6 Matrix Algebra

Assessment statement

1.9 Solutions of systems of linear equations (a maximum of three equations in three unknowns), including cases when there is a unique solution, an infinity of solutions or no solution.

Note: Sections 6.1 to 6.3 are not required for examinations. However, it is highly recommended that you review these sections because of their important applications. Sections 6.1 and 6.2 can be omitted. Special attention must be paid to the determinant concept in Section 6.3 because it will be used later in the book.

In Section 6.4 the Gauss-Jordan elimination method is required in its 'raw' form, i.e. using equations. However, for reasons of efficiency, and if you were to use a GDC to solve a system of equations, the matrix form is more appropriate. Even though it is not required for examination purposes, in exams, any 'mathematically sound' method is accepted.

Introduction

Ever since their first emergence, matrices have been and remain significant mathematical tools. Uses of matrices span several areas from simply solving systems of simultaneous linear equations, to describing atomic structure, designing computer game graphics, analyzing relationships, coding, and operations research, to mention a few. If you have ever used a spreadsheet such as Excel or Lotus, or have ever created a table, then you have used a matrix. Matrices make the presentation of data understandable and help make calculations easy to perform. For example, your teacher's grade book may look something like this:

Student	Quiz 1	Quiz 2	Test 1	Test 2	Homework	Grade
Tim	70	80	86	82	95	A
Maher	89	56	80	60	55	C
…	…	…	…	…	…	…

If we want to know Tim's grade on Test 2, we simply follow along the row 'Tim' to the column 'Test 2' and find that he received a score of 82. Take a look at the matrix below about the sale of cameras in a store according to location and type.

	City	Donau	Neubau	Moedling
Nikon	153	98	74	56
Canon	211	120	57	29
Olympus	82	31	12	5
Other	308	242	183	107

If we want to know how many Canon cameras were sold in the Neubau shop, we follow along the row 'Canon' to the column 'Neubau' and find that 57 Canons were sold.

6.1 Basic definitions

What is a matrix?

A matrix is a rectangular array of elements. The elements can be symbolic expressions or numbers.

Matrix $[A]$ is denoted by

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{matrix} \leftarrow \\ \leftarrow \\ \vdots \\ \leftarrow \end{matrix} \Bigg\} \; m \text{ rows}$$

$$\underbrace{\begin{matrix} \uparrow & \uparrow & \dots & \uparrow \end{matrix}}_{n \text{ columns}}$$

Row i of A has n elements and is $(a_{i1} \quad a_{i2} \quad \dots \quad a_{in})$.

Column j of A has m elements and is $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$.

The number of rows and columns of the matrix define its size (order). So, a matrix that has m rows and n columns is said to have an $m \times n$ (m by n) size (order). A matrix A with $m \times n$ order (size) is sometimes denoted as $[A]_{m \times n}$ or $[A]_{mn}$ to show that A is a matrix with m rows and n columns. (Some authors use $[a_{ij}]$ to represent a matrix.) The sales matrix has a 4×4 order. When $m = n$, the matrix is said to be a **square matrix** with order n, so the sales matrix is a square matrix of order 4.

Every entry in the matrix is called an **entry** or **element** of the matrix, and is denoted by a_{ij}, where i is the row number and j is the column number of that element. The ordered pair (i, j) is also called the **address** of the element. So, in the grades matrix example, the entry (2, 4) is 60, the student Maher's grade on Test 2, while (2, 4) in the sales matrix example is 29, Canon's sales in the Moedling shop.

Arthur Cayley (1821–1895)

Arthur Cayley entered Trinity College, Cambridge in 1838. While still an undergraduate, he published three papers in the *Cambridge Mathematical Journal*. Cayley graduated as Senior Wrangler in 1842 and won the first Smith's prize. Winning a fellowship enabled him to teach for four years at Cambridge. He published 28 papers in the *Cambridge Mathematical Journal* during these years. Since a fellowship had limited tenure, Cayley needed to find a profession. He spent 14 years as a lawyer but, although very skilled in his legal specialty, he always considered it as a means to make money so that he could pursue mathematics. During these 14 years as a lawyer he published around 250 mathematical papers.

His published work comprises over 900 papers and notes covering several fields of modern mathematics. The most important aspect of his work was in developing the algebra of matrices.

Vectors

A vector is a matrix that has only one row or one column. There are two types of vectors – row vectors and column vectors.

Row vector

If a matrix has one row, it is called a row vector.

$B = (b_1\ b_2\ \ldots\ b_m)$ is a row vector with dimension m.

$B = (1\ 2)$ could represent the position of a point in a plane and is an example of a row vector of dimension 2.

Column vector

If a matrix has one column, it is called a column vector.

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$ is a column vector with dimension n.

$C = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ again could represent the position of a point in a plane and is an example of a column vector of dimension 2.

As you see, vectors can be represented by row or column matrices.

Submatrix

If some row(s) and/or column(s) of a matrix A are deleted, the remaining matrix is called a submatrix of A.

For example, if we are interested in the sales of the three main types of cameras in the central part of the city, we can represent them with the following *submatrix* of the original matrix:

$$\begin{pmatrix} 153 & 98 \\ 211 & 120 \\ 82 & 31 \end{pmatrix}$$

Zero matrix

A matrix for which all entries are equal to zero ($a_{ij} = 0$ for all i and j).

$$(0\quad 0), \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ are zero matrices.

Diagonal

A square matrix where all entries except the diagonal entries are zero is called a **diagonal matrix**.

In a square matrix, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **diagonal elements** of the matrix. Sometimes the diagonal of the matrix is also called the **principal** or **main** of the matrix.

$$\begin{pmatrix} 153 & 0 & 0 & 0 \\ 0 & 120 & 0 & 0 \\ 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 107 \end{pmatrix}$$

What is the diagonal in our sales matrix? Here, $a_{11} = 153$, $a_{22} = 120$, $a_{33} = 12$ and $a_{44} = 107$.

Triangular matrix

You can use a matrix to present data showing distances between different cities.

Table 6.1

	Graz	Salzburg	Innsbruck	Linz
Vienna	191	298	478	185
Graz		282	461	220
Salzburg			188	135
Innsbruck				320

The data in Table 6.1 can be represented by a triangular matrix (upper triangular in this case).

$$\begin{pmatrix} 191 & 298 & 478 & 185 \\ 0 & 282 & 461 & 220 \\ 0 & 0 & 188 & 135 \\ 0 & 0 & 0 & 320 \end{pmatrix}$$

In a triangular matrix, the entries on one side of its diagonal are all zero.

Definition of a triangular matrix

A triangular matrix is a square matrix with order n for which $a_{ij} = 0$ when $i > j$ (upper triangular) or, alternatively, when $i < j$ (lower triangular).

Another way of representing the distance data is given by the following matrix:

	Vienna	*Graz*	*Salzburg*	*Innsbruck*	*Linz*
Vienna	0	191	298	478	185
Graz	191	0	282	461	220
Salzburg	298	282	0	188	135
Innsbruck	478	461	188	0	320
Linz	185	220	1325	320	0

Again the data in the table can be represented by a matrix called a **symmetric** matrix. In such matrices, $a_{ij} = a_{ji}$ for all i and j. All symmetric matrices are square!

$$\begin{pmatrix} 0 & 191 & 298 & 478 & 185 \\ 191 & 0 & 282 & 461 & 220 \\ 298 & 282 & 0 & 188 & 135 \\ 478 & 461 & 188 & 0 & 320 \\ 185 & 220 & 135 & 320 & 0 \end{pmatrix}$$

6.2 Matrix operations

When are two matrices considered to be equal?

Two matrices A and B are equal if the size of A and B is the same (number of rows and columns are the same for A and B) and $a_{ij} = b_{ij}$ for all i and j.

For example, $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ and $\begin{pmatrix} 2 & x \\ x^2 - 4 & 7 \end{pmatrix}$ can only be equal if $x = 3$ and $x^2 - 4 = 5$, which can only be true if $x = 3$.

How do you add/subtract two matrices?

Two matrices A and B can be added only if they have the *same size*. If C is the sum of the two matrices, then we write

$$C = A + B$$

where $c_{ij} = a_{ij} + b_{ij}$, i.e. we add 'corresponding' terms, one by one.

For example,

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} x & y \\ a & b \end{pmatrix} = \begin{pmatrix} 2 + x & 3 + y \\ 5 + a & 7 + b \end{pmatrix}$$

Subtraction is done similarly:

$$\begin{pmatrix} 2 & 3 & 1 \\ 5 & 7 & 0 \end{pmatrix} - \begin{pmatrix} x & y & 8 \\ a & b & 2 \end{pmatrix} = \begin{pmatrix} 2 - x & 3 - y & -7 \\ 5 - a & 7 - b & -2 \end{pmatrix}$$

The operations of addition and subtraction of matrices obey all rules of addition and subtraction of real numbers. That is,

$A + B = B + A$; $A + (B + C) = (A + B) + C$; $A - (B + C) = A - B - C$.

How do we multiply a scalar by a matrix?

A scalar is any object that is not a matrix. The multiplication by a scalar is straightforward. You multiply each term of the matrix by the scalar.

If A is an $m \times n$ matrix, and c is a scalar, the scalar product of c and A is another matrix $B = cA$ such that every entry b_{ij} of B is a multiple of its corresponding A entry, i.e. $b_{ij} = c \times a_{ij}$.

It is often convenient to rewrite the scalar multiple cA by factoring c out of every entry in the matrix. For instance, in the following example, the scalar $\frac{1}{2}$ has been factored out of the matrix.

$$\begin{pmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -3 \\ 5 & 1 \end{pmatrix}$$

Matrix multiplication

At first glance, the following definition may seem unusual. You will see later, however, that this definition of the product of two matrices has many practical applications.

Matrix multiplication

If $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix, the product AB is an $m \times p$ matrix, $AB = (c_{ij})$, where

$$c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

This definition means that each entry with an address ij AB is obtained by multiplying the entries in the ith row of A by the *corresponding* entries in the jth column of B and then adding the results. The following shows the process in detail:

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \ldots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

Example 1

Find $C = AB$ if $A = \begin{pmatrix} 3 & -5 & 2 \\ 2 & 1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 & 1 & 5 \\ 5 & 8 & -4 & 0 \\ -9 & 10 & 5 & 3 \end{pmatrix}$.

