Chapter 2

Exponential functions

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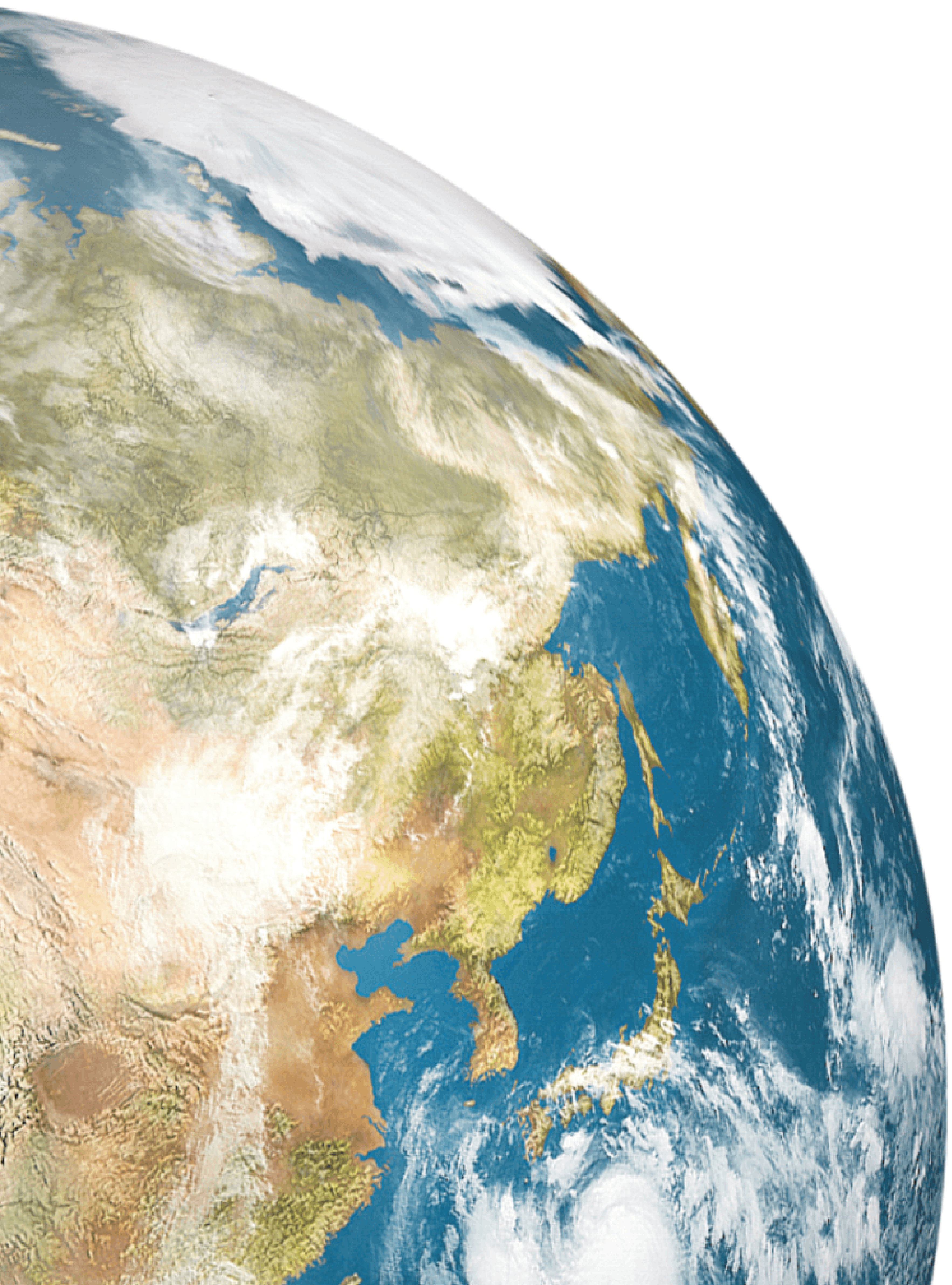


OPENING PROBLEM

At an antiques fair, Bernard purchases a clock for £500 and a vase for £400. The clock increases in value by 5% each year, and the vase increases in value by 7% each year.

Things to think about:

- **a** What is the value of each item 1 year after purchase?
- **b** Can you write a formula for the value of each item t years after purchase?
- **c** Which item is more valuable 15 years after purchase?
- **d** How can we determine when the items are equal in value?

We have seen previously how exponents are used to indicate when a number is raised to a power.

For a positive integer exponent, the exponent tells us how many of the base are multiplied together.

Any non-zero base to the power 0 is defined as 1, to give consistency to the exponent laws.

For a negative integer exponent, we take the reciprocal of the corresponding positive integer power.

$$\left\{\begin{array}{lll} 2^3 = 2\times 2\times 2 & = 8 \\ 2^2 = 2\times 2 & = 4 \\ 2^1 = 2 & = 2 \end{array}\right.$$

$$2^0 = 1 \qquad = 1$$

$$\left\{\begin{array}{lll} 2^{-1} = \frac{1}{2} & = \frac{1}{2} \\ 2^{-2} = \dfrac{1}{2\times 2} & = \frac{1}{4} \\ 2^{-3} = \dfrac{1}{2\times 2\times 2} & = \frac{1}{8} \end{array}\right.$$

In this Chapter we give meaning to exponents which are **rational**, allowing us to start filling in the gaps between the integer exponents. This will allow us to consider **exponential functions** for which the variable appears in an exponent.

A RATIONAL EXPONENTS

Using the definition $a^n = a\times a\times .... \times a$, the **laws of exponents** such as $a^n \times a^m = a^{n+m}$ can be proven for any integers n and m.

