1 Functions

CHAPTER OBJECTIVES:

2.1 Functions: domain, range, composite, identity and inverse functions

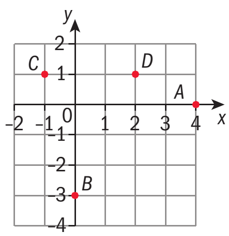
2.2 Graphs of functions, by hand and using GDC, their maxima and minima, asymptotes, the graph of $f^{-1}(x)$

2.3 Transformations of graphs, translations, reflections, stretches and composite transformations

Before you start

You should know how to:

1 Plot coordinates.
e.g. Plot the points $A(4, 0)$, $B(0, -3)$, $C(-1, 1)$ and $D(2, 1)$ on a coordinate plane.



2 Substitute values into an expression.
e.g. Given $x = 2$, $y = 3$ and $z = -5$, find the value of **a** $4x + 2y$ **b** $y^2 - 3z$

a $4x + 2y = 4(2) + 2(3) = 8 + 6 = 14$

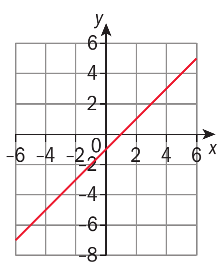
b $y^2 - 3z = (3)^2 - 3(-5) = 9 + 15 = 24$

3 Solve linear equations.
e.g. Solve $6 - 4x = 0$

$6 - 4x = 0 \Rightarrow 6 = 4x$

$1.5 = x \Rightarrow x = 1.5$

4 Use your GDC to graph a function.
e.g. Graph $f(x) = 2x - 1$, $-3 \le x \le 3$

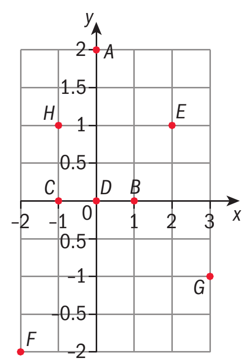


5 Expand linear binomials.
e.g. Expand $(x + 3)(x - 2)$
$= x^2 + x - 6$

Skills check

1 **a** Plot these points on a coordinate plane.
$A(1, 3)$, $B(5, -3)$, $C(4, 4)$, $D(-3, 2)$, $E(2, -3)$, $F(0, 3)$.

b Write down the coordinates of points A to H.



2 Given that $x = 4$, $y = 6$ and $z = -10$, find

a $4x + 3y$ **b** $z^2 - 3y$ **c** $y - z$ **d** $\frac{2x + 5}{yz}$

3 Solve

a $3x - 6 = 6$ **b** $5x + 7 = -3$ **c** $\frac{x}{2} + 6 = 11$

4 Graph these functions on your GDC within the given domain. Then sketch the functions on paper.

a $y = 2x - 3$, $-4 \le x \le 7$

b $y = 10 - 2x$, $-2 \le x \le 5$

c $y = x^2 - 3$, $-3 \le x \le 3$.

5 Expand

a $(x + 4)(x + 5)$ **b** $(x - 1)(x - 3)$

c $(x + 5)(x - 4)$

▲ International Space Station

The International Space Station (ISS) has been orbiting the Earth over 15 times a day for more than ten years, yet how many of us have actually seen it? Spotting the ISS with the naked eye is not as difficult as it might seem – provided you know in which direction to look. Although the ISS travels at a speed of $7.7\,\text{km}\,\text{s}^{-1}$, it is in one of the lowest orbits possible, at approximately 390 km above our heads. Thanks to its large solar wings it is one of the brightest 'stars', which makes it fairly easy to distinguish as it moves across the night sky.

The relation $t = \frac{d}{22\,744}$ gives the speed of the ISS, where t is the time measured in hours and d is the distance traveled in kilometres.

This is a mathematical relationship called a **function** and is just one example of how a mathematical function can be used to describe a situation.

In this chapter you will explore functions and how they can be applied to a wide variety of mathematical situations.

One of the first mathematicians to study the concept of function was French philosopher Nicole Oresme (1323–1382). He worked with independent and dependent variable quantities.

1.1 Introducing functions

Investigation – handshakes

In some countries it is customary at business meetings to shake hands with everybody in the meeting. If there are 2 people there is 1 handshake, if there are 3 people there are 3 handshakes and so on.

a How many handshakes are there for 4 people?

b Copy and complete this table.

Number of people	Number of handshakes
2	
3	
4	
5	
6	
7	
8	
9	
10	

You might find it helps to try this out with a group of your friends in class.

c **Plot** the points on a Cartesian coordinate plane with the number of people on the *x*-axis and the number of handshakes on the *y*-axis.

d Write a formula for the number of handshakes, *H*, in terms of the number of people, *n*.

Do not join the points in this case as we are dealing only with whole (discrete) numbers.

Relations and functions

Distance (m)	Time (s)
100	15
200	34
300	60
400	88

The table shows the amount of time it takes for a student to run certain distances.

Another way of showing this information is as **ordered pairs**: (100, 15), (200, 34), (300, 60) and (400, 88). Each ordered pair has two pieces of data in a specific order. They are separated by a comma and enclosed within brackets in the form (x, y).

→ A **relation** is a set of ordered pairs.

There is nothing special at all about the numbers that are in a relation. In other words, any group of numbers is a relation provided that these numbers come in pairs.

→ The **domain** is the set of all the first numbers (x-values) of the ordered pairs.

The domain of the ordered pairs above is {100, 200, 300, 400}.

The curly brackets, { }, mean 'the set of'.

→ The **range** is the set of the second numbers (y-values) in each pair.

The range of the ordered pairs above is {15, 34, 60, 88}.

Example 1

Find the domain and range of these relations.

a {(1, 4), (2, 7), (3, 10), (4, 13)}

b {(−2, 4), (−1, 1), (0, 0), (1, 1), (2, 4)}

Answers

a The domain is {1, 2, 3, 4} — *First elements in the ordered pairs*

The range is {4, 7, 10, 13} — *Second elements in the ordered pairs*

b The domain is {−2, −1, 0, 1, 2}
The range is {0, 1, 4} — *Do not repeat values even though there are two 4s and two 1s in the ordered pairs.*

→ A **function** is a mathematical relation such that each element of the domain of the function is associated with exactly one element of the range of the function. In order for a relation to be a function no two ordered pairs may have the same first element.

Example 2

Which of these sets of ordered pairs are functions?

a {(1, 4), (2, 6), (3, 8), (3, 9), (4, 10)}

b {(1, 3), (2, 5), (3, 7), (4, 9), (5, 11)}

c {(−2, 1), (−1, 1), (0, 2), (1, 4), (2, 6)}

Answers

a Not a function because the number 3 occurs twice in the domain.

b A function; all of the first elements are different.

c A function; all of the first elements are different. — *Note that it doesn't matter that some of the y-values are the same.*

Exercise 1A

1 Which of these sets of ordered pairs are functions?

a $\{(5,5), (4,4), (3,3), (2,2), (1,1)\}$

b $\{(-3,4), (-1,6), (0,5), (2,-1), (3,-1)\}$

c $\{(4,1), (4,2), (4,3), (4,4), (4,5)\}$

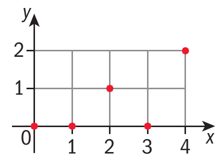
d $\{(-1,1), (0,3), (1,6), (1,7), (2,8)\}$

e $\{(-4,4), (-4,5), (-3,6), (-3,7), (-2,8)\}$

f $\{(1,2), (2,2), (3,2), (4,2), (5,2)\}$

2 For each diagram, identify the domain and range and say whether the relation is a function.

a



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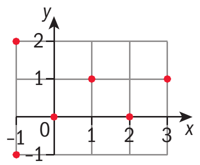


Write down the coordinates as ordered pairs.

3 Look back at the table on page 4 that shows the amount of time it takes for a student to run certain distances. Is the relationship between a distance traveled and time taken a function?

The vertical line test

You can represent relations and functions on a Cartesian plane. You can use the vertical line test to determine whether a particular relation is a function or not, by drawing vertical lines across the graph.

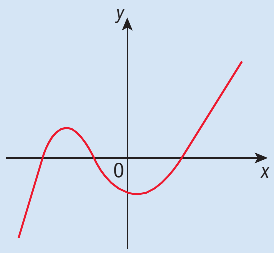
→ A relation is a function if any vertical line drawn will not intersect the graph more than once. This is called the **vertical line test**.

Cartesian coordinates and the Cartesian plane are named after Frenchman René Descartes (1596–1650).

Example 3

Which of these relations are functions?

a



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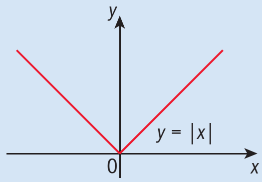

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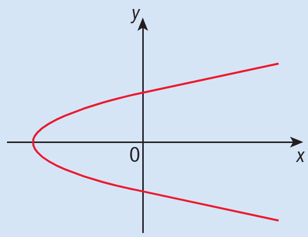

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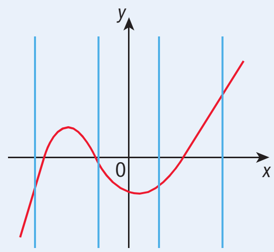
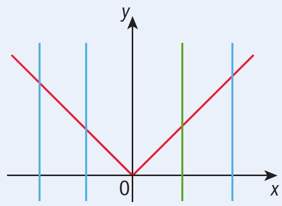
Answers

a



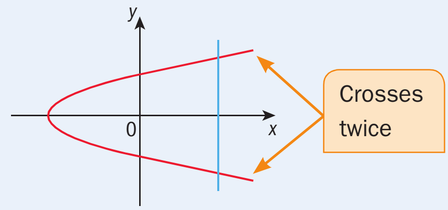
A function

b



A function

c

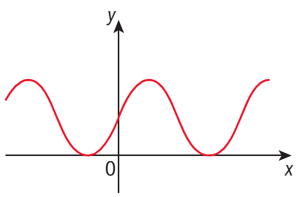


Not a function

Exercise 1B

1 Which of these relations are functions?

a



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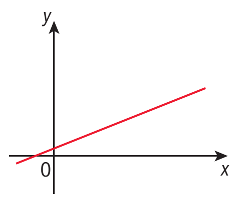

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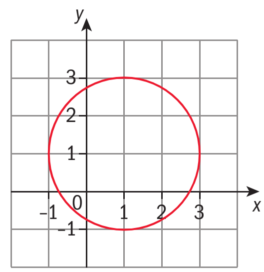
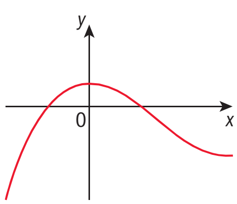


Draw, or imagine, vertical lines on the graph.

d



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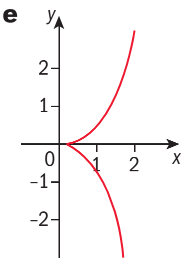

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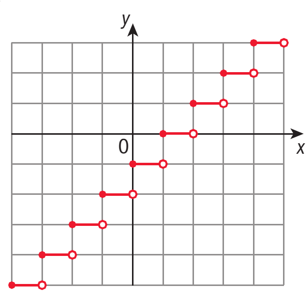
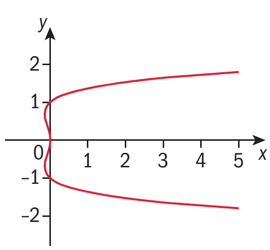


If the function has a 'solid dot' •, this indicates that the value is included in the function.
If the function has a 'hollow dot' ○, this indicates that the value is not included in the function.

g



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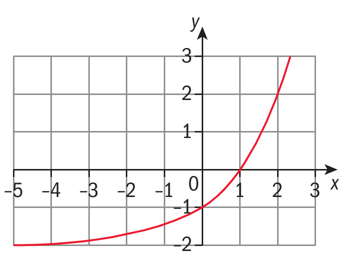

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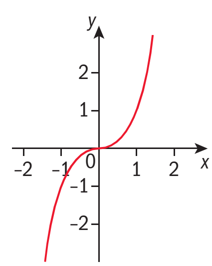


2 Use your GDC to sketch these straight line graphs.

a $y = x$ **b** $y = x + 2$ **c** $y = 2x - 3$ **d** $y = 4$

e Are they all functions? Explain your answer.

f Will all straight lines be functions? Why?

