Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2018

Mathematics

Higher level

Paper 2

18 pages

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Note: Accept answers that round to the correct 2sf unless otherwise stated in the markscheme.

Section A

1. (a) $z = \frac{(2+7\text{i})}{(6+2\text{i})} \times \frac{(6-2\text{i})}{(6-2\text{i})}$ ***(M1)***

$= \frac{26+38\text{i}}{40} \left(= \frac{13+19\text{i}}{20} = 0.65 + 0.95\text{i} \right)$ ***A1***

[2 marks]

(b) attempt to use $|z| = \sqrt{a^2 + b^2}$ ***(M1)***

$|z| = \sqrt{\frac{53}{40}} \left(= \frac{\sqrt{530}}{20} \right)$ or equivalent ***A1***

Note: ***A1*** is only awarded for the correct exact value.

[2 marks]

(c) **EITHER**

$\arg z = \arg(2+7\text{i}) - \arg(6+2\text{i})$ ***(M1)***

OR

$\arg z = \arctan\left(\frac{19}{13}\right)$ ***(M1)***

THEN

$\arg z = 0.9707$ (radians) (= 55.6197 degrees) ***A1***

Note: Only award the last ***A1*** if 4 decimal places are given.

[2 marks]

Total [6 marks]

2. **METHOD 1**

substitute each of $x = 1,\ 2$ and 3 into the quartic and equate to zero ***(M1)***

$p + q + r = -7$

$4p + 2q + r = -11$ or equivalent ***(A2)***

$9p + 3q + r = -29$

Note: Award ***A2*** for all three equations correct, ***A1*** for two correct.

attempting to solve the system of equations ***(M1)***

$p = -7,\ q = 17,\ r = -17$ ***A1***

Note: Only award ***M1*** when some numerical values are found when solving algebraically or using GDC.

continued…

Question 2 continued

METHOD 2

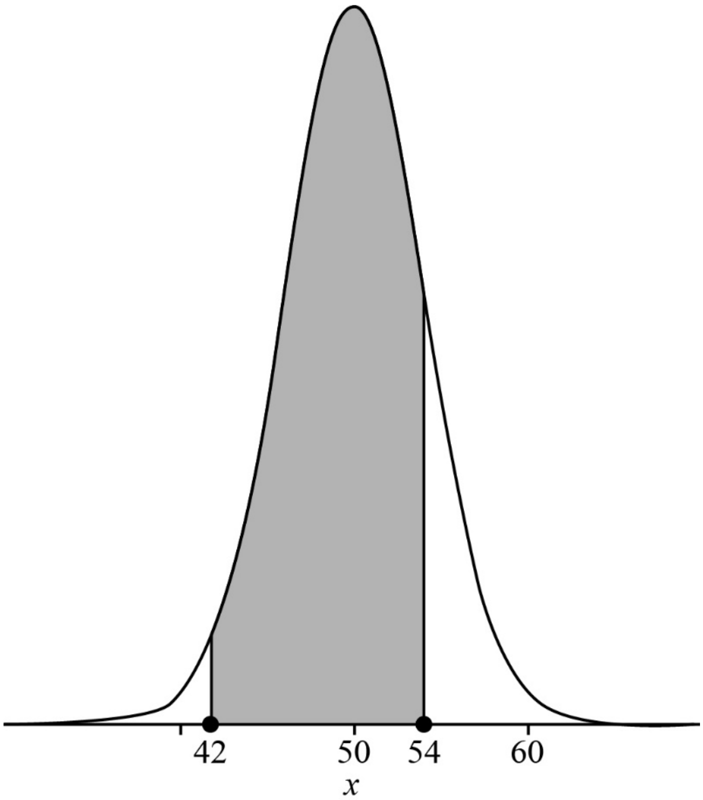
attempt to find fourth factor ***(M1)***
$(x-1)$ ***A1***
attempt to expand $(x-1)^2(x-2)(x-3)$ ***M1***
$= x^4 - 7x^3 + 17x^2 - 17x + 6 \left(p = -7,\ q = 17,\ r = -17 \right)$ ***A2***

Note: Award ***A2*** for all three values correct, ***A1*** for two correct.

