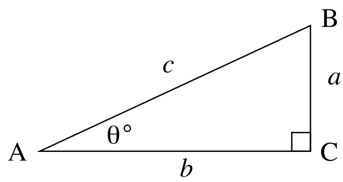
9.1 TRIGONOMETRIC RATIOS

9.1.1 REVIEW OF TRIGONOMETRIC FUNCTIONS FOR RIGHT-ANGLED TRIANGLES

The trigonometric functions are defined as **ratio functions** in a right-angled triangle. As such they are often referred to as the **trigonometric ratios**.

The trigonometric ratios are based on the right-angled triangle shown alongside. Such right-angled triangles are defined in reference to a nominated angle. In the right-angled triangle ABC the longest side [AB] (opposite the right-angle) is the **hypotenuse**. Relative to the angle $\angle BAC$ of size $\theta°$, the side BC is called the **opposite** side while the side AC is called the **adjacent** side.



The trigonometric ratios are defined as

$$\sin\theta° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}, \quad \cos\theta° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}, \quad \tan\theta° = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$

Note then, that $\tan\theta° = \dfrac{\sin\theta°}{\cos\theta°}, \cos\theta \neq 0$.

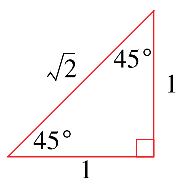
There also exists another important relation between the side lengths of a right-angled triangle.

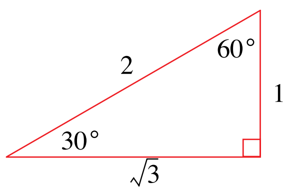
This relationship, using **Pythagoras's Theorem** is $a^2 + b^2 = c^2$

Do not forget to adjust the mode of your calculator to degree mode when necessary. On the TI–83, this is done by pressing **MODE** and then selecting the **Degree** mode. As angles can be quoted in degrees '°', minutes '′' and seconds '″' we make use of the **DMS** option under the **ANGLE** menu (accessed by pressing **2nd APPS**) to convert an angle quoted as a decimal into one quoted in degrees, minutes and seconds.

9.1.2 EXACT VALUES

There are a number of special right-angled triangles for which exact values of the trigonometric ratios exist. Two such triangles are shown below:





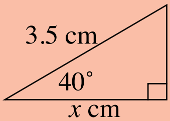
From these triangles we can tabulate the trigonometric ratios as follows:

θ	$\sin\theta°$	$\cos\theta°$	$\tan\theta°$
30	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

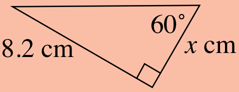
EXAMPLE 9.1

Find x in each of the following triangles (correct to 4 d.p)

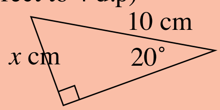
(a)



(b)

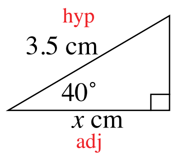


(c)



Solution

(a) We label the sides relative to the given angle:



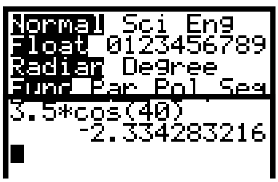
As the sides involved are the adjacent (adj) and the hypotenuse (hyp). The appropriate ratio is the cosine ratio, i.e., $\cos\theta = \frac{\text{adj}}{\text{hyp}}$. Then, substituting the information into the expression we can solve for x:

$$\cos 40° = \frac{x}{3.5} \Leftrightarrow 3.5 \times \cos 40° = x$$

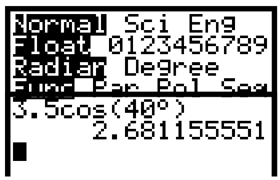
$$\therefore x = 2.6812 \text{ (to 4 d.p)}$$

A quick word about using the TI–83. Below we show that depending on the mode setting we obtain different values. In particular, note that in Case B, even though the mode setting was in radians, we were able to over ride this by using the degree measure, '°', under the **ANGLE** menu.

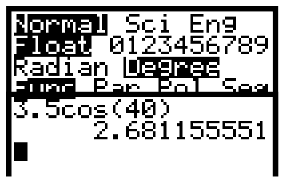
Case A:



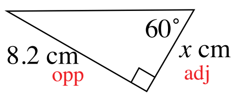
Case B:



Case C:



(b) We label the sides relative to the given angle.

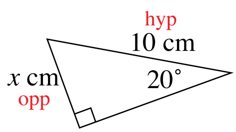


The sides involved are the adjacent (adj) and the opposite (opp) the appropriate ratio is the tangent ratio, i.e., $\tan\theta = \frac{\text{opp}}{\text{adj}}$. Then, substituting the information into the expression we can solve for x:

$$\tan 60° = \frac{8.2}{x} \Leftrightarrow x\tan 60° = 8.2 \Leftrightarrow x = \frac{8.2}{\tan 60°}$$

$$\therefore x = 4.7343 \text{ (to 4 d.p)}$$

(c) We label the sides relative to the given angle.



The sides involved are the opposite (opp) and the hypotenuse (hyp). The appropriate ratio is the sine ratio, i.e., $\sin\theta = \frac{\text{opp}}{\text{hyp}}$. Then, substituting the information into the expression we can solve for x:

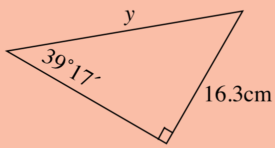
$$\sin 20° = \frac{x}{10} \Leftrightarrow 10 \times \sin 20° = x$$

$$\therefore x = 3.4202 \text{ (to 4 d.p)}$$

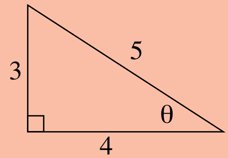
EXAMPLE 9.2

Find x and θ in the following triangles

(a)

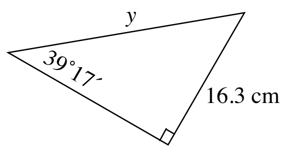


(b)



Solution

(a) The important sides are the opposite and hypotenuse. So,



$$\sin 39°17' = \frac{16.3}{y} \Leftrightarrow y \times \sin 39°17' = 16.3$$

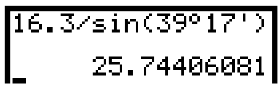
$$\Leftrightarrow y = \frac{16.3}{\sin 39°17'}$$

$$\therefore y = 25.7 \text{ cm}$$

The TI–82/83 calculators accept angle inputs using the **2nd ANGLE** menu. **Option 1** allows entry of angles in degrees irrespective of the **MODE** setting of the calculator. **Option 2** allows the entry of degrees, minutes, seconds.

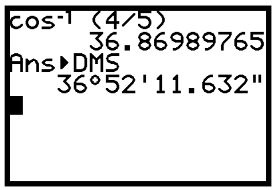
The problem would be solved using the keying sequence

16.3÷sin 39 **2nd ANGLE 1** 17 **2nd ANGLE 2** ENTER.



(b) When using a calculator to find an angle, **option 4** of the **2nd ANGLE** menu will allow you to display an answer in degree, minute, second format.

Any of the three trigonometric ratios will do, but when finding angles, it is generally best to use the cosine ratio. The reason for this should become apparent as this chapter progresses.



$$\cos\theta = \frac{4}{5} \Rightarrow \theta = \cos^{-1}\left(\frac{4}{5}\right) \therefore \theta \approx 36°52'12''$$

rounded to the nearest second.

EXAMPLE 9.3

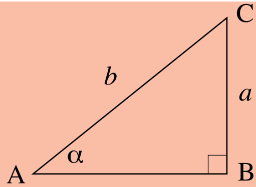
Using the triangle shown, find

(a) i. AB

ii. $\cos\alpha$

iii. $\tan\alpha$

(b) If $\cos\alpha = 0.2$ find $\sin(90° - \alpha)$.



