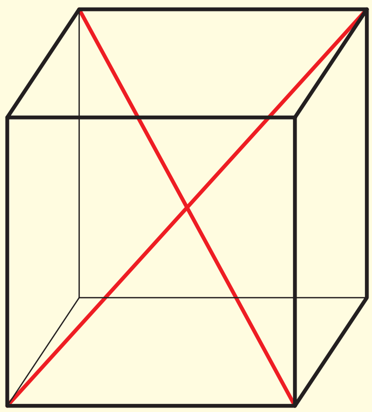
13 Vectors

In this chapter you will learn:

- to use vectors to represent displacements and positions in two and three dimensions
- to perform algebraic operations with vectors, and understand their geometric interpretation
- to calculate the distance between two points
- to use vectors to calculate the angle between two lines
- to use vectors to find areas of parallelograms and triangles
- to use two new operations on vectors, called scalar product and vector product.

Introductory problem

What is the angle between the diagonals of a cube?



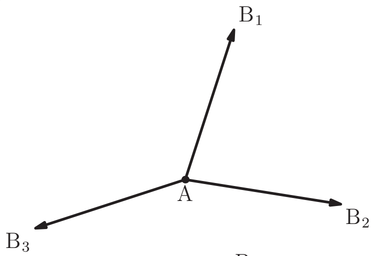
Solving problems in three dimensions can be difficult, as two-dimensional diagrams cannot always make clear what is happening. Vectors are a useful tool to describe geometrical properties using equations, which can often be analysed more easily. In this chapter we will develop techniques to calculate angles, distances and areas in two and three dimensions. We will apply those techniques to further geometrical problems in chapter 14.

13A Positions and displacements

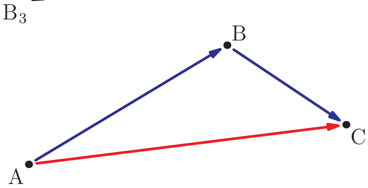
You may know from physics that **vectors** are used to represent quantities which have both magnitude (size) and direction, such as force or velocity. Vector quantities are different from **scalar** quantities which are fully described by a single number. In pure mathematics, vectors are used to represent displacements from one point to another, and so to describe geometrical figures.

If there is a fixed point A and a point B is 10 cm away from it, this information alone does not tell you where point B is.

Vectors are an example of abstraction in mathematics: a single concept that can be applied to many different situations. Force, velocity and displacements appear to have very little in common, yet they can all be described and manipulated using the rules of vectors. In the words of the French mathematician and physicist Henry Poincaré (1854–1912): 'Mathematics is the art of giving the same name to different things'.

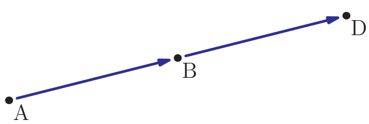


The position of B relative to A can be represented by the **vector displacement** $\overrightarrow{AB}$. The vector contains both distance and direction information; it describes a way of getting from A to B.

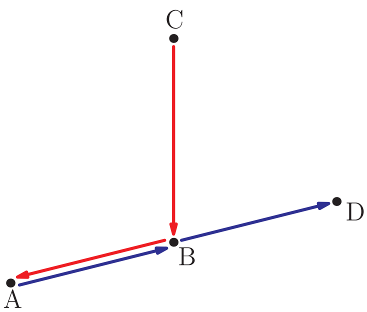


If we now add a third point, C, then there are two ways of getting from A to C: either directly, or via B. To express the second possibility using vectors, we use the addition sign to represent moving from A to B followed by moving from B to C: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

Always remember that a vector represents a way of getting from one point to another, but it does not tell us anything about the actual position of the starting and the end points.



If getting from B to D involves moving the same distance and in the same direction as getting from A to B, then $\overrightarrow{BD} = \overrightarrow{AB}$.



To return from the end point to the starting point, we use the minus sign and so $\overrightarrow{BA} = -\overrightarrow{AB}$.

We can also use the subtraction sign with vectors: $\overrightarrow{CB} - \overrightarrow{AB} = \overrightarrow{CB} + \overrightarrow{BA}$

To get from A to D we need to move in the same direction, but twice as far, as in getting from A to B. We can express this by writing $\overrightarrow{AD} = 2\overrightarrow{AB}$ or, equivalently, $\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AD}$.

It is convenient to give vectors letters, as we do with variables in algebra. To emphasise that something is a vector, rather than a scalar (number) we use either bold type or an arrow on top. When writing by hand, we use underlining instead of bold type. For example, we can denote vector $\overrightarrow{AB}$ by $\boldsymbol{a}$ (or $\vec{a}$). Then in the diagrams above, $\overrightarrow{BD} = \boldsymbol{a}$, $\overrightarrow{BA} = -\boldsymbol{a}$ and $\overrightarrow{AD} = 2\boldsymbol{a}$.

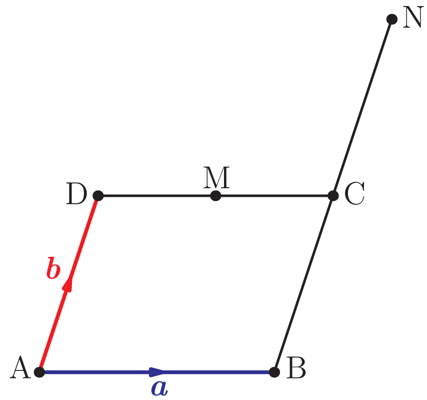
EXAM HINT

Fractions of a vector are usually written as multiples: $\frac{1}{2}\overrightarrow{AD}$, not $\frac{\overrightarrow{AD}}{2}$.

Worked example 13.1

The diagram shows a parallelogram $ABCD$.

Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. M is the midpoint of CD and N is the point on BC such that $CN = BC$.



Express vectors $\overrightarrow{CM}$, $\overrightarrow{BN}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

We can think of $\overrightarrow{CM}$ as describing a way of getting from C to M moving only along the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$

Going from C to M is the same as going half way from B to A, and $\overrightarrow{BA} = -\overrightarrow{AB}$

$$\overrightarrow{CM} = \frac{1}{2}\overrightarrow{BA} = -\frac{1}{2}\underline{a}$$

Going from B to N is twice the distance and in the same direction as from B to C, and $\overrightarrow{BC} = \overrightarrow{AD}$

$$\overrightarrow{BN} = 2\overrightarrow{BC} = 2\underline{b}$$

To get from M to N we can go from M to C and then from C to M

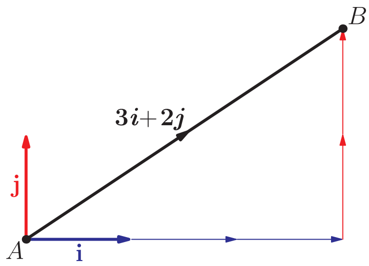
$$\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CN}$$

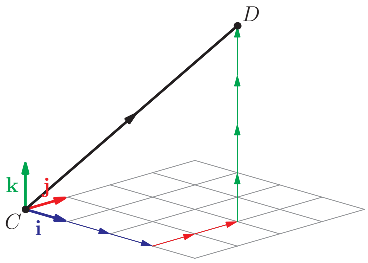
$\overrightarrow{MC} = -\overrightarrow{CM}$ and $\overrightarrow{CN} = \overrightarrow{BC}$

$$= -\overrightarrow{CM} + \overrightarrow{BC}$$

$$= \frac{1}{2}\underline{a} + \underline{b}$$

To do further calculations with vectors we also need to describe them with numbers, not just diagrams. You are already familiar with coordinates, which are used to represent positions of points. A similar idea can be used to represent vectors.





$\vec{CD} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$

Start in the two-dimensional plane and select two directions perpendicular to each other, and define vectors of length 1 in those two directions by ***i*** and ***j***. Then any vector in the plane can be expressed in terms of ***i*** and ***j***, as shown in the diagram. ***i*** and ***j*** are called **base or unit vectors**.

To represent displacements in three-dimensional space, we need three base vectors, all perpendicular to each other. They are conventionally called ***i***, ***j*** and ***k***.

An alternative notation is to use **column vectors**. In this notation, displacements shown above are written as:

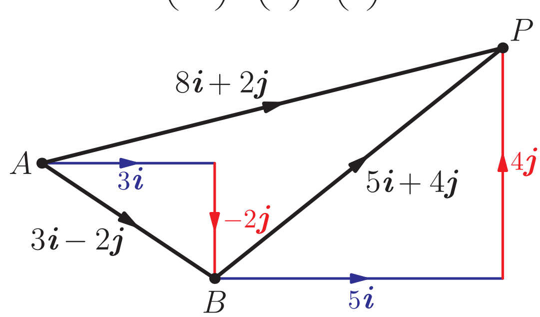
$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$$

The numbers in the column are called the **components** of a vector.

Using components in column vectors makes it easy to add displacements.

To get from A to B in the diagram below, we need to move 3 units in the ***i*** direction, and to get from B to P we need to move 5 units in the ***i*** direction; thus getting from A to P requires moving 8 units in the ***i*** direction. Similarly in the ***j*** direction we move –2 units from A to B and 4 units from B to P, making the total displacement from A to P equal to 2 units. As the total displacement from A to P is $\overrightarrow{AP} = \overrightarrow{AB} + \overrightarrow{BP}$, we can write it in component form as: $(3\boldsymbol{i} - 2\boldsymbol{j}) + (5\boldsymbol{i} + 4\boldsymbol{j}) = 8\boldsymbol{i} + 2\boldsymbol{j}$ or using the column vector notation as:

$$\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$$



Reversing the direction of a vector is also simple: to get from B to A we need to move –3 units in the ***i*** direction and 2 units in the ***j*** direction and so $\overrightarrow{BA} = -\overrightarrow{AB} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$.

These rules for adding and subtracting vectors also apply in three dimensions.

EXAM HINT

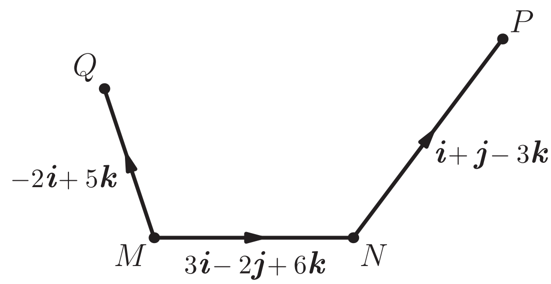
You must be familiar with both base vector and column vector notation, as both will be used in questions. When you write your answers, you can use whichever notation you prefer.

EXAM HINT

Vector diagrams do not have to be accurate or to scale to be useful. A two-dimensional sketch of a 3D situation is often enough to show you what is going on.

Worked example 13.2

The diagram shows points M, N, P and Q such that $\overrightarrow{MN} = 3\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}$, $\overrightarrow{NP} = \boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$ and $\overrightarrow{MQ} = -2\boldsymbol{j} + 5\boldsymbol{k}$.



Write the following vectors in component form:

(a) $\overrightarrow{MP}$ (b) $\overrightarrow{PM}$ (c) $\overrightarrow{PQ}$

We can get from M to P via N

(a) $\overrightarrow{MP} = \overrightarrow{MN} + \overrightarrow{NP}$
$= (3\boldsymbol{i} - 2\boldsymbol{j} + 6\boldsymbol{k}) + (\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k})$
$= 4\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

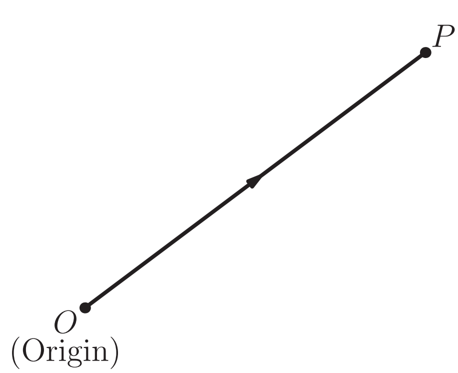
We have already found $\overrightarrow{MP}$

(b) $\overrightarrow{PM} = -\overrightarrow{MP} = -4\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$

We can get from P to Q via M using the answers to (a) and (b)

(c) $\overrightarrow{PQ} = \overrightarrow{PM} + \overrightarrow{MQ}$
$= (-4\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}) + (-2\boldsymbol{j} + 5\boldsymbol{k})$
$= -4\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k}$

Vectors represent displacements, but they can also represent the positions of points. If we use one fixed point, called the **origin**, then the position of a point can be described by its displacement from the origin. For example, the position of point P in the diagram can be represented by its **position vector**, $\overrightarrow{OP}$.