Solution

A is a 2×3 matrix and B is a 3×4 matrix, so the product must be a 2×4 matrix. Every entry in the product is the result of multiplying the entries in the rows of A and columns of B. For example:

$$c_{12} = \sum_{k=1}^{3} a_{1k}b_{k2} = (a_{11} \quad a_{12} \quad a_{13}) \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = (3 \quad -5 \quad 2) \begin{pmatrix} -2 \\ 8 \\ 10 \end{pmatrix}$$

$$= 3 \times (-2) - 5 \times 8 + 2 \times 10 = -26$$

or

$$c_{23} = \sum_{k=1}^{3} a_{2k}b_{k3} = (a_{21} \quad a_{22} \quad a_{23}) \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = (2 \quad 1 \quad 7) \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix}$$

$$= 2 \times 1 + 1 \times (-4) + 7 \times 5 = 33$$

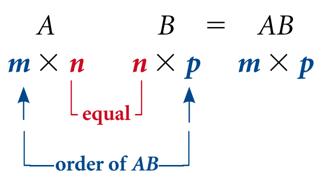
The operation is repeated eight times to get

$$C = AB = \begin{pmatrix} -34 & -26 & 33 & 21 \\ -52 & 74 & 33 & 31 \end{pmatrix}$$

This product can also be found using a GDC.

```
[A][B]
[[-34 -26 33 21…
 [-52 -74 33 31…
█
```

For the product of two matrices to be defined, the number of columns in the first matrix should be the same as the number of rows in the second matrix.



Examples – matrix multiplication

a) $\begin{pmatrix} 5 & 0 & 3 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 4 \\ 1 & -1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 14 \\ 11 & -13 \end{pmatrix}$

$2 \times 3 \qquad 3 \times 2 \qquad 2 \times 2$

b) $\begin{pmatrix} 4 & -5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ 1 & 7 \end{pmatrix}$

$2 \times 2 \qquad 2 \times 2 \qquad 2 \times 2$

c) $\begin{pmatrix} 5 & 0 & 3 \\ -2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{1}{7} & -\frac{3}{7} & \frac{3}{7} \\ -\frac{10}{7} & -\frac{9}{7} & \frac{16}{7} \\ \frac{4}{7} & \frac{5}{7} & -\frac{5}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3×3 $\quad 3 \times 3$ $\quad 3 \times 3$

As you see from part b) above, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ does not create a new value when it is multiplied by another matrix. This is why it is called the **identity** matrix of order 2.

The identity matrix

A $n \times n$ diagonal matrix where $a_{ij} = 1$ and $i = j$ is called the identity matrix of order n.

Examples – identity matrices

a) $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

c) $\begin{pmatrix} a & b & c & m \\ d & e & f & n \\ g & h & i & p \\ j & k & l & q \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c & m \\ d & e & f & n \\ g & h & i & p \\ j & k & l & q \end{pmatrix}$

Sometimes, the identity matrix is denoted by I_n, where n is the order. So, in parts a) and b) above, the identity is I_3, and in c) it is I_4.

Examples – comparing AB with BA

a) $\begin{pmatrix} 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} = (11)$

1×3 $\quad 3 \times 1$ $\quad 1 \times 1$

b) $\begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 6 \\ 10 & -5 & 15 \\ 8 & -4 & 12 \end{pmatrix}$

3×1 $\quad 1 \times 3$ $\quad 3 \times 3$

Notice the difference between the products in parts a) and b). Matrix multiplication, in general, is **not commutative**. It is usually not true that $AB = BA$.

Let $A = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix}$, then $AB = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 39 \\ -8 & 25 \end{pmatrix}$

but

$$BA = \begin{pmatrix} -2 & 3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 9 & -6 \\ 28 & 16 \end{pmatrix} \Rightarrow AB \neq BA$$

However, if we let

$$A = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}, \text{ then } AB = \begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 36 & 24 \\ 20 & 32 \end{pmatrix} \text{ and}$$

$$BA = \begin{pmatrix} 2 & 6 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} 36 & 24 \\ 20 & 32 \end{pmatrix} \Rightarrow AB = BA$$

Thus, in general, $AB \neq BA$. However, for some matrices A and B, it may happen that $AB = BA$.

Example 2

Find the average sales in each of the regions (City, Donau, Neubau and Moedling), given the following information.

	City	Donau	Neubau	Moedling
Nikon	153	98	74	56
Canon	211	120	57	29
Olympus	82	31	12	5
Other	308	242	183	107

The average selling price for each make of camera is as follows:
Nikon €1200, Canon €1100, Olympus €900, Other €600

Solution

We set up a matrix multiplication in which the individual camera sales are multiplied by the corresponding price. Since the rows represent the sales of the different makes of camera, create a row matrix of the different prices and perform the multiplication.

$$(1200 \quad 1100 \quad 900 \quad 600)\begin{pmatrix} 153 & 98 & 74 & 56 \\ 211 & 120 & 57 & 29 \\ 82 & 31 & 12 & 5 \\ 308 & 242 & 183 & 107 \end{pmatrix} = (674\,300 \quad 422\,700 \quad 272\,100 \quad 167\,800)$$

So, the regions' sales are:

	City	Donau	Neubau	Moedling
Sales	674 300	422 700	272 100	167 800

Remember that we are multiplying a 1×4 matrix with a 4×4 matrix and hence we get a 1×4 matrix.

Exercise 6.1 and 6.2

1 Consider the following matrices

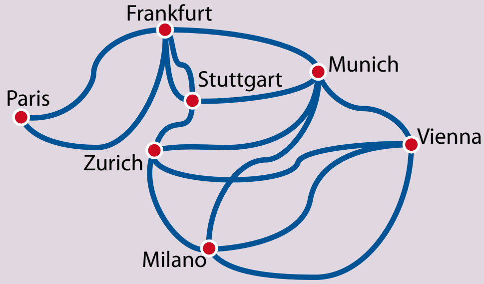
$$A = \begin{pmatrix} -2 & x \\ y-1 & 3 \end{pmatrix}, B = \begin{pmatrix} x+1 & -3 \\ 4 & y-2 \end{pmatrix}$$

a) Evaluate each of the following
   (i) $A + B$ (ii) $3A - B$.
b) Find x and y such that $A = B$.
c) Find x and y such that $A + B$ is a diagonal matrix.
d) Find AB and BA.

2 Solve for the variables.

a) $\begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$ b) $\begin{pmatrix} 2 & p \\ 3 & q \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix}$

3 The diagram below shows the major highways connecting some European cities: Vienna (V), Munich (M), Frankfurt (F), Stuttgart (S), Zurich (Z), Milano (L) and Paris (P).



a) Write the number of *direct* routes between each pair of cities into a matrix as started below:

$$\begin{array}{c} \\ V \\ M \\ F \\ S \\ Z \\ L \\ P \end{array}\begin{array}{c} \begin{array}{ccccccc} V & M & F & S & Z & L & P \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 2 & 0 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix} \end{array}$$

b) Multiply the matrix from part a) by itself and interpret what it signifies.

4 Consider the following matrices:

$$A = \begin{pmatrix} 2 & 5 & 1 \\ 0 & -3 & 2 \\ 7 & 0 & -1 \end{pmatrix}, B = \begin{pmatrix} m & -2 \\ 3m & -1 \\ 2 & 3 \end{pmatrix}, C = \begin{pmatrix} x-1 & 5 & y \\ 0 & -x & y+1 \\ 2x+y & x-3y & 2y-x \end{pmatrix}$$

a) Find $A + C$. b) Find AB. c) Find BA.

d) Solve for x and y if $A = C$. e) Find $B + C$.

f) Solve for m if $3B + 2\begin{pmatrix} -1 & m^2 \\ -5 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ 17 & 1 \\ 2m+2 & 7 \end{pmatrix}$.

5 Find a, b and c so that the following equation is true:

$$2 \cdot \begin{pmatrix} a-1 & b \\ c+2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -5 & 5 \\ 8 & c+9 \end{pmatrix}$$

6 Find x and y such that:

$$\begin{pmatrix} 2 & -3 \\ -5 & 7 \end{pmatrix}\begin{pmatrix} x-11 & 1-x \\ -5 & x+2y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

7 Find m and n if

$$\begin{pmatrix} m^2-1 & m+2 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 3 & n+1 \\ 5 & n-5 \end{pmatrix}.$$

8 There are two supermarkets in your area. Your shopping list consists of 2 kg of tomatoes, 500 g of meat and 3 litres of milk. Prices differ between the different shops, and it is difficult to switch between stores to make certain you are paying the least amount of money. A better strategy is to check and see where you pay less on *average*! The prices of the different items are given below. Which shop should you go to?

Product	Price in shop A	Price in shop B
Tomato	€1.66/kg	€1.58/kg
Meat	€2.55/100 g	€2.6/100 g
Milk	€0.90/litre	€0.95/litre

9 Consider the matrices

$A = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} -3 & 5 \\ 2 & 7 \end{pmatrix}$.

a) Find $A + (B + C)$ and $(A + B) + C$.

b) Make a conjecture about the addition of 2×2 matrices observed in a) above and prove it.

c) Find $A(BC)$ and $(AB)C$.

d) Make a conjecture about the multiplication of 2×2 matrices observed in c) above and prove it.

10 A company stores and sells air conditioning units, electric heaters and humidifiers. Row matrix A represents the number of each unit sold last year, and matrix B represents the profit margin for each unit. Find AB and describe what the product represents.

$A = (235 \quad 562 \quad 117)$, $B = \begin{pmatrix} €120 \\ €95 \\ €56 \end{pmatrix}$

11 Find r and s such that the following equation is true: $rA + B = A$, where

$A = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -4 & -6 \\ s - 8 & -14 \end{pmatrix}$.

12 Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

a) Find:

(i) A^2 (ii) A^3 (iii) A^4 (iv) A^n

Let $B = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$.

b) Find:

(i) B^2 (ii) B^3 (iii) B^4 (iv) B^n

13 Solve for x and y such that $\boldsymbol{AB} = \boldsymbol{BA}$ if $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} x & 2 \\ y & 3 \end{pmatrix}$.