Solution

(a) Using Pythagoras's Theorem we have $AC^2 = AB^2 + BC^2 \therefore b^2 = AB^2 + a^2$

$$\Leftrightarrow AB^2 = b^2 - a^2$$

$$\Rightarrow AB = \sqrt{b^2 - a^2}$$

(b) $\cos\alpha = \dfrac{AB}{AC} = \dfrac{\sqrt{b^2 - a^2}}{b}$

(c) $\tan\alpha = \dfrac{CB}{AB} = \dfrac{a}{\sqrt{b^2 - a^2}}$

(d) $\sin(90° - \alpha) = \dfrac{AB}{AC}$, but $\cos\alpha = \dfrac{AB}{AC} \therefore \sin(90° - \alpha) = \cos\alpha$.

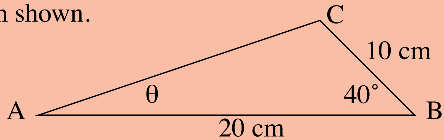
i.e., $\sin(90° - \alpha) = 0.2$

Often we are dealing with non right-angled triangles. However, these can be 'broken up' into at least two right-angled triangles, which then involves solving simultaneous equations. This is illustrated in the next example.

EXAMPLE 9.4

Find the angle θ in the diagram shown.

Note that $\angle ACB \neq 90°$.

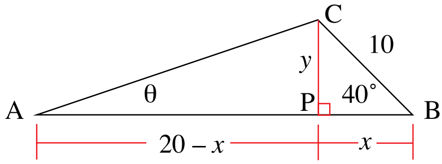


Solution

We start by 'breaking-up' the triangle into two right-angled triangles as follows:

Using ΔACP:

$$\tan\theta = \frac{PC}{AP} = \frac{y}{20 - x} \quad \text{– (1)}$$



We now need to determine x and y.

Using ΔBPC:

$$\sin 40° = \frac{PC}{BC} = \frac{y}{10}$$

$$\Leftrightarrow y = 10\sin 40° \quad \text{– (2)}$$

and $\quad \cos 40° = \dfrac{BP}{BC} = \dfrac{x}{10}$

$$\Leftrightarrow x = 10\cos 40° \quad \text{– (3)}$$

Therefore, substituting (3) and (2) into (1) we have:

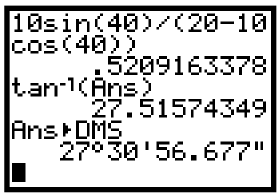
$$\tan\theta = \frac{10\sin 40°}{20 - 10\cos 40°}$$

$$= 0.5209$$

$$\therefore \theta = \tan^{-1}(0.5209)$$

$$= 27.5157$$

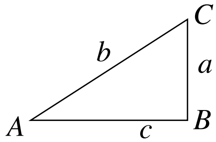
$$= 27°31'$$



Note that we have not rounded down our answer until the very last step.

EXERCISES 9.1

1. The parts of this question refer to the triangle shown. Complete the blank spaces in this table, giving lengths correct to three significant figures and angles correct to the nearest degree.

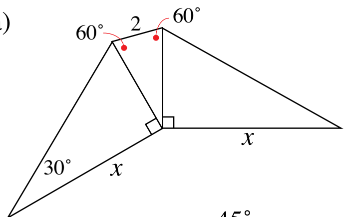


	a cm	*b* cm	*c* cm	*A*	*B*	*C*
1.			1.6		90°	23°
2.		98.3			90°	34°
3.			33.9		90°	46°
4.		30.7			90°	87°
5.	2.3				90°	33°
6.		77			90°	51°
7.	44.4		68.4		90°	57°
8.			12.7	13°	90°	
9.		94.4		52°	90°	

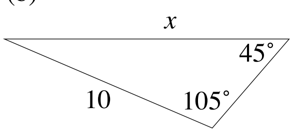
	a cm	*b* cm	*c* cm	*A*	*B*	*C*
10.	71.8		64.6	48°	90°	
11.		34.1		43°	90°	
12.			2.3	87°	90°	
13.	71.5			63°	90°	
14.	33.5		6.5		90°	
15.	6.1	7.2			90°	
16.		30	7.3		90°	
17.	29.0		2.0		90°	
18.	34.5	88.2			90°	
19.	24.0	29.7			90°	
20.		46.2			90°	27°
21.	59.6		41.8		90°	35°
22.		6.8			90°	37°
23.			14.9	41°	90°	49°
24.			16.1	41°	90°	49°
25.			33.3	68°	90°	22°

2. Find the exact value of x in each of the following

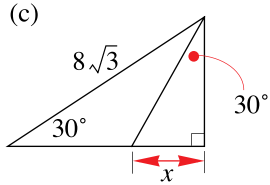
(a)



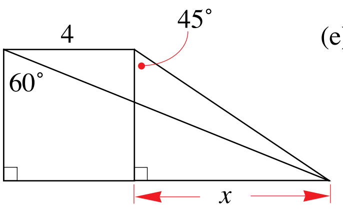
(b)



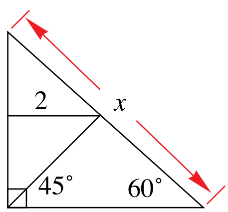
(c)



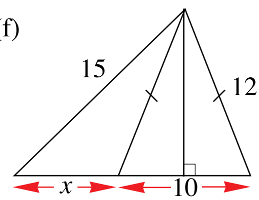
(d)



(e)



(f)

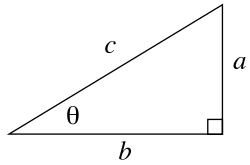


3. Using the triangle on the right, show that

(a) $\sin(90° - \theta) = \cos\theta$

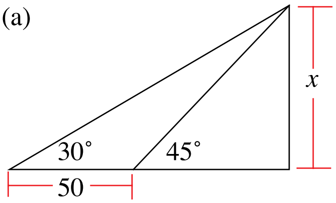
(b) $\cos(90° - \theta) = \sin\theta$

(c) $\tan(90° - \theta) = \dfrac{1}{\tan\theta}$

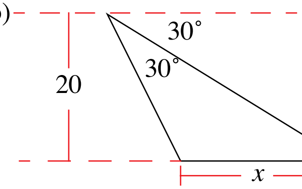


4. Find the exact value of x in each of the following.

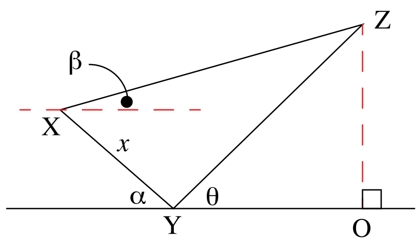
(a)



(b)



5. Show that $OZ = \dfrac{x\tan\theta(\sin\alpha + \cos\alpha\tan\beta)}{\tan\theta - \tan\beta}$

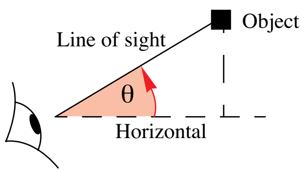


9.2 APPLICATIONS

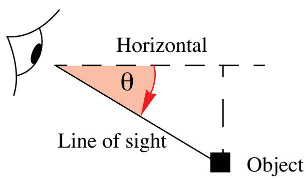
Applications that require the use of trigonometric ratios and right-angled triangles are many and varied. In this section we consider a number of standard applications to highlight this.

9.2.1 ANGLE OF ELEVATION AND DEPRESSION

The **angle of elevation** is the angle of the line of sight **above** the horizontal of an object seen above the horizontal



The **angle of depression** is the angle of the line of sight **below** the horizontal of an object seen below the horizontal



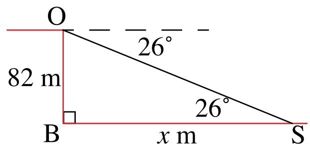
Note that the angle of depression and elevation for the same line of sight are **alternate angles**.