EXAM HINT

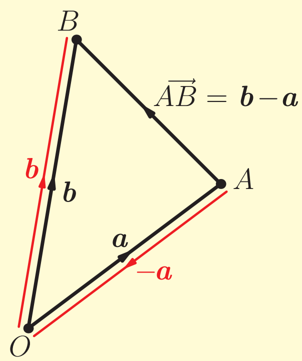
The position vector of point A is usually denoted by $\boldsymbol{a}$.

If we know position vectors of two points A and B we can find the displacement $\overrightarrow{AB}$ as shown in the diagram below: $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$.

KEY POINT 13.1

If points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ then

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$.



Position vectors are closely related to coordinates. If the base vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ have directions set along the coordinate axes, then the components of any position vector are simply the coordinates of the point.

Worked example 13.3

Points A and B have coordinates (3, –1, 2) and (5, 0, 3) respectively. Write as column vectors:

(a) the position vectors of A and B

(b) the displacement vector $\overrightarrow{AB}$.

The components of the position vectors are the coordinates of the point

(a) $\underline{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$\underline{b} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$

Use relationship $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

(b) $\overrightarrow{AB} = \underline{b} - \underline{a}$

$= \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

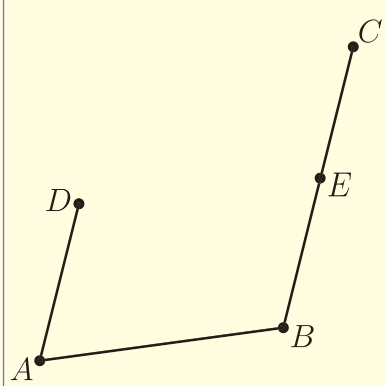
Worked example 13.4

Points A, B, C and D have position vectors $\boldsymbol{a} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} 7 \\ 8 \\ -3 \end{pmatrix}$, $\boldsymbol{d} = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$

Point E is the midpoint of the line BC.

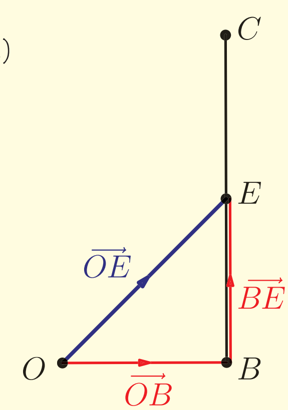
(a) Find the position vector of E.
(b) Show that $ABED$ is a parallelogram.

Draw a diagram to show what is going on



For this part, we only need to look at points B, C and E. It may help to show the origin on the diagram

(a)



$$\overrightarrow{OE} = \overrightarrow{OB} + \overrightarrow{BE}$$
$$= \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}$$

Use relationship $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$$= \underline{b} + \frac{1}{2}(\underline{c} - \underline{b})$$
$$= \frac{1}{2}\underline{b} + \frac{1}{2}\underline{c}$$
$$= \begin{pmatrix} 2.5 \\ 0 \\ 1.5 \end{pmatrix} + \begin{pmatrix} 3.5 \\ 4 \\ -1.5 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}$$

continued . . .

In a parallelogram, opposite sides are equal length and parallel, which means that the vectors corresponding to those sides are equal

We need to show that $\overrightarrow{AD} = \overrightarrow{BE}$

(b) $\overrightarrow{AD} = \underline{d} - \underline{a}$

$$= \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$\overrightarrow{BE} = \underline{e} - \underline{b}$

$$= \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$$

$\overrightarrow{AD} = \overrightarrow{BE}$

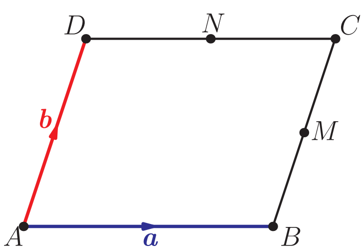
$ABED$ is a parallelogram.

In Worked example 13.4a we derived a general formula for finding the position vector of a midpoint of a line segment.

KEY POINT 13.2

The position vector of the midpoint of $[AB]$ is $\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b})$.

Exercise 13A

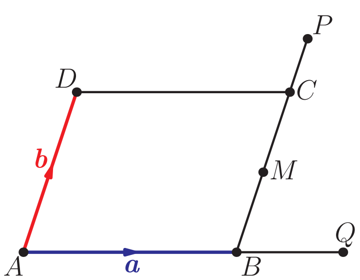


1. The diagram shows a parallelogram $ABCD$ with $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. M is the midpoint of BC and N is the midpoint of CD. Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(a) (i) $\overrightarrow{BC}$ (ii) $\overrightarrow{AC}$

(b) (i) $\overrightarrow{CD}$ (ii) $\overrightarrow{ND}$

(c) (i) $\overrightarrow{AM}$ (ii) $\overrightarrow{MN}$



2. In the parallelogram $ABCD$, $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$. M is the midpoint of BC, Q is the point on AB such that $BQ = \frac{1}{2}AB$ and P is the point on the extended line BC such that $BC : CP = 3 : 1$, as shown on the diagram.

Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(a) (i) $\overrightarrow{AP}$ (ii) $\overrightarrow{AM}$

(b) (i) $\overrightarrow{QD}$ (ii) $\overrightarrow{MQ}$

(c) (i) $\overrightarrow{DQ}$ (ii) $\overrightarrow{PQ}$

3. Write the following vectors in column vector notation (in three dimensions):

(a) (i) $4\boldsymbol{i}$ (ii) $-5\boldsymbol{j}$

(b) (i) $3\boldsymbol{i}+\boldsymbol{k}$ (ii) $2\boldsymbol{j}-\boldsymbol{k}$

4. Three points O, A and B are given. Let $\overrightarrow{OA}=a$ and $\overrightarrow{OB}=b$.

(a) Express $\overrightarrow{AB}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) C is the midpoint of AB. Express $\overrightarrow{OC}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(c) Point D lies on the line (AB) on the same side of B as A, so that $AD=3AB$. Express $\overrightarrow{OD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. *[5 marks]*

EXAM HINT

Remember that (AB) represents the infinite line through A and B, while $[AB]$ is the line segment (the part of the line between A and B).

5. Points A and B lie in a plane and have coordinates (3, 0) and (4, 2) respectively. C is the midpoint of $[AB]$.

(a) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as column vectors.

(b) Point D is such that $\overrightarrow{AD}=\begin{pmatrix}7\\-2\end{pmatrix}$.

Find the coordinates of D. *[5 marks]*

6. Points A and B have position vectors $\overrightarrow{OA}=\begin{pmatrix}3\\1\\-2\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}4\\-2\\5\end{pmatrix}$.

(a) Write $\overrightarrow{AB}$ as a column vector.

(b) Find the position vector of the midpoint of $[AB]$. *[5 marks]*

7. Point A has position vector $\boldsymbol{a}=2\boldsymbol{i}-3\boldsymbol{j}$ and point D is such that $\overrightarrow{AD}=\boldsymbol{i}-\boldsymbol{j}$. Find the position vector of point D. *[4 marks]*

8. Points A and B have position vectors $\boldsymbol{a}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}1\\-1\\3\end{pmatrix}$.

Point C lies on (AB) so that $AC:BC=2:3$. Find the position vector of C. *[5 marks]*

9. Points P and Q have position vectors $\boldsymbol{p}=2\boldsymbol{i}-\boldsymbol{j}-3\boldsymbol{k}$ and $\boldsymbol{q}=\boldsymbol{i}+4\boldsymbol{j}-\boldsymbol{k}$.

(a) Find the position vector of the midpoint M of $[PQ]$.

(b) Point R lies on the line (PQ) such that $QR=QM$. Find the coordinates of R ($R\neq M$). *[6 marks]*

10. Points A, B and C have position vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. Find the position vector of point D such that $ABCD$ is a parallelogram. *[5 marks]*

13B Vector algebra

Although the idea of representing forces by directed line segments dates back to antiquity, the algebra of vectors was first developed in the 19th Century and was originally used to study complex numbers, which you will meet in chapter 15.

In the previous section we used vectors to describe positions and displacements of points in space, but vectors can represent quantities other than displacements; for example velocities or forces. Whatever the vectors represent, they always follow the same algebraic rules. In this section we will summarise those rules, which can be expressed using either diagrams or equations.

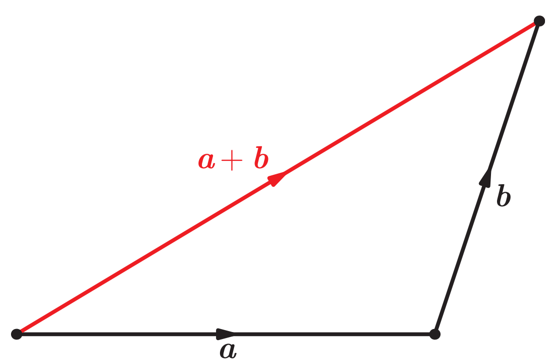
EXAM HINT

The ability to switch between diagrams and equations is essential for solving harder vector problems.

Vector addition can be done on a diagram by joining the starting point of the second vector to the end point of the first. In component form, it is carried out by adding corresponding components. When vectors represent displacements, vector addition represents one displacement followed by another.

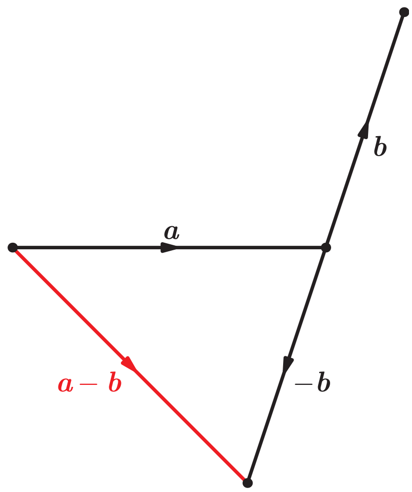
EXAM HINT

Remember that vectors only show relative positions of two points, they don't have a fixed starting point. So we are free to 'move' the second vector to the end point of the first.



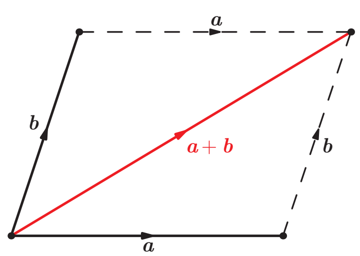
$$\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

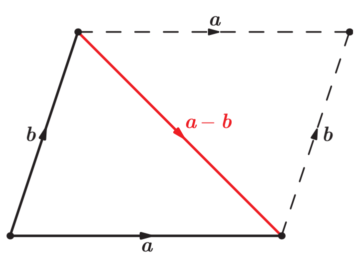
Vector subtraction is the same as adding a negative vector. ($-\boldsymbol{a}$ is the same length but the opposite direction to $\boldsymbol{a}$). In component form you simply subtract corresponding components.



$$\begin{pmatrix}5\\1\\-2\end{pmatrix}-\begin{pmatrix}3\\3\\-3\end{pmatrix}=\begin{pmatrix}5\\1\\-2\end{pmatrix}+\begin{pmatrix}-3\\-3\\3\end{pmatrix}=\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

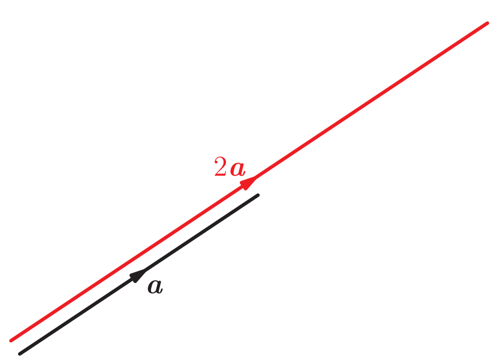
You can also consider vector addition as the diagonal of the parallelogram formed by the two vectors. The difference of two vectors can be represented by the other diagonal of the parallelogram formed by the two vectors.