14 Solve for x and y such that $\boldsymbol{AB} = \boldsymbol{BA}$ if $A = \begin{pmatrix} 3 & x \\ -2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ y & 1 \end{pmatrix}$.

15 Solve for x such that $\boldsymbol{AB} = \boldsymbol{BA}$ if $A = \begin{pmatrix} 1 & 2 & 3 \\ x & 2 & -3 \\ 1 & 0 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}$.

16 Solve for x and y such that $\boldsymbol{AB} = \boldsymbol{BA}$ if $A = \begin{pmatrix} y & 2 & y+2 \\ x & 2 & -3 \\ 1 & y-1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}$.

6.3 Applications to systems

There is a wide range of applications of matrices in solving systems of equations. Recall from your algebra that the equation of a straight line can take the form

$$ax + by = c$$

where a, b and c are constants and x and y are variables. We call this equation a **linear equation in two variables**. Similarly, the equation of a plane in three-dimensional space has the form

$$ax + by + cz = d$$

where a, b, c and d are constants. We call this equation a **linear equation in three variables**.

A **solution** of a linear equation in n variables (in this case two or three) is an ordered set of real numbers (x_0, y_0, z_0) so that the equation in question is satisfied when these values are substituted for the corresponding variables. For example, the equation

$$x + 2y = 4$$

is satisfied when $x = 2$ and $y = 1$. Some other solutions are $x = -4$ and $y = 4$, $x = 0$ and $y = 2$, and $x = -2$ and $y = 3$.

The set of all solutions of a linear equation is its **solution set**, and when this set is found, the equation is said to have been **solved**. To describe the entire solution set we often use a **parametric representation** as illustrated in the following examples.

Example 3

Solve the linear equation $x + 2y = 4$.

Solution

To find the solution set of an equation in two variables, we solve for one variable in terms of the other. For instance, if we solve for x, we obtain

$$x = 4 - 2y.$$

In this form, y is **free**, in the sense that it can take on any real value, while x is not free, since its value depends on that of y. To represent this solution set in general terms, we introduce a third variable, for example, t, called a **parameter**, and by letting $y = t$ we represent the solution set as

$$x = 4 - 2t, y = t, t \text{ is any real number.}$$

Particular solutions can then be obtained by assigning values to the parameter t. For instance, $t = 1$ yields the solution $x = 2$ and $y = 1$, and $t = 3$ yields the solution $x = -2$ and $y = 3$.

Note that the solution set of a linear equation can be represented parametrically in several ways. For instance, in this example, if we solve for y in terms of x, the parametric representation would take the following form:

$$x = m, y = 2 - \tfrac{1}{2}m, m \text{ is a real number.}$$

Also, by choosing $m = 2$, one particular solution would be $(x, y) = (2, 1)$, and by choosing $m = -2$, another particular solution would be $(-2, 3)$.

Example 4

Solve the linear equation $3x + 2y - z = 3$.

Solution

Choosing x and y as the *free* variables, we solve for z.

$$z = 3x + 2y - 3$$

Letting $x = p$ and $y = q$, we obtain the parametric representation:

$$x = p, y = q, z = 3x + 2y - 3, p \text{ and } q \text{ any real numbers.}$$

A particular solution $(x, y, z) = (1, 1, 2)$.

Parametric representation is very important when we study vectors and lines later on in the book.

Systems of linear equations – refresher

A system of k equations in n variables is a set of k linear equations in the same n variables. For example,

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 4 \end{aligned}$$

is a system of two linear equations in two variables, while

$$\begin{aligned} x - 2y + 3z &= 9 \\ x - 3y &= 4 \end{aligned}$$

is a system with two equations and three variables, and

$$\begin{aligned} x - 2y + 3z &= 9 \\ x - 3y &= 4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

is a system with three equations and three variables.

A **solution** of a system of equations is an ordered set of numbers x_0, y_0, … which satisfy every equation in the system. For example, $(3, -1)$ is a solution of

$$\begin{aligned} 2x + 3y &= 3 \\ x - y &= 4 \end{aligned}$$

Both equations in the system are satisfied when $x = 3$ and $y = -1$ are substituted into the equations. On the contrary, $(0, 1)$ is not a solution of the system, even though it satisfies the first equation, as it does not satisfy the second.

As you already know, there are several ways of finding solutions to systems. In this chapter, we will consider using matrix methods to solve systems of equations.

Taking our example above, notice how we can write the system of equations in matrix form:

$$\begin{cases} 2x + 3y = 3 \\ x - y = 4 \end{cases} \Rightarrow \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

The representation of the system of equations in this way enables us to use matrix operations in solving systems. This matrix equation can be written as

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow AX = C$$

where A is the coefficient matrix, X is the variables matrix and C is the constants matrix. However, to solve this equation, the inverse of a matrix has to be defined as the solution of the system in the form

$$X = A^{-1}C$$

where A^{-1} is the inverse of the matrix A.

Matrix inverse (Optional)

To solve the equation $2x = 6$ for x, we need to multiply both sides of the equation by $\frac{1}{2}$:

$\frac{1}{2} \times 2x = \frac{1}{2} \times 6 \Rightarrow x = 3$. This is so, because $\frac{1}{2} \times 2 = 2 \times \frac{1}{2} = 1$.

$\frac{1}{2}$ is called the multiplicative inverse of 2. The inverse of a matrix is defined in a similar manner and plays a similar role in solving a matrix equation, such as $AX = C$.

Inverse of a matrix

A square matrix B is the inverse of a square matrix A if $AB = BA = I$, where I is the identity matrix.

The notation A^{-1} is used to denote the inverse of a matrix A. Thus, $B = A^{-1}$. Note that only square matrices can have multiplicative inverses.

Example – matrix inverse

$A = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix}$ are multiplicative inverses since

$$AB = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix} = \begin{pmatrix} 21 - 20 & -35 + 35 \\ 12 - 12 & -20 + 21 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 3 & -5 \\ -4 & 7 \end{pmatrix}\begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 21 - 20 & 15 - 15 \\ -28 + 28 & -20 + 21 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Finding the inverse can also be achieved using a GDC.

```
[A]-1
                 [[3   -5]
                  [-4  7 ]]
[A]-1[A]
                   [[1 0]
                    [0 1]]
```

There are a few methods available for finding the inverse of a 2 × 2 matrix. We will be using the following method only, since the other methods are beyond the scope of this textbook.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and assume $A^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and then solve the following matrix equation for e, f, g and h in terms of a, b, c and d.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now we can set up two systems to solve for the required variables, i.e.:

$$\begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left.\begin{matrix} ae + bg = 1 \\ ce + dg = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} dae + \mathbf{dbg} = d \\ bce + \mathbf{bdg} = 0 \end{matrix}\right\} \Rightarrow e = \frac{d}{ad - bc},\ g = \frac{-c}{ad - bc}$$

$$\left.\begin{matrix} af + bh = 0 \\ cf + dh = 1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} daf + \mathbf{dbh} = 0 \\ bcf + \mathbf{bdh} = b \end{matrix}\right\} \Rightarrow f = \frac{-b}{ad - bc},\ h = \frac{a}{ad - bc}$$

Therefore, $A^{-1} = \begin{pmatrix} \frac{d}{ad - bc} & \frac{-b}{ad - bc} \\ \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{pmatrix}$ or $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Example 5

Find the inverse of $\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}$.

Solution

Here $a = 4$, $b = 7$, $c = 3$ and $d = 5$, so $ad - bc = -1$. Thus,

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 5 & -7 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} -5 & 7 \\ 3 & -4 \end{pmatrix}.$$

```
[A]
              [[4  7]
               [3  5]]
[A]-1
              [[-5  7]
               [3  -4]]
```

The determinant

The number $ad - bc$ is called the **determinant** of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The notation we will use for this number is **det *A***, so $\det A = ad - bc$.

The determinant plays an important role in determining whether a matrix has an inverse or not.

If the determinant is zero, i.e. $ad - bc = 0$, the matrix does not have an inverse. If a matrix has no inverse, it is called a **singular matrix**; if it is invertible, it is called **non-singular**.

Example 6

Solve the system of equations.

$$2x + 3y = 3$$
$$x - y = 4$$

Solution

In matrix form, the system can be written as

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-5}\begin{pmatrix} -1 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-5}\begin{pmatrix} -15 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

```
[A]-1 [C]
               [[3  ]
                [-1]]
```

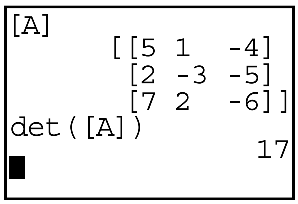
Solving systems of equations in three variables follows similar procedures. However, finding the inverse of a 3×3 matrix will be delegated to the GDC at this level. As in the case of a 2×2 matrix, the existence of an inverse for a 3×3 matrix depends on the value of its determinant.

The determinant of a 3×3 matrix A can be achieved in one of two ways:

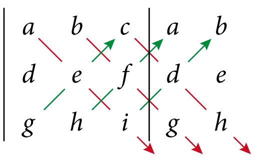
1. $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow \det A = a(ei - fh) - b(di - fg) + c(dh - eg)$

   For example, if

   $$A = \begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix} \Rightarrow \det A = 5(18 + 10) - 1(-12 + 35) - 4(4 + 21) = 17$$

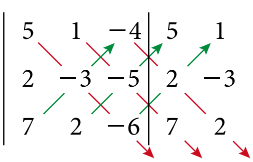


2. A practical method is to use a 'special' set up as follows:

   det A = 

$= aei + bfg + cdh - gec - hfa - idb$

   This is done by 'copying' the first two columns and adding them to the end of the matrix, multiplying down the main diagonals and adding the products, and then multiplying up the second diagonals and subtracting them from the previous product, as shown. In the example above:



$= 5(-3)(-6) + 1(-5)(7) + (-4) \cdot 2 \cdot 2 - 7(-3)(-4) - 2(-5) \cdot 5 - (-6) \cdot 2 \cdot 1$

$= 90 - 35 - 16 - 84 + 50 + 12$

$= 152 - 135$

$= 17$

   In fact, this arrangement is simply a reordering of the calculations involved in the previous method.

Example 7

Solve the system of equations.

$$\begin{aligned} 5x + y - 4z &= 5 \\ 2x - 3y - 5z &= 2 \\ 7x + 2y - 6z &= 5 \end{aligned}$$

Solution

We write this system in matrix form:

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

Since det $A \neq 0$, we can find the solution in the same way we did for the 2×2 matrix, i.e.