EXAMPLE 9.5

An observer standing on the edge of a cliff 82 m above sea level sees a ship at an angle of depression of 26°. How far from the base of the cliff is the ship situated?

Solution

We first draw a diagram to represent this situation:
Let the ship be at point S, x metres from the base of the cliff, B, and let O be where the observer is standing.



Using the right-angled triangle OBS we have:

$$\tan 26° = \frac{82}{x} \Leftrightarrow x\tan 26° = 82$$

$$\Leftrightarrow x = \frac{82}{\tan 26°}$$

$$= 168.1249\ldots$$

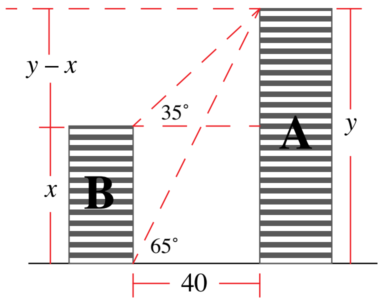
Therefore, the ship is 168 m from the base of the cliff.

EXAMPLE 9.6

The angle of depression from the roof of building A to the foot of a second building, B, across the same street and 40 metres away is 65°. The angle of elevation of the roof of building B to the roof of building A is 35°. How tall is building B?

Solution

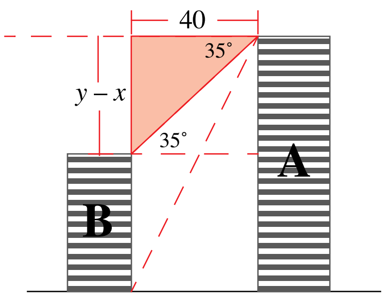
Let the height of building B be x m and that of building A be y m.

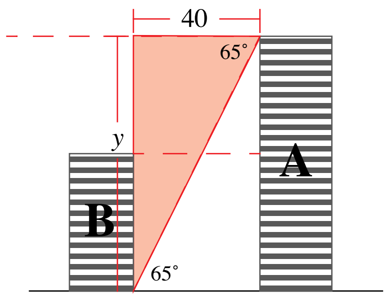


Note that we are using the fact that for the same line of sight, the angle of depression and elevation is equal.

The height difference between the two buildings must then be $(y - x)$ m.

We now have two right-angled triangles to work with:





$$\tan 35° = \frac{y-x}{40}$$

$$\Leftrightarrow y - x = 40\tan 35° \quad -(1)$$

$$\tan 65° = \frac{y}{40}$$

$$\Leftrightarrow y = 40\tan 65° \quad -(2)$$

Substituting (2) into (1) we have:

$$40\tan 65° - x = 40\tan 35°$$

$$\Leftrightarrow x = 40\tan 65° - 40\tan 35°$$

$$\therefore x = 57.7719\ldots$$

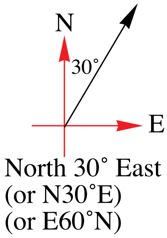
That is, building B is 57.77 m

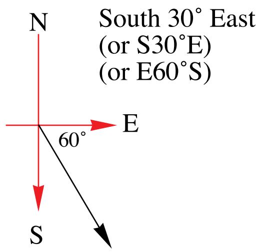
9.2.2 BEARINGS

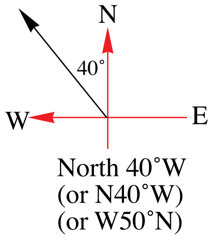
In the sport of orienteering, participants need to be skilled in handling bearings and reading a compass. Bearings can be quoted by making reference to the North, South, East and West directions or using true bearings. We look at each of these.

Compass bearings

These are quoted in terms of an angle measured East, West, North or South, or somewhere in between. For example, North 30˚ East, expressed as N30˚E, informs us that from the North direction we rotate 30˚ towards the East and then follow that line of direction. The following diagrams display this for a number of bearings.

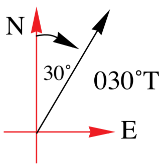


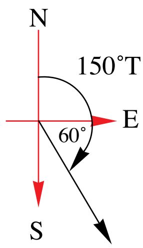


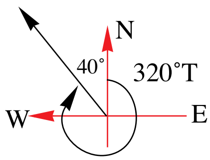


True bearings

These are quoted in terms of an angle measured in a clockwise direction from North (and sometimes a capital T is attached to the angle to highlight this fact). So, for example, a bearing of 030˚T would represent a bearing of 30˚ in a clockwise direction from the North – this corresponds to a compass bearing of N30˚E. Using the above compass bearings we quote the equivalent true bearings:







EXAMPLE 9.7

Janette walks for 8 km North-East and then 11 km South-East. Find the distance and bearing from her starting point.

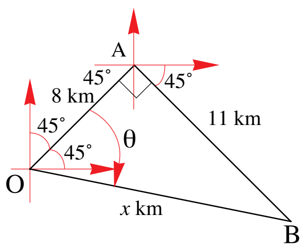
Solution

First step is to draw a diagram.

As $\angle OAB = 90°$ we can make use of Pythagoras's Theorem:

$$x^2 = 8^2 + 11^2$$

$$\therefore x = 13.60 \text{ [taking +ve square root]}$$



Let $\theta = \angle AOB$ so that $\tan\theta = \frac{11}{8}$. $\therefore \theta = \tan^{-1}\left(\frac{11}{8}\right)$

$$= 53.97°$$

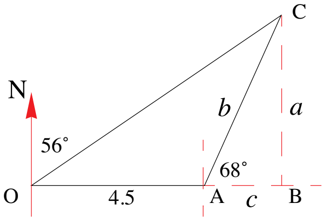
Therefore, bearing is $45° + \theta = 45° + 53.97° = 98.97°$

That is, B has a bearing of 98.97˚T from O and is (approx) 13.6 km away.

EXAMPLE 9.8 The lookout, on a ship sailing due East, observes a light on a bearing of 056˚T. After the ship has travelled 4.5 km, the lookout now observes the light to be on a bearing of 022˚T. How far is the light source from the ship at its second sighting?

Solution

As always, we start with a diagram.



Using ΔOBC we have, $\tan 34° = \frac{BC}{OB} = \frac{a}{4.5+c}$

$$\therefore a = (4.5+c)\tan 34° \quad -(1)$$

Using ΔABC we have, $\tan 68° = \frac{BC}{AB} = \frac{a}{c}$

$$\therefore a = c\tan 68° \quad -(2)$$

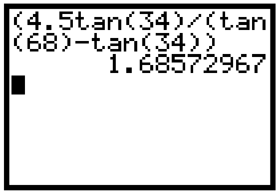
Equating (1) and (2) we have, $c\tan 68° = (4.5+c)\tan 34°$

$$c\tan 68° = 4.5\tan 34° + c\tan 34°$$

$$\Leftrightarrow c(\tan 68° - \tan 34°) = 4.5\tan 34°$$

$$\Leftrightarrow c = \frac{4.5\tan 34°}{(\tan 68° - \tan 34°)}$$

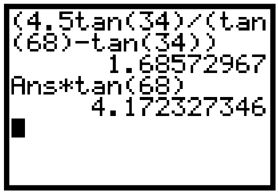
$$\therefore c = 1.6857$$



Substituting this result into (2) we have,

$$a = \frac{4.5\tan 34°}{(\tan 68° - \tan 34°)} \times \tan 68°$$

$$\therefore a = 4.1723$$



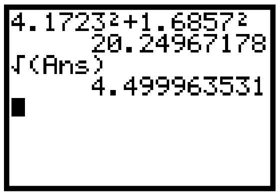
Then, using ΔABC and Pythagoras's Theorem, we have

$$b^2 = a^2 + c^2$$

$$= 4.1723^2 + 1.6857^2$$

$$\therefore b = \sqrt{20.2496}$$

$$= 4.4999$$



That is, the light is 4.5 km from the ship (at the second sighting).
Can you see a much quicker solution? Hint – think isosceles triangle.