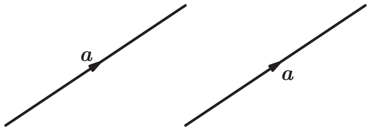


Scalar multiplication changes the magnitude (length) of the vector, leaving the direction the same. In component form, each component is multiplied by the scalar.

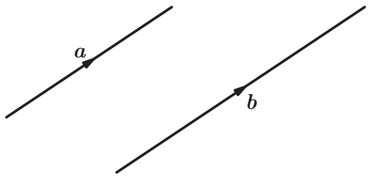
For any vector $\boldsymbol{a}$, $k\boldsymbol{a}$ represents a displacement in the same direction but with distance multiplied by k.



$$2\begin{pmatrix}3\\-5\\0\end{pmatrix}=\begin{pmatrix}6\\-10\\0\end{pmatrix}$$



Two vectors are **equal** if they have the same magnitude and direction. All their components are equal. They represent the same displacements but may have different start and end points.



If two vectors are in the same direction then they are **parallel**. Parallel vectors are scalar multiples of each other. This is because multiplying a vector by a scalar does not change its direction.

$$\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \text{ is parallel to } \begin{pmatrix} 6 \\ -9 \\ 3 \end{pmatrix} \text{ because } \begin{pmatrix} 6 \\ -9 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}.$$

KEY POINT 13.3

If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel we can write $\boldsymbol{b} = t\boldsymbol{a}$ for some scalar t.

The next example illustrates the vector operations we have just described.

Worked example 13.5

Given vectors $\boldsymbol{a} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} -2 \\ p \\ q \end{pmatrix}$:

(a) Find $2\boldsymbol{a} - 3\boldsymbol{b}$.

(b) Find the values of p and q such that $\boldsymbol{c}$ is parallel to $\boldsymbol{a}$.

(c) Find the value of scalar k such that $\boldsymbol{a} + k\boldsymbol{b}$ is parallel to vector $\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$.

(a) $2\underline{a} - 3\underline{b} = 2\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} - 3\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 2 \\ 4 \\ 14 \end{pmatrix} - \begin{pmatrix} -9 \\ 12 \\ 6 \end{pmatrix} = \begin{pmatrix} 11 \\ -8 \\ 8 \end{pmatrix}$

continued . . .

If vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are parallel we can write $\mathbf{v_2} = t\mathbf{v_1}$

(b) Write $\underline{c} = t\underline{a}$ for some scalar t

Then: $\begin{pmatrix} -2 \\ p \\ q \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} t \\ 2t \\ 7t \end{pmatrix}$

If two vectors are equal then all their components are equal

$$\Rightarrow \begin{cases} -2 = t \\ p = 2t \\ q = 7t \end{cases}$$

$\therefore p = -4,\ q = -14$

Write vector $\mathbf{a} + k\mathbf{b}$ in terms of k

(c) $\underline{a} + k\underline{b} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} + \begin{pmatrix} -3k \\ 4k \\ 2k \end{pmatrix} = \begin{pmatrix} 1-3k \\ 2+4k \\ 7+2k \end{pmatrix}$

Then use $\mathbf{a} + k\mathbf{b} = t\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$

Parallel to $\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix} \Rightarrow \begin{pmatrix} 1-3k \\ 2+4k \\ 7+2k \end{pmatrix} = t\begin{pmatrix} 0 \\ 10 \\ 23 \end{pmatrix}$

$$\Rightarrow \begin{cases} 1-3k = 0 & (1) \\ 2+4k = 10t & (2) \\ 7+2k = 23t & (3) \end{cases}$$

Find k from the first equation, but check that all three equations are satisfied

(1) $1-3k = 0 \Rightarrow k = \frac{1}{3}$

(2) $2+4\left(\frac{1}{3}\right) = 10t \Rightarrow t = \frac{1}{3}$

(3) $7+2\left(\frac{1}{3}\right) = 23\left(\frac{1}{3}\right)$ (correct)

$\therefore k = \frac{1}{3}$

Exercise 13B

1. Let $\boldsymbol{a}=\begin{pmatrix}7\\1\\12\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}5\\-2\\3\end{pmatrix}$ and $\boldsymbol{c}=\begin{pmatrix}1\\1\\2\end{pmatrix}$. Find the following vectors:

(a) (i) $3\boldsymbol{a}$ (ii) $4\boldsymbol{b}$

(b) (i) $\boldsymbol{a}-\boldsymbol{b}$ (ii) $\boldsymbol{b}+\boldsymbol{c}$

(c) (i) $2\boldsymbol{b}+\boldsymbol{c}$ (ii) $\boldsymbol{a}-2\boldsymbol{b}$

(d) (i) $\boldsymbol{a}+\boldsymbol{b}-2\boldsymbol{c}$ (ii) $3\boldsymbol{a}-\boldsymbol{b}+\boldsymbol{c}$

2. Let $\boldsymbol{a}=\boldsymbol{i}+2\boldsymbol{j}$, $\boldsymbol{b}=\boldsymbol{i}-\boldsymbol{k}$ and $\boldsymbol{c}=2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$. Find the following vectors:

(a) (i) $-5\boldsymbol{b}$ (ii) $4\boldsymbol{a}$

(b) (i) $\boldsymbol{c}-\boldsymbol{a}$ (ii) $\boldsymbol{a}-\boldsymbol{b}$

(c) (i) $\boldsymbol{a}-\boldsymbol{b}+2\boldsymbol{c}$ (ii) $4\boldsymbol{c}-3\boldsymbol{b}$

3. Given that $\boldsymbol{a}=4\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$, find the vector $\boldsymbol{b}$ such that:

(a) $\boldsymbol{a}+\boldsymbol{b}$ is the zero vector (b) $2\boldsymbol{a}+3\boldsymbol{b}$ is the zero vector

(c) $\boldsymbol{a}-\boldsymbol{b}=\boldsymbol{j}$ (d) $\boldsymbol{a}+2\boldsymbol{b}=3\boldsymbol{i}$

4. Given that $\boldsymbol{a}=\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}5\\3\\3\end{pmatrix}$ find vector $\boldsymbol{x}$ such that $3\boldsymbol{a}+4\boldsymbol{x}=\boldsymbol{b}$. *[4 marks]*

5. Given that $\boldsymbol{a}=3\boldsymbol{i}-2\boldsymbol{j}+5\boldsymbol{k}$, $\boldsymbol{b}=\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k}$ and $\boldsymbol{c}=\boldsymbol{i}+\boldsymbol{k}$, find the value of the scalar t such that $\boldsymbol{a}+t\boldsymbol{b}=\boldsymbol{c}$. *[4 marks]*

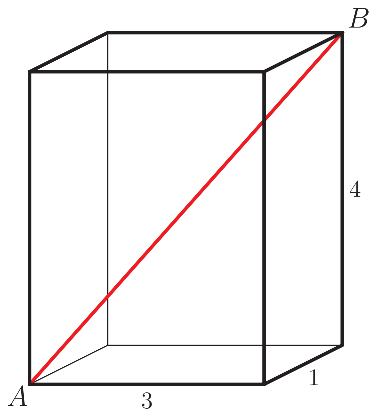
6. Given that $\boldsymbol{a}=\begin{pmatrix}2\\0\\2\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}3\\1\\3\end{pmatrix}$ find the value of the scalar p such that $\boldsymbol{a}+p\boldsymbol{b}$ is parallel to the vector $\begin{pmatrix}3\\2\\3\end{pmatrix}$. *[5 marks]*

7. Given that $\boldsymbol{x}=2\boldsymbol{i}+3\boldsymbol{j}+\boldsymbol{k}$ and $\boldsymbol{y}=4\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$ find the value of the scalar λ such that $\lambda x+y$ is parallel to vector $\mathbf{j}$. *[5 marks]*

8. Given that $\boldsymbol{a}=\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k}$ and $\boldsymbol{b}=2q\boldsymbol{i}+\boldsymbol{j}+q\boldsymbol{k}$ find the values of scalars p and q such that $p\boldsymbol{a}+\boldsymbol{b}$ is parallel to vector $\boldsymbol{i}+\boldsymbol{j}+2\boldsymbol{k}$. *[6 marks]*

13C Distances

Geometry problems often involve finding distances between points. In this section we will see how to use vectors to do this.



Consider two points, A and B such that the displacement $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$. The distance AB can be found by using Pythagoras' theorem in three dimensions: $AB = \sqrt{3^2 + 1^2 + 4^2} = \sqrt{26}$. This quantity is called the **magnitude** of $\overrightarrow{AB}$, and written as $\left|\overrightarrow{AB}\right|$.

To find the distance between A and B, using their position vectors, we first need to find the displacement vector $\overrightarrow{AB}$ and then calculate its magnitude.

Worked example 13.6

Points A and B have position vectors $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$. Find the exact distance AB.

The distance is the magnitude of the displacement vector, so find $\overrightarrow{AB}$ first

$$\overrightarrow{AB} = \underline{\boldsymbol{b}} - \underline{\boldsymbol{a}}$$

$$= \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$$

Now use the formula for the magnitude

$$\left|\overrightarrow{AB}\right| = \sqrt{3^2 + 3^2 + 2^2} = \sqrt{22}$$

KEY POINT 13.4

The magnitude of a vector, $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, is $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

The distance between points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $|\boldsymbol{b} - \boldsymbol{a}|$.

EXAM HINT

Don't forget that squaring a negative number gives a positive value.

The symbol $\gtreqless$ means 'greater than, equal to or less than'. This may appear to be a useless symbol, but it highlights an important idea in vectors – they cannot be put into order. So while it is correct to say that $|v| \gtreqless |u|$ it is not possible to say the same about the vectors themselves.

We saw in Section 13B that multiplying a vector by a scalar produces a vector in the same direction but of different magnitude. In more advanced applications of vectors it is useful to be able to use vectors of length 1, called **unit vectors**. The base vectors $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are examples of unit vectors.

Worked example 13.7

(a) Find the unit vector in the same direction as $\boldsymbol{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

(b) Find a vector of magnitude 5 parallel to $\boldsymbol{a}$.

(a) Call the required unit vector $\hat{a}$.

To produce a vector in the same direction but of different magnitude as ***a***, we need to multiply ***a*** by a scalar. We need to find the value of the scalar

Then $\hat{a} = k\hat{a}$ and $\hat{a} = 1$

$|k\underline{a}| = k|\underline{a}| = 1$

$\Rightarrow k = \dfrac{1}{|\underline{a}|}$

$|\underline{a}| = \sqrt{2^2 + 2^2 + 1^2} = 3$

$\therefore k = \dfrac{1}{3}$

Find the vector ***â***

The unit vector is

$\hat{a} = \dfrac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{pmatrix}$

To get a vector of magnitude 5 we need to multiply the unit vector by 5

(b) Let $\underline{b}$ be parallel to $\underline{a}$ and $|\underline{b}| = 5$

Then $\underline{b} = 5\hat{a}$

$\therefore \underline{b} = \begin{pmatrix} \frac{10}{3} \\ \frac{-10}{3} \\ \frac{5}{3} \end{pmatrix}$

EXAM HINT

Note that part (b) has two possible answers, as ***b*** could be in the opposite direction. To get the second answer we would take the scalar to be –5 instead of 5.

The last example showed the general method for finding the unit vector in a given direction.