3 Sketch the region $y < 3x - 2$. Is this a function? Why?

4 Use an algebraic method to show that $x^2 + y^2 = 4$ is not a function.

Indicate where the line crosses the x- and/or y-axis on your sketch.

When using your GDC, aim to have the ends of your graph near the corners of the view window.

Try substituting positive and negative values of x.

1.2 The domain and range of a relation on a Cartesian plane

You can often write the domain and range of a relation using interval notation. This is another method of writing down a set of numbers. For example, for the set of numbers that are all less than 3, you can write the inequality $x < 3$, where x is a number in the set. In interval notation, this set of numbers is written $(-\infty, 3)$

Interval notation uses only five symbols.

Brackets	( )
Square brackets	[ ]
Infinity	∞
Negative infinity	$-\infty$
Union	$\cup$

To use interval notation:

→ Use the round brackets (,) if the value is not included in the graph as in $(-\infty, 3)$ or when the graph is undefined at that point (a hole or **asymptote**, or a jump).
Use the square brackets [,] if the value is part of the graph.

Whenever there is a break in the values, write the interval up to the point. Then write another interval for the values after that point. Put a union sign between each interval to 'join' them together. For example $(-\infty, 3) \cup (4, \infty)$

If a graph goes on forever to the left, the domain (x-values) starts with $(-\infty$. If it goes on forever to the right then the domain ends with $\infty)$. If a graph travels downward forever, the range (y-values) starts with $(-\infty$. And if a graph goes up forever, then the range ends with $\infty)$.

Usually we use interval notation to describe a set of values along the x- or y-axis. However, you can use it to describe any group of numbers. For example, in interval notation $x \geq 6$ is $[6, \infty)$.

R
A
DOMAIN
G
E

▲ A function maps the domain (horizontal, x-values) onto the range (vertical, y-values)

How many numbers are there in the sequence 0, 1, 2, 3, 4,… if we go on forever?
How many numbers are there in the sequence 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4,… if we go on forever?

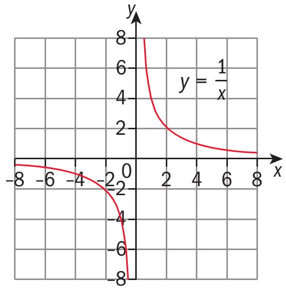
Why do we call infinity undefined?

Asymptotes

Asymptotes are visible on your GDC for some functions. An asymptote is a line that a graph approaches, but does not intersect.

For example, in the graph of $y = \frac{1}{x}$, the line approaches the x-axis ($y = 0$), but never touches it. As we go to infinity the line will not actually reach $y = 0$, but will always get closer and closer. The x-axis or $y = 0$ is callled the horizontal asymptote.

The y-axis or $x = 0$ is the vertical asymptote for the same reasons. There will be a more in-depth treatment of asymptotes in the chapter on rational functions.



Finding asymptotes by looking at the graph is called locating asymptotes by inspection.

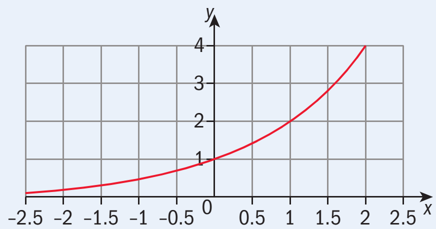
Example 4

Identify the horizontal and vertical asymptotes for these functions if they exist.

a $y = 2^x$ **b** $y = \frac{2x}{x+1}$ **c** $y = \frac{x+2}{(x+1)(x-2)}$

Answers

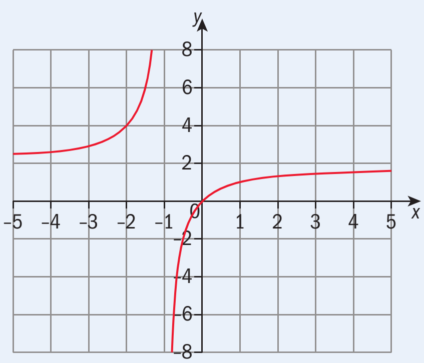
a



Horizontal asymptote $y = 0$

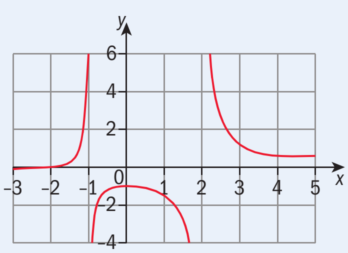
As we go along the x-axis to the left the curve gets closer but never actually meets the x-axis.

b



Horizontal asymptote $y = 2$
Vertical asympote $x = -1$

c



Horizontal asymptote $y = 0$
Vertical asympote $x = -1$ and $x = 2$

Exercise 1C

Identify the horizontal and vertical asympotes for these functions, if they exist.

1 $y = 3^x$ **2** $y = \frac{3}{x}$ **3** $y = \frac{4}{x+1}$

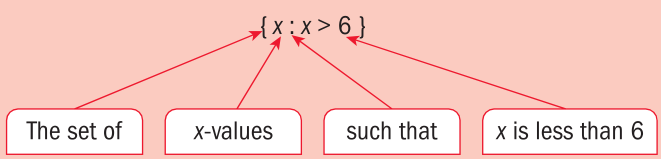
4 $y = \frac{2x}{x+2}$ **5** $y = \frac{2x+1}{x-1}$ **6** $y = \frac{6}{x^2-9}$

Set builder notation

In set builder notation we use curly brackets { } and variables to express the domain and range. We can compile sets of inequalities using inequality and other symbols.

the set of	{ }
less than	$<$
less than or equal to	$\leq$
greater than	$>$
greater than or equal to	$\geq$
is a member of the set of real numbers	$\in \mathbb{R}$

→ Set notation:



Interval notation	Description	Set builder notation
$(-2, +\infty)$	x is greater than -2	$\{x : x > -2\}$
$(-\infty, 4]$	x is less than or equal to 4	$\{x : x \leq 4\}$
$[-3, 3)$	x lies between -3 and 3 including -3 but not 3	$\{x : -3 \leq x < 3\}$
$(-\infty, 5) \cup [6, +\infty)$	x is less than 5 or greater than or equal to 6	$\{x : x < 5, x \geq 6\}$
$(-\infty, +\infty)$	x may be any real number	$x \in \mathbb{R}$

Some people use 'backwards square brackets' to show greater than or less than. For example:] 2, ∞ [is equivalent to $x > 2$, and] ∞, −4[is equivalent to $x < -4$.

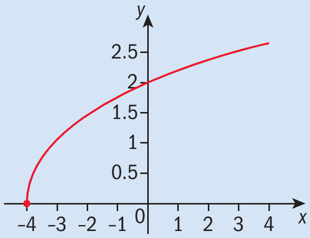
You may wish to explore the 'internationalism' of symbols in the language of mathematics.

Interval notation is often considered more efficient than set builder notation

Around the world there are many different words for the same symbol. Brackets are also called parentheses. Radicals are also called surds. How does this affect understanding? Can you find some more examples?

Example 5

Find the domain and range of this function.



Answer

The domain of the function is $\{x: x \geq -4\}$ or $[-4, +\infty)$ — *x only takes values greater than or equal to −4.*

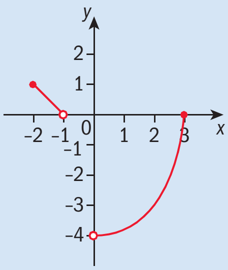
The range of the function is $\{y: y \geq 0\}$ or $[0, +\infty)$ — *The function only takes y-values greater than or equal to 0.*

You may wish to explore the influence of technology on notation and vice versa.

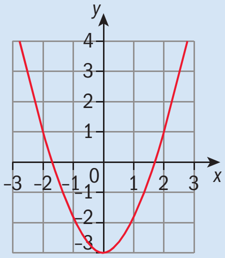
Example 6

Find the domain and range of each function.

a



b



What values are included in the domain $0 \leq x \leq 1$? How many values are there?

Answers

a The domain is $\{x: -2 \leq x < -1$ and $0 < x \leq 3\}$ or $[-2, -1) \cup (0, 3]$.
The range is $\{y: -4 < y \leq 1\}$ or $(-4, 1]$.

b The domain of the function is $x \in \mathbb{R}$ or $(-\infty, +\infty)$. — *x can take any real value.*
The range of the function is $\{y: y \geq -3\}$ or $[-3, +\infty)$.

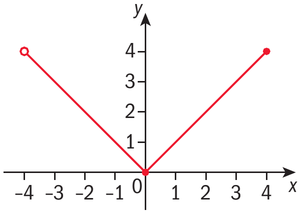
Do we all use the same notation in mathematics? We are using an empty dot to indicate that $x = -1$ is *not* included. Different countries have different notations to represent the same thing. Furthermore, different teachers from the same country use different notations!

Exercise 1D

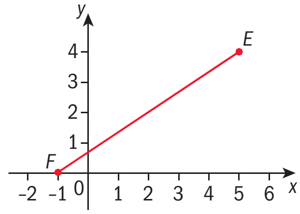
1 Look back to page 4 at the graph and the formula for the numbers of handshakes for various numbers of people. Is this a function? If so, what is the domain and range?

2 Find the domain and range for each of these relations.

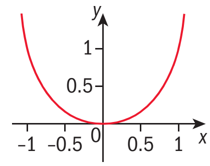
a



b



c



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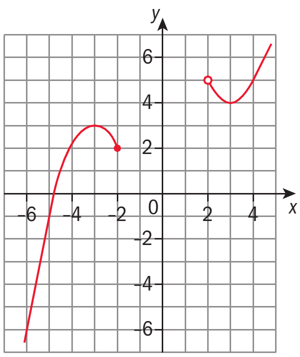

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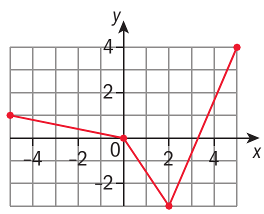

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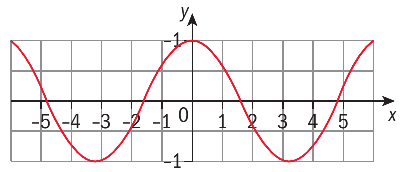

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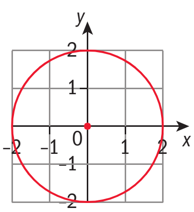

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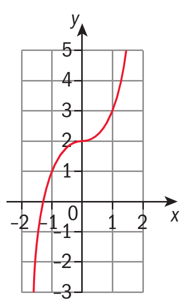
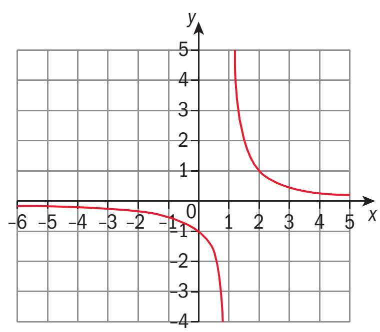


i



EXAM-STYLE QUESTION

3 Use your GDC to sketch these graphs. Write down the domain and range of each.

a $y = 2x - 3$ **b** $y = x^2$

c $y = x^2 + 5x + 6$ **d** $y = x^3 - 4$

e $y = \sqrt{x}$ **f** $y = \sqrt{4 - x}$

g $y = \frac{1}{x}$ **h** $y = e^x$

i $y = \frac{1}{x+2}$ **j** $y = \frac{x+4}{x-2}$

k $y = \frac{x^2 - 9}{x+3}$ **l** $y = \frac{2}{x^2 + 1}$

Your GDC will find the *x*- and *y*-intercepts. To do this algebraically, use the fact that a function crosses the *x*-axis when $y = 0$ and crosses the *y*-axis when $x = 0$. For example, the function $y = 2x - 4$ crosses the *x*-axis where $2x - 4 = 0$, $x = 2$. It crosses the *y*-axis where $y = 2(0) - 4 = -4$.

3k gives a most unusual answer. Look carefully for a hole where $x = -3$.

1.3 Function notation

Functions are often described by equations. For example, the equation $y = 2x + 1$ describes y as a function of x. By giving the function the symbol 'f' we write this equation in function notation as $f(x) = 2x + 1$ and so $y = f(x)$.

→ $f(x)$ is read as 'f of x' and means the value of function f at x.

$f(x)$ can also be written like this: $f: x \rightarrow 2x + 1$.
An ordered pair (x, y) can be written as $(x, f(x))$.
Finding $f(x)$ for a particular value of x means evaluating the function f at that value.

$f: (x) \rightarrow 2x + 1$ means that f is a function that maps x to $2x + 1$.