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}^{-1}\begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

Using a GDC:

```
[A]-1 [C]
              [[3  ]
               [-2]
               [2  ]]
```

To check your work, you can store the answer matrix as D and then substitute the values into the system:

$$\begin{pmatrix} 5 & 1 & -4 \\ 2 & -3 & -5 \\ 7 & 2 & -6 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 - 2 - 8 \\ 6 + 6 - 10 \\ 21 - 4 - 12 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}, \text{ or}$$

```
[A] [D]
              [[5]
               [2]
               [5]]
```

Area of a triangle

An interesting application of determinants that you may find helpful is finding the area of a triangle whose vertices are given as points in a coordinate plane. The following result will become obvious as you study Chapter 14.

Area of a triangle

The area of a triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) is equal to $\left|\frac{1}{2}|A|\right|$ where

$$A = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$

Example 8

Find the area of triangle ABC whose vertices are $A(1, 3)$, $B(5, -1)$ and $C(-2, 5)$.

Solution

We let $(x_1, y_1) = (1, 3)$, $(x_2, y_2) = (5, -1)$, and $(x_3, y_3) = (-2, 5)$. To find the area, we evaluate the determinant:

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 5 & -1 & 1 \\ -2 & 5 & 1 \end{vmatrix} = -4.$$

Using this value, we can conclude that the area of the triangle is given by:

$$\text{Area} = \left| \frac{1}{2} \begin{vmatrix} 1 & 3 & 1 \\ 5 & -1 & 1 \\ -2 & 5 & 1 \end{vmatrix} \right| = \left| \frac{1}{2} \cdot -4 \right| = 2$$

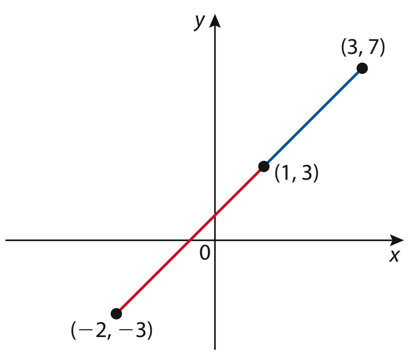
● **Hint:** Try using determinants to find the area of triangle ABC with $A(2, 3)$, $B(12, 3)$, and $C(12, 9)$. Confirm your answer by using the usual area formula of a triangle, $\frac{1}{2}(\textit{base} \times \textit{height})$.

Lines in planes

In our previous discussion, what if the three points are collinear? The answer is very simple. The triangle would collapse into a line segment and the area becomes zero. This fact helps us develop two techniques that are very helpful in dealing with questions of collinearity and equations of lines.

For example, take the points $A(-2, -3)$, $B(1, 3)$ and $C(3, 7)$. Find the area of 'triangle' ABC.

$$\text{Area} = \left| \frac{1}{2} \begin{vmatrix} -2 & -3 & 1 \\ 1 & 3 & 1 \\ 3 & 7 & 1 \end{vmatrix} \right| = \left| \frac{1}{2} \cdot -0 \right| = 0$$



This result can be stated in general as given below:

Test for collinearity

The three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Example 9

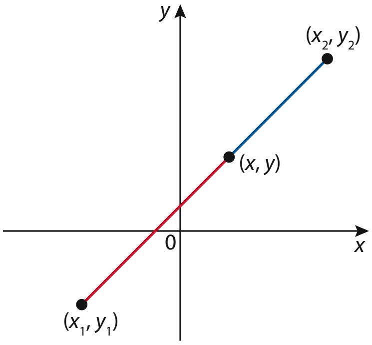
Determine whether the points $(-2, 3)$, $(2, 5)$ and $(5, 7)$ lie on the same line.

Solution

By setting up the matrix as suggested by the rule above, we have

$$\begin{vmatrix} -2 & 3 & 1 \\ 2 & 5 & 1 \\ 5 & 7 & 1 \end{vmatrix} = 2 \neq 0.$$

Because the value of the determinant is not equal to zero, the points cannot lie on a line.



Two-point equation of a line

The test for collinearity leads us to the following result, which enables us to find the equation of a line containing two points. Consider two points (x_1, y_1), (x_2, y_2) which lie on a given line. To find the equation of the line through these two points, we introduce a general point (x, y) on the line. These three points (x_1, y_1), (x_2, y_2) and (x, y) are collinear, and hence they satisfy the determinant equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

which gives us the equation of the line in the form:

$$(y_1 - y_2)x + (x_2 - x_1)y + (x_1y_2 - y_1x_2) = 0$$

which in turn is of the form: $Ax + By + C = 0$.

Example 10

Find the equation of the line through $(-2, 3)$ and $(3, 7)$.

Solution

Applying the determinant formula for the equation of a line produces

$$\begin{vmatrix} x & y & 1 \\ -2 & 3 & 1 \\ 3 & 7 & 1 \end{vmatrix} = (3 - 7)x + (3 + 2)y + (-14 - 9) = 0$$

$$-4x + 5y - 23 = 0$$

Exercise 6.3

1 Consider the matrix M which satisfies the matrix equation

$$\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix} M = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}.$$

a) Write out the inverse of matrix $\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix}$.

b) Hence, write M as a product of two matrices.

c) Evaluate M.

d) Now consider the equation containing the matrix N:

$$N\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$$

(i) Write N as a product of two matrices.

(ii) Evaluate N.

e) Write a short paragraph describing your work on this problem.

2 Find the matrix E in the following equation:

$$\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} E \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$$

3 a) Prove that the matrix $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -3 \\ 3 & -2 & -3 \end{pmatrix}$ should have an inverse.

b) Write out A^{-1}.

c) Hence, solve the system of equations:

$$\begin{cases} 2x - 3y + z = 4.2 \\ x + y - 3z = -1.1 \\ 3x - 2y - 3z = 2.9 \end{cases}$$

4 Find the inverse for each matrix.

a) $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ b) $B = \begin{pmatrix} a & 1 \\ a + 2 & \frac{3}{a} + 1 \end{pmatrix}$

5 For what values of x is the following matrix singular?

$A = \begin{pmatrix} x + 1 & 3 \\ 3x - 1 & x + 3 \end{pmatrix}$

6 Find n such that $\begin{pmatrix} 2 & -1 & 4 \\ 2n & 2 & 0 \\ 2 & 1 & 4n \end{pmatrix}$ is the inverse of $\begin{pmatrix} -2 & -3 & 4 \\ 1 & 2 & -2 \\ 3n & 2 & -5n \end{pmatrix}$.

7 Consider the two matrices $A = \begin{pmatrix} 4 & 2 \\ 0 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$.

a) Find X such that $XA = B$.

b) Find Y such that $AY = B$.

c) Is $X = Y$? Explain.

8 Consider the two matrices

$P = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 5 & 4 \\ 1 & 0 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 3 & -1 & 1 \\ 4 & 0 & 0 \\ 1 & 2 & -1 \end{pmatrix}$.

a) Find PQ and QP.

b) Find P^{-1}, Q^{-1}, $P^{-1}Q^{-1}$, $Q^{-1}P^{-1}$ $(PQ)^{-1}$, and $(QP)^{-1}$.

c) Write a few sentences about your observations in parts a) and b).

9 Consider the matrices A and B.

$A = \begin{pmatrix} 3 & -2 & 1 \\ -4 & 1 & -3 \\ 1 & -5 & 1 \end{pmatrix}$; $B = \begin{pmatrix} -29 \\ 37 \\ -24 \end{pmatrix}$

a) Find the matrix C if $AC = B$.

b) Solve the system of equations:

$$\begin{cases} 3x - 2y + z = -29 \\ 4x - y + 3z = -37 \\ -x + 5y - z = 24 \end{cases}$$

10 Solve the matrix equation

$$\begin{pmatrix} 2 & 2 + x \\ 5 & 4 + x \end{pmatrix}\begin{pmatrix} 3 & x \\ x - 4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & x \\ x - 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 + x \\ 5 & 4 + x \end{pmatrix}$$

11 Consider the matrices A and B below. Find x and y such that $AB = BA$.

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix};\ B = \begin{pmatrix} 2 - x & 1 \\ 5x & y \end{pmatrix}$$

12 Consider the matrices A and B below. Find x and y such that $AB = BA$.

$$A = \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix};\ B = \begin{pmatrix} 1 - x & x \\ 5x & y \end{pmatrix}$$

13 Consider the matrices A and B below. Find x and y such that $AB = BA$.

$$A = \begin{pmatrix} 3 + x & 1 \\ -5 & 2 \end{pmatrix};\ B = \begin{pmatrix} y - x & x \\ 5x - y + 1 & y + x \end{pmatrix}$$

14 In each case, you are given two points in the plane. Use matrix methods to find an equation of a line that contains the given points.

a) $A(-5, -6), B(3, 11)$

b) $A(5, -2), B(3, -2)$

c) $A(-5, 3), B(-5, 8)$

15 Find the area of the parallelogram with the given points as three of its vertices:

a) $A(-5, -6), B(3, 11), C(8, 1)$

b) $A(3, -5), B(3, 11), C(8, 11)$

c) $A(4, -6), B(-3, 9), C(7, 7)$

16 Find x such that the area of triangle ABC is 10 square units.

a) $A(x, -6), B(3, 11), C(8, 3)$

b) $A(-5, x), B(3, x + 2), C(x^2 + 2x - 3, 1)$

17 Find the value of k such that the points P, Q, and R are collinear.

a) $P(2, -5), Q(4, k), R(5, -2)$

b) $P(-6, 2), Q(-5, k), R(-3, 5)$

18 Exploration:

Consider the matrix $A = \begin{pmatrix} 2 & 7 \\ 5 & 5 \end{pmatrix}$. Define $f(x) = \det(xI - A)$ where x is any real number and I is the identity matrix.

a) Find $\det(A)$.

b) Expand $f(x)$ and compare the constant term to your answer in a).

c) How is the coefficient of x in the expansion of $f(x)$ related to A?

d) Find $f(A)$ and simplify it.

e) Now repeat parts a)–d) with matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

• **Hint:** $f(x)$ is called the characteristic polynomial of A.