KEY POINT 13.5

The unit vector in the same direction as $\boldsymbol{a}$ is $\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Exercise 13C

1. Find the magnitude of the following vectors in two dimensions.

$\boldsymbol{a} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ $\quad \boldsymbol{b} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ $\quad \boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j}$ $\quad \boldsymbol{d} = -\boldsymbol{i} + \boldsymbol{j}$

2. Find the magnitude of the following vectors in three dimensions.

$\boldsymbol{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ $\quad \boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\quad \boldsymbol{c} = 2\boldsymbol{i} - 4\boldsymbol{j} + \boldsymbol{k}$ $\quad \boldsymbol{d} = \boldsymbol{j} - \boldsymbol{k}$

3. Find the distance between the following pairs of points in the plane.

(a) (i) A (1, 2) and B (3, 7) (ii) C (2, 1) and D (1, 2)

(b) (i) P (−1, −5) and Q (−4, 2) (ii) M (1, 0) and N (0, −2)

4. Find the distance between the following pairs of points in three dimensions.

(a) (i) A (1, 0, 2) and B (2, 3, 5)

(ii) C (2, 1, 7) and D (1, 2, 1)

(b) (i) P (3, −1,−5) and Q (−1, −4, 2)

(ii) M (0, 0, 2) and N (0, −3, 0)

5. Find the distance between the points with the given position vectors.

(a) $\boldsymbol{a} = 2\boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}$

(b) $\boldsymbol{a} = \begin{pmatrix} 3 \\ 7 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$

(c) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$

(d) $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{j} - \boldsymbol{k}$

6. (a) (i) Find a unit vector parallel to $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

(ii) Find a unit vector parallel to $6\boldsymbol{i} + 6\boldsymbol{j} - 3\boldsymbol{k}$.

(b) (i) Find a unit vector in the same direction as $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

(ii) Find a unit vector in the same direction as $\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$.

7. Find the possible values of the constant c such that the vector $\begin{pmatrix} 2c \\ c \\ -c \end{pmatrix}$ has magnitude 12. *[4 marks]*

8. Points A and B have position vectors $\boldsymbol{a} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

C is the midpoint of $[AB]$. Find the exact distance AC. *[4 marks]*

9. Let $\boldsymbol{a} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$. Find the possible values of λ such that $|\mathbf{a} + \lambda\mathbf{b}| = 5\sqrt{2}$. *[6 marks]*

10. (a) Find a vector of magnitude 6 parallel to $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$.

(b) Find a vector of magnitude 3 in the same direction as $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$. *[6 marks]*

11. Points A and B are such that $\overrightarrow{OA} = \begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$ and

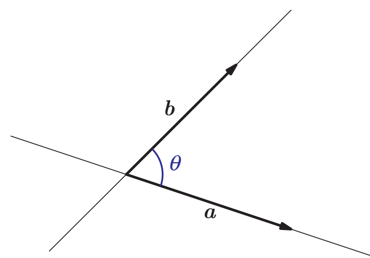
$\overrightarrow{OB} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$ where O is the origin.

Find the possible values of t such that $AB = 3$. *[5 marks]*

12. Points P and Q have position vectors $\boldsymbol{p} = \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{q} = (2+t)\boldsymbol{i} + (1-t)\boldsymbol{j} + (1+t)\boldsymbol{k}$. Find the value of t for which the distance PQ is the minimum possible and find this minimum distance. *[6 marks]*

13D Angles

In geometry problems you are often asked to find angles between two lines. The diagram shows two lines with angle θ between them. $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors in the directions of the two lines. Note that both arrows are pointing *away* from the intersection point. It turns out that $\cos\theta$ can be expressed in terms of the components of the two vectors. This result can be derived using the cosine rule. See Fill-in proof sheet 12 'Deriving scalar products' on the CD-ROM.



KEY POINT 13.6

If θ is the angle between vectors, $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{|\boldsymbol{a}||\boldsymbol{b}|}.$$

The expression in the numerator of the above fraction has many important uses, and is called the **scalar product**.

KEY POINT 13.7

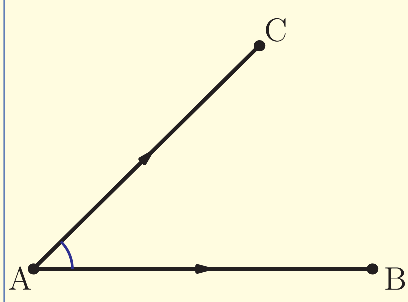
Scalar product

The quantity $a_1b_1 + a_2b_2 + a_3b_3$ is called the scalar product (**inner product**, or **dot product**) of $\boldsymbol{a}$ and $\boldsymbol{b}$ and denoted by $\boldsymbol{a}\bullet\boldsymbol{b}$.

Worked example 13.8

Given points A (3, −5, 2), B (4, 1, 1) and C (−1, 1, 2) find the size of the angle $B\hat{A}C$ in degrees.

It's always a good idea to draw a diagram to be sure which vectors the angle lies between



The required angle is between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

Let $\theta = B\hat{A}C$

$$\cos\theta = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|}$$

We need the components of vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix}4\\1\\1\end{pmatrix} - \begin{pmatrix}3\\-5\\2\end{pmatrix} = \begin{pmatrix}1\\6\\-1\end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix}-1\\1\\2\end{pmatrix} - \begin{pmatrix}3\\-5\\2\end{pmatrix} = \begin{pmatrix}-4\\6\\0\end{pmatrix}$$

$$\cos\theta = \frac{1\times(-4)+6\times 6+(-1)\times 0}{\sqrt{1^2+6^2+1^2}\sqrt{4^2+6^2+0^2}}$$

$$= \frac{32}{\sqrt{38}\sqrt{52}} = 0.7199$$

$$\therefore \theta = \arccos(0.7199) = 44.0°$$

It is very straightforward to check whether two vectors are perpendicular. If $\theta = 90°$ then $\cos\theta = 0$, so the top of the fraction in the formula for $\cos\theta$ must be zero. We do not even have to calculate the magnitudes of the two vectors.

KEY POINT 13.8

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a}\bullet\boldsymbol{b} = 0$.

Worked example 13.9

If $\boldsymbol{p} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ find the value of the scalar t such that $\boldsymbol{p} + t\boldsymbol{q}$ is perpendicular to $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$.

Two vectors are perpendicular if their dot product equals 0

$$(\underline{p} + t\underline{q}) \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 0$$

Find the components of $\boldsymbol{p} + t\boldsymbol{q}$ in terms of t

$$\underline{p} + t\underline{q} = \begin{pmatrix} 4+2t \\ -1+t \\ 2+t \end{pmatrix}$$

So

$$\begin{pmatrix} 4+2t \\ -1+t \\ 2+t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = 0$$

Form and solve the equation

$$\Leftrightarrow 3(4+2t) + 5(-1+t) + 1(2+t) = 0$$

$$\Leftrightarrow 9 + 12t = 0$$

$$\Leftrightarrow \quad t = -\frac{3}{4}$$

Exercise 13D

1. Calculate the angle between the following pairs of vectors, giving your answers in radians.

(a) (i) $\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

(b) (i) $2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

(ii) $3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{i} - 2\boldsymbol{k}$

(c) (i) $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 4 \end{pmatrix}$ (ii) $\boldsymbol{i} - \boldsymbol{j}$ and $2\boldsymbol{i} + 3\boldsymbol{j}$

2. The angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is θ. Find the exact value of $\cos\theta$ in the following cases:

(a) (i) $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$

(ii) $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 5\boldsymbol{j} - 2\boldsymbol{k}$

(b) (i) $\boldsymbol{a}=\begin{pmatrix}2\\2\\3\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}1\\1\\-2\end{pmatrix}$ (ii) $\boldsymbol{a}=\begin{pmatrix}5\\1\\-3\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}2\\-1\\2\end{pmatrix}$

(c) (i) $\boldsymbol{a}=-2\boldsymbol{k}$ and $\boldsymbol{b}=4\boldsymbol{i}$ (ii) $\boldsymbol{a}=5\boldsymbol{i}$ and $\boldsymbol{b}=3\boldsymbol{j}$

3. (a) The vertices of a triangle have position vectors

$$\boldsymbol{a}=\begin{pmatrix}1\\1\\3\end{pmatrix},\ \boldsymbol{b}=\begin{pmatrix}2\\-1\\1\end{pmatrix} \text{ and } \boldsymbol{c}=\begin{pmatrix}5\\1\\2\end{pmatrix}.$$

Find, in degrees, the angles of the triangle.

(b) Find, in degrees, the angles of the triangle with vertices (2, 1, 2), (4, –1, 5) and (7, 1, –2).

4. Which of the following pairs of vectors are perpendicular?

(a) (i) $\begin{pmatrix}2\\1\\-3\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\2\end{pmatrix}$ (ii) $\begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\6\\0\end{pmatrix}$

(b) (i) $5\boldsymbol{i}-2\boldsymbol{j}+\boldsymbol{k}$ and $3\boldsymbol{i}+4\boldsymbol{j}-7\boldsymbol{k}$

(ii) $\boldsymbol{i}-3\boldsymbol{k}$ and $2\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$

5. Points A and B have position vectors $\overrightarrow{OA}=\begin{pmatrix}2\\2\\3\end{pmatrix}$ and $\overrightarrow{OB}=\begin{pmatrix}-1\\7\\2\end{pmatrix}$.

Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{OA}$. *[5 marks]*

6. Four points are given with coordinates A (2, –1, 3), B (1, 1, 2), C (6, –1, 2) and D (7, –3, 3).
Find the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$. *[5 marks]*

7. Four points have coordinates A (2, 4, 1), B (k, 4, $2k$), C ($k+4$, $2k+4$, $2k+2$) and D (6, $2k+4$, 3).

(a) Show that $ABCD$ is a parallelogram for all values of k.

(b) When $k=1$ find the angles of the parallelogram.

(c) Find the value of k for which $ABCD$ is a rectangle.

[8 marks]

8. Vertices of a triangle have position vectors $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j} + 7\boldsymbol{k}$ and $\boldsymbol{c} = 5\boldsymbol{i}$.

(a) Show that the triangle is right-angled.

(b) Calculate the other two angles of the triangle.

(c) Find the area of the triangle. *[8 marks]*

13E Properties of the scalar product

In the last section we defined the scalar product of vectors

$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ as

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

and saw that if θ is the angle between the directions of $\boldsymbol{a}$ and $\boldsymbol{b}$ then:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta$$

In this section we look at various properties of the scalar product in more detail; in particular, its algebraic rules. The scalar product has many properties similar to multiplication of numbers. These properties can be proved by using components of the vectors.

KEY POINT 13.9

Algebraic properties of the scalar product

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

$$(-\boldsymbol{a}) \cdot \boldsymbol{b} = -(\boldsymbol{a} \cdot \boldsymbol{b})$$

$$(k\boldsymbol{a}) \cdot \boldsymbol{b} = k(\boldsymbol{a} \cdot \boldsymbol{b})$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{b}) + (\boldsymbol{a} \cdot \boldsymbol{c})$$

But there are some properties of multiplication of numbers which do *not* apply to scalar product. For example, it is not possible to calculate the scalar product of three vectors: the expression $(\boldsymbol{a} \cdot \boldsymbol{b}) \cdot \boldsymbol{c}$ has no meaning, as $\boldsymbol{a} \cdot \boldsymbol{b}$ is a scalar (and so has no direction), and scalar product involves multiplying two vectors.

Two important properties of scalar product concern perpendicular and parallel vectors. These are very useful when solving geometry problems.

All the operations with vectors work in both two and three dimensions. If there were a fourth dimension, the position of each point could be described using four numbers. We could use analogous rules to calculate 'distances' and 'angles'. Does this mean that we can acquire knowledge about a four-dimensional world which we can't see, or even imagine?

EXAM HINT

These are not in the formula booklet!