Example 7

a Evaluate the function $f(x) = 2x + 1$ at $x = 3$.
b If $f(x) = x^2 + 4x - 3$, find **i** $f(2)$ **ii** $f(0)$ **iii** $f(-3)$ **iv** $f(x + 1)$

Answers

a $f(3) = 2(3) + 1 = 7$ — *For x, substitute 3.*

b **i** $f(2) = (2)^2 + 4(2) - 3 = 4 + 8 - 3 = 9$

ii $f(0) = (0)^2 + 4(0) - 3 = 0 + 0 - 3 = -3$

iii $f(-3) = (-3)^2 + 4(-3) - 3$
$= 9 - 12 - 3 = -6$

iv $f(x + 1) = (x + 1)^2 + 4(x + 1) - 3$
$= x^2 + 2x + 1 + 4x + 4 - 3$
$= x^2 + 6x + 2$

The German mathematician and philosopher Gottfried Leibniz first used the mathematical term 'function' in 1673.

Exercise 1E

1 Find **i** $f(7)$ **ii** $f(-3)$ **iii** $f(\frac{1}{2})$ **iv** $f(0)$ **v** $f(a)$
for these functions.

a $f(x) = x - 2$ **b** $f(x) = 3x$ **c** $f(x) = \frac{1}{4}x$

d $f(x) = 2x + 5$ **e** $f(x) = x^2 + 2$

2 If $f(x) = x^2 - 4$, find
a $f(-a)$ **b** $f(a + 5)$ **c** $f(a - 1)$
d $f(a^2 - 2)$ **e** $f(5 - a)$

EXAM-STYLE QUESTION

3 If $g(x) = 4x - 5$ and $h(x) = 7 - 2x$
a find x when $g(x) = 3$
b find x when $h(x) = -15$
c find x when $g(x) = h(x)$.

4 **a** If $h(x) = \frac{1}{x - 6}$ find $h(-3)$.
b Is there a value where $h(x)$ does not exist? Explain.

Notice that we do not always use the letter f for a function. Here we have used g and h. When considering velocity in terms of time we often use $v(t)$.

5 The volume of a cube with edges of length x is given by the function $f(x) = x^3$.

a Find $f(5)$.

b Explain what $f(5)$ represents.

6 $g(x) = \frac{3x+1}{x-2}$

a Evaluate

i $g(6)$ ii $g(-2)$ iii $g(0)$ iv $g\left(-\frac{1}{3}\right)$

b Evaluate

i $g(1)$ ii $g(1.5)$ iii $g(1.9)$ iv $g(1.99)$

v $g(1.999)$ vi $g(1.9999)$

c What do you notice about your answers to **b**?

d Is there a value of x for which $g(x)$ does not exist?

e Graph the function on your GDC and look what happens when $x = 2$. Explain.

You can use mathematical functions to represent things from your own life. For example, suppose the number of pizzas your family eats depends on the number of football games you watch. If you eat 3 pizzas during every football game, the function would be 'number of pizzas' (p) = 3 times 'number of football games' (g) or $p = 3g$. Can you think of another real-life function? It could perhaps be about the amount of money you spend or the number of minutes you spend talking on the phone.

EXAM-STYLE QUESTION

7 The velocity of a particle is given by $v(t) = t^2 - 9\,\text{m}\,\text{s}^{-1}$.

a Find the initial velocity.

b Find the velocity after 4 seconds.

c Find the velocity after 10 seconds.

d At what time does the particle come to rest?

The initial velocity means the velocity at the start, when $t = 0$.

The particle comes to rest when $v = 0$.

8 Given $f(x) = \frac{f(x+h) - f(x)}{h}$ find

a $f(2 + h)$ b $f(3 + h)$

Extension material on CD:
Worksheet 1 - Polynomials

1.4 Composite functions

A **composite function** is a combination of two functions. You apply one function to the result of another.

→ The composition of the function f with the function g is written as $f(g(x))$, which is read as 'f of g of x', or $(f \circ g)(x)$, which is read as 'f composed with g of x'.

When you evaluate a function $f(x)$, you substitute a number or another variable for x.
For example, if $f(x) = 2x + 3$ then $f(5) = 2(5) + 3 = 13$
You can find $f(x^2 + 1)$ by substituting $x^2 + 1$ for x to get
$f(x^2 + 1) = 2(x^2 + 1) + 3 = 2x^2 + 5$

→ A **composite function** applies one function to the result of another and is defined by $(f \circ g)(x) = f(g(x))$.

Example 8

If $f(x) = 5 - 3x$ and $g(x) = x^2 + 4$, find $(f \circ g)(x)$.

Answer

$(f \circ g)(x) = 5 - 3(x^2 + 4)$ — *Substitute $x^2 + 4$ into $f(x)$.*

$= 5 - 3x^2 - 12$

$= -3x^2 - 7$

$g(x)$ goes in here

You may need to evaluate a composite function for a particular value of x.

Example 9

$f(x) = 5 - 3x$ and $g(x) = x^2 + 4$. Find $(f \circ g)(3)$.

Answer

Method 1

$(f \circ g)(x) = 5 - 3(x^2 + 4)$ — *Work out the composite function.*

$= -3x^2 - 7$

$(f \circ g)(3) = -3(3)^2 - 7$ — *Then substitute 3 for x.*

$= -27 - 7$

$= -34$

Method 2

$g(3) = (3)^2 + 4 = 13$ — *Substitute 3 into $g(x)$.*

$f(13) = 5 - 3(13) = -34$ — *Substitute that value into $f(x)$.*

Both methods give the same result – you can use the one you prefer.

Example 10

Given $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, find

a $(f \circ g)(x)$ **b** $(f \circ g)(4)$

Answers

a $(f \circ g)(x) = 2(x^2 - 2) + 1$ — *Substitute $x^2 - 2$ into $f(x)$.*

$= 2x^2 - 3$

b $(f \circ g)(4) = 2(4)^2 - 3 = 29$ — *Substitute 4 for x.*

Or use Method 2:
$g(4) = (4)^2 - 2 = 14$
and then
$f(14) = 2(14) + 1 = 29$

Exercise 1F

1 Given $f(x) = 3x$, $g(x) = x + 1$ and $h(x) = x^2 + 2$, find

a $(f \circ g)(3)$	**b** $(f \circ g)(0)$	**c** $(f \circ g)(-6)$	**d** $(f \circ g)(x)$
e $(g \circ f)(4)$	**f** $(g \circ f)(5)$	**g** $(g \circ f)(-6)$	**h** $(g \circ f)(x)$
i $(f \circ h)(2)$	**j** $(h \circ f)(2)$	**k** $(f \circ h)(x)$	**l** $(h \circ f)(x)$
m $(g \circ h)(3)$	**n** $(h \circ g)(3)$	**o** $(g \circ h)(x)$	**p** $(h \circ g)(x)$

$(f \circ h)(2) \neq (h \circ f)(2)$

2 Given $f(x) = x^2 - 1$ and $g(x) = 3 - x$, find

a $(g \circ f)(1)$ **b** $(g \circ f)(2)$ **c** $(g \circ f)(4)$ **d** $(f \circ g)(3)$

e $(g \circ f)(3)$ **f** $(f \circ g)(-4)$ **g** $(f \circ g)(x + 1)$ **h** $(f \circ g)(x + 2)$

EXAM-STYLE QUESTIONS

3 Given the functions $f(x) = x^2$ and $g(x) = x + 2$ find

a $(f \circ g)(x)$ **b** $(f \circ g)(3)$

4 Given the functions $f(x) = 5x$ and $g(x) = x^2 + 1$ find

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

5 $g(x) = x^2 + 3$ and $h(x) = x - 4$

a Find $(g \circ h)(x)$.

b Find $(h \circ g)(x)$.

c Hence solve the equation $(g \circ h)(x) = (h \circ g)(x)$.

'Hence' means 'Use the preceding work to obtain the required result'.

6 If $r(x) = x - 4$ and $s(x) = x^2$, find $(r \circ s)(x)$ and state its domain and range.

1.5 Inverse functions

→ The **inverse** of a function $f(x)$ is $f^{-1}(x)$. It reverses the action of that function.

If $f(x) = 3x - 4$ and $g(x) = \frac{x + 4}{3}$, then

$f(10) = 3(10) - 4 = 26$ and $g(26) = \frac{26 + 4}{3} = 10$, so we are back to where we started.

$(f \circ g)\ (10) = 10$

So $g(x)$ is the inverse of $f(x)$.

Not all functions have an inverse.

If g is the inverse function of f, then g will reverse the action of f for all values in the domain of f and f will also be the inverse of g.

When f and g are inverse functions, we write $g(x) = f^{-1}(x)$.

Note that f^{-1} means the inverse of f; the '–1' is not an exponent (power).

→ Functions $f(x)$ and $g(x)$ are inverses of one another if:

$(f \circ g)(x) = x$ for all of the x-values in the domain of g

$(g \circ f)(x) = x$ for all of the x-values in the domain of f.

The horizontal line test

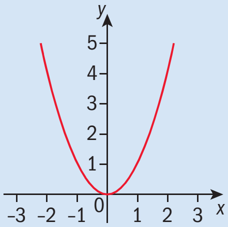
→ You can use the **horizontal line test** to identity inverse functions.

If a horizontal line crosses the graph of a function more than once, there is no inverse function.

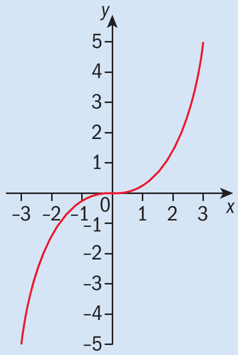
Example 11

Which of these functions have inverse functions?

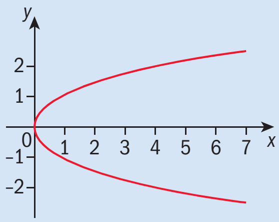
a



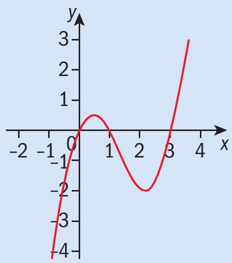
b



c

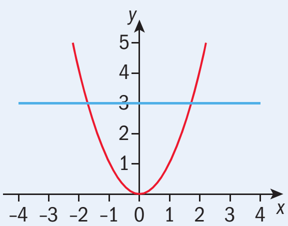


d



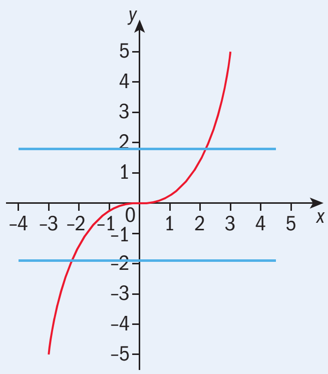
Answers

a



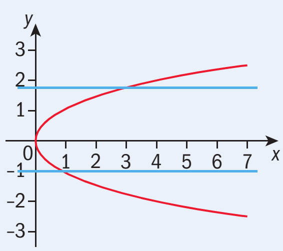
No inverse function

b



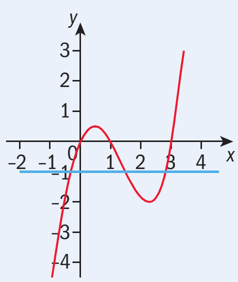
Inverse function

c



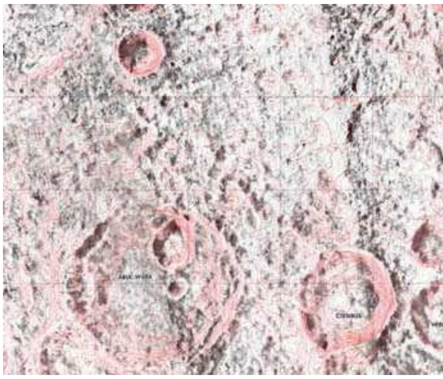
Inverse function

d



No inverse function

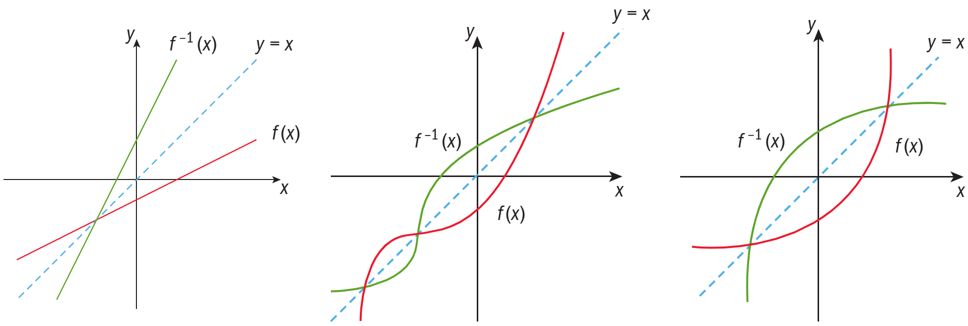
Did you know that Abul Wafa Buzjani, a Persian mathematician from the 10th century, used functions? There is a crater on the moon named after him.