EXERCISES 9.2

1. (a) Change the following compass bearings into true bearings

i. N30˚E   ii. N30˚W   iii. S15˚W   iv. W70˚S

(b) Change the following true bearings into compass bearings

i. 025˚T   ii. 180˚T   iii. 220˚T   iv. 350˚T

2. The angle of depression from the top of a building 60 m high to a swing in the local playground is 58˚. How far is the swing from the foot of the building?

3. From a point A on the ground, the angle of elevation to the top of a tree is 52˚. If the tree is 14.8 m away from point A, find the height of the tree.

4. Find the angle of elevation from a bench to the top of an 80 m high building if the bench is 105 m from the foot of the building.

5. Patrick runs in a direction N60˚E and after 45 minutes finds himself 3900 m North of his starting point. What is Patrick's average speed in ms^{-1}.

6. A ship leaves Oldport and heads NW. After covering a distance of 16 km it heads in a direction of N68˚22'W travelling a distance of 22 km where it drops anchor. Find the ship's distance and bearing from Oldport after dropping anchor.

7. From two positions 400 m apart on a straight road, running in a northerly direction, the bearings of a tree are N36˚40'E and E33˚22'S. What is the shortest distance from the tree to the road?

8. A lamp post leaning away from the sun and at 6˚ from the vertical, casts a shadow 12 m long when the sun's angle of elevation is 44˚. Assuming that the level of the ground where the pole is situated is horizontal, find its length.

9. From a window, 29.6 m above the ground, the angle of elevation of the top of a building is 42˚, while the angle of depression to the foot of the building is 32˚. Find the height of the building.

10. Two towns P and Q are 50 km apart, with P due west of Q. The bearing of a station from town P is 040˚T while the bearing of the station from town Q is 300˚T. How far is the station from town P?

11. When the sun is 74˚ above the horizon, a vertical flagpole casts a shadow 8.5 m onto a horizontal ground. Find the shadow cast by the sun when it falls to 62˚ above the horizontal.

12. A hiker walks for 5km on a bearing of 053˚ true (North 53˚ East). She then turns and walks for another 3km on a bearing of 107˚ true (East 17˚ South).

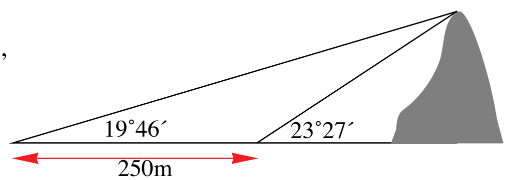
(a) Find the distance that the hiker travels North/South and the distance that she travels East/West on the first part of her hike.

(b) Find the distance that the hiker travels North/South and the distance that she travels East/West on the second part of her hike.

(c) Hence find the total distance that the hiker travels North/South and the distance that she travels East/West on her hike.

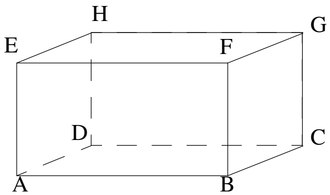
(d) If the hiker intends to return directly to the point at which she started her hike, on what bearing should she walk and how far will she have to walk?

13. A surveying team are trying to find the height of a hill. They take a 'sight' on the top of the hill and find that the angle of elevation is 23˚27´. They move a distance of 250 metres on level ground directly away from the hill and take a second 'sight'. From this point, the angle of elevation is 19˚46´. Find the height of the hill, correct to the nearest metre.

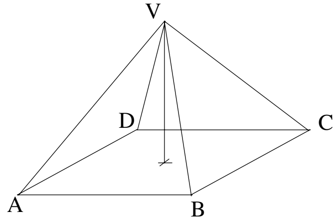


9.3 RIGHT ANGLES IN 3–DIMENSIONS

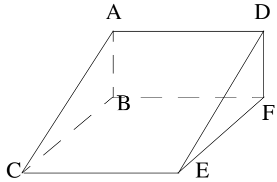
When dealing with problems in three dimensions, we draw the figures in perspective, so that a model can be more accurately visualised. This does not mean that you must be an artist, simply that you take a little time (and a lot of practice) when drawing such diagrams. The key to many 3–D problems is locating the relevant right–angled triangles within the diagram. Once this is done, all of the trigonometric work that has been covered in the previous two sections can be applied. As such, we will not be learning new theory, but rather developing new drawing and modelling skills. Some typical examples of solids that may be encountered are:



Cuboid ABCD, EFGH



Right Pyramid V, ABCD

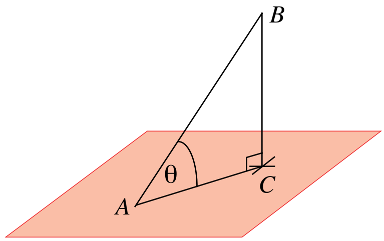


Wedge ABC, DEF

We look at two basic concepts and drawing techniques to help us.

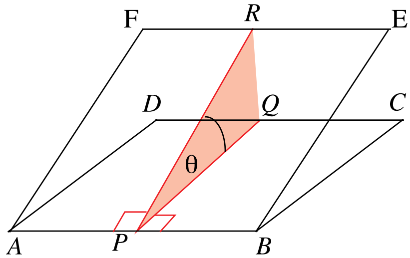
1. A line and a plane:

A line will always cut a plane at some point (unless the line is parallel to the plane). To find the angle between a line and a plane construct a perpendicular from the line to the plane and complete a right–angled triangle. In our diagram, we have that the segment $[AB]$ is projected onto the plane. A perpendicular, $[BC]$ is drawn, so that a right–angled triangle, ABC is completed. The angle that the line then makes with the plane is given by θ (which can be found by using the trig–ratios).



2. A plane and a plane:

To find the angle between two planes ABCD and ABEF (assuming that they intersect), take any point P on the intersecting segment $[AB]$ and draw $[PQ]$ and $[PR]$ on each plane in such a way that they are perpendicular to $[AB]$. Then, the angle between $[PQ]$ and $[PR]$ (θ) is the angle between the two planes.



EXAMPLE 9.9

A cube ABCD, EFGH has a side length measuring 6 cm.

(a) Find the length of the segment $[AC]$.

(b) The length of the diagonal $[AG]$.

(c) The angle that the diagonal $[AG]$ makes with the base.

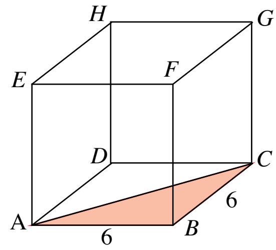
Solution

First we need to draw a cube:

(a) Now the base of the cube is a square, so that $\angle ABC = 90°$, i.e., we have a right–angled triangle, so we can use Pythagoras's Theorem:

$$AC^2 = AB^2 + BC^2 = 6^2 + 6^2 = 72$$

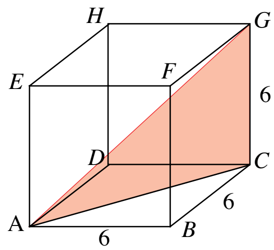
$$\therefore AC = \sqrt{72} \approx 8.49$$



(b) This time we have that $\angle ACG = 90°$, therefore,

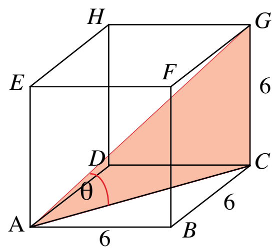
$$AG^2 = AC^2 + CG^2 = (\sqrt{72})^2 + 6^2 = 108$$

$$\therefore AC = \sqrt{108} \approx 10.39$$



(c) Using triangle ACG, $\tan\theta = \dfrac{CG}{AC} \therefore \tan\theta = \dfrac{6}{\sqrt{72}}$

$$\theta = \tan^{-1}\left(\frac{6}{\sqrt{72}}\right) = 35.26° = 35°16'$$



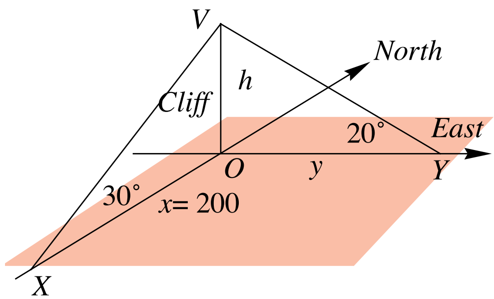
EXAMPLE 9.10

From a point *X*, 200 m due South of a cliff, the angle of elevation of the top of the cliff is 30°. From a point *Y*, due East of the cliff, the angle of elevation of the top of the cliff is 20°. How far apart are the points *X* and *Y*?