KEY POINT 13.10

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors then $\boldsymbol{a}\cdot\boldsymbol{b} = 0$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors then $\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$, in particular, $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$.

The following two examples show you how you can use these rules.

Worked example 13.10

Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors such that $|\boldsymbol{a}| = 5$ and $|\boldsymbol{b}| = 3$, evaluate $(2\boldsymbol{a} - \boldsymbol{b})\cdot(\boldsymbol{a} + 4\boldsymbol{b})$.

Multiply out the brackets as we would with numbers $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\boldsymbol{c}$

$(2\underline{a} - \underline{b})\cdot(\underline{a} + 4\underline{b}) = 2\underline{a}\cdot\underline{a} + 8\underline{a}\cdot\underline{b} - \underline{b}\cdot\underline{a} - 4\underline{b}\cdot\underline{b}$

$\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular so $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a} = 0$

$= 2\underline{a}\cdot\underline{a} - 4\underline{b}\cdot\underline{b}$

Use the fact that $\boldsymbol{a}\cdot\boldsymbol{a} = |\boldsymbol{a}|^2$

$= 2|\underline{a}|^2 - 4|\underline{b}|^2$

$= 2\times 5^2 - 4\times 3^2$

$= 14$

Worked example 13.11

Points A, B and C have position vectors $\boldsymbol{a} = k\begin{pmatrix}3\\-1\\1\end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix}3\\4\\-2\end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix}1\\1\\5\end{pmatrix}$.

(a) Find $\overrightarrow{BC}$.

(b) Find $\overrightarrow{AB}$ in terms of k.

(c) Find the value of k for which (AB) is perpendicular to (BC).

Use $\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$

(a) $\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b}$

$= \begin{pmatrix}1\\1\\5\end{pmatrix} - \begin{pmatrix}3\\4\\-2\end{pmatrix} = \begin{pmatrix}-2\\-3\\7\end{pmatrix}$

Use $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

(b) $\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$

$= \begin{pmatrix}3\\4\\-2\end{pmatrix} - \begin{pmatrix}3k\\-k\\k\end{pmatrix} = \begin{pmatrix}3-3k\\4+k\\-2-k\end{pmatrix}$

continued . . .

If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular then $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$

(c) $\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$

$$\begin{pmatrix} 3-3k \\ 4+k \\ -2-2k \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = 0$$

$\Rightarrow -6+6k-12-3k-14-14k=0$

$-11k = 32$

$\Rightarrow \quad k = -\frac{11}{32}$

Exercise 13E

1. Evaluate $\boldsymbol{a} \cdot \boldsymbol{b}$ in the following cases:

(a) (i) $\boldsymbol{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -12 \\ 4 \\ -8 \end{pmatrix}$

(b) (i) $\boldsymbol{a} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix}$

(c) (i) $\boldsymbol{a} = 4\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

(ii) $\boldsymbol{a} = 4\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

(d) (i) $\boldsymbol{a} = -3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 4\boldsymbol{k}$

(ii) $\boldsymbol{a} = -3\boldsymbol{j}$ and $\boldsymbol{b} = 4\boldsymbol{k}$

2. Given that θ is the angle between vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ find the exact value of $\cos\theta$.

(a) (i) $\boldsymbol{p} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{p} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(b) (i) $\boldsymbol{p} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ (ii) $\boldsymbol{p} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\boldsymbol{q} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

3. (a) Given that $|\boldsymbol{a}| = 3, |\boldsymbol{b}| = 5$ and $\boldsymbol{a} \cdot \boldsymbol{b} = 10$, find, in degrees, the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Given that $|\boldsymbol{c}| = 9, |\boldsymbol{d}| = 12$ and $\boldsymbol{c} \cdot \boldsymbol{d} = -15$, find, in degrees, the angle between $\boldsymbol{c}$ and $\boldsymbol{d}$.

4. (a) Given that $|\boldsymbol{a}| = 6$, $|\boldsymbol{b}| = 4$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $37°$, calculate $\boldsymbol{a} \cdot \boldsymbol{b}$.

(b) Given that $|\boldsymbol{a}| = 8$, $\boldsymbol{a} \cdot \boldsymbol{b} = 12$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $60°$, find the exact value of $|\boldsymbol{b}|$.

5. Given that $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{c} = 5\boldsymbol{i} - 3\boldsymbol{k}$ and $\boldsymbol{d} = -2\boldsymbol{j} + \boldsymbol{k}$ verify that:

(a) $\boldsymbol{b} \cdot \boldsymbol{d} = \boldsymbol{d} \cdot \boldsymbol{b}$

(b) $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$

(c) $(\boldsymbol{c} - \boldsymbol{d}) \cdot \boldsymbol{c} = |\boldsymbol{c}|^2 - \boldsymbol{c} \cdot \boldsymbol{d}$

(d) $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{b}) = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 + 2\boldsymbol{a} \cdot \boldsymbol{b}$

6. Find the values of t for which the following pairs of vectors are perpendicular.

(a) (i) $\begin{pmatrix} 2t \\ 1 \\ -3t \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ (ii) $\begin{pmatrix} t+1 \\ 2t-1 \\ 2t \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$

(b) (i) $5t\boldsymbol{i} - (2+t)\boldsymbol{j} + \boldsymbol{k}$ and $3\boldsymbol{i} + 4\boldsymbol{j} - t\boldsymbol{k}$

(ii) $t\boldsymbol{i} - 3\boldsymbol{k}$ and $2t\boldsymbol{i} + \boldsymbol{j} + t\boldsymbol{k}$

We will meet equations of this type in the next chapter. See Worked example 14.17.

7. In this question, we will introduce a method using scalar product to find x and y such that $\begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 2x \\ -3x \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + \begin{pmatrix} 3y \\ y \end{pmatrix}$, and then use it to solve other similar equations.

(a) Use the usual method of simultaneous equations to find x and y.

(b) (i) Find the scalar product of both sides of the equation with $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and hence find x.

(ii) Find the scalar product of both sides of the equation with $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and hence find y.

(iii) Can you see why the vectors $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ were selected?

(c) (i) Find a vector perpendicular to $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and a vector perpendicular to $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$.

(ii) Hence find x and y such that

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + x\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + y\begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

8. Given that $\boldsymbol{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\boldsymbol{c} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$ and $\boldsymbol{d} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$: calculate:

(a) $\boldsymbol{a} \bullet (\boldsymbol{b} + \boldsymbol{c})$

(b) $(\boldsymbol{b} - \boldsymbol{a}) \bullet (\boldsymbol{d} - \boldsymbol{c})$

(c) $(\boldsymbol{b} + \boldsymbol{d}) \bullet (2\boldsymbol{a})$ *[7 marks]*

9. (a) If $\boldsymbol{a}$ is a unit vector perpendicular to $\boldsymbol{b}$, find the value of $\boldsymbol{a} \bullet (2\boldsymbol{a} - 3\boldsymbol{b})$.

(b) If $\boldsymbol{p}$ is a unit vector making a 45° angle with vector $\boldsymbol{q}$ and $\boldsymbol{p} \bullet \boldsymbol{q} = 3\sqrt{2}$, find $|\boldsymbol{q}|$. *[6 marks]*

10. (a) $\boldsymbol{a}$ is a vector of magnitude 3 and $\boldsymbol{b}$ makes an angle of 60° with $\boldsymbol{a}$. Given that $\boldsymbol{a} \bullet (\boldsymbol{a} - \boldsymbol{b}) = \frac{1}{3}$, find the exact value of $|\boldsymbol{b}|$.

(b) Given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are two vectors of equal magnitude such that $(3\boldsymbol{a} + \boldsymbol{b})$ is perpendicular to $(\boldsymbol{a} - 3\boldsymbol{b})$, prove that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular. *[6 marks]*

11. Points A, B and C have position vectors $\boldsymbol{a} = \boldsymbol{i} - 19\boldsymbol{j} + 5\boldsymbol{k}$, $\boldsymbol{b} = 2\lambda\boldsymbol{i} + (\lambda + 2)\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{c} = -6\boldsymbol{i} - 15\boldsymbol{j} + 7\boldsymbol{k}$.

(a) Find the value of λ for which BC is perpendicular to AC.

For the value of λ found above:

(b) find the angles of the triangle ABC

(c) find the area of the triangle ABC. *[8 marks]*

12. $ABCD$ is a parallelogram with AB parallel to DC. Let $\overrightarrow{AB} = \boldsymbol{a}$ and $\overrightarrow{AD} = \boldsymbol{b}$.

(a) Express $\overrightarrow{AC}$ and $\overrightarrow{BD}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

(b) Simplify $(\boldsymbol{a} + \boldsymbol{b}) \bullet (\boldsymbol{b} - \boldsymbol{a})$.

(c) Hence show that if $ABCD$ is a rhombus then its diagonals are perpendicular. *[8 marks]*

13. Points A and B have position vectors $\begin{pmatrix}2\\1\\4\end{pmatrix}$ and $\begin{pmatrix}2\lambda\\\lambda\\4\lambda\end{pmatrix}$.

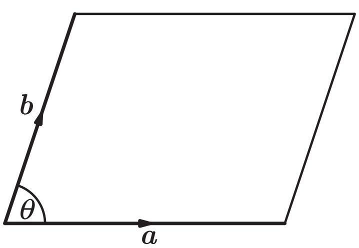
(a) Show that B lies on the line OA for all values of λ.

Point C has position vector $\begin{pmatrix}12\\2\\4\end{pmatrix}$.

(b) Find the value of λ for which $C\hat{B}A$ is a right angle.

(c) For the value of λ found above, calculate the exact distance from C to the line OA. *[8 marks]*

13F Areas



Given the coordinates of the four vertices of a parallelogram, how can we calculate its area? The area of the parallelogram is given by $ab\sin\theta$ where a and b are the lengths of the sides and θ is the angle between them. We could use the coordinates of the vertices to find the lengths of the sides, and then use the cosine rule to find angle θ. However, using vectors gives a quicker way to calculate the area.

KEY POINT 13.11

The area of the parallelogram with sides defined by vectors

$\boldsymbol{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$ is equal to the magnitude of the

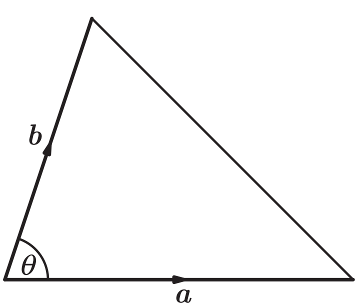
vector $\begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix}$.

This vector is called the **vector product** (or **cross product**) of $\boldsymbol{a}$ and $\boldsymbol{b}$ and denoted by $\boldsymbol{a} \times \boldsymbol{b}$.

EXAM HINT

Notice that Notice that vectors a and b form two adjacent sides of the parallelogram. We can use any pair of adjacent sides.

A parallelogram can be divided in half to form two triangles, so we can also use the vector product to calculate the area of a triangle.



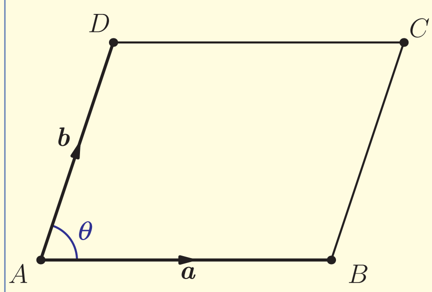
KEY POINT 13.12

The area of the triangle with two sides defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to $\frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}|$.

Worked example 13.12

Find the area of the parallelogram with vertices $A\begin{pmatrix}1\\4\\2\end{pmatrix}$, $B\begin{pmatrix}3\\-3\\0\end{pmatrix}$, $C\begin{pmatrix}1\\1\\7\end{pmatrix}$, $D\begin{pmatrix}-1\\8\\9\end{pmatrix}$.