The graphs of inverse functions

→ The graph of the inverse of a function is a reflection of that function in the line $y = x$.

Here are some examples of functions and their inverse functions.

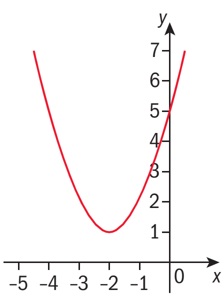


If (x, y) lies on the line $f(x)$, then (y, x) lies on $f^{-1}(x)$. Reflecting the function in the line $y = x$ 'swaps' x and y, so the point $(1, 3)$ reflected in the line $y = x$ becomes point $(3, 1)$.

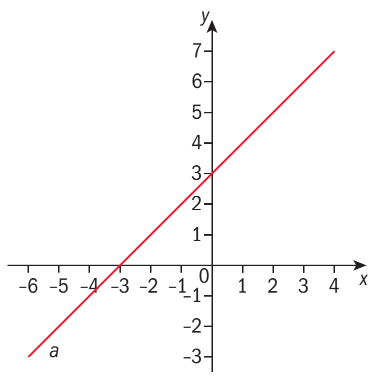
Exercise 1G

1 Use the horizontal line test to determine which of these functions have inverse functions.

a

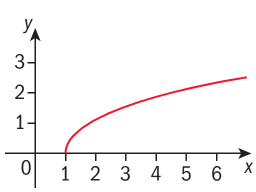


b

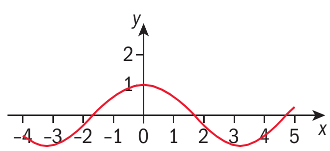


Pioneering work by Indian scientist Panini in the 6th century BCE included functions.

c

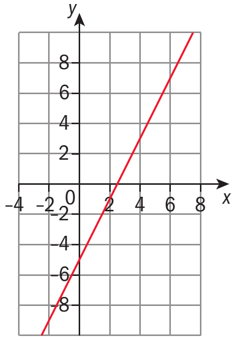


d

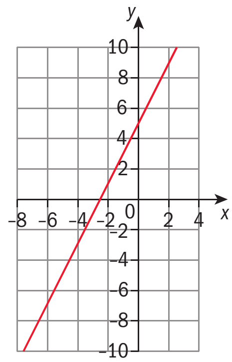


2 Copy the graphs of these functions. For each, draw the line $y = x$ and the graph of the inverse function.

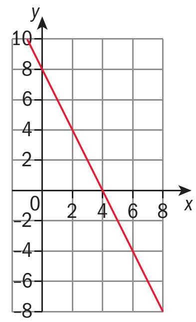
a



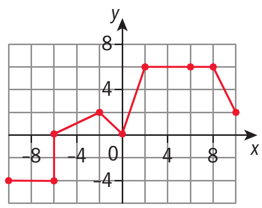
b



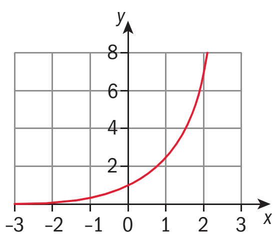
c



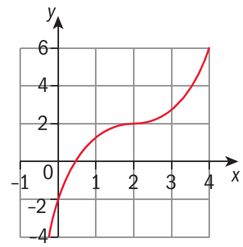
d



e



f



Finding inverse functions algebraically

Look at how the function $f(x) = 3x - 2$ is made up. We start with x on the left.

x → [× 3] → [–2] → $3x - 2$

To form the inverse function we reverse the process, using inverse operations.

$\frac{x+2}{3}$ ← [÷ 3] ← [+2] ← x

The inverse of +2 is –2
The inverse of ×3 is ÷3

So $f^{-1}(x) = \frac{x+2}{3}$

The next example shows you how to do this without diagrams.

Example 12

If $f(x) = 3x - 2$, find the inverse function $f^{-1}(x)$.

Answer

$y = 3x - 2$	*Replace f(x) with y.*
$x = 3y - 2$	*Replace every x with y and every y with x.*
$x + 2 = 3y$	*Make y the subject.*
$y = \frac{x+2}{3}$	
$f^{-1}(x) = \frac{x+2}{3}$	*Replace y with $f^{-1}(x)$.*

As you saw in the graphs of functions and their inverses, the inverse function of a given function f is the reflection of the graph $y = f(x)$ in the line $y = x$, which 'swaps' x and y. So in Example 12 we swapped x and y, and then made y the subject.

→ To find the inverse function algebraically, replace $f(x)$ with y and solve for y.

Example 13

If $f(x) = 4 - 3x$, find $f^{-1}(x)$.

Answer

$y = 4 - 3x$	*Replace f(x) with y.*
$x = 4 - 3y$	*Replace every x with y and every y with x.*
$x - 4 = -3y$	
$\frac{x-4}{-3} = y$	*Make y the subject.*
$y = \frac{4-x}{3}$	
$f^{-1}(x) = \frac{4-x}{3}$	*Replace y with $f^{-1}(x)$.*

To check that the inverse function in Example 13 is correct, combine the functions

$$(f \circ f^{-1})(x) = 4 - 3\left(\frac{4-x}{3}\right) = 4 - (4 - x) = x$$

So $(f \circ f^{-1})(x) = x$ and f and f^{-1} are inverses of each other.

→ The function $I(x) = x$ is called the identity function.
It leaves x unchanged.
So $f \circ f^{-1} = I$

Exercise 1H

EXAM-STYLE QUESTION

1 If $f(x) = \frac{x+4}{2}$ and $g(x) = 2x - 4$, find

a **i** $g(1)$ and $(f \circ g)(1)$ **ii** $f(-3)$ and $(g \circ f)(-3)$

iii $(f \circ g)(x)$ **iv** $(g \circ f)(x)$

b What does this tell you about functions f and g?

2 Find the inverse for each of these functions.

a $f(x) = 3x - 1$ **b** $g(x) = x^3 - 2$ **c** $h(x) = \frac{1}{4}x + 5$

d $f(x) = \sqrt[3]{x} - 3$ **e** $g(x) = \frac{1}{x} - 2$ **f** $h(x) = 2x^3 + 3$

g $f(x) = \frac{x}{3+x}, x \neq -3$ **h** $g(x) = \frac{2x}{5-x}, x \neq 5$

3 What is $f^{-1}(x)$ if

a $f(x) = 1 - x$ **b** $f(x) = x$ **c** $f(x) = \frac{1}{x}, x \neq 0$

Self-inverse functions are such that a function and its inverse are the same. Look for self-inverse functions in question 3.

4 Evaluate $f^{-1}(5)$ where

a $f(x) = 6 - x$ **b** $f(x) = \frac{10}{x+7}$ **c** $f(x) = \frac{2}{4x-3}$

5 If $f(x) = \frac{x+1}{x-2}$, find $f^{-1}(x)$.

Note that the image of point $(a, -b)$ after a reflection in the line $y = x$ is the point $(b, -a)$.

EXAM-STYLE QUESTION

6 **a** Draw the graph of $f(x) = 2^x$ by making a table of values and plotting several points.

b Draw the line $y = x$ on the same graph.

c Draw the graph of f^{-1} by reflecting the graph of f in the line $y = x$.

d State the domain and range of f and f^{-1}.

7 The function $f(x) = x^2$ has no inverse function. However, the square root function $g(x) = \sqrt{x}$ does have an inverse function. Find this inverse.
By comparing the range and domain explain why the inverse of $g(x) = \sqrt{x}$ is not the same as $f(x) = x^2$.

8 Prove that the graphs of a linear function and its inverse can never be perpendicular.

Extension material on CD:
Worksheet 1 - Polynomials

1.6 Transforming functions

Investigation – functions

You should use your GDC to sketch all the graphs in this investigation.

1 Sketch $y = x$, $y = x + 1$, $y = x - 4$, $y = x + 4$ on the same axes.
Compare and contrast your functions.
What effect do the constant (number) terms have on the graphs of $y = x + b$?

You will also find this standard equation of a line written as $y = mx + b$ or $y = mx + c$

2 Sketch $y = x + 3$, $y = 2x + 3$, $y = 3x + 3$, $y = -2x + 3$, $y = 0.5x + 3$ on the same axes.
Compare and contrast your functions.
What effect does changing the x-coefficient have?

The coefficient of x is the number that multiplies the x-value.

3 Sketch $y = |x|$, $y = |x + 2|$, $y = |x - 3|$ on the same axes.
Compare and contrast your functions.
What effect does changing the values of h have on the graphs of $y = |x + h|$?

$|x|$ means the modulus of x. See chapter 18 for more explanation.

4 Sketch $y = x^2$, $y = -x^2$, $y = 2x^2$, $y = 0.5x^2$ on the same axes.
Compare and contrast your functions.
What effect does the negative sign have on the graph?
What effect does changing the value of a have on the graphs of $y = ax^2$?

In the investigation you should have found that your graphs in parts 1, 2 and 3 were all the same shape but the position of the graphs changed. The graphs in part 4 should have been reflected or changed by stretching.

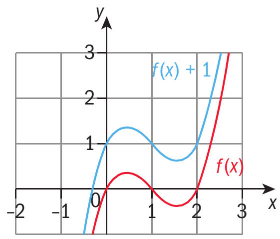
These are examples of 'transformations' of graphs. We will now look at these transformations in detail.

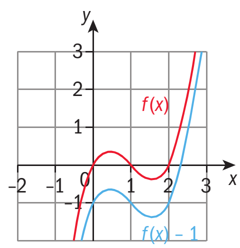
Translations

Shift upward or downward

→ $f(x) + k$ translates $f(x)$ vertically a distance of k units upward.

→ $f(x) - k$ translates $f(x)$ vertically a distance of k units downward.

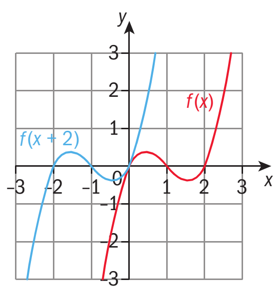


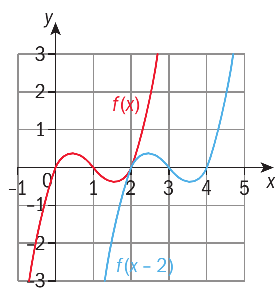


Shift to the right or left

→ $f(x + k)$ translates $f(x)$ horizontally k units to the **left**, when $k > 0$.

→ $f(x - k)$ translates $f(x)$ horizontally k units to the **right**, when $k > 0$.





Translations can be represented by vectors in the form $\begin{pmatrix} a \\ b \end{pmatrix}$ where a is the horizontal component and b is the vertical component.

$\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ is a horizontal shift of 3 units right. $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ is a vertical shift of 2 units down.

Translation by the vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ denotes a horizontal shift of 3 units to the right, and a vertical shift of 2 units down.

Try transforming some functions with different values of k on your GDC.

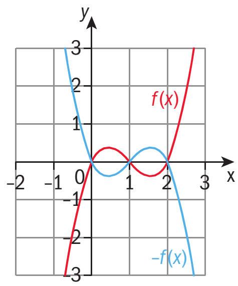
Reflections

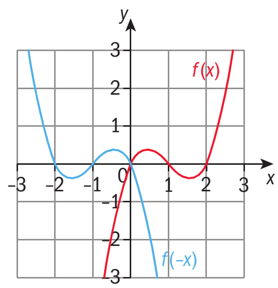
Reflection in the x-axis

→ $-f(x)$ reflects $f(x)$ in the x-axis.

Reflection in the y-axis

→ $f(-x)$ reflects $f(x)$ in the y-axis.