Solution

We start by illustrating this information on a 3–D diagram (Note that North–South and West–East are drawn on a plane. It is necessary to do this otherwise the diagram will not make sense).

Let the cliff be h metres high. The distance from X to the base of the cliff be x metres and the distance from Y to the base of the cliff be y metres.



As $\angle XOY = 90°$, then $XY^2 = x^2 + y^2 = 200^2 + y^2$

But, $\tan 20° = \dfrac{h}{y}$, of which we know neither h or y.

However, using triangle XOV, we have that $\tan 30° = \dfrac{h}{200} \Rightarrow h = 200 \times \tan 30°$.

Therefore, we have that $\tan 20° = \dfrac{200 \times \tan 30°}{y} \Leftrightarrow y = \dfrac{200 \times \tan 30°}{\tan 20°}$

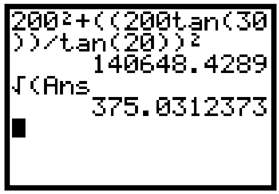
That is, $y = 317.25$

Therefore, $XY^2 = x^2 + y^2 = 200^2 + \left(\dfrac{200 \times \tan 30°}{\tan 20°}\right)^2$

$= 140648.4289$

$XY = 375.0312$.

That is, X and Y are approximately 375 m apart.



EXERCISES 9.3

1. For the diagram shown, determine the angle of inclination between the plane
(a) ABCD and the base, EABH (Figure 1).
(b) ABC and the base EBFA (Figure 2).

Figure 1.

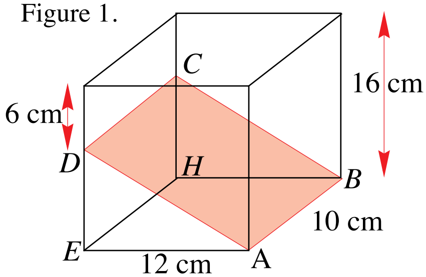
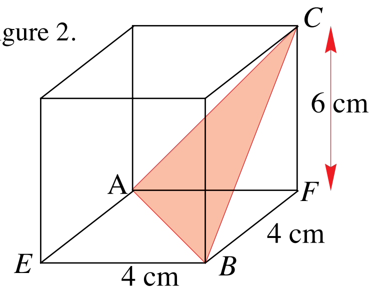
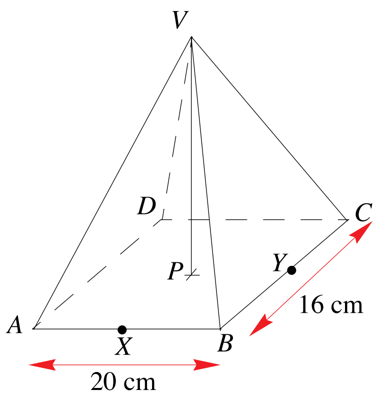


Figure 2.

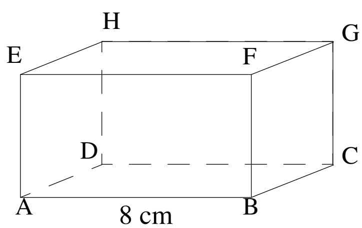


2. A right pyramid with a rectangular base and a vertical height of 60 cm is shown in the diagram alongside. The points X and Y are the midpoints of the sides [AB] *and* [BC] *respectively*
Find
(a) the length, AP.
(b) the length of the edge [AV].
(c) the angle that the edge AV makes with the base ABCD.
(d) the length, YV .
(e) The angle that the plane BCV makes with the base.



3. The diagram alongside shows a rectangular box with side lengths AB = 8 cm, BC = 6 cm and CG = 4 cm.
Find the angle between

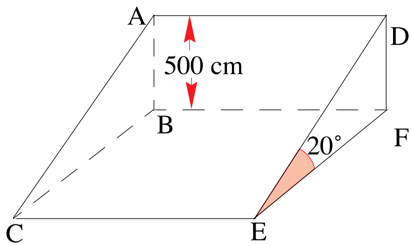
(a) the line [BH] and the plane $ABCD$.
(b) the lines [BH and [BA].
(c) the planes $ADGF$ and $ABCD$.



4. For the wedge shown alongside, given that the angle between the lines EA and ED is 50°find

(a) the length of [AE].

(b) the $\angle AEB$.



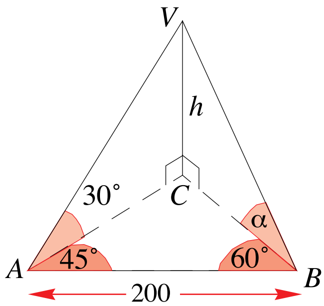
5. From a point A, 100 m due South of a tower, the angle of elevation of the top of the tower is 40°. From a point B, due East of the tower, the angle of elevation of the top of the tower is 20°. How far apart are the points A and B?

6. For the triangular prism shown alongside find

(a) the value of h

(b) the value of α

(c) the angle that the plane ABV makes with the base ABC.



7. The angle of depression from the top of a tower to a point X South of the tower, on the ground and 120 m from the foot of the tower is 24°. From point Y due West of X the angle of elevation to the top of the tower is 19°.

(a) Illustrate this information on a diagram.

(b) Find the height of the tower.

(c) How far is Y from the foot of the tower?

(d) How far apart are the points X and Y?

8. A mast is held in a vertical position by four ropes of length 60 metres. All four ropes are attached at the same point at the top of the mast so that their other ends form the vertices of a square when pegged into the (level) ground. Each piece of rope makes an angle of 54° with the ground.

(a) Illustrate this information on a diagram.

(b) How tall is the mast?

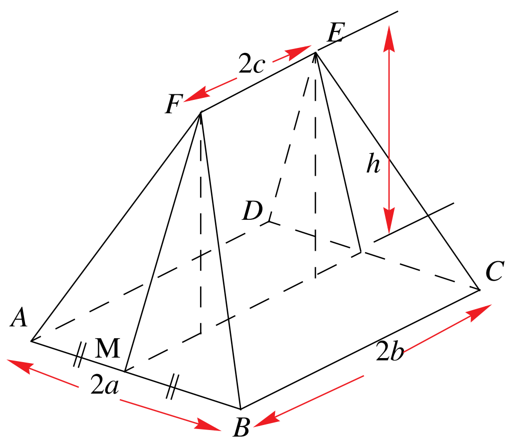
9. A symmetrical sloping roof has dimensions as shown in the diagram. Find

(a) the length of [FM].

(b) the angle between the plane BCEF and the ground.

(c) the angle between the plane ABF and the ground

(d) the total surface area of the roof.



10. The angle of elevation of the top of a tower from a point A due South of it is 68°. From a point B, due East of A, the angle of elevation of the top is 54°. If A is 50 m from B, find the height of the tower.

11. A tower has been constructed on the bank of a long straight river. From a bench on the opposite bank and 50 m downstream from the tower, the angle of elevation of the top of the tower is 30°. From a second bench on the same side of the tower and 100 m upstream from the tower, the angle of elevation of the top of the tower is 20°. Find
(a) the height of the tower.
(b) the width of the river.

12. A right pyramid of height 10 m stands on a square base of side lengths 5 m. Find
(a) the length of the slant edge.
(b) the angle betwen a sloping face and the base.
(c) the angle between two sloping faces.