Draw a diagram to show which two vectors to use: we can choose any two adjacent sides of the parallelogram, e.g. *AB* and *AD*



$$\boldsymbol{a} = \overrightarrow{AB} = \begin{pmatrix}2\\-7\\-2\end{pmatrix}$$

$$\boldsymbol{b} = \overrightarrow{AD} = \begin{pmatrix}-2\\4\\7\end{pmatrix}$$

Calculate $\boldsymbol{a}\times\boldsymbol{b}$ first, and then find its magnitude

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{pmatrix}2\\-7\\-2\end{pmatrix}\times\begin{pmatrix}-2\\4\\7\end{pmatrix}$$

$$= \begin{pmatrix}-49+8\\4-14\\8-14\end{pmatrix} = \begin{pmatrix}-41\\-10\\-6\end{pmatrix}$$

$$\text{Area} = |\boldsymbol{a}\times\boldsymbol{b}| = \sqrt{41^2+10^2+6^2} = 55.4$$

Exercise 13F

1. Calculate $\boldsymbol{a}\times\boldsymbol{b}$ for the following pairs of vectors:

(a) (i) $\boldsymbol{a} = \begin{pmatrix}2\\1\\2\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}5\\2\\2\end{pmatrix}$ (ii) $\boldsymbol{a} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix}-12\\4\\-8\end{pmatrix}$

(b) (i) $\boldsymbol{a} = 4\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$

(ii) $\boldsymbol{a} = -3\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 4\boldsymbol{k}$

2. Calculate the area of the triangle with vertices:

(a) (i) (1, 3, 3), (−1, 1, 2) and (1, −2, 4)
(ii) (3, −5, 1), (−1, 1, 3) and (−1, 5, 2)

(b) (i) (−3, −5, 1), (4, 7, 2) and (−1, 2, 2)
(ii) (4, 0, 2), (4, 1, 5) and (4, −3, 2)

3. Given points A, B and C with coordinates (3, −5, 1), (7, 7, 2) and (−1, 1, 3).

(a) calculate $\boldsymbol{p} = \overrightarrow{AB} \times \overrightarrow{AC}$ and $\boldsymbol{q} = \overrightarrow{BA} \times \overrightarrow{BC}$.

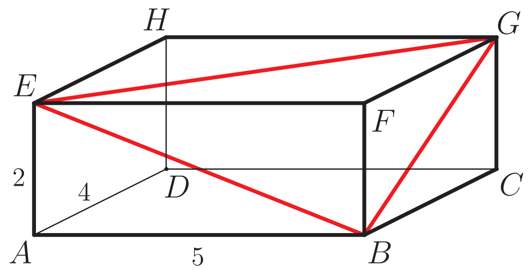
(b) What can you say about vectors $\boldsymbol{p}$ and $\boldsymbol{q}$?

4. The points A (3, 1, 2), B (−1, 1, 5) and C (7, 2, 3) are vertices of a parallelogram $ABCD$.

(a) Find the coordinates of D.

(b) Calculate the area of the parallelogram.

5. A cuboid $ABCDEFGH$ is shown in the diagram. The coordinates of four of the vertices are A (0, 0, 0), B (5, 0, 0), C (5, 4, 0) and E (0, 0, 2).



(a) Find the coordinates of the remaining four vertices.

Face diagonals BE, BG and EG are drawn as shown.

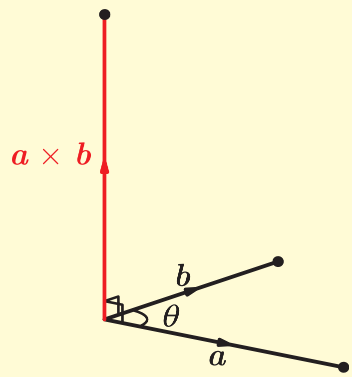
(b) Find the area of the triangle BEG.

13G Properties of the vector product

In this section we look at more properties of the vector product $\boldsymbol{a} \times \boldsymbol{b}$. We have already seen that the magnitude of this vector is equal to the area of the parallelogram defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

KEY POINT 13.13

The vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$, denoted by $\boldsymbol{a}\times\boldsymbol{b}$, has magnitude $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The direction of $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown in the diagram.



In component form, $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix} a_2b_3-a_3b_2 \\ a_3b_1-a_1b_3 \\ a_1b_2-a_2b_1 \end{pmatrix}$.

If you study physics, you may have come across the 'right-hand rule' for the direction of the magnetic field. This is just one example of application of vector product; others include circular motion, fluid dynamics and Maxwell's theory of electromagnetism.

EXAM HINT

The Formula booklet gives you the equations, but not the diagrams.

Worked example 13.13

Find the exact value of the sine of the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ given that

$|\boldsymbol{a}|=3$, $|\boldsymbol{b}|=2$ and $\boldsymbol{a}\times\boldsymbol{b}=\begin{pmatrix}3\\6\\1\end{pmatrix}$.

We are not given the components of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ so need to use the definition of the vector product involving magnitudes

$$|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$$
$$\sqrt{3^2+6^2+1^2}=(3\times2)\sin\theta$$
$$\sqrt{46}=6\sin\theta$$
$$\therefore \sin\theta=\frac{\sqrt{46}}{6}$$

The fact that the vector product is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ is very useful.

Worked example 13.14

Find a vector perpendicular to both $\boldsymbol{a} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$.

The vector product of two vectors is perpendicular to both of them

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -5 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 7 \\ 6 \end{pmatrix}$$

You may find the two different ways of multiplying vectors confusing. However, if you think about normal multiplication, you will realise that it can have at least two very different interpretations: It can be considered as repeated addition, taking two numbers and producing a third number as the answer; or the result can represent the area of a rectangle with given lengths of sides. The two 'types' of multiplication of numbers just happen to give the same numerical answer.

The vector product has many properties similar to multiplication of numbers, but the most important difference is that $\boldsymbol{a} \times \boldsymbol{b}$ is not the same as $\boldsymbol{b} \times \boldsymbol{a}$.

KEY POINT 13.14

Algebraic properties of vector product

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$
$$(k\boldsymbol{a}) \times \boldsymbol{b} = k(\boldsymbol{a} \times \boldsymbol{b})$$
$$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{a} \times \boldsymbol{c})$$

With the vector product it is possible to multiply three vectors together, but $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ is not the same as $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$.

Again, there are special results concerning parallel and perpendicular vectors.

KEY POINT 13.15

If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\boldsymbol{a} \times \boldsymbol{b} = 0$. In particular, $\boldsymbol{a} \times \boldsymbol{a} = \mathbf{0}$.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular then $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|$.

Worked example 13.15

Given that $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, evaluate $|(2\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + 3\boldsymbol{b})|$.

Expand the brackets as we would with numbers
$\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{C}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{C}$

$$(2\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + 3\boldsymbol{b}) = 2\boldsymbol{a} \times \boldsymbol{a} + 6\boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{b} \times \boldsymbol{a} - 3\boldsymbol{b} \times \boldsymbol{b}$$

continued . . .

$\boldsymbol{a} \times \boldsymbol{a} = 0$

$= 6\boldsymbol{a} \times \boldsymbol{b} - \boldsymbol{b} \times \boldsymbol{a}$

$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$

$= 7\boldsymbol{a} \times \boldsymbol{b}$

For perpendicular vectors, $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}||\boldsymbol{b}|$

$\therefore |(2\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + 3\boldsymbol{b})| = 7|\boldsymbol{a} \times \boldsymbol{b}|$

$= 7|\boldsymbol{a}||\boldsymbol{b}| = 140$

Exercise 13G

1. Find, in radians, the acute angle between the directions of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ given that:

   (a) $|\boldsymbol{a}| = 2$, $|\boldsymbol{b}| = 5$ and $|\boldsymbol{a} \times \boldsymbol{b}| = 7$

   (b) $|\boldsymbol{a}| = 12$, $|\boldsymbol{b}| = 3$ and $\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

   (c) $|\boldsymbol{a}| = 7$, $|\boldsymbol{b}| = 1$ and $\boldsymbol{a} \times \boldsymbol{b} = 2\boldsymbol{i} - 3\boldsymbol{j} - 2\boldsymbol{k}$

   (d) $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 4$ and $|\boldsymbol{a} \times \boldsymbol{b}| = 0$

2. Find a vector perpendicular to the following pairs of vectors:

   (a) (i) $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$

   (b) (i) $\begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$

3. Find the unit vector perpendicular to the following pairs of vectors:

   (a) $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

4. Given that $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 7$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$ find the exact value of $|\boldsymbol{a} \times \boldsymbol{b}|$. *[4 marks]*

5. (a) Prove that $(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{a} + \boldsymbol{b}) = 2\boldsymbol{a} \times \boldsymbol{b}$.

   (b) Simplify $(2\boldsymbol{a} - 3\boldsymbol{b}) \times (3\boldsymbol{a} + 2\boldsymbol{b})$. *[6 marks]*

6. (a) Explain why $(\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{a} = 0$.

(b) Evaluate $(\boldsymbol{a} \times \boldsymbol{b}) \bullet (\boldsymbol{a} - \boldsymbol{b})$. *[5 marks]*

7. Prove that for any two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$,
$|\boldsymbol{a} \times \boldsymbol{b}|^2 + (\boldsymbol{a} \bullet \boldsymbol{b})^2 = |\boldsymbol{a}|^2|\boldsymbol{b}|^2$. *[5 marks]*

Summary

- The position of B relative to A can be represented by the **vector displacement** $\overrightarrow{AB}$.
- Vectors can be expressed in terms of **base vectors** $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ or as **column vectors** using **components**. For example, $\overrightarrow{AB}$ can be represented by $(3\boldsymbol{i} + 2\boldsymbol{j})$ or $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. The numbers 3 and 2 are the components.
- Vectors can represent the position of points relative to the **origin**; the displacement of a point from the origin is the point's **position vector**.
- The position vector of the midpoint of $[AB]$ is $\frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$.
- The displacement between points A and B with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{b} - \boldsymbol{a}$.
- The distance between the points with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by $|\boldsymbol{b} - \boldsymbol{a}|$.
- The **unit vector** in the same direction as $\boldsymbol{a}$ is $\hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.
- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel we can write $\boldsymbol{b} = t\boldsymbol{a}$ for some scalar t.
- The magnitude of a vector $\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- The angle θ, between the directions of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, is given by: $\cos\theta = \frac{\boldsymbol{a} \bullet \boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ where $\boldsymbol{a} \bullet \boldsymbol{b}$ is the **scalar product** (dot product), given in terms of the components by: $\boldsymbol{a} \bullet \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \bullet \boldsymbol{b} = 0$.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel vectors then $\boldsymbol{a} \bullet \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$, in particular, $\boldsymbol{a} \bullet \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Algebraic properties of scalar product:

$$\boldsymbol{a} \bullet \boldsymbol{b} = \boldsymbol{b} \bullet \boldsymbol{a}$$
$$(-\boldsymbol{a}) \bullet \boldsymbol{b} = -(\boldsymbol{a} \bullet \boldsymbol{b})$$
$$(k\boldsymbol{a}) \bullet \boldsymbol{b} = k(\boldsymbol{a} \bullet \boldsymbol{b})$$
$$\boldsymbol{a} \bullet (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} \bullet \boldsymbol{b}) + (\boldsymbol{a} \bullet \boldsymbol{c})$$

- The area of the parallelogram with sides defined by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is equal to the magnitude of the vector $\boldsymbol{a} \times \boldsymbol{b}$, which is called the **vector product** (or cross product). Its direction is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$, and it is given by:

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

- The vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ has magnitude $|\boldsymbol{a}||\boldsymbol{b}|\sin\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. The direction of $\boldsymbol{a}\times\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Algebraic properties of vector product:

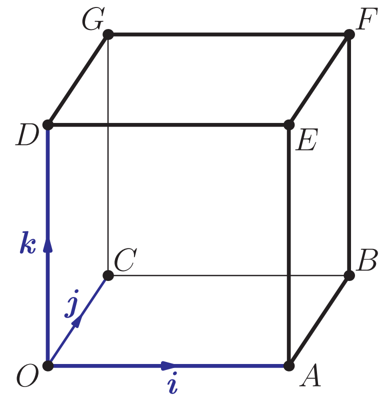
$$\boldsymbol{a}\times\boldsymbol{b}=-\boldsymbol{b}\times\boldsymbol{a}$$
$$(k\boldsymbol{a})\times\boldsymbol{b}=k\ (\boldsymbol{a}\times\boldsymbol{b})$$
$$\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b})+(\boldsymbol{a}\times\boldsymbol{c})$$

- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel then $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{0}$. In particular, $\boldsymbol{a}\times\boldsymbol{a}=0$. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular then $|\boldsymbol{a}\times\boldsymbol{b}|=|\boldsymbol{a}||\boldsymbol{b}|$.

