1. [Maximum points: 30]

In this problem you will use the Binomial Distribution to derive the Poisson Distribution, and use it to solve a probability problem.

Let $L = \left(1 - \frac{\lambda}{n}\right)^n$ where λ is a constant and $0 < \lambda < n$ where $n \in \mathbb{N}$.

(a) Find the first four terms of the expansion of L. [4]

(b) Show that $\lim_{n \to \infty} L = \mathrm{e}^{-\lambda}$. [5]

Let $X \sim B(n,p)$ and $\lambda = E(X)$.

(c) Write down an expression for [2]

(i) $P(X = x)$

(ii) λ

(d) Show that [4]

$$P(X = x) = \frac{\lambda^x}{x!} \frac{n!}{(n-x)!} \left(\frac{1}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-x}$$

(e) Evaluate [7]

(i) $\lim_{n \to \infty} \frac{n!}{(n-x)!} \left(\frac{1}{n}\right)^x$

(ii) $\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^{-x}$

(f) Hence show that [2]

$$\lim_{n \to \infty} P(X = x) = \frac{\lambda^x \mathrm{e}^{-\lambda}}{x!}$$

This is called the *Poisson Distribution*.

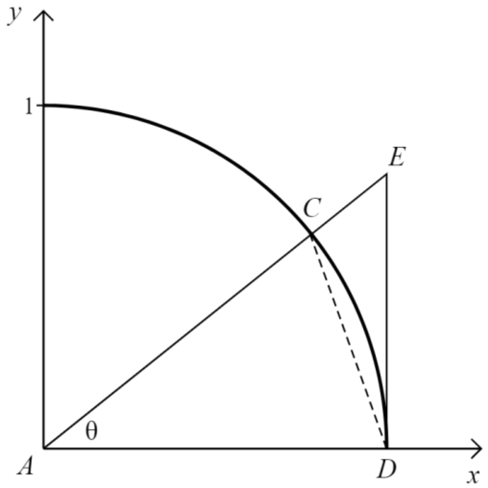
(g) A pizza delivery restaurant receives an average of 10 online orders per hour. [6]

(i) Explain how we can apply the result from (f) to approximate the probability of the restaurant receiving a specific number of orders in an hour.

(ii) Find the probability that between 12 pm and 1 pm the restaurant will receive more than 12 orders.

2. [Maximum points: 31]

The diagram below contains a quarter of a circle of radius 1. Triangle ADE is right-angled. Angle θ is measured in radians.



(a) Determine an expression for length DE in terms of θ. [2]

(b) By considering the area of triangle ADE, the area of sector ACD and the area of triangle ADC show that [7]

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1$$

(c) Consider the inequality [4]

$$\lim_{\theta\to 0}\cos\theta \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq \lim_{\theta\to 0} 1$$

(i) Write down the value of $\lim_{\theta\to 0}\cos\theta$.

(ii) Write down the value of $\lim_{\theta\to 0} 1$.

(iii) Hence deduce the value of $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$.

(d) Show that [6]

(i) $\frac{\cos\theta - 1}{\theta} = -\frac{\sin^2\theta}{\theta(\cos\theta + 1)}$

(ii) $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$

(e) Hence differentiate the following from first principles [12]

(i) $f(x) = \sin x$

(ii) $f(x) = \cos x$