Note: Accept long / synthetic division.

[5 marks]

3. (a)



normal curve centred on 50 ***A1***
vertical lines at $x = 42$ and $x = 54$, with shading in between ***A1***

[2 marks]

(b) $\mathrm{P}(42 < X < 54) \left(= \mathrm{P}(-2 < Z < 1) \right)$ ***(M1)***
$= 0.819$ ***A1***

[2 marks]

continued…

Question 3 continued

(c) $\mathrm{P}(\mu - k\sigma < X < \mu + k\sigma) = 0.5 \Rightarrow \mathrm{P}(X < \mu + k\sigma) = 0.75$ ***(M1)***

$k = 0.674$ ***A1***

Note: Award ***M1A0*** for $k = -0.674$.

[2 marks]

Total [6 marks]

4. (a) (i) **METHOD 1**

$\mathrm{PC} = \frac{\sqrt{3}}{2}$ or 0.8660 ***(M1)***

$\mathrm{PM} = \frac{1}{2}\mathrm{PC} = \frac{\sqrt{3}}{4}$ or 0.4330 ***(A1)***

$\mathrm{AM} = \sqrt{\frac{1}{4} + \frac{3}{16}}$

$= \frac{\sqrt{7}}{4}$ or 0.661 (m) ***A1***

Note: Award ***M1*** for attempting to solve triangle AMP.

METHOD 2

using the cosine rule

$\mathrm{AM}^2 = 1^2 + \left(\frac{\sqrt{3}}{4}\right)^2 - 2 \times \frac{\sqrt{3}}{4} \times \cos(30°)$ ***M1A1***

$\mathrm{AM} = \frac{\sqrt{7}}{4}$ or 0.661 (m) ***A1***

(ii) $\tan\left(\mathrm{A\hat{M}P}\right) = \frac{2}{\sqrt{3}}$ or equivalent ***(M1)***

$= 0.857$ ***A1***

[5 marks]

continued…

Question 4 continued

(b) **EITHER**

$\frac{1}{2}\text{AM}^2\left(2\text{A}\hat{\text{M}}\text{P} - \sin(2\text{A}\hat{\text{M}}\text{P})\right)$ ***(M1)A1***

OR

$\frac{1}{2}\text{AM}^2 \times 2\text{A}\hat{\text{M}}\text{P} - \frac{\sqrt{3}}{8}$ ***(M1)A1***

$= 0.158(\text{m}^2)$ ***A1***

Note: Award ***M1*** for attempting to calculate area of a sector minus area of a triangle.

[3 marks]

Total [8 marks]

5. (a) $\binom{3n+1}{3n-2} = \frac{(3n+1)!}{(3n-2)!3!}$ ***(M1)***

$= \frac{(3n+1)3n(3n-1)}{3!}$ ***A1***

$= \frac{9}{2}n^3 - \frac{1}{2}n$ or equivalent ***A1***

[3 marks]

(b) attempt to solve $\frac{9}{2}n^3 - \frac{1}{2}n > 10^6$ ***(M1)***

$n > 60.57\ldots$ ***(A1)***

Note: Allow equality.

$\Rightarrow n = 61$ ***A1***

[3 marks]

Total [6 marks]

6. let P_n be the statement: $(1-a)^n > 1-na$ for some $n \in \mathbb{Z}^+, n \geq 2,$ where $0 < a < 1$

consider the case $n = 2$: $(1-a)^2 = 1 - 2a + a^2$ ***M1***

$> 1 - 2a$ because $a^2 > 0$. Therefore P_2 is true ***R1***

assume P_n is true for some $n = k$

$(1-a)^k > 1-ka$ ***M1***

Note: Assumption of truth must be present. Following marks are not dependent on this ***M1***.