Stretches

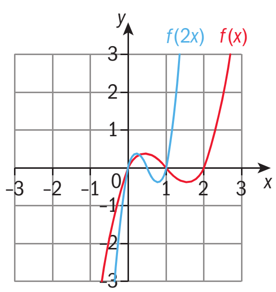
Horizontal stretch (or compress)

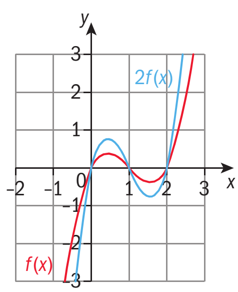
→ $f(qx)$ stretches or compresses $f(x)$ horizontally with scale factor $\frac{1}{q}$.

Vertical stretch (or compress)

→ $pf(x)$ stretches $f(x)$ vertically with scale factor p.

A stretch with a scale factor p where $0 < p < 1$ will actually compress the graph.





The transformation is a **horizontal stretch of scale factor** $\frac{1}{q}$.

When $q > 1$ the graph is compressed towards the y-axis

When $0 < q < 1$ the graph is stretched away from the y-axis.

The transformation is a **vertical stretch of scale factor** p.

When $p > 1$ the graph stretches away from the x-axis.

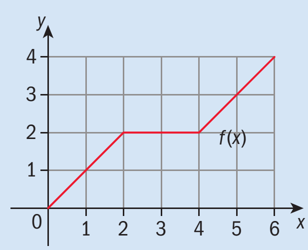
When $0 < p < 1$ the graph is compressed towards the x-axis.

Students often make mistakes with stretches. It is important to remember the different effects of, for example, $2f(x)$ and $f(2x)$.

Example 14

1 Given the graph of the function $f(x)$ shown here, sketch the graphs of:

a $f(x+1)$ **b** $f(x) - 2$ **c** $f(-x)$ **d** $-f(x)$ **e** $2f(x)$



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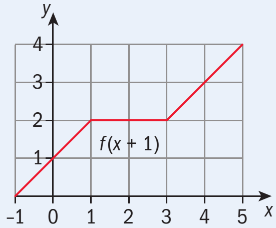
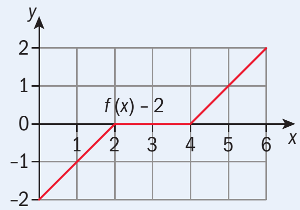
Answers

a



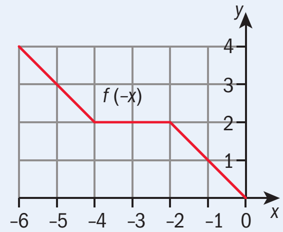
Translated one unit to the left

b



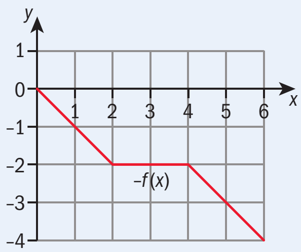
Translated two units down

c



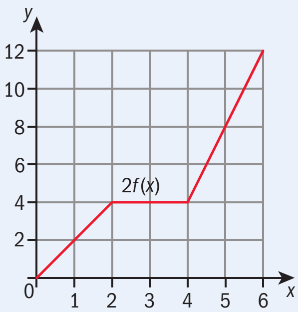
Reflected in the *y*-axis

d



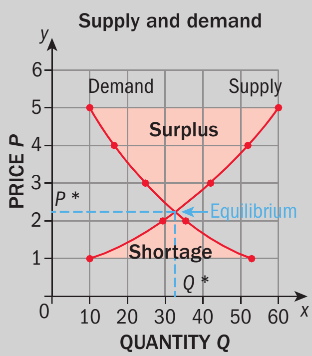
Reflected in the *x*-axis

e

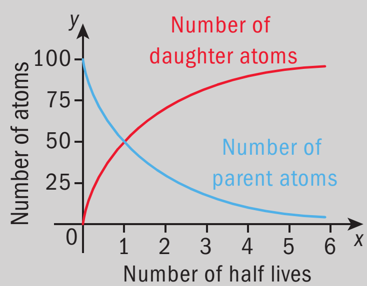


Vertical stretch of scale factor 2

Supply and demand curves in business and economics are reflections.



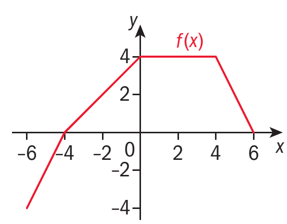
Radioactive decay curves are reflections.



Exercise 1I

EXAM-STYLE QUESTION

1 Copy the graph. Draw these functions on the same axes.

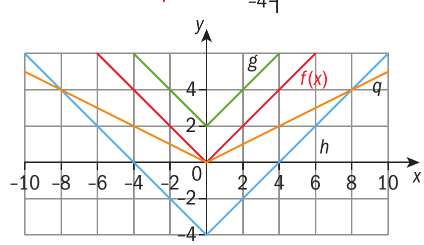


a $f(x) + 4$ **b** $f(x) - 2$ **c** $-f(x)$

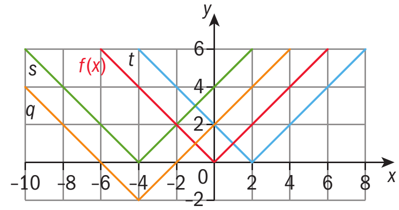
d $f(x + 3)$ **e** $f(x - 4)$ **f** $2f(x)$

g $f(2x)$

2 Functions *g*, *h* and *q* are transformations of $f(x)$. Write each transformation in terms of $f(x)$.



3 Functions q, s and t are transformations of $f(x)$. Write each transformation in terms of $f(x)$.

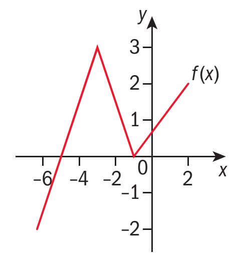


EXAM-STYLE QUESTION

4 Copy the graph of $f(x)$. Sketch the graph of each of these functions, and state the domain and range for each.

a $2f(x-5)$

b $-f(2x)+3$

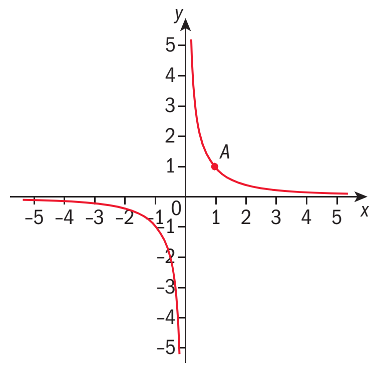


5 The graph of $f(x)$ is shown. A is the point $(1, 1)$. Make separate copies of the graph and draw the function after each transformation. On each graph, label the new position of A as A_1.

a $f(x+1)$ **b** $f(x)+1$

c $f(-x)$ **d** $2f(x)$

e $f(x-2)+3$



6 In each case, describe the transformation that would change the graph of $f(x)$ into the graph of $g(x)$.

a $f(x) = x^3$, $g(x) = -(x^3)$

b $f(x) = x^2$, $g(x) = (x-3)^2$

c $f(x) = x$, $g(x) = -2x + 5$

EXAM-STYLE QUESTION

7 Let $f(x) = 2x + 1$.

a Draw the graph of $f(x)$ for $0 \le x \le 2$.

b Let $g(x) = f(x+3) - 2$. On the same graph draw $g(x)$ for $-3 \le x \le -1$.

> If a domain is given in the question, you must only draw the function for that domain.

Review exercise

1 **a** If $g(a) = 4a - 5$, find $g(a-2)$.

b If $h(x) = \frac{1+x}{1-x}$, find $h(1-x)$.

2 **a** Evaluate $f(x-3)$ when $f(x) = 2x^2 - 3x + 1$.

b For $f(x) = 2x + 7$ and $g(x) = 1 - x^2$, find the composite function defined by $(f \circ g)(x)$.

3 Find the inverses of these functions.

a $f(x) = \frac{3x+17}{2}$

b $g(x) = 2x^3 + 3$

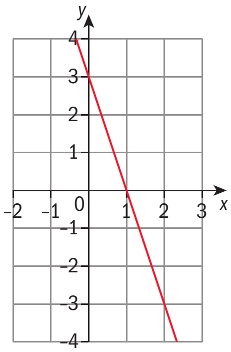
4 Find the inverse of $f(x) = -\frac{1}{5}x - 1$. Then graph the function and its inverse.

5 Find the inverse functions for

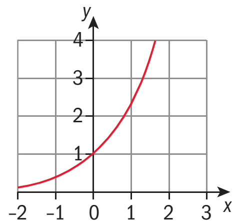
a $f(x) = 3x + 5$ **b** $f(x) = \sqrt[3]{x+2}$

6 Copy each graph and draw the inverse of each function.

a

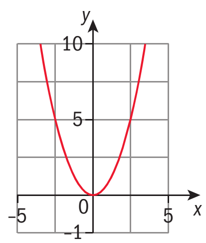


b

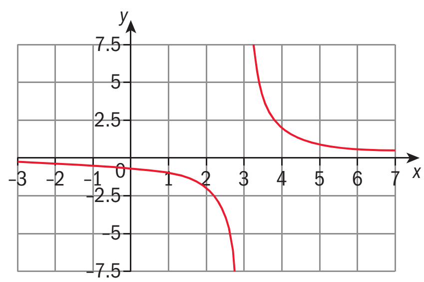


7 Find the domain and range for each of these graphs.

a



b



EXAM-STYLE QUESTION

8 For each function, write a single equation to represent the given combination of transformations.

a $f(x) = x$, reflected in the y-axis, stretched vertically by a factor of 2, horizontally by a factor of $\frac{1}{3}$ and translated 3 units left and 2 units up.

b $f(x) = x^2$, reflected in the x-axis, stretched vertically by a factor of $\frac{1}{4}$, horizontally by a factor of 3, translated 5 units right and 1 unit down.

9 **a** Explain how to draw the inverse of a function from its graph.

b Graph the inverse of $f(x) = 2x + 3$.

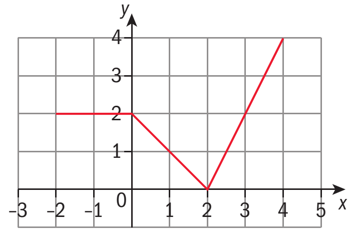
EXAM-STYLE QUESTION

10 Let $f(x) = 2x^3 + 3$ and $g(x) = 3x - 2$.

a Find $g(0)$. **b** Find $(f \circ g)(0)$. **c** Find $f^{-1}(x)$.

EXAM-STYLE QUESTIONS

11 The graph shows the function $f(x)$, for $-2 \le x \le 4$.



a Let $h(x) = f(-x)$. Sketch the graph of $h(x)$.

b Let $g(x) = \frac{1}{2} f(x - 1)$. The point $A(3, 2)$ on the graph of f is transformed to the point P on the graph of g. Find the coordinates of P.

12 The functions f and g are defined as $f(x) = 3x$ and $g(x) = x + 2$.

a Find an expression for $(f \circ g)(x)$.

b Show that $f^{-1}(12) + g^{-1}(12) = 14$.

13 Let $g(x) = 2x - 1$, $h(x) = \frac{3x}{x-2}, x \ne 2$

a Find an expression for $(h \circ g)(x)$. Simplify your answer.

b Solve the equation $(h \circ g)(x) = 0$.

The instruction 'Show that…' means 'Obtain the required result (possibly using information given) without the formality of proof'.
For 'Show that' questions you do not usually need to use a calculator.
A good method is to cover up the right-hand side of the equation and then work out the left-hand side until your answer is the same as the right-hand side.

Review exercise

1 Use your GDC to sketch the function and state the domain and range of $f(x) = \sqrt{x+2}$.

2 Sketch the function $y = (x + 1)(x - 3)$ and state its domain and range.

3 Sketch the function $y = \frac{1}{x+2}$ and state its domain and range.

EXAM-STYLE QUESTIONS

4 The function $f(x)$ is defined as $f(x) = 2 + \frac{1}{x+1}, x \ne -1$.

a Sketch the curve $f(x)$ for $-3 \le x \le 2$.

b Use your GDC to help you write down the value of the x-intercept and the y-intercept.

5 **a** Sketch the graph of $f(x) = \frac{1}{x^2}$

b For what value of x is $f(x)$ undefined?

c State the domain and range of $f(x)$.

6 Given the function $f(x) = \frac{2x-5}{x+2}$

a write down the equations of the asymptotes

b sketch the function

c write down the coordinates of the intercepts with both axes.

7 Let $f(x) = 2 - x^2$ and $g(x) = x^2 - 2$.

a Sketch both functions on one graph with $-3 \le x \le 3$.

b Solve $f(x) = g(x)$.