19 Exploration:

Consider the matrix $A = \begin{pmatrix} 2 & 7 & 1 \\ -1 & 3 & 2 \\ 5 & 5 & -4 \end{pmatrix}$. Define $f(x) = \det(xI - A)$ where x is any real number and I is the identity matrix.

a) Find $\det(A)$.

b) Expand $f(x)$ and compare the constant term to your answer in a).

c) How is the coefficient of x^2 in the expansion of $f(x)$ related to A?

d) Find $f(A)$ and simplify it.

e) Now repeat parts a)–d) with matrix $B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

6.4 Further properties and applications

Pages 267–269 are optional material. You can choose not to work on them. However, starting with Gauss-Jordan elimination (on page 269) the material is required in examinations.

In question 8 of Exercise 6.3, you were asked to make some observations concerning the answers to parts a) and b). The purpose is for you to discover some properties of inverse matrices.

Let us take the following matrices, for example:

Consider the two matrices A and B, where $A = \begin{pmatrix} -1 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & -2 & -1 \end{pmatrix}$,
$B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 3 \end{pmatrix}$.

Find A^{-1}, B^{-1}, AB, BA, $(AB)^{-1}$, $A^{-1}B^{-1}$, $B^{-1}A^{-1}$, and $(BA)^{-1}$.

As shown below,

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{3} & \frac{1}{12} & -\frac{7}{12} \\ \frac{2}{3} & \frac{1}{12} & \frac{5}{12} \end{pmatrix}, B^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ \frac{2}{3} & 0 & -\frac{1}{3} \end{pmatrix}$$

```
 [[0.0  .3  .3 ]
  [-.3  .1  -.6]
  [.7   .1  .4 ]]
Ans▸Frac
[[0.0   1/4   1/4...
 [-1/3  1/12  -7/...
 [2/3   1/12  5/1...
```

```
[[1.0   -2.0 1.0...
 [-1.0  1.0  0.0...
 [.7    0.0  -.3...
Ans▸Frac
[[1.0   -2.0 1.0...
 [-1.0  1.0  0.0...
 [2/3   0.0  -1/...
```

Also,

$$AB = \begin{pmatrix} 4 & 9 & 6 \\ 7 & 16 & 18 \\ -3 & -8 & -6 \end{pmatrix}, BA = \begin{pmatrix} 8 & -1 & 1 \\ 11 & 1 & 2 \\ 13 & 4 & 5 \end{pmatrix}$$

```
[[4.0   9.0   6.0...
 [7.0   16.0  18....
 [-3.0  -8.0  -6....
[B][A]
[[8.0   -1.0  1.0...
 [11.0  1.0   2.0...
 [13.0  4.0   5.0...
```

```
([A][B])-1
 [[1.3  .2   1.8]
  [-.3  -.2  -.8]
  [-.2  .1   .0 ]]
```

```
Ans▸Frac
[[4/3   1/6   11/...
 [-1/3  -1/6  -5/...
 [-2/9  5/36  1/3...
```

$$(AB)^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{1}{6} & \frac{11}{6} \\ -\frac{1}{3} & -\frac{1}{6} & -\frac{5}{6} \\ -\frac{2}{9} & \frac{5}{36} & \frac{1}{36} \end{pmatrix}, \text{ also}$$

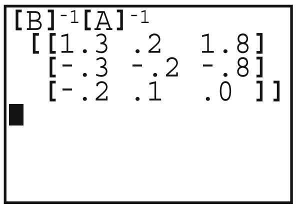
```
[A]-1[B]-1
[[-.1  .3    -.1]
 [-.8  .7    -.1]
 [.9   -1.3  .5 ]]
```

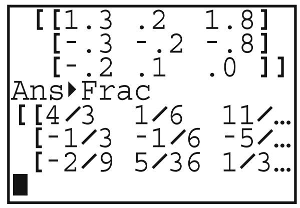
```
[[-.1  .3    -.1]
 [-.8  .7    -.1]
 [.9   -1.3  .5 ]]
Ans▸Frac
[[-1/12   1/4   -...
 [-29/36  3/4   -...
 [31/36   -5/4  1...
```

$$A^{-1}B^{-1} = \begin{pmatrix} -\frac{1}{12} & \frac{1}{4} & -\frac{1}{12} \\ -\frac{29}{36} & \frac{3}{4} & -\frac{5}{36} \\ \frac{31}{36} & -\frac{5}{4} & \frac{19}{36} \end{pmatrix}.$$

This last result shows that $(AB)^{-1} \neq A^{-1}B^{-1}$. However, as you notice below $(AB)^{-1} = B^{-1}A^{-1}$:

$$B^{-1}A^{-1} = \begin{pmatrix} \frac{4}{3} & \frac{1}{6} & \frac{11}{6} \\ -\frac{1}{3} & -\frac{1}{6} & -\frac{5}{6} \\ -\frac{2}{9} & \frac{5}{36} & \frac{1}{36} \end{pmatrix}.$$





Finally, we also have

$$(BA)^{-1} = \begin{pmatrix} -\frac{1}{12} & \frac{1}{4} & -\frac{1}{12} \\ -\frac{29}{36} & \frac{3}{4} & -\frac{5}{36} \\ \frac{31}{36} & -\frac{5}{4} & \frac{19}{36} \end{pmatrix}.$$

This in turn is nothing but $A^{-1}B^{-1}$.

So, in general we have the following result:

If A and B are non-singular matrices of order n, then AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

The proof of this theorem is straightforward:

To show that $B^{-1}A^{-1}$ is the inverse of AB, we need only show that it conforms to the definition of an inverse matrix. That is,

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I.$$

Now, $(AB)(B^{-1}A^{-1}) = A(B\,B^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I.$

Similarly, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}B = I.$

Hence, AB is non-singular (invertible) and its inverse is $B^{-1}A^{-1}$.

The following properties will be listed without proof:

$(A^{-1})^{-1} = A$

$(cA)^{-1} = \frac{1}{c}A^{-1}; c \neq 0$

$\det(AB) = \det A \cdot \det B$

This last result is helpful in proving the following property.

If A is non-singular, then $\det A^{-1} = \dfrac{1}{\det A}$.

Proof: Since $AA^{-1} = I$, then

$$\det(AA^{-1}) = \det I \Rightarrow \det A \cdot \det A^{-1} = 1 \Rightarrow \det A^{-1} = \frac{1}{\det A}.$$

In the previous section, we solved a system of equations using inverse matrices. However, that method works as long as the system is consistent with a unique solution. In many cases, the solution either has an infinite number of solutions or is inconsistent. There is another method of solution which we want to introduce you to.

Some terminology

As we have seen before, it is usual to represent a system of equations using matrix notation. In the previous section you learned how to solve a system of equations by writing the system in matrix form. For example, to solve the system

$$\begin{cases} 2x + 3y - 4z = 8 \\ \quad\;\; 2y + 4z = -3 \\ x \qquad\; - 2z = 4 \end{cases}$$

we wrote

$$\begin{pmatrix} 2 & 3 & -4 \\ 0 & 2 & 4 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ 4 \end{pmatrix}$$

The first matrix is called the **coefficient matrix** (or **matrix of coefficients**) and the matrix on the right is called the **constants matrix** or the **answers matrix**. If the system has a unique solution then it can be solved. As you see, the method is limited and it has a strict constraint. Thanks to a slightly different arrangement, we can use matrices to arrive at our solution regardless of whether it is unique, has an infinite number of solutions, or simply no solution. To that end we need to write the system as follows:

$$\left(\begin{array}{ccc|c} 2 & 3 & -4 & 8 \\ 0 & 2 & 4 & -3 \\ 1 & 0 & -2 & 4 \end{array}\right)$$

This is called the **augmented** matrix of the system. It is customary to put a bar between the coefficients and the answers. However, this bar is not necessary and we will not be using it in this book. Just remember that the last column is the answers column!

Gauss-Jordan elimination

The idea behind this method is very simple. We successively apply certain simple operations to the system of equations reducing them into a special form that is easy to solve. The operations are called **elementary row**

operations and they can be applied to the system without changing the solution to the system. That is, the solution to the reduced system (**reduced row echelon form**) is the same as that for the original system. We can apply the operations either to the system itself or to its augmented matrix. Since the latter is easier to work with, we recommend that you first write the augmented matrix, reduce it, and then write the equivalent system to read the solution from.

There are three types of **elementary row operations**.

1. **Multiply any row by non-zero real number**.
2. **Interchange any two rows**.
3. **Add a multiple of one row to another row**.

Note: The order with which we apply the operations is not unique!

We will demonstrate the method with an example.

Consider the following system and its associated matrix:

$$\begin{cases} 2x + y - z = 2 \\ x + 3y + 2z = 1 \\ 2x + 4y + 6z = 6 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 2 & 1 & -1 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 4 & 6 & 6 \end{array}\right)$$

Switch row 1 and row 2 – type 2 operation:

$$\begin{cases} x + 3y + 2z = 1 \\ 2x + y - z = 2 \\ 2x + 4y + 6z = 6 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 2 & 4 & 6 & 6 \end{array}\right)$$

Multiply row 3 by $\frac{1}{2}$ − type 1 operation:

$$\begin{cases} x + 3y + 2z = 1 \\ 2x + y - z = 2 \\ x + 2y + 3z = 3 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 2 & 3 & 3 \end{array}\right)$$

Multiply row 1 by −2 and add it to row 2, and multiply row 1 by −1 and add it to row 3 – type 3 operations:

$$\begin{cases} x + 3y + 2z = 1 \\ -5y - 5z = 0 \\ -y + z = 2 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & -5 & -5 & 0 \\ 0 & -1 & 1 & 2 \end{array}\right)$$

Notice here that row 1 did not change and rows 2 and three were replaced with the result of the elementary operation.