EITHER

consider $(1-a)^{k+1} = (1-a)(1-a)^k$ ***M1***

$> 1-(k+1)a + ka^2$ ***A1***

$> 1-(k+1)a \Rightarrow P_{k+1}$ is true (as $ka^2 > 0$) ***R1***

OR

multiply both sides by $(1-a)$ (which is positive) ***M1***

$(1-a)^{k+1} > (1-ka)(1-a)$

$(1-a)^{k+1} > 1-(k+1)a + ka^2$ ***A1***

$(1-a)^{k+1} > 1-(k+1)a \Rightarrow P_{k+1}$ is true (as $ka^2 > 0$) ***R1***

THEN

P_2 is true and P_k is true $\Rightarrow P_{k+1}$ is true so P_n true for all $n \geq 2$ (or equivalent) ***R1***

Note: Only award the last ***R1*** if at least four of the previous marks are gained including the ***A1***.

[7 marks]

7. (a) attempt to solve $v(t) = 0$ for t or equivalent ***(M1)***

$t_1 = 0.441$(s) ***A1***

[2 marks]

(b) (i) $a(t) = \frac{dv}{dt} = -e^{-t} - 16te^{-2t} + 16t^2e^{-2t}$ ***M1A1***

Note: Award ***M1*** for attempting to differentiate using the product rule.

(ii) $a(t_1) = -2.28\left(\text{ms}^{-2}\right)$ ***A1***

[3 marks]

Total [5 marks]

8. (a) $np = 3.5$ ***(A1)***

$p \leq 1 \Rightarrow$ least $n = 4$ ***A1***

[2 marks]

continued…

Question 8 continued

(b) $(1-p)^n + np(1-p)^{n-1} = 0.09478$ ***M1A1***

attempt to solve above equation with $np = 3.5$ ***(M1)***

$n = 12,\ p = \frac{7}{24}\ (= 0.292)$ ***A1A1***

Note: Do not accept n as a decimal.

[5 marks]

Total [7 marks]

Section B

9. (a) (i) $X \sim \mathrm{Po}(5.3)$

$\mathrm{P}(X = 4) = \mathrm{e}^{-5.3}\dfrac{5.3^4}{4!}$ ***(M1)***

$= 0.164$ ***A1***

(ii) **METHOD 1**

listing probabilities (table or graph) ***M1***

mode $X = 5$ (with probability 0.174) ***A1***

Note: Award ***M0A0*** for 5 (taxis) or mode $= 5$ with no justification.

METHOD 2

mode is the integer part of mean ***R1***

$E(X) = 5.3 \Rightarrow \text{mode} = 5$ ***A1***

Note: Do not allow ***R0A1***.

(iii) attempt at conditional probability ***(M1)***

$\dfrac{\mathrm{P}(X = 7)}{\mathrm{P}(X \geq 6)}$ or equivalent $\left(= \dfrac{0.1163...}{0.4365...}\right)$ ***A1***

$= 0.267$ ***A1***

[7 marks]

(b) **METHOD 1**

the possible arrivals are $(2,0)$, $(1,1)$, $(0,2)$ ***(A1)***

$Y \sim \mathrm{Po}(0.65)$ ***A1***

attempt to compute, using sum and product rule, ***(M1)***

$0.070106\ldots \times 0.52204\ldots + 0.026455\ldots \times 0.33932\ldots + 0.0049916\ldots \times 0.11028\ldots$

(A1)(A1)

Note: Award ***A1*** for one correct product and ***A1*** for two other correct products.