13. A camera sits on a tripod with legs 1.5 m long. The feet rest on a horizontal flat surface and form an equilateral triangle of side lengths 0.75 m. Find
(a) the height of the camera above the ground.
(b) the angles made by the legs with the ground.
(c) the angle between the sloping faces formed by the tripod legs.

14. From a point A due South of a mountain, the angle of elevation of the mountaintop is α. When viewed from a point B, x m due East of A, the angle of elevation of the mountaintop is β. Show that the height, h m, of the mountain is given by $h = \dfrac{x\sin\alpha\sin\beta}{\sqrt{\sin^2\alpha - \sin^2\beta}}$.

9.4 AREA OF A TRIANGLE

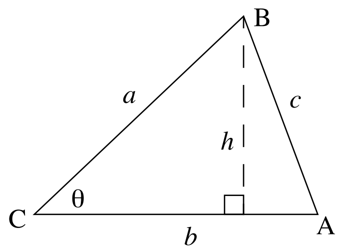
Given **any** triangle with sides a and b, and the included angle θ, the area, A, is given by

$$A = \frac{1}{2}bh$$

However, $\sin\theta = \frac{h}{a} \Leftrightarrow h = a \times \sin\theta$ and so, we have that

$$\boxed{A = \frac{1}{2}b \times a \times \sin\theta}$$

where θ is the angle between sides a and b.



Note that the triangle need not be a right-angled triangle.

Because of the standard labelling system for triangles, the term $\sin\theta$ is often replaced by $\sin C$, given the expresion Area $= \frac{1}{2}ab\sin C$.

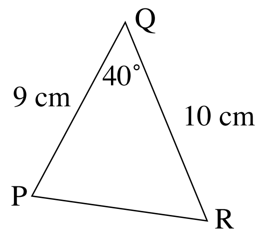
A similar argument can be used to generate the formulas: Area $= \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$

EXAMPLE 9.11

Find the area of the triangle PQR given that PQ = 9 cm, QR = 10 cm and $\angle PQR = 40°$.

Solution

Based on the given information we can contruct the following triangle:



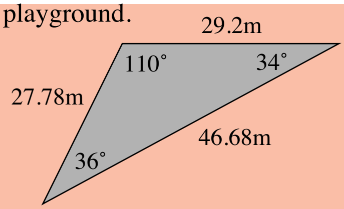
Meaning that, the required area, A, is given by

$$A = \frac{1}{2}ab\sin\theta = \frac{1}{2} \times 9 \times 10 \times \sin 40°$$
$$= 28.9$$

That is, the area is 28.9 cm^2.

EXAMPLE 9.12

The diagram shows a triangular children's playground. Find the area of the playground



Solution

Since all the measurements of the triangle are known, any one of the three formulas could be used. Many people remember the formula as 'Area equals half the product of the lengths of two sides times the sine of the angle between them'.

$$\text{Area} = \frac{1}{2} \times 27.78 \times 46.68 \times \sin 36° = 381\,\text{m}^2$$

$$\text{Area} = \frac{1}{2} \times 27.78 \times 29.2 \times \sin 110° = 381\,\text{m}^2$$

$$\text{Area} = \frac{1}{2} \times 29.2 \times 46.68 \times \sin 34° = 381\,\text{m}^2$$

EXERCISES 9.4

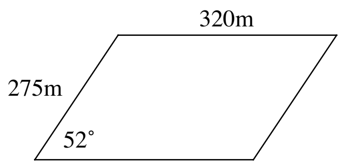
1. Find the areas of these triangles that are labelled using standard notation.

	a cm	*b* cm	*c* cm	*A*	*B*	*C*
(a)	35.94	128.46	149.70	12°	48°	120°
(b)	35.21	54.55	81.12	20°	32°	128°
(c)	46.35	170.71	186.68	14°	63°	103°
(d)	33.91	159.53	163.10	12°	78°	90°
(e)	42.98	25.07	48.61	62°	31°	87°
(f)	39.88	24.69	34.01	84°	38°	58°
(g)	43.30	30.26	64.94	34°	23°	123°
(h)	12.44	2.33	13.12	68°	10°	102°

	a cm	*b* cm	*c* cm	*A*	*B*	*C*
(i)	43.17	46.44	24.15	67°	82°	31°
(j)	23.16	32.71	24.34	45°	87°	48°
(k)	50.00	52.91	38.64	64°	72°	44°
(l)	44.31	17.52	48.77	65°	21°	94°
(m)	12.68	23.49	22.34	32°	79°	69°
(n)	42.37	42.37	68.56	36°	36°	108°
(o)	40.70	15.65	41.26	77°	22°	81°

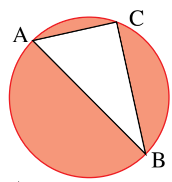
2. A car park is in the shape of a parallelogram. The lengths of the sides of the car park are given in metres.

What is the area of the car park?

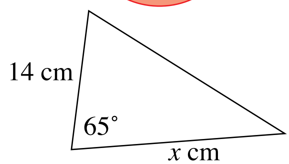


3. The diagram shows a circle of radius 10 cm. AB is a diameter of the circle. AC = 6 cm.

Find the area of the shaded region, giving an exact answer.

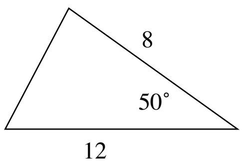


4. The triangle shown has an area of 110 cm^2. Find x.

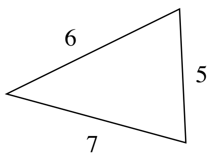


5. Find the area of the following

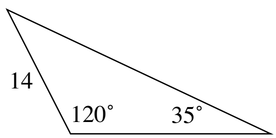
(a)



(b)



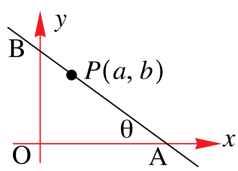
(c)



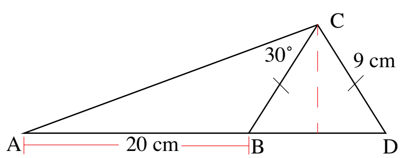
6. A napkin is in the shape of a quadrilateral with diagonals 9 cm and 12 cm long. The angle between the diagonals is 75°. What area does the napkin cover when laid out flat?

7. A triangle of area 50 cm^2 has side lengths 10 cm and 22 cm. What is the magnitude of the included angle?

8. A variable triangle OAB is formed by a straight line passing through the point $P(a, b)$ on the Cartesian plane and cutting the x–axis and y–axis at A and B respectively.
If $\angle OAB = \theta$, find the area of ΔOAB in terms of a, b and θ.



9. Find the area of ΔABC for the given diagram.



5. (a) $I = \frac{a}{n^k}$ **6.** (a) 0.10 (b) $\lambda = \lambda_0 \times 10^{-kx}$ (c) 16.82% (d) $k = -\frac{1}{x}\log\left(\frac{\lambda}{\lambda_0}\right)$

EXERCISE 8.1.1

1. i. (b) 4 (c) $t_n = 4n - 2$ ii. (b) –3 (c) $t_n = -3n + 23$ iii. (b) –5 (c) $t_n = -5n + 6$ iv. (b) 0.5 (c) $t_n = 0.5n$ v. (b) 2 (c) $t_n = y + 2n - 1$ vi. (b) –2 (c) $t_n = x - 2n + 4$ **2.** –28 **3.** 9,17 **4.** –43 **5.** 7 **6.** 7 **7.** –5 **8.** 0 **9.** (a) 41 (b) 31st **10.** 2, $\sqrt{3}$ **11.** (a) i. 2 ii. –3 (b) i. 4 ii. 11 **12.** $x - 8y$ **13.** $t_n = 5 + \frac{10}{3}(n-1)$ **14.** (a) –1 (b) 0