EXAM-STYLE QUESTIONS

8 Let $f(x) = x^3 - 3$.

a Find the inverse function $f^{-1}(x)$.

b Sketch both $f(x)$ and $f^{-1}(x)$ on the same axes.

c Solve $f(x) = f^{-1}(x)$.

9 $f(x) = e^{2x-1} + \frac{2}{x+1}, x \neq 1.$

Sketch the curve of $f(x)$ for $-5 \leq x \leq 2$, including any asymptotes.

10 Consider the functions f and g where $f(x) = 3x - 2$ and $g(x) = x - 3$.

a Find the inverse function, f^{-1}.

b Given that $g^{-1}(x) = x + 3$, find $(g^{-1} \circ f)(x)$.

c Show that $(f^{-1} \circ g)(x) = \frac{x-1}{3}$.

d Solve $(f^{-1} \circ g)(x) = (g^{-1} \circ f)(x)$

Let $h(x) = \frac{f(x)}{g(x)}, x \neq 2.$

d **Sketch** the graph of h for $-6 \leq x \leq 10$ and $-4 \leq y \leq 10$, including any asymptotes.

e Write down the **equations** of the asymptotes.

When IB exams have words in **bold** script, it means that you must do exactly what is required. For example the equation could be given as $x = 3$ but not just as 3.

CHAPTER 1 SUMMARY

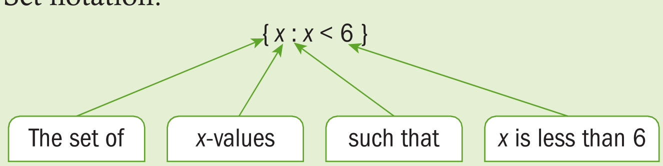
Introducing functions

- A **relation** is a set of ordered pairs.
- The **domain** is the set of all the first numbers (x-values) of the ordered pairs.
- The **range** is the set of the second numbers (y-values) in each pair.
- A **function** is a relation where every x-value is related to a unique y-value.
- A relation is a function if any vertical line drawn will not intersect the graph more than once. This is called the **vertical line test**.

The domain and range of a relation on a Cartesian plane

Interval notation:
Use round brackets (,) if the value is not included in the graph or when the graph is undefined at that point (a hole or **asymptote**, or a jump).
Use square brackets [,] if the value is included in the graph.

- Set notation:



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Function notation

- $f(x)$ is read as 'f of x' and means 'the value of function f at x'.

Composite functions

- The composition of the function f with the function g is written as $f(g(x))$, which is read as 'f of g of x', or $(f \circ g)(x)$, which is read as 'f composed with g of x'.
- A **composite function** applies one function to the result of another and is defined by $(f \circ g)(x) = f(g(x))$.

Inverse functions

- The **inverse** of a function $f(x)$ is $f^{-1}(x)$. It reverses the action of the function.
- Functions $f(x)$ and $g(x)$ are inverses of one another if:
  $(f \circ g)(x) = x$ for all of the x-values in the domain of g and
  $(g \circ f)(x) = x$ for all of the x-values in the domain of f.
- You can use the **horizontal line test** to identify inverse functions. If a horizontal line crosses a function more than once, there is no inverse function.

The graphs of inverse functions

- The graph of the inverse of a function is a reflection of that function in the line $y = x$.
- To find the inverse function algebraically, replace $f(x)$ with y and solve for y.
- The function $I(x) = x$ is called the identity function. It leaves x unchanged. So $f \circ f^{-1} = I$.

Transformations of functions

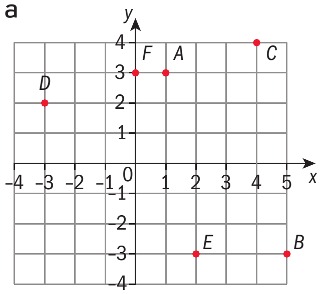
- $f(x) + k$ translates $f(x)$ vertically a distance of k units upward.
- $f(x) - k$ translates $f(x)$ vertically a distance of k units downward.
- $f(x + k)$ translates $f(x)$ horizontally k units to the left, where $k > 0$.
- $f(x - k)$ translates $f(x)$ horizontally k units to the right, where $k > 0$.
- $-f(x)$ reflects $f(x)$ in the x-axis.
- $f(-x)$ reflects $f(x)$ in the y-axis.
- $f(qx)$ stretches $f(x)$ horizontally with scale factor $\frac{1}{q}$.
- $pf(x)$ stretches $f(x)$ vertically with scale factor p.

Answers

Chapter 1

Skills check

1 a



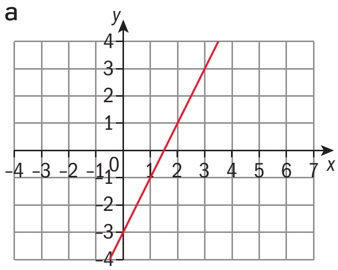
b $A(0, 2)$, $B(1, 0)$, $C(-1, 0)$, $D(0, 0)$, $E(2, 1)$, $F(-2, -2)$, $G(3, -1)$, $H(-1, 1)$

2 a 34 b 82

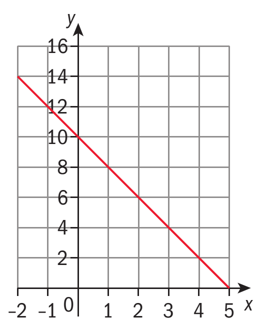
c 16 d $-\frac{13}{60}$

3 a 4 b -2 c 10

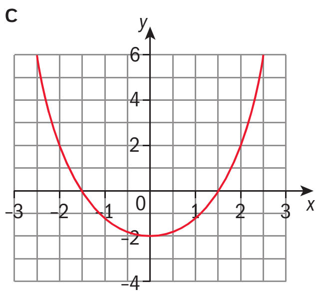
4 a



b



c



5 a $x^2 + 9x = 20$

b $x^2 - 4x + 3$

c $x^2 + x - 20$

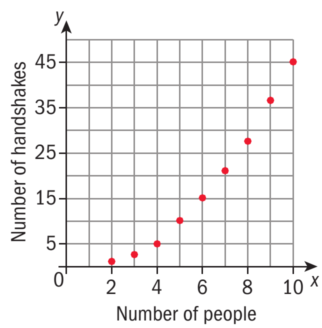
Investigation - handshakes

a 6

b

Number of people	Number of handshakes
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45

c



d $H = \frac{1}{2}n(n-1)$

Exercise 1A

1 Functions: **a b f**

2 a Function: domain $\{0, 1, 2, 3, 4\}$ range $\{0, 1, 2\}$

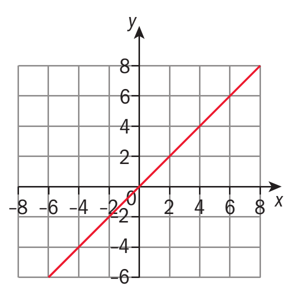
b Relation: domain $\{-1, 0, 1, 2, 3\}$ range $\{-1, 0, 1, 2\}$

3 Yes, function.

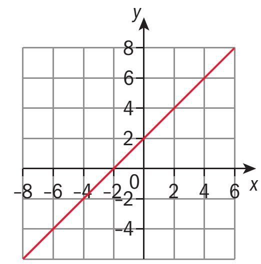
Exercise 1B

1 a b d f h i

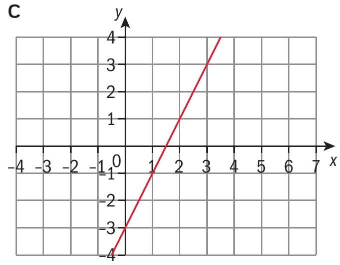
2 a



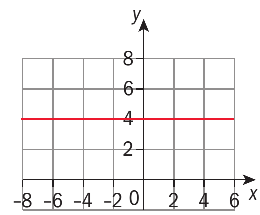
b



c



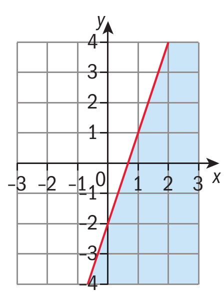
d



e Yes, vertical line will only cross once.

f No, vertical lines such as $x = 3$ are not functions.

3 Not a function as a vertical line crosses the region in many places



4 $y^2 = 4 - x^2$, $y = \pm\sqrt{4 - x^2}$

When $x = 1$, $y = \pm\sqrt{3}$. two possible values so not a function.

Exercise 1C

1 Horizontal asymptote: $y = 0$

2 Horizontal asymptote: $y = 0$, Vertical asymptote: $x = 0$

3 Horizontal asymptote: $y - 0$, Vertical asymptote: $x = -1$

4 Horizontal asymptote: $y = 2$, Vertical asymptote: $x = -2$

5 Horizontal asymptote: $y = 2$, Vertical asymptote: $x = 1$

6 Horizontal asymptote: $y = 0$, Vertical asymptote: $x = -3$

Exercise 1D

1 Function, domain $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$, range $\{1, 3, 6, 10, 15, 21, 28, 36, 45\}$.

2 a domain $\{x{:}-4 < x \le 4\}$, range $\{y{:}\ 0 \le y \le 4\}$

b domain $\{x{:}-1 \le x \le 5\}$, range $\{y{:}\ 0 \le y \le 4\}$

c domain $\{x{:}\ -\infty < x < \infty\}$, range $\{y{:}\ 0 \le y < \infty\}$

d domain $\{x{:}-\infty < x \le -2 < x < \infty\}$, range $\{y{:}\ -\infty < y \le 3\ 4 \le y < 8\}$

e domain $\{x{:}-5 \le x \le 5\}$, range $\{y{:}\ -3 \le y \le 4\}$

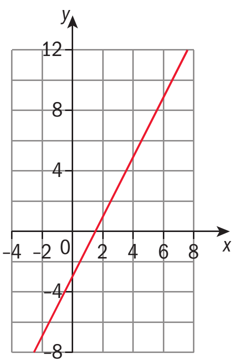
f domain $\{x{:}-\infty < x \le \infty\}$, range $\{y{:}\ -1 \le y \le 1\}$

g domain $\{x{:}-2 \le x \le 2\}$, range $\{y{:}\ -2 \le y \le 2\}$

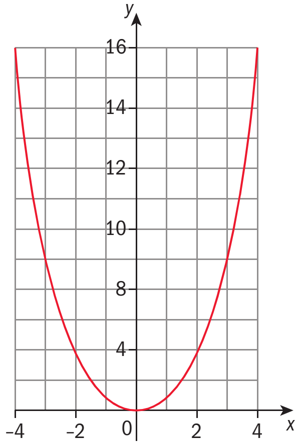
h domain $\{x{:}\ -\infty < x \le \infty\}$, range $\{y{:}\ -\infty < x \le \infty\}$

i domain $x \in \mathbb{R}$, $x \ne 1$, range $y \in \mathbb{R}$, $y \ne 0$

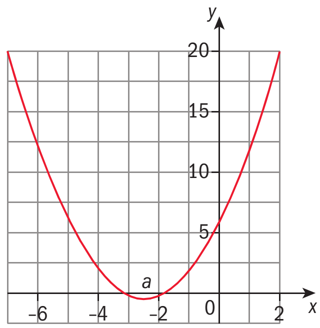
3 a $x \in \mathbb{R}$, $y \in \mathbb{R}$



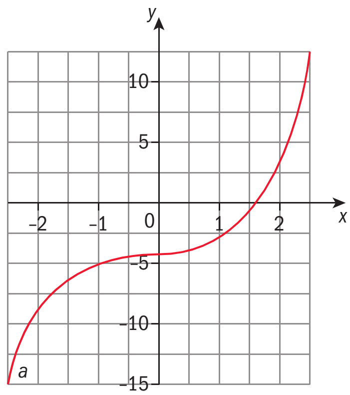
b $x \in \mathbb{R}$, $y \ge 0$



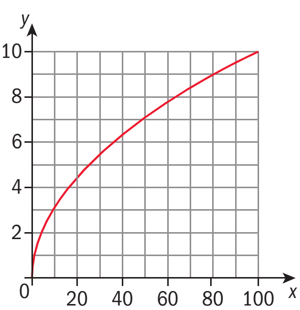
c $x \in \mathbb{R}$, $y \ge -0.25$



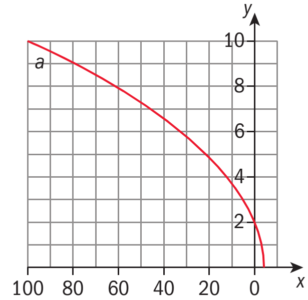
d $x \in \mathbb{R}$, $y \in \mathbb{R}$



