ***M1***

$P_i = 2n\sin\left(\frac{\pi}{n}\right)$ ***AG***

[3 marks]

continued…

Question 1 continued

(e) consider $\lim_{n\to\infty} 2n\sin\left(\frac{\pi}{n}\right)$

use of $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ ***M1***

$2n\sin\left(\frac{\pi}{n}\right) = 2n\left(\frac{\pi}{n} - \frac{\pi^3}{6n^3} + \frac{\pi^5}{120n^5} - \dots\right)$ ***(A1)***

$= 2\left(\pi - \frac{\pi^3}{6n^2} + \frac{\pi^5}{120n^4} - \dots\right)$ ***A1***

$\Rightarrow \lim_{n\to\infty} 2n\sin\left(\frac{\pi}{n}\right) = 2\pi$ ***A1***

as $n \to \infty$ polygon becomes a circle of radius 1 and $P_i = 2\pi$ ***R1***

[5 marks]

(f) consider an n-sided polygon of side length x

$2n$ right-angled triangles with angle $\frac{2\pi}{2n} = \frac{\pi}{n}$ at centre ***M1A1***

opposite side $\frac{x}{2} = \tan\left(\frac{\pi}{n}\right) \Rightarrow x = 2\tan\left(\frac{\pi}{n}\right)$ ***M1A1***

Perimeter $P_c = 2n\tan\left(\frac{\pi}{n}\right)$ ***AG***

[4 marks]

(g) consider $\lim_{n\to\infty} 2n\tan\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\left(\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}\right)$

$= \lim_{n\to\infty}\left(\frac{2\tan\left(\frac{\pi}{n}\right)}{\frac{1}{n}}\right) = \frac{0}{0}$ ***R1***

attempt to use L'Hopital's rule ***M1***

$= \lim_{n\to\infty}\left(\frac{-\frac{2\pi}{n^2}\sec^2\left(\frac{\pi}{n}\right)}{-\frac{1}{n^2}}\right)$ ***A1A1***

$= 2\pi$ ***A1***

[5 marks]

continued…

Question 1 continued

(h) $P_i < 2\pi < P_c$

$2n\sin\left(\frac{\pi}{n}\right) < 2\pi < 2n\tan\left(\frac{\pi}{n}\right)$ ***M1***

$n\sin\left(\frac{\pi}{n}\right) < \pi < n\tan\left(\frac{\pi}{n}\right)$ ***A1***

[2 marks]

(i) attempt to find the lower bound and upper bound approximations within 0.005 of π ***(M1)***

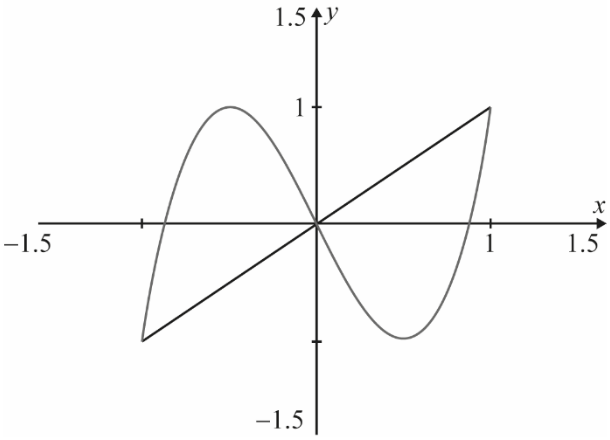
$n = 46$ ***A2***

[3 marks]

Total [30 marks]

2. (a) correct graph of $y = f_1(x)$ ***A1***

correct graph of $y = f_3(x)$ ***A1***



[2 marks]

(b) (i) graphical or tabular evidence that n has been systematically varied ***M1***

eg $n = 3$, 1 local maximum point and 1 local minimum point

$n = 5$, 2 local maximum points and 2 local minimum points

$n = 7$, 3 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-1}{2}$ local maximum points ***A1***

(ii) $\frac{n-1}{2}$ local minimum points ***A1***

Note: Allow follow through from an incorrect local maximum formula expression.

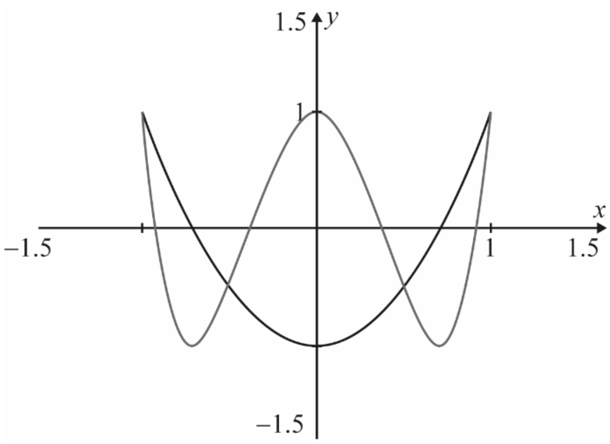
[4 marks]

continued…

Question 2 continued

(c) correct graph of $y = f_2(x)$ ***A1***

correct graph of $y = f_4(x)$ ***A1***



[2 marks]

(d) (i) graphical or tabular evidence that n has been systematically varied ***M1***

eg $n = 2$, 0 local maximum point and 1 local minimum point
$n = 4$, 1 local maximum points and 2 local minimum points
$n = 6$, 2 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-2}{2}$ local maximum points ***A1***

(ii) $\frac{n}{2}$ local minimum points ***A1***

[4 marks]

(e) $f_n(x) = \cos\left(n\arccos(x)\right)$

$$f_n'(x) = \frac{n\sin\left(n\arccos(x)\right)}{\sqrt{1-x^2}}$$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f_n'(x) = 0 \Rightarrow n\sin\left(n\arccos(x)\right) = 0$ ***M1***

$n\arccos(x) = k\pi$ ($k \in \mathbb{Z}^+$) ***A1***

leading to

$x = \cos\frac{k\pi}{n}$ ($k \in \mathbb{Z}^+$ and $0 < k < n$) ***AG***

[4 marks]

continued…

Question 2 continued

(f) $f_2(x) = \cos(2\arccos x)$

$= 2(\cos(\arccos x))^2 - 1$ ***M1***

stating that $(\cos(\arccos x)) = x$ ***A1***

so $f_2(x) = 2x^2 - 1$ ***AG***

[2 marks]

(g) $f_{n+1}(x) = \cos((n+1)\arccos x)$

$= \cos(n\arccos x + \arccos x)$ ***A1***

use of $\cos(A+B) = \cos A\cos B - \sin A\sin B$ leading to ***M1***

$= \cos(n\arccos x)\cos(\arccos x) - \sin(n\arccos x)\sin(\arccos x)$ ***AG***

[2 marks]

(h) (i) $f_{n-1}(x) = \cos((n-1)\arccos x)$ ***A1***

$= \cos(n\arccos x)\cos(\arccos x) + \sin(n\arccos x)\sin(\arccos x)$ ***M1***

$f_{n+1}(x) + f_{n-1}(x) = 2\cos(n\arccos x)\cos(\arccos x)$ ***A1***

$= 2xf_n(x)$ ***AG***

(ii) $f_3(x) = 2xf_2(x) - f_1(x)$ ***(M1)***

$= 2x(2x^2 - 1) - x$

$= 4x^3 - 3x$ ***A1***

[5 marks]

Total [25 marks]