Multiply row 2 by $-\frac{1}{5}$:

$$\begin{cases} x + 3y + 2z = 1 \\ y + z = 0 \\ -y + z = 2 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 2 \end{array}\right)$$

Now, add row 2 to row 3, and multiply row 2 by −3 and add it to row 1:

$$\begin{cases} x - z = 1 \\ y + z = 0 \\ 2z = 2 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

Now multiply row 3 by $\frac{1}{2}$:

$$\begin{cases} x \quad\quad - z = 1 \\ \quad y + z = 0 \\ \quad\quad\quad z = 1 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Lastly, add row 3 to row 1, and multiply row 3 by -1 and add it to row 2:

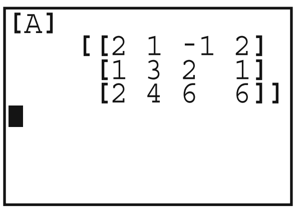
$$\begin{cases} x \quad\quad\quad = 2 \\ \quad y \quad\quad = -1 \\ \quad\quad\quad z = 1 \end{cases} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

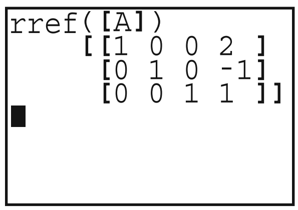
As you notice, from this last system it is easy to read the solution of $(2, -1, 1)$. You can verify that this solution is also the solution to the original system.

The simplified matrix is in its reduced row echelon form (to be defined later).

Of course, when we do the work, we do not have to show the processes in parallel. We just perform the operation on the matrix and then translate it into the equation form.

Note: This whole operation can easily be performed using a GDC.





Example 11

Solve the following system:

$$\begin{cases} x + y + 2z = 1 \\ x + \quad\quad z = 2 \\ \quad y + \quad z = 0 \end{cases}$$

Solution

The augmented matrix is:

$$\begin{cases} x + y + 2z = 1 \\ x + y \quad\quad z = 2 \\ \quad y + z = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Multiply row 1 with -1 and add to row 2:

$$\begin{cases} x + y + 2z = 1 \\ \quad -y - z = 1 \\ \quad y + z = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

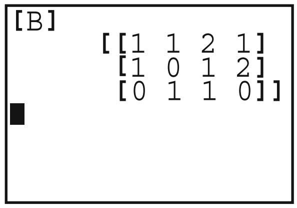
Add row 2 to row 1 and row 2 to row 3:

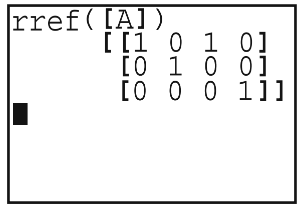
$$\begin{cases} x \quad\quad + z = 2 \\ \quad -y - z = 1 \\ \quad\quad\quad 0 = 1 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

At this stage, work can stop because if you write the last row as an equation, it reads

$$0x + 0y + 0z = 1.$$

This statement cannot be true for any value, and hence the system is inconsistent.





Example 12

Solve the following system:

$$\begin{cases} 2x + y - z = 4 \\ x + 3y + 7z = 7 \\ 2x + 4y + 8z = 10 \end{cases}$$

Solution

The augmented matrix is:

$$\begin{cases} 2x + y - z = 4 \\ x + 3y + 7z = 7 \\ 2x + 4y + 8z = 10 \end{cases} \Leftrightarrow \begin{pmatrix} 2 & 1 & -1 & 4 \\ 1 & 3 & 7 & 7 \\ 2 & 4 & 8 & 10 \end{pmatrix}$$

$$\begin{cases} x + 3y + 7z = 7 \\ 2x + y - z = 4 \\ 2x + 4y + 8z = 10 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 3 & 7 & 7 \\ 2 & 1 & -1 & 4 \\ 2 & 4 & 8 & 10 \end{pmatrix} \quad R_1 \Leftrightarrow R_2$$

$$\begin{cases} x + 3y + 7z = 7 \\ -5y - 15z = -10 \\ 3y + 9z = 6 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 3 & 7 & 7 \\ 0 & -5 & -15 & -10 \\ 0 & 3 & 9 & 6 \end{pmatrix} \quad \begin{cases} -R_2 + R_3 \\ -2R_1 + R_2 \end{cases}$$

$$\begin{cases} x + 3y + 7z = 7 \\ y + 3z = 2 \\ y + 3z = 2 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 3 & 7 & 7 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 \end{pmatrix} \quad \begin{cases} -\frac{1}{5}R_2 \\ \frac{1}{3}R_3 \end{cases}$$

$$\begin{cases} x - 2z = 1 \\ y + 3z = 2 \\ 0 = 0 \end{cases} \Leftrightarrow \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{cases} -R_2 + R_3 \\ -3R_2 + R_1 \end{cases}$$

Since the last row is all zeros, there is not much that we can do. The conclusion is that this last row is true for any choice of values for the variables. Now we are left with a system of two equations and three variables.

$$\begin{cases} x - 2z = 4 \\ y + 3z = 2 \end{cases}$$

We need to solve for two of the variables in terms of the third. A wise choice here would be to solve for x and y in terms of z. That is,

$$x = 1 + 2z, y = 2 - 3z.$$

This means that for every choice of a value for z, we have a corresponding solution for the system. For example, if $z = 0$, then the solution would be $(1, 2, 0)$, for $z = 2$, the solution is $(5, -4, 2)$, and so on. This means that we have an infinite number of solutions. It is customary to present the solution in terms of a parameter, t for example. We let $z = t$, and our general solution would then be

$$(1 + 2t, 2 - 3t, t).$$

So, what is a **reduced row echelon form** (rref)?

We are confident that by now, you have a feel for what it is:

A matrix is in rref if it satisfies the following properties:

1. If there are any rows consisting entirely of zeros, they appear at the bottom of the matrix.
2. In any non-zero row, the first non-zero entry is 1. This entry is called the **pivot** of the row.
3. For any consecutive rows, the pivot of the lower row must be to the right of the pivot of the preceding row.
4. Any column that contains a pivot, has zeros everywhere else.

See the demonstration below; A is in rref while B is not.

$$A = \begin{pmatrix} \boxed{1} & 0 & 3 & 0 & 5 & 8 \\ 0 \rightarrow & \boxed{1} & 4 & 0 & 4 & 2 \\ 0 & 0 \rightarrow & 0 \rightarrow & \boxed{1} & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 3 & 6 & 7 \\ 0 & 0 & 1 \leftarrow & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Curve fitting

Another application of matrices (systems) is to help fit specific models to sets of points.

Example 13

Fit a quadratic model to pass through the points $(-1, 10)$, $(2, 4)$, and $(3, 14)$.

Solution

The problem is to find parameters a, b, and c that will force the curve representing the function $f(x) = ax^2 + bx + c$ to contain the given points. This means

$$f(-1) = 10, f(2) = 4, \text{ and } f(3) = 14.$$

Since we need to find the three unknown parameters, we need three equations which are offered by the conditions above:

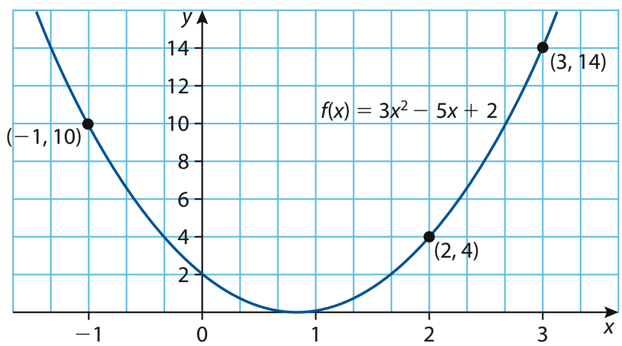
$$\begin{aligned} f(x) &= ax^2 + bx + c \\ f(-1) &= a - b + c = 10 \\ f(2) &= 4a + 2b + c = 4 \\ f(3) &= 9a + 3b + c = 14 \end{aligned}$$

This is clearly a system of three equations which can be solved using matrix methods, among other methods of course.

Using *rref*, we get the following result:

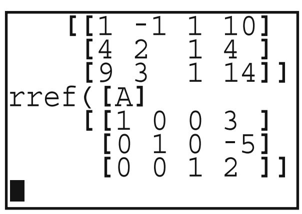
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 10 \\ 4 & 2 & 1 & 4 \\ 9 & 3 & 1 & 14 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

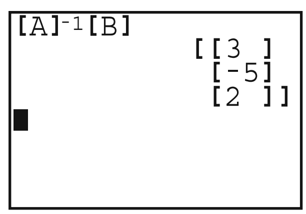
Which means that $a = 3$, $b = -5$, and $c = 2$; so the function is $f(x) = 3x^2 - 5x + 2$.



Equivalently, we can use the inverse matrix directly:

$$\begin{pmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \\ 14 \end{pmatrix} \Leftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 10 \\ 4 \\ 14 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$





Exercise 6.4

1 Given the matrix $A = \begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}$ find the value of the real number m such that $\det(A - mI) = 0$, where I is the 2×2 multiplication identity matrix.

2 a) Find the values of a and b, given that the matrix $A = \begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$ is the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}$.

b) For the values of a and b found in part a), solve the system of linear equations:

$x + 2y - 2z = 5$

$3x + by + z = 0$

$-x + y - 3z = a - 1$

3 Find the value(s) of m so that the matrix $\begin{pmatrix} 1 & m & 1 \\ 3 & 1-m & 2 \\ m & -3 & m-1 \end{pmatrix}$ is singular.

4 Solve each system of equations. If a solution does not exist, justify why not.

a) $\begin{cases} 4x - y + z = -5 \\ 2x + 2y + 3z = 10 \\ 5x - 2y + 6z = 1 \end{cases}$

b) $\begin{cases} 4x - 2y + 3z = -2 \\ 2x + 2y + 5z = 16 \\ 8x - 5y - 2z = 4 \end{cases}$

c) $\begin{cases} 5x - 3y + 2z = 2 \\ 2x + 2y - 3z = 3 \\ x - 7y + 8z = -4 \end{cases}$

d) $\begin{cases} 3x - 2y + z = -29 \\ -4x + y - 3z = 37 \\ x - 5y + z = -24 \end{cases}$

e) $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 13 \end{cases}$

f) $\begin{cases} 2x + 3y + 5z = 4 \\ 3x + 5y + 9z = 7 \\ 5x + 9y + 17z = 1 \end{cases}$

g) $\begin{cases} -x + 4y - 2z = 12 \\ 2x - 9y + 5z = -25 \\ -x + 5y - 4z = 10 \end{cases}$

h) $\begin{cases} x - 3y - 2z = 8 \\ -2x + 7y + 3z = -19 \\ x - y - 3z = 3 \end{cases}$

5 a) Find the values of k such that the following matrix is not singular

$$A = \begin{pmatrix} 1 & 1 & k-1 \\ k & 0 & -1 \\ 6 & 2 & -3 \end{pmatrix}.$$

b) Find the value(s) of k such that A is the inverse of B, where

$$B = \begin{pmatrix} k-3 & -3 & k \\ 3 & k+2 & -1 \\ -2 & -4 & 1 \end{pmatrix}.$$

c) For the value of k found in b), apply elementary row operations to reduce the matrix $\begin{pmatrix} 1 & 1 & k-1 & 1 & 0 & 0 \\ k & 0 & -1 & 0 & 1 & 0 \\ 6 & 2 & -3 & 0 & 0 & 1 \end{pmatrix}$ into $\begin{pmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{pmatrix}$ where $a, \ldots, i$ are to be determined.