$= 0.0461$ ***A1***

[6 marks]

continued…

Question 9 continued

METHOD 2

recognising a sum of 2 independent Poisson variables *eg* $Z = X + Y$ ***R1***

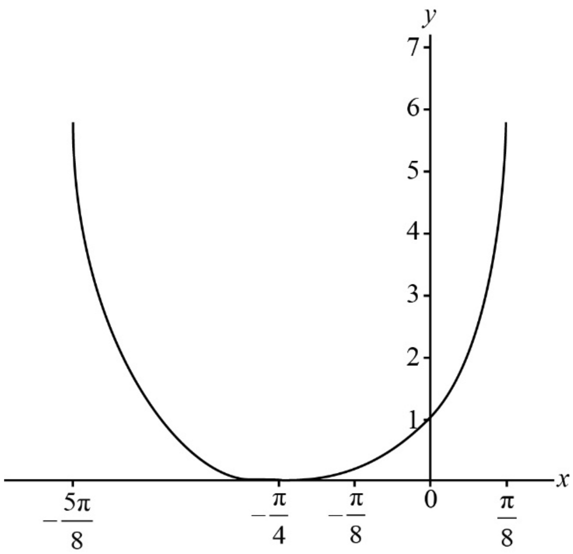
$\lambda = 5.3 + \frac{1.3}{2}$ ***A1***

$\mathrm{P}(Z = 2) = 0.0461$ ***(M1)A3***

[6 marks]

Total [13 marks]

10. (a) (i)



A1A1

A1 for correct concavity, many to one graph, symmetrical about the midpoint of the domain and with two axes intercepts.

Note: Axes intercepts and scales not required.

A1 for correct domain

(ii) for each value of x there is a unique value of $f(x)$ ***A1***

Note: Accept "passes the vertical line test" or equivalent.

(iii) no inverse because the function fails the horizontal line test or equivalent ***R1***

Note: No **FT** if the graph is in degrees (one-to-one).

(iv) the expression is not valid at either of $x = \frac{\pi}{4}$ $\left(\text{or } -\frac{3\pi}{4}\right)$ ***R1***

[5 marks]

continued…

Question 10 continued

(b) **METHOD 1**

$$f(x) = \frac{\tan\left(x + \frac{\pi}{4}\right)}{\tan\left(\frac{\pi}{4} - x\right)}$$ ***M1***

$$= \frac{\frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}}{\frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}}$$ ***M1A1***

$$= \left(\frac{1+t}{1-t}\right)^2$$ ***AG***

METHOD 2

$$f(x) = \tan\left(x + \frac{\pi}{4}\right)\tan\left(\frac{\pi}{2} - \frac{\pi}{4} + x\right)$$ ***(M1)***

$$= \tan^2\left(x + \frac{\pi}{4}\right)$$ ***A1***

$$g(t) = \left(\frac{\tan x + \tan\frac{\pi}{4}}{1 - \tan x \tan\frac{\pi}{4}}\right)^2$$ ***A1***

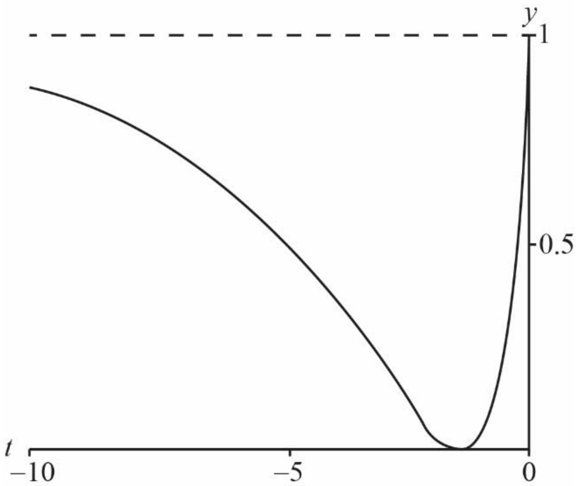
$$= \left(\frac{1+t}{1-t}\right)^2$$ ***AG***

[3 marks]

continued…

Question 10 continued

(c)



for $t \le 0$, correct concavity with two axes intercepts and with asymptote $y = 1$ ***A1***

t intercept at $(-1, 0)$ ***A1***

y intercept at $(0, 1)$ ***A1***

[3 marks]