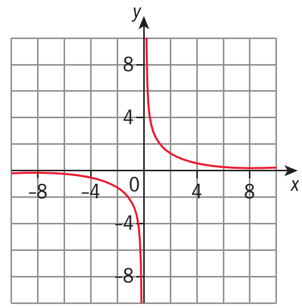
e $x \ge 0$, $y \ge 0$



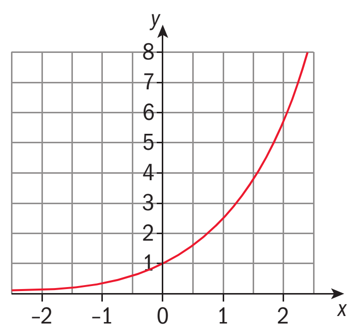
f $x \le 4$, $y \ge 0$



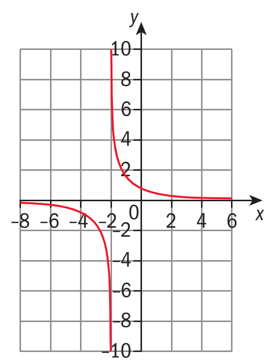
g $x \in \mathbb{R}\ x \ne 0$, $y \in \mathbb{R}\ y \ne 0$,



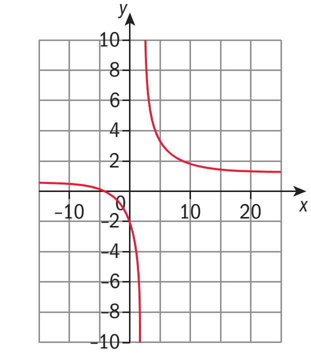
h $x \in \mathbb{R}$, $y > 0$



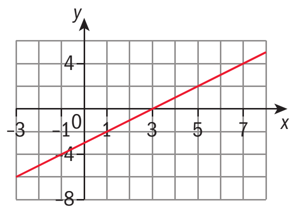
i $x \in \mathbb{R}\ x \ne -2$, $y \in \mathbb{R}\ y \ne 0$,



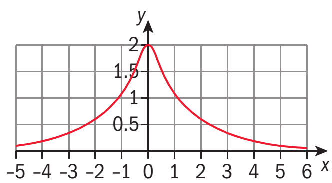
j $x \in \mathbb{R}\ x \neq 2, y \in \mathbb{R}\ y \neq 1$



k $x \in \mathbb{R}\ x \neq -3, y \in \mathbb{R}\ y \neq -6$



l $x \in \mathbb{R}, 0 < y \leq 2$



Exercise 1E

1 a i 5 ii -5 iii $-1\frac{1}{2}$
iv -2, v $a-2$
b i 21 ii -9 iii $1\frac{1}{2}$
iv 0 v $3a$
c i $\frac{7}{4}$ ii $\frac{-3}{4}$ iii $\frac{1}{8}$
iv 0 v $\frac{1}{4}a$
d i 19 ii -1 iii 6
iv 5 v $2a+5$
e i 51 ii 11 iii $2\frac{1}{4}$
iv 2 v a^2+2

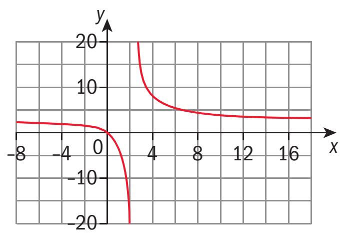
2 a a^2-4 b $a^2+10a+21$
c a^2-2a-3 d a^4-4a^2
e $21-10a+a^2$

3 a 2 b 11 c 2

4 a $-\frac{1}{9}$
b $x = 6$, denominator $= 0$ and $h(x)$ undefined.

5 a 125
b The volume of a cube of side 5.

6 a i $-\frac{1}{9}$ ii $\frac{5}{4}$
iii $-\frac{1}{2}$ iv 0.
b i -4 ii -11 iii -67
iv -697 v -6997
vi -69997
c The value of $g(x)$ is getting increasingly smaller as x approaches 2.
d 2
e asymptote at $x = 2$.



7 a $-9\ \text{m s}^{-1}$ b $7\ \text{m s}^{-1}$
c $91\ \text{m s}^{-1}$ d 3 s

8 a $\dfrac{f(2+2h) - f(2+h)}{h}$
b $\dfrac{f(3+2h) - f(3+h)}{h}$

Exercise 1F

1 a 12 b 3 c -15
d $3x+3$ e 13 f 16
g -17 h $3x+1$ i 18
j 38 k $3x^2+6$
l $9x^2+2$ m 12 n 18
o x^2+3 p x^2+2x+3

2 a 3 b 0 c -12
d -1 e -5 f 48
g $3-4x+x^2$ h $-2x+x^2$

3 a x^2+4x+4 b 25

4 a $5x^2+5$ b $25x^2+1$

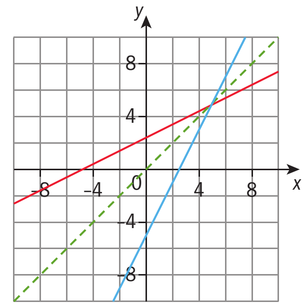
5 a $x^2-8x+19$ b x^2-1
c 2.5

6 $(r \circ s)(x) = x^2 - 4, x \in \mathbb{R}, y \geq -4$

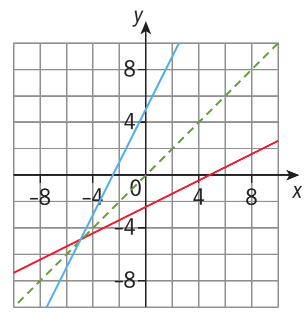
Exercise 1G

1 b, c

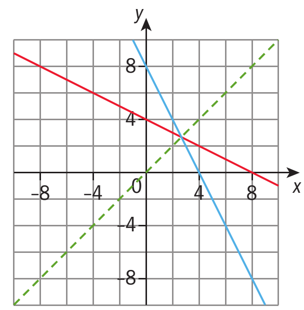
2 a



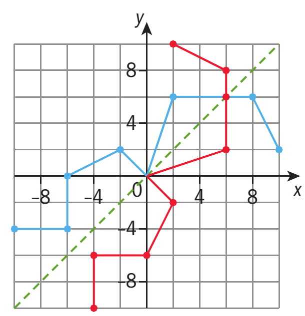
b



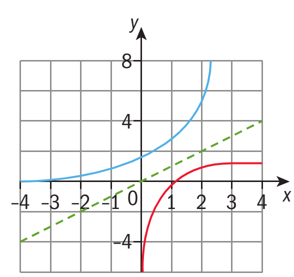
c



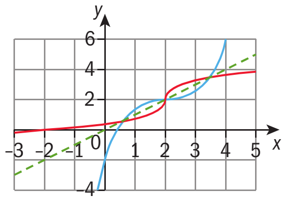
d



e



f



Exercise 1H

1 a i -2 and 1

ii $\frac{1}{2}$ and -3

iii x iv x

b They are inverses of each other.

2 a $\frac{x+1}{3}$ b $\sqrt[3]{x+2}$

c $4(x-5)$ d $(x+3)^3$

e $\frac{1}{x+2}$ f $\sqrt[3]{\frac{x-3}{2}}$

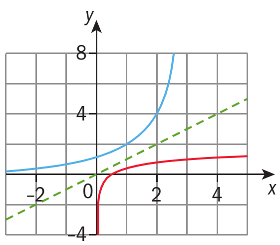
g $\frac{3x}{1-x}$ h $\frac{5x}{x+2}$

3 a $1-x$ b x c $\frac{1}{x}$

4 a 1 b -5 c $\frac{17}{20}$

5 $\frac{1+2x}{x-1}$

6 a–c



d $f(x)$: $x \in \mathbb{R}, y > 0$
$f^{-1}(x)$: $x > 0, y \in \mathbb{R}$

7 $g^{-1}(x) = x^2$. The range of $g(x)$ is $x \geq 0$ so the domain of $g^{-1}(x)$ is $x \geq 0$. The domain of $f(x)$ is $x \in \mathbb{R}$ so $g^{-1}(x) \neq f(x)$

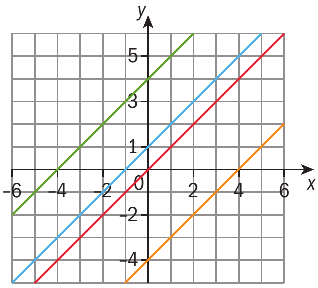
8 If $f(x) = mx + c$ then

$f^{-1}(x) = \frac{1}{m}x - \frac{c}{m}$

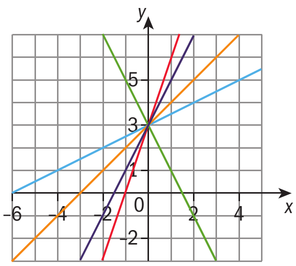
$m \times \frac{1}{m} = 1$ not -1 so not perpendicular.

Investigation – functions

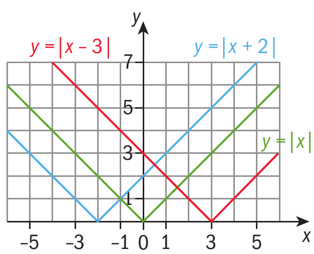
1 Changing the constant term translates $y = x$ along the y-axis.



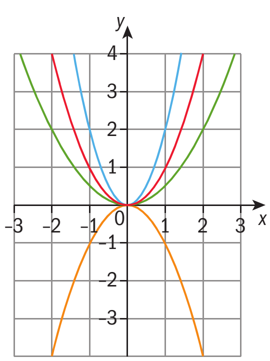
2 Changing the x-coefficient alters the gradient of the line.



3 $y = |x + h|$ is a translation of $-h$ along the x-axis

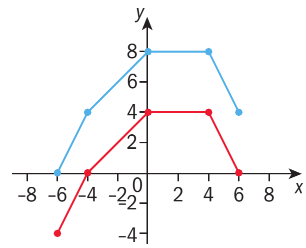


4 The negative sign reflects the graph in the x-axis. Increasing the value of a means the graph increases more steeply.

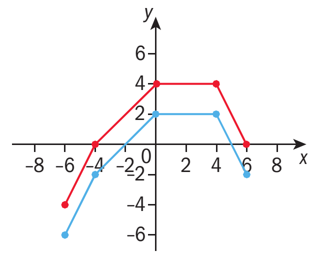


Exercise 1I

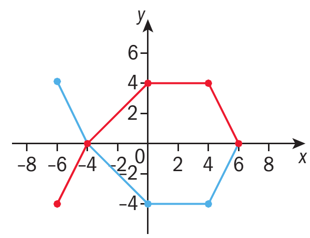
1 a



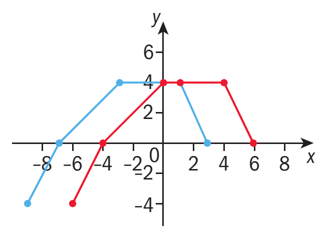
b



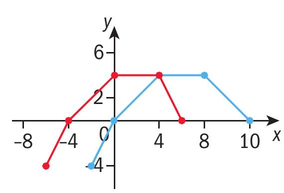
c



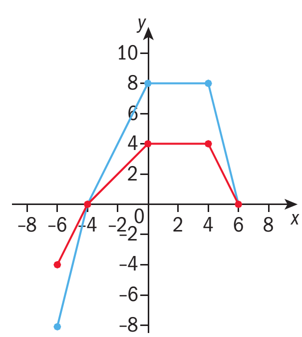
d



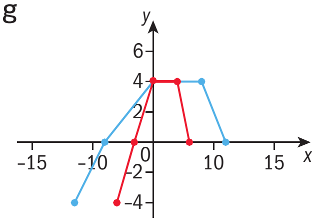
e



f



g



2 $g(x) = f(x) + 2$

$h(x) = f(x) - 4$

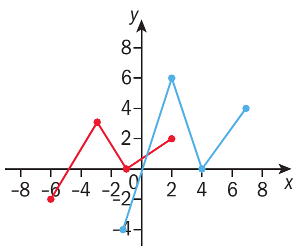
$q(x) = \frac{1}{2}f(x)$

3 $q(x) = f(x + 4) - 2$

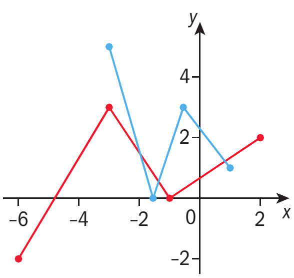
$s(x) = f(x + 4)$

$t(x) = f(x - 2)$

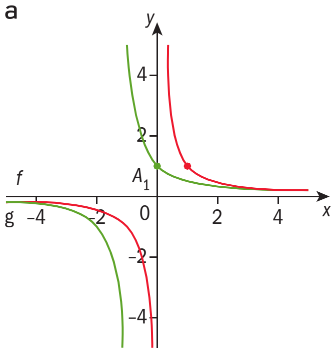
4 a Domain $-1 \le x \le 7$, range $-4 \le y \le 6$