6 a) Find the values of k such that the following matrix is not singular.

$$A = \begin{pmatrix} \frac{2}{5} & -\frac{17}{5} & \frac{k+9}{5} \\ -\frac{1}{5} & \frac{21}{5} & -\frac{13}{5} \\ k-2 & 3 & -2 \end{pmatrix}$$

b) Find the value(s) of k such that A is the inverse of B, where

$$B = \begin{pmatrix} k+1 & 1 & k \\ 2 & k+2 & -3 \\ 3 & 6 & -5 \end{pmatrix}.$$

c) For the value of k found in b), apply elementary row operations to reduce the matrix $\begin{pmatrix} 2 & -17 & k+9 & 1 & 0 & 0 \\ -1 & 21 & -13 & 0 & 1 & 0 \\ 5(k-2) & 15 & -10 & 0 & 0 & 1 \end{pmatrix}$ into $\begin{pmatrix} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{pmatrix}$ where $a, \ldots, i$ are to be determined.

7 Use elementary row operations to transform the matrix $[A\vdots I]$ to a matrix in the form $[I\vdots B]$. Comment on the relationship between A and B and support your conclusion.

a) $\left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$

b) $\left(\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -1 & 8 & 6 & 0 & 0 & 1 \end{array}\right)$

8 Determine the function f so that the curve representing it contains the indicated points.

a) $f(x) = ax^2 + bx + c$ to contain $(-1, 5)$, $(2, -1)$, and $(4, 35)$.

b) $f(x) = ax^2 + bx + c$ to contain $(-1, 12)$ and $(2, -3)$.

● **Hint:** there is more than one curve!

c) $f(x) = ax^3 + bx^2 + cx + d$ to contain the points $(-1, 5)$, $(1, -3)$, $(2, 5)$, and $(3, 45)$. [optional material]

d) $f(x) = ax^3 + bx^2 + cx + d$ to contain the points $(-3, 4)$, $(-1, 4)$, and $(2, 4)$.

9 Consider the following system of equations:

$$\begin{cases} 2x + y + 3z = -5 \\ 3x - y + 4z = 2 \\ 5x + 7z = m - 5 \end{cases}$$

Find the value(s) of m for which this system is consistent. For the value of m found, find the most general solution of the system.

10 Consider the following system of equations:

$$\begin{cases} -3x + 2y + 3z = 1 \\ 4x - y - 5z = -5 \\ x + y - 2z = m - 3 \end{cases}$$

Find the value(s) of m for which this system is consistent. For the value of m found, find the most general solution of the system.

11 Consider the matrix $A = \begin{pmatrix} 3 & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$.

a) Find $\det(A)$.

b) Use the third elementary row operation to transform the matrix A into matrix B in triangular form (i.e. **add a multiple of one row to another row**).

c) Find $\det(B)$.

d) Use a GDC to find $\det(C)$ for $C = \begin{pmatrix} 2 & 1 & -3 & 5 \\ 4 & 3 & -4 & -6 \\ 6 & -8 & 5 & 7 \\ -6 & -5 & 3 & 4 \end{pmatrix}$.

e) Repeat b) and c) for C.

Practice questions

1 If $\begin{pmatrix} 2x & 3 \\ -4x & x \end{pmatrix}$ and det $A = 14$, find x.

2 Let $M = \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix}$, where $a \in \mathbb{Z}$.

a) Find M^2 in terms of a.

b) If M^2 is equal to $\begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$, find the value of a.

Using this value of a, find M^{-1} and hence solve the system of equations:

$$-x + 2y = -3$$
$$2x - y = 3$$

3 Two matrices are given, where $A = \begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix}$ and $BA = \begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}$. Find B.

4 The matrices A, B, and X are given, where

$A = \begin{pmatrix} 3 & 1 \\ -5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 8 \\ 0 & -3 \end{pmatrix}$, $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$.

Find the values of a, b, c and d such that $AX + X = B$.

5 $A = \begin{pmatrix} 5 & -2 \\ 7 & 1 \end{pmatrix}$ is a 2×2 matrix.

a) Write out A^{-1}.

b) **(i)** If $XA + B = C$, where B, C, and X are 2×2 matrices, express X in terms of A^{-1}, B, and C.

(ii) Find X if $B = \begin{pmatrix} 6 & 7 \\ 5 & -2 \end{pmatrix}$ and $C = \begin{pmatrix} -5 & 0 \\ -8 & 7 \end{pmatrix}$.

6 Given $A = \begin{pmatrix} a & b \\ c & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ d & c \end{pmatrix}$,

a) write out $A + B$;

b) find AB.

7 **a)** Write out the inverse of the matrix $\begin{pmatrix} 1 & -3 & 1 \\ 2 & 2 & -1 \\ 1 & -5 & 3 \end{pmatrix}$.

b) Hence, solve the system of simultaneous equations:

$$x - 3y + z = 1$$
$$2x + 2y - z = 2$$
$$x - 5y + 3z = 3$$

8 Given the two matrices C and D, where

$C = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix}$ and $D = \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$,

the matrix Q is given such that $3Q = 2C - D$.

b) Find Q.

b) Find CD.

c) Find D^{-1}.

9 a) Find the values of a and b given that the matrix $A = \begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$ is the inverse of the matrix $B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}$.

b) For the values of a and b found in part a), solve the system of linear equations:

$x + 2y - 2z = 5$

$3x + by + z = 0$

$-x + y - 3z = a - 1$

10 a) Given matrices A, B, C for which $AB = C$ and $\det A \neq 0$, express B in terms of A and C.

b) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 2 \\ 3 & -3 & 2 \end{pmatrix}$, $D = \begin{pmatrix} -4 & 13 & -7 \\ -2 & 7 & -4 \\ 3 & -9 & 5 \end{pmatrix}$ and $C = \begin{pmatrix} 5 \\ 7 \\ 10 \end{pmatrix}$.

(i) Find the matrix DA.

(ii) Find B if $AB = C$.

c) Find the coordinates of the point of intersection of the planes $x + 2y + 3z = 5$, $2x - y + 2z = 7$ and $3x - 3y + 2z = 10$. (This can be answered after Chapter 14.)

11 a) Find the determinant of the matrix $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}$.

b) Find the value of λ for which the following system of equations can be solved.

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ \lambda \end{pmatrix}$$

c) For this value of λ, find the general solution to the system of equations.

12 The square matrix X is such that $X^3 = 0$. Show that the inverse of the matrix $(I - X)$ is $I + X + X^2$.

19 $x = \sqrt{e}, x = e$

20 a) $V = \$265.33$ b) 235 months

21 $x = 5^{\frac{5}{3}}$ or $x = 5^{\frac{-5}{3}}$

22 $x = e - 3$ or $x = \frac{1}{e} - 3$

23 $x = -2.50, -1.51$ or 0.440 (3 s.f.)

24 $k = \frac{\ln 2}{20}$

25 a) $f(x) = \ln\left(\frac{x}{x+2}\right)$ b) $f^{-1}(x) = -\frac{2e^x}{e^x - 1}$ or $\frac{2e^x}{1 - e^x}$

26 a) (i) Minimum value of f is 0.

(ii) From part (i) $f(x) \geqslant 0 \Rightarrow e^x - 1 - x \geqslant 0 \Rightarrow e^x \geqslant 1 + x$

d) $n > e^{100}$

Chapter 6

Exercise 6.1 and 6.2

1 a) (i) $\begin{pmatrix} x-1 & x-3 \\ y+3 & y+1 \end{pmatrix}$ (ii) $\begin{pmatrix} -x-7 & 3x+3 \\ 3y-7 & 11-y \end{pmatrix}$

b) $x = -3, y = 5$ c) $x = 3, y = -3$

d) $AB = \begin{pmatrix} 2x-2 & xy-2x+6 \\ xy-x+y+11 & -3 \end{pmatrix}$;

$BA = \begin{pmatrix} -2x-3y+1 & x^2+x-9 \\ y^2-3y-6 & 4x+3y-6 \end{pmatrix}$

2 a) $x = 2, y = -10$ b) $p = 2, q = -4$

3 a) $\begin{bmatrix} 0&1&0&0&1&2&0 \\ 1&0&1&1&1&1&0 \\ 0&1&0&2&0&0&2 \\ 0&1&2&0&1&0&0 \\ 1&1&0&1&0&1&0 \\ 2&1&0&0&1&0&0 \\ 0&0&2&0&0&0&0 \end{bmatrix}$ b) $\begin{bmatrix} 6&3&1&2&3&2&0 \\ 3&5&2&3&3&3&2 \\ 1&2&9&1&3&1&0 \\ 2&3&1&6&1&2&4 \\ 3&3&3&1&4&3&0 \\ 2&3&1&2&3&6&0 \\ 0&2&0&4&0&0&4 \end{bmatrix}$

Matrix signifies the number of routes between each pair that go via one other city.

4 a) $A + C = \begin{pmatrix} x+1 & 10 & y+1 \\ 0 & -x-3 & y+3 \\ 2x+y+7 & x-3y & -x+2y-1 \end{pmatrix}$

b) $\begin{pmatrix} 17m+2 & -6 \\ 4-9m & 9 \\ 7m-2 & -17 \end{pmatrix}$

c) Not possible d) $x = 3, y = 1$

e) Not possible f) $m = 3$

5 $a = -3, b = 3, c = 2$

6 $x = 4, y = -3$

7 $m = 2, n = 3$

8 Shop A: €18.77

9 a) $\begin{pmatrix} 2 & 4 \\ -2 & 12 \end{pmatrix}$ b) associative

c) $\begin{pmatrix} -22 & 16 \\ 60 & -7 \end{pmatrix}$ d) associative

10 $AB = [88\ 142]$, which represents total profit.