Introductory problem revisited

What is the angle between the diagonals of a cube?

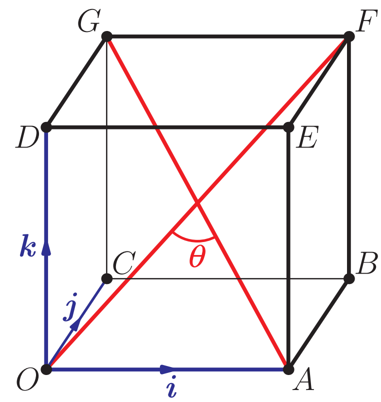
You can solve this problem by using the cosine rule in a triangle made by the diagonals and one side. However, using vectors gives a slightly faster solution, as we do not have to find the lengths of the sides of the triangle.



The angle between two lines can be found by using the vectors in the directions of two lines and the formula involving the scalar product. We do not know the actual positions of the vertices of the cube, or even the lengths of its sides, but as the answer does not depend on the size of the cube, we can look at the cube with side length 1, set with one vertex at the origin and sides parallel to the base vectors.

We want to find the angle between the diagonals OF and AG, so we need the coordinates of those four vertices. They are: $O(0, 0, 0)$, $A(1, 0, 0)$, $F(1, 1, 1)$, $G(0, 1, 1)$.

The required angle θ is between the lines OF and AG.



The corresponding vectors are: $\overrightarrow{OF}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\overrightarrow{AG}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$

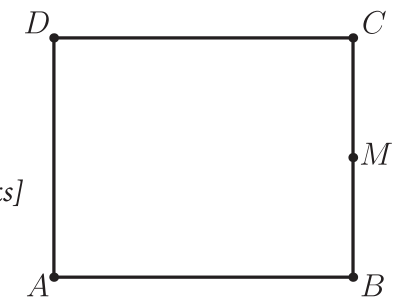
$$\cos\theta=\frac{\overrightarrow{OF}\cdot\overrightarrow{AG}}{|\overrightarrow{OF}||\overrightarrow{AG}|}$$
$$=\frac{-1+1+1}{\sqrt{3}\sqrt{3}}=\frac{1}{3}$$
$$\therefore\theta=70.5°$$

Mixed examination practice 13

Short questions

1. Given that $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 3\boldsymbol{k}$, $\boldsymbol{b} = -3\boldsymbol{i} + 2\boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} - 3\boldsymbol{j} + 4\boldsymbol{k}$, find $(\boldsymbol{a} \times \boldsymbol{b}) \bullet \boldsymbol{c}$. *[6 marks]*

2. The diagram shows a rectangle $ABCD$. M is the midpoint of BC.



(a) Express MD in terms of $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

(b) Given that $AB = 6$ and $AD = 4$, show that $\overrightarrow{MD} \bullet \overrightarrow{MC} = 4$. *[5 marks]*

3. The position vectors of points N and L are:

$$\boldsymbol{n} = 2\boldsymbol{i} - 5\boldsymbol{j} + \boldsymbol{k}$$

$$\boldsymbol{l} = \boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

(a) Find the vector product $\boldsymbol{n} \times \boldsymbol{l}$.

(b) Using your answer to part (a), or otherwise, find the area of the parallelogram with two sides $\overline{ON}$ and $\overline{OL}$. *[6 marks]*

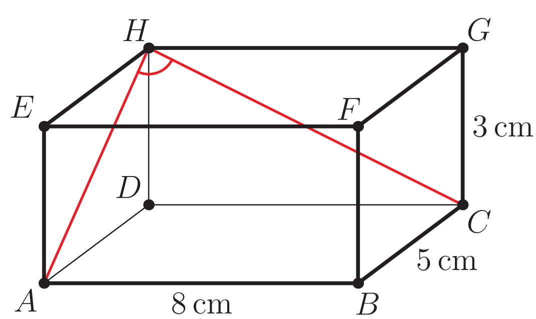
4. Let $\boldsymbol{a} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} -1 \\ 5 \\ p \end{pmatrix}$ and $\boldsymbol{c} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}$.

(a) Find $\boldsymbol{a} \times \boldsymbol{b}$.

(b) Find the value of p, given that $\boldsymbol{a} \times \boldsymbol{b}$ is parallel to $\boldsymbol{c}$. *[6 marks]*

5. The rectangle box shown in the diagram has dimensions 8 cm × 5 cm × 3 cm.

Find, correct to the nearest one-tenth of a degree, the size of the angle $A\hat{H}C$. *[6 marks]*



6. Let α be the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, where $\boldsymbol{a} = (\cos\theta)\boldsymbol{i} + (\sin\theta)\boldsymbol{j}$ and $\boldsymbol{b} = (\sin\theta)\boldsymbol{i} + (\cos\theta)\boldsymbol{j}$ and $0 < \theta < \pi/4$. Express α in terms of θ.
(*© IB Organization 2000*) *[6 marks]*

7. Given two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, such that $|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}-\boldsymbol{b}|$, find the value of $\boldsymbol{a}\cdot\boldsymbol{b}$.
(*© IB Organization 2002*) *[6 marks]*

8. (a) Show that $(\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{b}-\boldsymbol{a}) = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2\boldsymbol{a}\cdot\boldsymbol{b}$.

(b) In triangle MNP, $M\hat{P}N = \theta$. Let $\overrightarrow{PM} = \boldsymbol{a}$ and $\overrightarrow{PN} = \boldsymbol{b}$. Use the result from part (a) to prove the cosine rule: $MN^2 = PM^2 + PN^2 - 2PM\,PN\cos\theta$. *[6 marks]*

Long questions

1. Points A, B and D have coordinates (1, 1, 7), (−1, 6, 3) and (3, 1, k), respectively. AD is perpendicular to AB.

(a) Write down, in terms of $\boldsymbol{k}$, the vector $\overrightarrow{AD}$.

(b) Show that $\boldsymbol{k} = 6$.

Point C is such that $\overrightarrow{BC} = 2\overrightarrow{AD}$.

(c) Find the coordinates of C.

(d) Find the exact value of $\cos(A\hat{D}C)$. *[10 marks]*

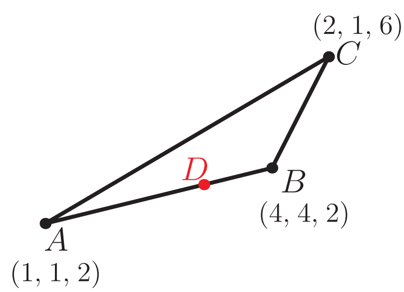
2. A triangle has vertices $A(1, 1, 2)$, $B(4, 4, 2)$ and $C(2, 1, 6)$. Point D lies on the side AB and $AD:DB = 1:k$.

(a) Find $\overrightarrow{CD}$ in terms of k.

(b) Find the value of k such that CD is perpendicular to AB.

(c) For the above value of k, find the coordinates of D.

(d) Hence find the length of the altitude from vertex C. *[10 marks]*



3. Point P lies on the parabola $y = x^2$ and has x-coordinate $a\,(a > 0)$.

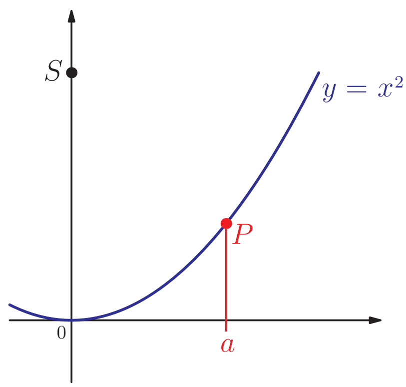
(a) Write down, in terms of a, the coordinates of P.

Point S has coordinates (0, 4) and O is the origin.

(b) Write down the vectors $\overrightarrow{PO}$ and $\overrightarrow{PS}$.

(c) Use scalar product to find the value of a for which OP is perpendicular to PS.

(d) For the value of a found above, calculate the exact area of the triangle OPS.



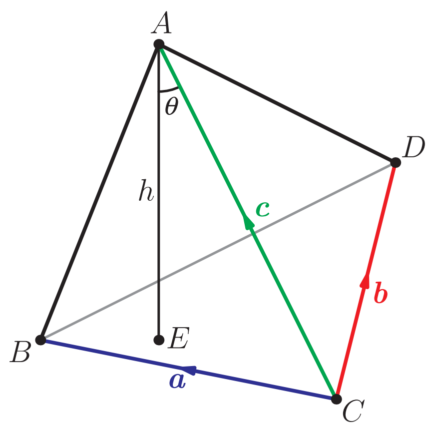
[10 marks]

4. Consider the tetrahedron shown in the diagram and define vectors $\boldsymbol{a} = \overrightarrow{CB}$, $\boldsymbol{b} = \overrightarrow{CD}$ and $\boldsymbol{c} = \overrightarrow{CA}$.

(a) Write down an expression for the area of the base in terms of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ only.

(b) AE is the height of the tetrahedron, $|AE| = h$ and $C\hat{A}E = \theta$. Express h in terms of $\boldsymbol{c}$ and θ.

(c) Use the results of (a) and (b) to prove that the volume of the tetrahedron is given by $\frac{1}{6}|\boldsymbol{a}\times\boldsymbol{b}|\cdot\boldsymbol{c}$.



(d) Find the volume of the tetrahedron with vertices (0, 4, 0), (0, 6, 0) (1, 6, 1) and (3, –1, 2).