(d) (i) **METHOD 1**

α, β satisfy $\frac{(1+t)^2}{(1-t)^2} = k$ ***M1***

$1 + t^2 + 2t = k(1 + t^2 - 2t)$ ***A1***

$(k-1)t^2 - 2(k+1)t + (k-1) = 0$ ***A1***

attempt at using quadratic formula ***M1***

$\alpha, \beta = \frac{k + 1 \pm 2\sqrt{k}}{k - 1}$ or equivalent ***A1***

METHOD 2

α, β satisfy $\frac{1+t}{1-t} = (\pm)\sqrt{k}$ ***M1***

$t + \sqrt{k}t = \sqrt{k} - 1$ ***M1***

$t = \frac{\sqrt{k}-1}{\sqrt{k}+1}$ (or equivalent) ***A1***

$t - \sqrt{k}t = -\left(\sqrt{k}+1\right)$ ***M1***

$t = \frac{\sqrt{k}+1}{\sqrt{k}-1}$ (or equivalent) ***A1***

so for *eg*, $\alpha = \frac{\sqrt{k}-1}{\sqrt{k}+1}$, $\beta = \frac{\sqrt{k}+1}{\sqrt{k}-1}$

continued…

Question 10 continued

(ii) $\alpha + \beta = 2\frac{(k+1)}{(k-1)} \left(= -2\frac{(1+k)}{(1-k)}\right)$ ***A1***

since $1 + k > 1 - k$ ***R1***

$\alpha + \beta < -2$ ***AG***

Note: Accept a valid graphical reasoning.

[7 marks]

Total [18 marks]

11. (a) attempt at implicit differentiation ***M1***

$1 + \frac{dy}{dx} + (y + x\frac{dy}{dx})\sin(xy) = 0$ ***A1M1A1***

Note: Award ***A1*** for first two terms. Award ***M1*** for an attempt at chain rule ***A1*** for last term.

$(1 + x\sin(xy))\frac{dy}{dx} = -1 - y\sin(xy)$ or equivalent ***A1***

$\frac{dy}{dx} = -\left(\frac{1 + y\sin(xy)}{1 + x\sin(xy)}\right)$ ***AG***

[5 marks]

(b) (i) **EITHER**

when $xy = -\frac{\pi}{2}$, $\cos xy = 0$ ***M1***

$\Rightarrow x + y = 0$ ***(A1)***

OR

$x - \frac{\pi}{2x} - \cos\left(\frac{-\pi}{2}\right) = 0$ or equivalent ***M1***

$x - \frac{\pi}{2x} = 0$ ***(A1)***

THEN

therefore $x^2 = \frac{\pi}{2}$ $\left(x = \pm\sqrt{\frac{\pi}{2}}\right)(x = \pm 1.25)$ ***A1***

$P\left(\sqrt{\frac{\pi}{2}}, -\sqrt{\frac{\pi}{2}}\right), Q\left(-\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$ or $P(1.25, -1.25)$, $Q(-1.25, 1.25)$ ***A1***

continued…

Question 11 continued

(ii) $m_1 = -\left(\dfrac{1-\sqrt{\dfrac{\pi}{2}} \times -1}{1+\sqrt{\dfrac{\pi}{2}} \times -1}\right)$ ***M1A1***

$m_2 = -\left(\dfrac{1+\sqrt{\dfrac{\pi}{2}} \times -1}{1-\sqrt{\dfrac{\pi}{2}} \times -1}\right)$ ***A1***

$m_1 m_2 = 1$ ***AG***

Note: Award ***M1A0A0*** if decimal approximations are used.

Note: No ***FT*** applies.

[7 marks]

(c) equate derivative to -1 ***M1***

$(y - x)\sin(xy) = 0$ ***(A1)***

$y = x$, $\sin(xy) = 0$ ***R1***

in the first case, attempt to solve $2x = \cos\left(x^2\right)$ ***M1***

$(0.486, 0.486)$ ***A1***

in the second case, $\sin(xy) = 0 \Rightarrow xy = 0$ and $x + y = 1$ ***(M1)***

$(0,1)$, $(1,0)$ ***A1***

[7 marks]

Total [19 marks]