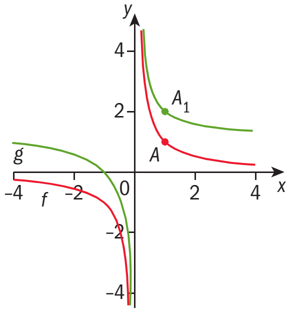
b Domain $-3 \le x \le 1$, range $0 \le y \le 5$



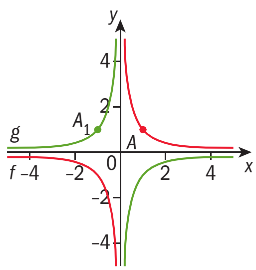
5 a



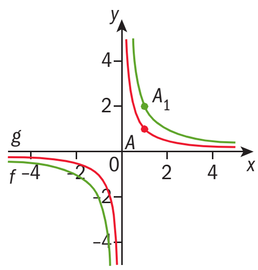
b



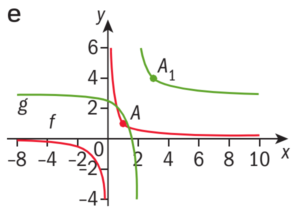
c



d



e

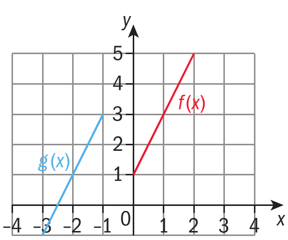


6 a Reflection in x-axis.

b Horizontal translation 3 units.

c Vertical stretch SF2, reflection x-axis, vertical translation of 5 units.

7 a, b



Review exercise non-GDC

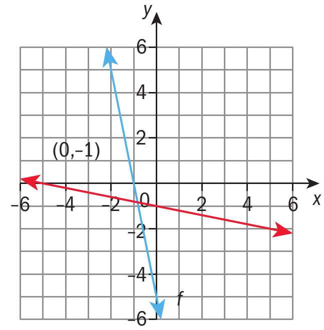
1 a $4a - 13$ b $\frac{2-x}{x}$

2 a $2x^2 - 15x + 28$

b $-2x^2 + 9$

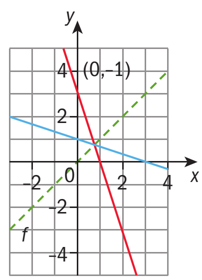
3 a $\frac{2x-17}{3}$ b $\sqrt[3]{\frac{x-3}{2}}$

4 $f^{-1}(x) = -5x - 5$

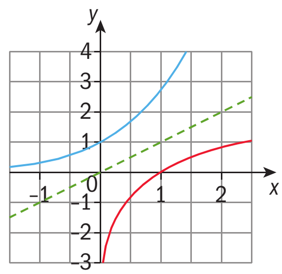


5 a $\frac{x-5}{3}$ b $x^3 - 2$

6 a



b



7 a Domain $x \in \mathbb{R}$, $y \ge 0$

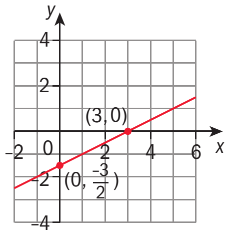
b Domain $x \in \mathbb{R}$, $x \ne 3$, Range $y \in \mathbb{R}$, $y \ne 0$

8 a $f(x) = 2\sqrt{-3x-9} + 2$

b $f(x) = -\frac{1}{4}\left(3^{\frac{x-5}{3}}\right) - 1$

9 a Inverse function graph is the reflection in $y = x$.

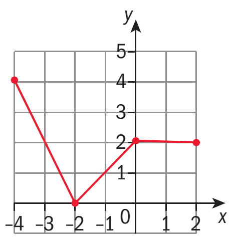
b



10 a -2 b -13

c $f^{-1}(x) = \sqrt[3]{\frac{x-3}{2}}$

11 a



b P is (4, 1)

12 a $(f \circ g)(x) = 3x + 6$

b $f^{-1}(x) = \frac{x}{3}$ $g^{-1}(x) = x - 2$

$f^{-1}(12) = \frac{12}{3} = 4$

$g^{-1}(12) = 12 - 2 = 10$

$f^{-1}(12) + g^{-1}(12) = 4 + 10$

$f^{-1}(12) + g^{-1}(12) = 14$

13 a $(h \circ g)(x) = \frac{3(2x-1)}{(2x-1)-2}$

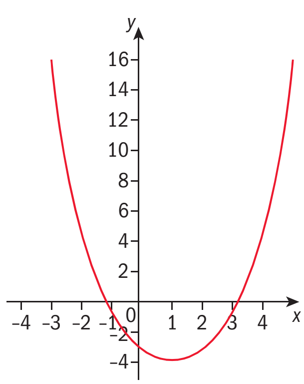
$= \frac{6x-3}{2x-3}$

b $x = \frac{1}{2}$

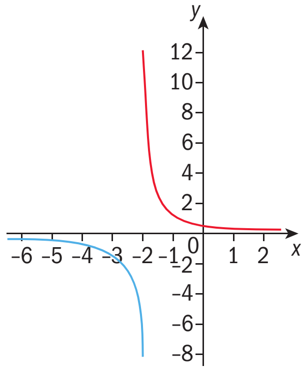
Review exercise GDC

1 Domain: $x \geq -2$, range: $y \geq 0$

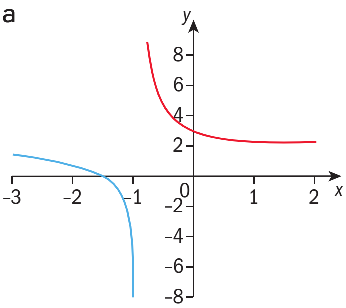
2 Domain: $x \in \mathbb{R}$, range: $y \geq -4$



3 Domain $x \in \mathbb{R}$, $x \neq -2$, range $y \in \mathbb{R}$, $x \neq 0$

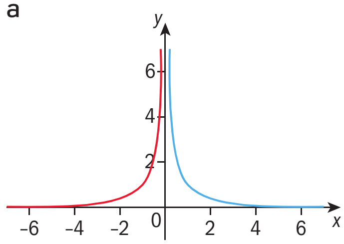


4 a



b x-intercept -1.5, y-intercept 3.

5 a

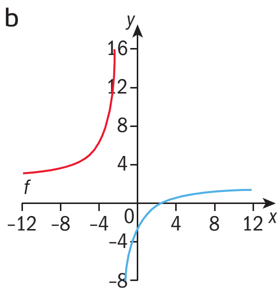


b 0

c Domain $x \in \mathbb{R}$, $x \neq \mathbb{R}$, range $y > 0$

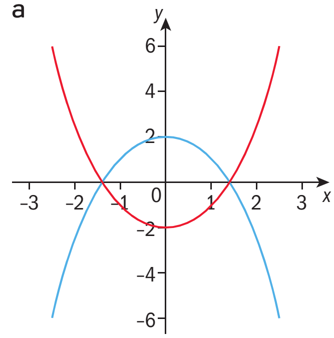
6 a $x = -2$, $y = 2$

b



c $(2.5, 0)$, $(0, -2.5)$

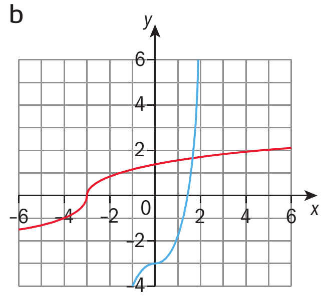
7 a



b $x = \pm\sqrt{2}$

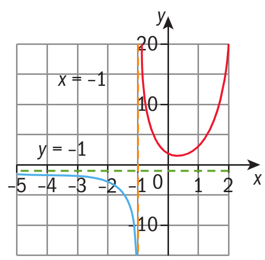
8 a $\sqrt[3]{x+3}$

b



c 1.67

9



10 a $f^{-1}(x) = \frac{x+2}{3}$

b $(g^{-1} \circ f)(x) = (3x - 2) + 3$
$= 3x + 1$

c $(f^{-1} \circ g) = \frac{(x-3)+2}{3} = \frac{x-1}{3}$

$\frac{x-1}{3} = 3x + 1$

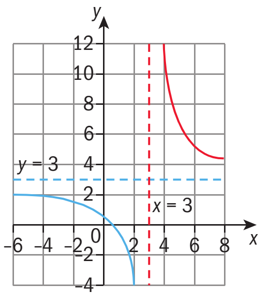
$x - 1 = 3(3x + 1)$

$x - 1 = 9x + 3$

$8x = -4$

$x = -\frac{1}{2}$

d



e $x = 3, y = 3$

Chapter 2

Skills check

1 a $a = 6$

b $x = \pm\sqrt{5}$

c $n = -11$

2 a $2k(k-5)$

b $7a(2a^2 + 3a - 7)$

c $(2x+3)(x+2y)$

d $(5a-b)(a-2)$

e $(n+1)(n+3)$

f $(2x-3)(x+1)$

g $(m+6)(m-6)$

h $(5x+9y)(5x-9y)$

Exercise 2A

1 a $1, 2$

b $-8, 7$

c $5, 6$

d $-5, 5$

e $-8, 6$

f -3

2 a $-\frac{4}{3}, \frac{1}{2}$

b $-2, \frac{4}{5}$

c $-1, \frac{5}{2}$

d $-\frac{1}{2}, \frac{9}{2}$

e $-4, -\frac{2}{3}$

f $-\frac{3}{2}, \frac{4}{3}$

Exercise 2B

1 a $-5, 4$

b $-2, -\frac{2}{3}$

c $-\frac{3}{2}$

d $-2, \frac{25}{2}$

e $-9, 4$

f $-\frac{1}{4}, 1$

2 -3 or 4

3 $\frac{2}{5}$ or 3

Investigation – perfect square trinomials

1 -5

2 -3

3 -7

4 4

5 9

6 10

Exercise 2C

1 $-4 \pm \sqrt{19}$

2 $\frac{5 \pm \sqrt{37}}{2}$

3 $3 \pm 2\sqrt{2}$

4 $\frac{-7 \pm \sqrt{65}}{2}$

5 $1 \pm \sqrt{7}$

6 $\frac{-1 \pm \sqrt{13}}{2}$

Exercise 2D

1 $-3 \pm 2\sqrt{3}$

2 $1 \pm \sqrt{2}$

3 $1 \pm \sqrt{\frac{3}{5}}$

4 $\frac{-3 \pm \sqrt{29}}{4}$

5 $-\frac{3}{2}, 2$

6 $\frac{-2 \pm 3\sqrt{6}}{10}$

Exercise 2E

1 $\frac{-9 \pm \sqrt{193}}{8}$

2 $-2, \frac{4}{3}$

3 $-1, -\frac{1}{5}$

4 $3 \pm \sqrt{5}$

5 no solution

6 $\frac{-5 \pm 2\sqrt{10}}{3}$

7 $\frac{3 \pm \sqrt{17}}{4}$

8 $\frac{9 \pm \sqrt{113}}{4}$

9 $x = \frac{-9 \pm \sqrt{129}}{4}$.

10 $\frac{3 \pm \sqrt{21}}{4}$

Exercise 2F

1 18, 32

2 24 m, 11 m

3 10

4 18 cm, 21 cm

5 2.99 seconds

Investigation – roots of quadratic equations

1 a 4 b $\frac{3}{2}$

c $-\frac{1}{5}$

2 a $-7, 2$ b $\frac{4 \pm \sqrt{10}}{3}$

c $\frac{3 \pm \sqrt{89}}{10}$

3 a No solution

b No solution

c No solution

Exercise 2G

1 a 37; two different real roots

b 8; two different real roots

c -79; no real roots

d 0; two equal real roots

e -23; no real roots

f -800; no real roots