[14 marks]

(e) 1.06

(f) 0.058, 0.557

4. (a) $r = 2, \alpha = \frac{\pi}{6}$

(b) $[-2, 2]$

(c) $\frac{\pi}{2}, \frac{7\pi}{6}$

5. (a) $(t+1)(t^2 - 4t + 1)$

(c) 1

(d) $\tan 15° = 2 - \sqrt{3}$,

$\tan 75° = 2 + \sqrt{3}$

Chapter 13

Exercise 13A

1. (a) (i) $\boldsymbol{b}$ (ii) $\boldsymbol{a} + \boldsymbol{b}$

(b) (i) $-\boldsymbol{a}$ (ii) $-\frac{1}{2}\boldsymbol{a}$

(c) (i) $\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

(ii) $\frac{1}{2}\boldsymbol{b} - \frac{1}{2}\boldsymbol{a}$

2. (a) (i) $\boldsymbol{a} + \frac{4}{3}\boldsymbol{b}$ (ii) $\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

(b) (i) $-\frac{3}{2}\boldsymbol{a} + \boldsymbol{b}$

(ii) $-\frac{1}{2}\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}$

(c) (i) $\frac{3}{2}\boldsymbol{a} - \boldsymbol{b}$

(ii) $-\frac{4}{3}\boldsymbol{b} + \frac{1}{2}\boldsymbol{a}$

3. (a) (i) $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix}$

(b) (i) $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$

4. (a) $\boldsymbol{b} - \boldsymbol{a}$

(b) $\frac{1}{2}\boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

(c) $4\boldsymbol{a} - 3\boldsymbol{b}$

5. (a) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$

(b) $(10, -2)$

6. (a) $\begin{pmatrix} 1 \\ -3 \\ 7 \end{pmatrix}$ (b) $\begin{pmatrix} 3.5 \\ -0.5 \\ 1.5 \end{pmatrix}$

7. $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$

8. $\begin{pmatrix} 1.6 \\ 0.8 \\ 1.8 \end{pmatrix}$

9. (a) $\frac{3}{2}\boldsymbol{i} + \frac{3}{2}\boldsymbol{j} - 2\boldsymbol{k}$

(b) $\left(\frac{1}{2}, \frac{13}{2}, 0\right)$

10. $\begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}$

Exercise 13B

1. (a) (i) $\begin{pmatrix} 21 \\ 3 \\ 36 \end{pmatrix}$ (ii) $\begin{pmatrix} 20 \\ -8 \\ 12 \end{pmatrix}$

(b) (i) $\begin{pmatrix} 2 \\ 3 \\ 9 \end{pmatrix}$ (ii) $\begin{pmatrix} 6 \\ -1 \\ 5 \end{pmatrix}$

(c) (i) $\begin{pmatrix} 11 \\ -3 \\ 8 \end{pmatrix}$ (ii) $\begin{pmatrix} -3 \\ 5 \\ 6 \end{pmatrix}$

(d) (i) $\begin{pmatrix} 10 \\ -3 \\ 11 \end{pmatrix}$ (ii) $\begin{pmatrix} 17 \\ 6 \\ 35 \end{pmatrix}$

2. (a) (i) $-5\boldsymbol{i} + 5\boldsymbol{k}$ (ii) $4\boldsymbol{i} + 8\boldsymbol{j}$

(b) (i) $\boldsymbol{i} - 3\boldsymbol{j} + 3\boldsymbol{k}$ (ii) $2\boldsymbol{j} + \boldsymbol{k}$

(c) (i) $4\boldsymbol{i} + 7\boldsymbol{k}$

(ii) $5\boldsymbol{i} - 4\boldsymbol{j} + 15\boldsymbol{k}$

3. (a) $-4\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$

(b) $-\frac{8}{3}\boldsymbol{i} + \frac{4}{3}\boldsymbol{j} - \frac{2}{3}\boldsymbol{k}$

(c) $4\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$

(d) $-\frac{1}{2}\boldsymbol{i} + \boldsymbol{j} - \frac{1}{2}\boldsymbol{k}$

4. $\begin{pmatrix} 2 \\ 0 \\ \frac{-3}{4} \end{pmatrix}$

5. -2

6. $-\frac{4}{3}$

7. -2

8. $p=\frac{3}{8}, q=\frac{1}{8}$

Exercise 13C

1. $|\boldsymbol{a}|=2\sqrt{5}$ $|\boldsymbol{b}|=\sqrt{26}$ $|\boldsymbol{c}|=2\sqrt{5}$ $|\boldsymbol{d}|=\sqrt{2}$

2. $|\boldsymbol{a}|=\sqrt{21}$ $|\boldsymbol{b}|=\sqrt{2}$ $|\boldsymbol{c}|=\sqrt{21}$ $|\boldsymbol{d}|=\sqrt{2}$

3. (a) (i) $\sqrt{29}$ (ii) $\sqrt{2}$
   (b) (i) $\sqrt{58}$ (ii) $\sqrt{5}$

4. (a) (i) $\sqrt{19}$ (ii) $\sqrt{38}$
   (b) (i) $\sqrt{74}$ (ii) $\sqrt{13}$

5. (a) $\sqrt{53}$ (b) $\sqrt{94}$
   (c) $\sqrt{53}$ (d) $\sqrt{2}$

6. (a) (i) $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ (ii) $\frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$
   (b) (i) $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
   (ii) $\frac{1}{5}\begin{pmatrix} 4 \\ -1 \\ 2\sqrt{2} \end{pmatrix}$

7. $\pm 2\sqrt{6}$

8. $\frac{3}{2}$

9. $3, -\frac{5}{3}$

10. (a) $\begin{pmatrix} 4\sqrt{2} \\ -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$ (b) $\frac{\sqrt{6}}{2}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

11. $-2, -\frac{23}{15}$

12. $t=\frac{1}{3}$, $d=\sqrt{\frac{14}{3}}$

Exercise 13D

1. (a) (i) 1.12 (ii) 1.17
   (b) (i) 1.88 (ii) 1.13
   (c) (i) 1.23 (ii) 1.77

2. (a) (i) $-\frac{5}{2\sqrt{21}}$
   (ii) $-\frac{20}{\sqrt{570}}$
   (b) (i) $-\frac{2}{\sqrt{102}}$
   (ii) $\frac{1}{\sqrt{35}}$
   (c) (i) $\frac{1}{\sqrt{50}}$ (ii) 0

3. (a) $61.0°, 74.5°, 44.5°$
   (b) $94.3°, 54.2°, 31.5°$

4. (a) (i) No (ii) Yes
   (b) (i) Yes (ii) No

5. $92.3°$

6. $40.0°$

7. (b) $107°, 73.2°$
   (c) $\frac{5}{4}$

8. (b) $41.8°, 48.2°$
   (c) $6\sqrt{5}$

Exercise 13E

1. (a) (i) 16 (ii) -56
   (b) (i) 16 (ii) -16
   (c) (i) 9 (ii) 9
   (d) (i) -4 (ii) 0

2. (a) (i) $\frac{7}{3\sqrt{6}}$ (ii) $\frac{5}{\sqrt{39}}$
   (b) (i) $\frac{2}{3}$ (ii) $\frac{1}{\sqrt{10}}$

3. (a) (i) $48.2°$ (ii) $98.0°$

4. (a) 19.2 (b) 3

6. (a) (i) $-\frac{1}{2}$ (ii) $\frac{2}{7}$
   (b) (i) $\frac{4}{5}$ (ii) $0, \frac{3}{2}$

7. (a) $x = -\frac{18}{11}$ $y = \frac{10}{11}$
(b) (i) $1 + 11x = -17$
(ii) $14 = 4 + 11y$
(iii) They are perpendicular to $\begin{pmatrix} 3y \\ y \end{pmatrix}$ and $\begin{pmatrix} 2x \\ -3x \end{pmatrix}$, respectively.
(c) (i) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$, and $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ or $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$
(ii) $x = -\frac{13}{11}$ $y = -\frac{3}{11}$

8. (a) 19 (b) 7
(c) 32

9. (a) 2 (b) 6

10. (a) $\frac{52}{9}$

11. (a) 1.6
(b) 68.7°, 21.3°, 90°
(c) 88.7

12. (a) $\boldsymbol{a} + \boldsymbol{b}$, $\boldsymbol{b} - \boldsymbol{a}$
(b) $|\boldsymbol{b}|^2 - |\boldsymbol{a}|^2$

13. (b) 2
(c) $4\sqrt{5}$

Exercise 13F

1. (a) (i) $\begin{pmatrix} -2 \\ 6 \\ -1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
(b) (i) $\begin{pmatrix} -5 \\ -11 \\ -2 \end{pmatrix}$ (ii) $\begin{pmatrix} 12 \\ 2 \\ 6 \end{pmatrix}$

2. (a) (i) $\frac{1}{2}\sqrt{153}$ (ii) $\sqrt{117}$
(b) (i) $\frac{15\sqrt{3}}{2}$ (ii) $\frac{9}{2}$

3. (a) $\begin{pmatrix} 18 \\ -12 \\ 72 \end{pmatrix}, \begin{pmatrix} -18 \\ 12 \\ -72 \end{pmatrix}$
(b) $\boldsymbol{p} = -\boldsymbol{q}$

4. (a) (11,2,0) (b) 16.8

5. (a) $C(5,4,0), F(5,0,2), G(5,4,2), H(0,4,2)$
(b) 11.9

Exercise 13G

1. (a) 0.775 (b) 0.128
(c) 0.630 (d) 0

2. (a) (i) $\begin{pmatrix} -1 \\ -10 \\ 7 \end{pmatrix}$ (ii) $\begin{pmatrix} -9 \\ -19 \\ 2 \end{pmatrix}$
(b) (i) $\begin{pmatrix} -23 \\ 1 \\ 8 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$

3. (a) $\frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$
(b) $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

4. 17.5

5. (b) $13\boldsymbol{a} \times \boldsymbol{b}$

6. (b) 0

Mixed examination practice 13

Short questions

1. –1

2. (a) $\frac{1}{2}\overrightarrow{AD} - \overrightarrow{AB}$

3. (a) $9\boldsymbol{i} + 5\boldsymbol{j} + 7\boldsymbol{k}$
(b) $\sqrt{155}$

4. (a) $\begin{pmatrix} -5 \\ -3p - 1 \\ 15 \end{pmatrix}$
(b) $\frac{19}{3}$

5. 74.4°

6. $\frac{\pi}{2} - 2\theta$

7. 0

Long questions

1. (a) $\begin{pmatrix} 2 \\ 0 \\ k - 7 \end{pmatrix}$ (c) $(3,6,1)$
(d) $-\frac{1}{\sqrt{10}}$

2. (a) $\begin{pmatrix} -1+\dfrac{3}{k+1} \\ \dfrac{3}{4+1} \\ -4 \end{pmatrix}$ (b) 5

(c) $\left(\frac{3}{2},\frac{3}{2},2\right)$ (d) $\sqrt{\frac{33}{2}}$

3. (a) (a,a^2)

(b) $\begin{pmatrix} -a \\ -a^2 \end{pmatrix},\begin{pmatrix} -a \\ 4-a^2 \end{pmatrix}$

(c) $\sqrt{3}$

(d) $2\sqrt{3}$

4. (a) $\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}|$

(b) $|\boldsymbol{c}|\cos\theta$

(d) $\frac{1}{3}$

Chapter 14

Exercise 14A

1. (a) (i) $\boldsymbol{r}=\begin{pmatrix} 4 \\ -1 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ (ii) $\boldsymbol{r}=\begin{pmatrix} 4 \\ 1 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -3 \end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$

(c) (i) $\boldsymbol{r}=\begin{pmatrix} 4 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} 0 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

(d) (i) $\boldsymbol{r}=\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

2. (a) (i) $\boldsymbol{r}=\begin{pmatrix} 4 \\ 1 \end{pmatrix}+\lambda\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ (ii) $\boldsymbol{r}=\begin{pmatrix} 2 \\ 7 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -9 \end{pmatrix}$

(b) (i) $\boldsymbol{r}=\begin{pmatrix} -5 \\ -2 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

(ii) $\boldsymbol{r}=\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}+\lambda\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

3. (a) (i) Yes (ii) Yes
(b) (i) Yes (ii) No

4. (b) $(0,3,0)$

5. (a) $\boldsymbol{r}=\begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}+\lambda\begin{pmatrix} -4 \\ -2 \\ 3 \end{pmatrix}$

(b) $(-5,-5,-11)$ or $(19,7,-7)$

6. (a) $\boldsymbol{r}=\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}+\lambda\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

(b) 7
(c) $(-8,16,-26),(12,-14,34)$

Exercise 14B

1. (a) (i) 44.5° (ii) 56.5°
(b) (i) 26.6° (ii) 82.1°

2. (a) Perpendicular (b) Parallel
(c) Parallel (d) Same line

3. (a) (i) $(10,-7,-2)$
(ii) $(4.5,0,0)$
(b) (i) No intersection
(ii) No intersection

4. $\left(\frac{64}{9},\frac{4}{9},\frac{19}{9}\right)$

5. $\sqrt{\frac{6}{11}}$

6. (a) $(4,1,-2)$
(c) $(1,1,2)$ (d) $\frac{5\sqrt{26}}{2}$

7. (a) $\begin{pmatrix} 3t \\ 4t \end{pmatrix}$

(b) $\begin{pmatrix} 3t \\ 18-5t \end{pmatrix}$

(d) $t=2$
(e) 2 hours

8. (a) $\begin{pmatrix} 3t \\ 5-4t \\ t \end{pmatrix}$

(d) 30 km

9. 3