EXERCISE 8.1.2

1. (a) 145 (b) 300 (c) –170 **2.** (a) –18 (b) 690 (c) 70.4 **3.** (a) –105 (b) 507 (c) 224 **4.** (a) 126 (b) 3900 (c) 14th week **5.** 855 **6.** (a) 420 (b) –210 **7.** $a = 9, b = 7$

EXERCISE 8.1.3

1. 123 **2.** –3, –0.5, 2, 4.5, 7, 9.5, 12 **3.** 3.25 **4.** $a = 3$ $d = -0.05$ **5.** 10 000 **6.** 330 **7.** –20 **8.** 328 **9.** $725, 37wks **10.** i. $55 ii. 2750 **11.** (a) (i) 8m (ii) 40m (b) 84m (c) Dist $= 2n^2 - 2n = 2n(n-1)$ (d) 8 (e) 26 players, 1300m **12.** (a) 5050 (b) 10200 (c) 4233 **13.** (a) 145 (b) 390 (c) –1845 **14.** (b) $3n - 2$

EXERCISE 8.2.1

1. (a) $r = 2, u_5 = 48, u_n = 3 \times 2^{n-1}$ (b) $r = \frac{1}{3}, u_5 = \frac{1}{27}, u_n = 3 \times \left(\frac{1}{3}\right)^{n-1}$ (c) $r = \frac{1}{5}, u_5 = \frac{2}{625}, u_n = 2 \times \left(\frac{1}{5}\right)^{n-1}$ (d) $r = -4, u_5 = -256, u_n = -1 \times (-4)^{n-1}$ (e) $r = \frac{1}{b}, u_5 = \frac{a}{b^3}, u_n = ab \times \left(\frac{1}{b}\right)^{n-1}$ (f) $r = \frac{b}{a}, u_5 = \frac{b^4}{a^2}, u_n = a^2 \times \left(\frac{b}{a}\right)^{n-1}$ **2.** (a) ±12 (b) $\frac{\pm\sqrt{5}}{2}$ **3.** (a) ±96 (b) 15th **4.** (a) $u_n = 10 \times \left(\frac{5}{6}\right)^{n-1}$ (b) $\frac{15625}{3888} \cong 4.02$ (c) $n = 5$ (4 times) **5.** $-2, \frac{4}{3}$ **6.** (a) i. $4096 ii. $2097.15 (b) 6.2 yrs **7.** $\left(u_n = \frac{1000}{169} \times \left(\frac{12}{5}\right)^{n-1}\right), \frac{1990656}{4225} \cong 471.16$ **8.** 2.5,5,10 or 10,5,2.5 **9.** 53757 **10.** 108 952 **11.** (a) $56 156 (b) $299 284

EXERCISE 8.2.2

1. (a) 3 (b) $\frac{1}{3}$ (c) –1 (d) $-\frac{1}{3}$ (e) 1.25 (f) $-\frac{2}{3}$ **2.** (a) 216513 (b) 1.6384×10^{-10} (c) $\frac{256}{729}$ (d) $\frac{729}{2401}$ (e) $-\frac{81}{1024}$ **3.** (a) 11; 354292 (b) 7; 473 (c) 8; 90.90909 (d) 8; 172.778 (e) 5; 2.256 (f) 13; 111.1111111111 **4.** (a) $\frac{127}{128}$ (b) $\frac{63}{8}$ (c) $\frac{130}{81}$ (d) 60 (e) $\frac{63}{64}$ **5.** 4; 118096 **6.** $2109.50 **7.** 9.28cm **8.** (a) $V_n = V_0 \times 0.7^n$ (b) 7 **9.** 54 **10.** 53.5gms; 50 weeks. **11.** 7 **12.** 9 **13.** –0.5, –0.7797 **14.** $r = 5$, 1.8×10^{10} **15.** $8407.35 **16.** 1.8×10^{19} or about 200 billion tonnes.

EXERCISE 8.2.3

1. Term 9 AP = 180, GP = 256. Sum to 11 terms AP = 1650, GP = 2047. **2.** 18. **3.** 12 **4.** 7, 12 **5.** 8 weeks (Ken $220 & Bo-Youn $255) **6.** (a) week 8 (b) week 12 **7.** (a) 1.618 (b) 121379 [~121400, depends on rounding errors]

EXERCISE 8.2.4

1. (i) $\frac{81}{2}$ (ii) $\frac{10}{13}$ (iii) 5000 (iv) $\frac{30}{11}$ **2.** $23\frac{23}{99}$ **3.** 6667 fish. [Nb: $t_{43} < 1$. If we use $n = 43$ then ans is 6660 fish]; 20 000 fish. Overfishing means that fewer fish are caught in the long run. [*An alternate estimate for the total catch is* 1665 *fish*.] **4.** 27 **5.** 48,12,3 or 16,12,9 **6.** (a) $\frac{11}{30}$ (b) $\frac{37}{99}$ (c) $\frac{191}{90}$ **7.** 128 cm **8.** $\frac{121}{9}$ **9.** $2 + \frac{4}{3}\sqrt{3}$ **10.** $\frac{1-(-t)^n}{1+t}$ $\frac{1}{1+t}$ **11.** $\frac{1-(-t^2)^n}{1+t^2}$ $\frac{1}{1+t^2}$

EXERCISE 8.2.5

1. 3, –0.2 **2.** $\frac{2560}{93}$ **3.** $\frac{10}{3}$ **4.** (a) $\frac{43}{18}$ (b) $\frac{458}{99}$ (c) $\frac{413}{990}$ **5.** 9900 **6.** 3275 **7.** 3 **8.** $t_n = 6n - 14$ **9.** 6 **10.** $-\frac{1}{6}$ **11.** i. 12 ii. 26 **12.** 9, 12 **13.** ±2 **14.** (5, 5, 5), (5, –10, 20) **15.** (a) 2, 7 (b) 2, 5, 8 (c) $3n - 1$ **16.** (a) 5 (b) 2 m

EXERCISE 8.3

1. $2773.08 **2.** $4377.63 **3.** $1781.94 **4.** $12216 **5.** $35816.95 **6.** $40349.37 **7.** $64006.80 **8.** $276971.93, $281325.41 **9.** $63762.25 **10.** $98.62, $9467.14, interest $4467.14. Flat interest = $6000 **11.** $134.41, $3790.44, 0.602% /month (or 7.22% p.a)

EXERCISE 9.1

1.	*a* cm	*b* cm	*c* cm	*A*	*B*	*C*
1	3.8	4.1	1.6	67˚	90˚	23˚
2	81.5	98.3	55.0	56˚	90˚	34˚
3	32.7	47.1	33.9	44˚	90˚	46˚
4	1.61	30.7	30.7	3˚	90˚	87˚
5	2.3	2.74	1.49	57˚	90˚	33˚
6	48.5	77	59.8	39˚	90˚	51˚
7	44.4	81.6	68.4	33˚	90˚	57˚
8	2.93	13.0	12.7	13˚	90˚	77˚
9	74.4	94.4	58.1	52˚	90˚	38˚
10	71.8	96.5	64.6	48˚	90˚	42˚
11	23.3	34.1	24.9	43˚	90˚	47˚
12	43.1	43.2	2.3	87˚	90˚	3˚
13	71.5	80.2	36.4	63˚	90˚	27˚
14	33.5	34.1	6.5	79˚	90˚	11˚
15	6.1	7.2	3.82	58˚	90˚	32˚
16	29.1	30	7.3	76˚	90˚	14˚
17	29.0	29.1	2.0	86˚	90˚	4˚
18	34.5	88.2	81.2	23˚	90˚	67˚
19	24.0	29.7	17.5	54˚	90˚	36˚
20	41.2	46.2	21.0	63˚	90˚	27˚
21	59.6	72.9	41.8	55˚	90˚	35˚
22	5.43	6.8	4.09	53˚	90˚	37˚