11 $r = 3, s = -2$

12 a) (i) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 1^n & n \\ 0 & 3^n \end{pmatrix}$

b) (i) $\begin{pmatrix} 9 & 18 \\ 0 & 9 \end{pmatrix}$ (ii) $\begin{pmatrix} 27 & 81 \\ 0 & 27 \end{pmatrix}$

(iii) $\begin{pmatrix} 81 & 324 \\ 0 & 81 \end{pmatrix}$ (iv) $\begin{pmatrix} 3^n & 3^{n+1} \\ 0 & 3^n \end{pmatrix}$

13 $\left(\frac{11}{3}, \frac{8}{3}\right)$ **14** $(1, -4)$

15 5 **16** $(5, 1)$

Exercise 6.3

1 a) $\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$ b) $M = \begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$

c) $\begin{pmatrix} -39 & -44 \\ 17 & 19 \end{pmatrix}$

d) (i) $N = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$ (ii) $N = \begin{pmatrix} -14 & -11 \\ -7 & -6 \end{pmatrix}$

e) If $AB = C$ then $B = A^{-1}C$, while if $BA = C$, then $B = CA^{-1}$. Also, $A^{-1}C \neq CA^{-1}$.

2 $\begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & 0 \end{pmatrix}$

3 a) $|A| = -5 \neq 0$ b) $\begin{pmatrix} \frac{9}{5} & \frac{11}{5} & -\frac{8}{5} \\ \frac{6}{5} & \frac{9}{5} & -\frac{7}{5} \\ 1 & 1 & -1 \end{pmatrix}$ c) $\begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{5} \end{pmatrix}$

4 a) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ b) $\begin{pmatrix} \frac{3}{a}+1 & -1 \\ -a-2 & a \end{pmatrix}$

5 $x = 2$ or $x = 3$

6 $n = 0.5$

7 a) $X = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{3}{4} & -\frac{7}{6} \end{pmatrix}$ b) $Y = \begin{pmatrix} 1 & \frac{13}{12} \\ -1 & -\frac{5}{3} \end{pmatrix}$

c) $X \neq Y$ – not commutative

8 a) $PQ = \begin{pmatrix} 5 & -4 & 3 \\ 33 & 5 & -1 \\ 2 & -3 & 2 \end{pmatrix}$, $QP = \begin{pmatrix} 4 & -5 & -8 \\ 8 & 0 & -4 \\ 7 & 10 & 8 \end{pmatrix}$

b) $P^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -\frac{7}{5} & \frac{1}{5} & \frac{11}{5} \\ 1 & 0 & -2 \end{pmatrix}$, $Q^{-1} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 1 & -1 & 1 \\ 2 & -\frac{7}{4} & 1 \end{pmatrix}$

$P^{-1}Q^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ \frac{23}{5} & -\frac{22}{5} & \frac{12}{5} \\ -4 & \frac{15}{4} & -2 \end{pmatrix}$

$Q^{-1}P^{-1} = \begin{pmatrix} -\frac{7}{20} & \frac{1}{20} & \frac{11}{20} \\ \frac{17}{5} & -\frac{1}{5} & -\frac{26}{5} \\ \frac{109}{20} & -\frac{7}{20} & -\frac{157}{20} \end{pmatrix}$

$(PQ)^{-1} = \begin{pmatrix} -\frac{7}{20} & \frac{1}{20} & \frac{11}{20} \\ \frac{17}{5} & -\frac{1}{5} & -\frac{26}{5} \\ \frac{109}{20} & -\frac{7}{20} & -\frac{157}{20} \end{pmatrix}$

$(QP)^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ \frac{23}{5} & -\frac{22}{5} & \frac{12}{5} \\ -4 & \frac{15}{4} & -2 \end{pmatrix}$

9 a) $\begin{pmatrix} -7 \\ 3 \\ -2 \end{pmatrix}$ b) $\begin{pmatrix} -7 \\ 3 \\ -2 \end{pmatrix}$

10 $x = -1$ **11** $x = 1, y = 2$

12 $(0, 1)$ **13** $(-3, -29), (0, 1)$

14 $17x - 8y + 37 = 0;\ y + 2 = 0;\ x + 5 = 0$ **15** 165; 80; 136

16 $x = \frac{89}{2}$ or $x = \frac{129}{8}$; $x = -4$ or $x = -2$ or $x = -3 \pm \sqrt{21}$

17 $-3; 3$

18 a) -25
b) $x^2 - 7x - 25$, constant $= \det(A)$
c) $-(a + d)$
d) $f(A) = 0$
e) $ad - bc;\ x^2 - (a + d)x + (ad - bc)$,
constant $= \det(A)$; $f(A) = 0$

19 a) -22
b) $x^3 - x^2 - 22x + 22$, constant $= -\det(A)$
c) Opposite of the sum of the main diagonal
d) $f(A) = 0$

Exercise 6.4

1 $\mathbf{m} = 2$ or $\mathbf{m} = 3$

2 a) $a = 7, b = 2$ b) $(-1, 2, -1)$

3 $\mathbf{m} = 2$

4 a) $(-1, 3, 2)$ b) $(5, 8, -2)$
c) $\left(\frac{13}{16} + \frac{5}{16}t, \frac{11}{16} + \frac{19}{16}t, t\right)$ d) $(-7, 3, -2)$
e) $(-1 + 2t, 2 - 3t, t)$ f) inconsistent
g) $(-2, 4, 3)$ h) $(4, -2, 1)$

5 a) $k \neq \frac{-1 \pm \sqrt{33}}{4}$ b) $k = 1$
c) $\begin{pmatrix} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & 3 & 3 & -1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{pmatrix}$

6 a) $\frac{71 \pm i\sqrt{251}}{42}$ b) $k = 2$
c) $\begin{pmatrix} 1 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{2}{5} & \frac{4}{5} & \frac{-3}{5} \\ 0 & 0 & 1 & \frac{3}{5} & \frac{6}{5} & -1 \end{pmatrix}$

7 $\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{2}{3} & -\frac{5}{6} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$; $\begin{pmatrix} 1 & 0 & 0 & 2 & \frac{-16}{13} & \frac{-19}{13} \\ 0 & 1 & 0 & 1 & \frac{-11}{13} & \frac{-9}{13} \\ 0 & 0 & 1 & -1 & \frac{12}{13} & \frac{11}{13} \end{pmatrix}$
B is the inverse of A

8 a) $f(x) = 4x^2 - 6x - 5$
b) $f(x) = \frac{1}{2}(m - 27)x^2 + \frac{3}{2}(17 - m)x + m,\ m \in \mathbb{R}$
c) $f(x) = 3x^3 - 2x^2 - 7x + 3$
d) $f(x) = \frac{1}{6}(4 - m)x^3 + \frac{1}{3}(4 - m)x^2 - \frac{5}{6}(4 - m)x + m,\ m \in \mathbb{R}$

9 $m = 2, \begin{pmatrix} -t - \frac{3}{5} \\ -t - \frac{19}{5} \\ 5t \end{pmatrix}$ **10** $m = -1, \begin{pmatrix} 7t - \frac{9}{5} \\ \frac{3}{5} - 11t \\ 5t \end{pmatrix}$

11 a) 3 b) $\begin{pmatrix} 3 & -4 & -6 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$ c) 3
d) -1672 e) $\begin{pmatrix} 2 & 1 & -3 & 5 \\ 0 & 1 & 2 & -16 \\ 0 & 0 & 36 & -184 \\ 0 & 0 & 0 & -\frac{209}{9} \end{pmatrix}$ f) -1672

Practice questions

1 $x = -7$ or $x = 1$

2 a) $\begin{pmatrix} a^2 + 4 & 2a - 2 \\ 2a - 2 & 5 \end{pmatrix}$
b) $a = -1$; $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

3 $B = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$

4 $a = \frac{28}{33}$; $b = \frac{59}{33}$; $c = \frac{20}{33}$; $d = \frac{28}{33}$

5 a) $A^{-1} = \begin{pmatrix} \frac{1}{19} & \frac{2}{19} \\ \frac{-7}{19} & \frac{5}{19} \end{pmatrix}$
b) (i) $X = (C - B)A^{-1}$ (ii) $X = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$

6 a) $A + B = \begin{pmatrix} a + 1 & b + 2 \\ c + d & 1 + c \end{pmatrix}$
b) $AB = \begin{pmatrix} a + bd & 2a + bc \\ c + d & 3c \end{pmatrix}$

7 a) $\begin{pmatrix} 0.1 & 0.4 & 0.1 \\ -0.7 & 0.2 & 0.3 \\ -1.2 & 0.2 & 0.8 \end{pmatrix}$
b) $x = 1.2, y = 0.6, z = 1.6$

8 a) $Q = \begin{pmatrix} -3 & 2 \\ 1 & \frac{14 - a}{3} \end{pmatrix}$
b) $CD = \begin{pmatrix} -14 & -4 + 4a \\ -2 & 2 + 7a \end{pmatrix}$
c) $D^{-1} = \frac{1}{5a + 2}\begin{pmatrix} a & -2 \\ 1 & 5 \end{pmatrix}$

9 a) $(7, 2)$ b) $(-1, 2, -1)$

10 a) $B = A^{-1}C$ b) $DA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$
c) $(1, -1, 2)$

11 a) Det $= 0$ b) $\lambda = 5$ c) $(2 - 3t, 1 + t, t)$

12 No answer required – proof

Chapter 7

Exercise 7.1

1 $\frac{\pi}{3}$ **2** $\frac{5\pi}{6}$ **3** $-\frac{3\pi}{2}$ **4** $\frac{\pi}{5}$

5 $\frac{3\pi}{4}$ **6** $\frac{5\pi}{18}$ **7** $-\frac{\pi}{4}$ **8** $\frac{20\pi}{9}$

9 $-\frac{8\pi}{3}$

10 135° **11** -630° **12** 115° **13** 210°

14 -143° **15** 300° **16** 115° **17** $89.95^\circ \approx 90^\circ$