23	13.0	19.8	14.9	41°	90°	49°
24	14.0	21.3	16.1	41°	90°	49°
25	82.4	88.9	33.3	68°	90°	22°

2. (a) $2\sqrt{3}$ (b) $5(1+\sqrt{3})$ (c) 4 (d) $2(1+\sqrt{3})$ (e) $\frac{4}{3}(3+\sqrt{3})$ (f) $\sqrt{106}-5$ **4.** (a) $25(1+\sqrt{3})$ (b) $\frac{40\sqrt{3}}{3}$

EXERCISE 9.2

1. (a) i. 030°T ii. 330°T iii. 195°T iv. 200°T (b) i. N25°E ii. S iii. S40°W iv. N10°W
2. 37.49m **3.** 18.94m **4.** 37° 18' **5.** $\frac{26}{9}$ m/s **6.** N58° 33'W, 37.23 km **7.** 199.82 m **8.** 10.58 m
9. 72.25 m **10.** 25.39 km **11.** 15.76 m **12.** (a) 3.01km N, 3.99km E (b) 2.87km E 0.88km S (c) 6.86km E 2.13km N (d) 7.19km 253°T **13.** 524m

EXERCISE 9.3

1. (a) 39°48' (b) 64°46' **2.** (a) 12.81 cm (b) 61.35 cm (c) 77°57' (d) 60.83 cm (e) 80° 32'
3. (a) 21°48' (b) 42°2' (c) 26°34' **4.** (a) 2274 (b) 12.7° **5.** 251.29 m **6.** (a) 103.5 m (b) 35.26° (c) 39.23° **7.** (b) 53.43 (c) 155.16 m (d) 145.68 m **8.** (b) 48.54 m **9.** (a) $\sqrt{(b-c)^2+h^2}$ (b) $\tan^{-1}\left(\frac{h}{a}\right)$ (c) $\tan^{-1}\left(\frac{h}{b-c}\right)$ (d) $2(b+c)\sqrt{h^2+a^2}+2a\sqrt{(b-c)^2+h^2}$ **10.** 82.80 m
11. (a) 40.61 m (b) 49.46 m **12.** (a) 10.61 cm (b) 75° 58' (c) 93° 22' **13.** (a) 1.44 m (b) 73° 13' (c) 62° 11'

EXERCISE 9.4

1. (a) 1999.2 cm^2 (b) 756.8 cm^2 (c) 3854.8 cm^2 (d) 2704.9 cm^2 (e) 538.0 cm^2 (f) 417.5 cm^2 (g) 549.4 cm^2 (h) 14.2 cm^2 (i) 516.2 cm^2 (j) 281.5 cm^2 (k) 918.8 cm^2 (l) 387.2 cm^2 (m) 139.0 cm^2 (n) 853.7 cm^2 (o) 314.6 cm^2 **2.** 69345 m^2 **3.** $100\pi - 6\sqrt{91}$ cm^2 **4.** 17.34 cm
5. (a) 36.77sq units (b) 14.70 sq units (c) 62.53 sq units **6.** 52.16 cm^2 **7.** 27° 2'
8. $\frac{(b+a\times\tan\theta)^2}{2\tan\theta}$ **9.** Area of $\Delta ACD = 101.78$ cm^2, Area of $\Delta ABC = 61.38$ cm^2

EXERCISE 9.5.1

	a cm	b cm	c cm	A	B	C
1	13.3	37.1	48.2	10°	29°	141°
2	2.7	1.2	2.8	74°	25°	81°
3	11.0	0.7	11.3	60°	3°	117°
4	31.9	39.1	51.7	38°	49°	93°
5	18.5	11.4	19.5	68°	35°	77°
6	14.6	15.0	5.3	75°	84°	21°
7	26.0	7.3	26.4	79°	16°	85°
8	21.6	10.1	28.5	39°	17°	124°
9	0.8	0.2	0.8	82°	16°	82°
10	27.7	7.4	33.3	36°	9°	135°
11	16.4	20.7	14.5	52°	84°	44°
12	21.4	45.6	64.3	11°	24°	145°
13	30.9	27.7	22.6	75°	60°	45°

14	29.3	45.6	59.1	29°	49°	102°
15	9.7	9.8	7.9	65°	67°	48°
16	21.5	36.6	54.2	16°	28°	136°
17	14.8	29.3	27.2	30°	83°	67°
18	10.5	0.7	10.9	52°	3°	125°
19	11.2	6.9	17.0	25°	15°	140°
20	25.8	18.5	40.1	30°	21°	129°

EXERCISE 9.5.2

	a	b	c	$A°$	$B°$	$C°$	$c*$	$B*°$	$C*°$
1	7.40	18.10	21.06	20.00	56.78	103.22	12.95	123.22	36.78
2	13.30	19.50	31.36	14.00	20.77	145.23	6.49	159.23	6.77
3	13.50	17.00	25.90	28.00	36.24	115.76	4.12	143.76	8.24
4	10.20	17.00	25.62	15.00	25.55	139.45	7.22	154.45	10.55
5	7.40	15.20	19.55	20.00	44.63	115.37	9.02	135.37	24.63
6	10.70	14.10	21.41	26.00	35.29	118.71	3.94	144.71	9.29
7	11.50	12.60	22.94	17.00	18.68	144.32	1.16	161.32	1.68
8	8.30	13.70	18.67	24.00	42.17	113.83	6.36	137.83	18.17
9	13.70	17.80	30.28	14.00	18.32	147.68	4.27	161.68	4.32
10	13.40	17.80	26.19	28.00	38.58	113.42	5.24	141.42	10.58
11	12.10	16.80	25.63	23.00	32.85	124.15	5.30	147.15	9.85
12	12.00	14.50	24.35	21.00	25.66	133.34	2.72	154.34	4.66
13	12.10	19.20	29.34	16.00	25.94	138.06	7.57	154.06	9.94
14	7.20	13.10	19.01	15.00	28.09	136.91	6.30	151.91	13.09
15	12.20	17.70	23.73	30.00	46.50	103.50	6.93	133.50	16.50
16	9.20	20.90	27.97	14.00	33.34	132.66	12.59	146.66	19.34
17	10.50	13.30	21.96	20.00	25.67	134.33	3.03	154.33	5.67
18	9.20	19.20	26.29	15.00	32.69	132.31	10.80	147.31	17.69
19	7.20	13.30	18.33	19.00	36.97	124.03	6.82	143.03	17.97
20	13.50	20.40	25.96	31.00	51.10	97.90	9.01	128.90	20.10
21	10.80	20.80	24.48	26.00	57.59	96.41	12.91	122.41	31.59
22	13.00	12.20	23.91	19.00	17.79	143.21	0.84	162.21	1.21
23	13.60	20.40	22.92	36.00	61.85	82.15	10.09	118.15	25.85
24	11.40	12.50	22.88	16.00	17.59	146.41	1.15	162.41	1.59
25	8.00	16.80	23.99	10.00	21.39	148.61	9.10	158.61	11.39

26. (a-d) no triangles exist.

EXERCISE 9.5.3

1. 30.64 km **2.** 4.57 m **3.** 476.4 m **4.** 201°47'T **5.** 222.9 m **6.** (a) 3.40 m (b) 3.11 m
7. (b) 1.000 m (c) 1.715 m **8.** (a) 51.19 min (b)1 hr 15.96 min (c) 14.08 km **9.** \$4886 **10.** 906 m

EXERCISE 9.5.4

	a cm	b cm	c cm	A	B	C
1	13.5	9.8	16.7	54°	36°	90°
2	8.9	10.8	15.2	35°	44°	101°
3	22.8	25.6	12.8	63°	87°	30°
4	21.1	4.4	21.0	85°	12°	83°
5	15.9	10.6	15.1	74°	40°	66°
6	8.8	13.6	20.3	20°	32°	128°
7	9.2	9.5	13.2	44°	46°	90°
8	23.4	62.5	58.4	22°	89°	69°