For a positive base a, we choose to define a raised to a rational exponent so that the laws of exponents still hold.

So, for any $a > 0$, notice that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$ {exponent laws}

and $\sqrt{a}\times\sqrt{a} = a$ also.

Likewise, $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$

and $\sqrt[3]{a}\times\sqrt[3]{a}\times\sqrt[3]{a} = a$ also.

By direct comparison, we conclude that $\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$ and $\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$.

In general, $\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$ where $\sqrt[n]{a}$ reads "the nth root of a" for $n \in \mathbb{Z}^+$.

We can now determine that $\sqrt[n]{a^m} = (a^m)^{\frac{1}{n}}$
$= a^{\frac{m}{n}}$

$\therefore$ $\boldsymbol{a^{\frac{m}{n}} = \sqrt[n]{a^m}}$ for $a > 0,\ n \in \mathbb{Z}^+,\ m \in \mathbb{Z}$

Example 1 **Self Tutor**

Write as a single power of 2:

a $\sqrt[3]{2}$ **b** $\dfrac{1}{\sqrt{2}}$ **c** $\sqrt[5]{4}$

a $\sqrt[3]{2}$
$= 2^{\frac{1}{3}}$

b $\dfrac{1}{\sqrt{2}}$
$= \dfrac{1}{2^{\frac{1}{2}}}$
$= 2^{-\frac{1}{2}}$

c $\sqrt[5]{4}$
$= (2^2)^{\frac{1}{5}}$
$= 2^{2 \times \frac{1}{5}}$
$= 2^{\frac{2}{5}}$

EXERCISE 2A

1 Write as a single power of 2:

a $\sqrt[5]{2}$ **b** $\dfrac{1}{\sqrt[5]{2}}$ **c** $2\sqrt{2}$ **d** $4\sqrt{2}$ **e** $\dfrac{1}{\sqrt[3]{2}}$

f $2 \times \sqrt[3]{2}$ **g** $\dfrac{4}{\sqrt{2}}$ **h** $(\sqrt{2})^3$ **i** $\dfrac{1}{\sqrt[3]{16}}$ **j** $\dfrac{1}{\sqrt{8}}$

2 Write as a single power of 3:

a $\sqrt[3]{3}$ **b** $\dfrac{1}{\sqrt[3]{3}}$ **c** $\sqrt[4]{3}$ **d** $3\sqrt{3}$ **e** $\dfrac{1}{9\sqrt{3}}$

3 Write in the form a^k, where a is a prime number and k is rational:

a $\sqrt[3]{7}$ **b** $\sqrt[4]{27}$ **c** $\sqrt[5]{16}$ **d** $\sqrt[3]{32}$ **e** $\sqrt[7]{49}$

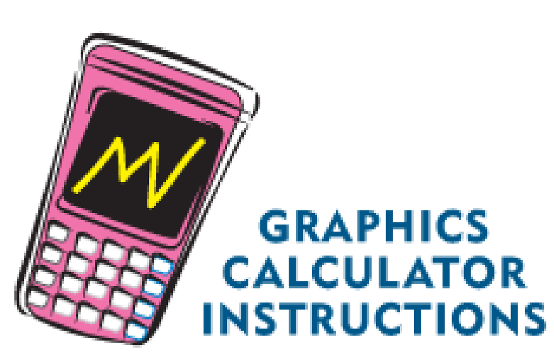
f $\dfrac{1}{\sqrt[3]{7}}$ **g** $\dfrac{1}{\sqrt[4]{27}}$ **h** $\dfrac{1}{\sqrt[5]{16}}$ **i** $\dfrac{1}{\sqrt[3]{32}}$ **j** $\dfrac{1}{\sqrt[7]{49}}$

4 Write in the form x^k, where k is rational:

a $\sqrt{x}$ **b** $x\sqrt{x}$ **c** $\dfrac{1}{\sqrt{x}}$ **d** $x^2\sqrt{x}$ **e** $\dfrac{1}{x\sqrt{x}}$

5 Use your calculator to find, correct to 3 significant figures:

a $3^{\frac{3}{4}}$ **b** $4^{-\frac{3}{5}}$ **c** $\sqrt[4]{8}$ **d** $\sqrt[5]{27}$ **e** $\dfrac{1}{\sqrt[3]{7}}$



6 Write *without* rational exponents:

a $5^{\frac{1}{3}}$ **b** $3^{-\frac{1}{2}}$ **c** $3^{\frac{5}{2}}$ **d** $m^{\frac{3}{2}}$ **e** $x^{\frac{7}{2}}$

Example 2 Self Tutor

Without using a calculator, write in simplest rational form:

a $8^{\frac{4}{3}}$ **b** $27^{-\frac{2}{3}}$

a $8^{\frac{4}{3}}$
$= (2^3)^{\frac{4}{3}}$
$= 2^{3\times\frac{4}{3}}$ $\{(a^m)^n = a^{mn}\}$
$= 2^4$
$= 16$

b $27^{-\frac{2}{3}}$
$= (3^3)^{-\frac{2}{3}}$
$= 3^{3\times-\frac{2}{3}}$
$= 3^{-2}$
$= \frac{1}{9}$

7 Without using a calculator, write in simplest rational form:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $16^{\frac{3}{4}}$ **d** $25^{\frac{3}{2}}$ **e** $32^{\frac{2}{5}}$

f $4^{-\frac{1}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $8^{-\frac{4}{3}}$ **i** $27^{-\frac{4}{3}}$ **j** $125^{-\frac{2}{3}}$

B ALGEBRAIC EXPANSION AND FACTORISATION

We can use the standard rules of algebra, together with the laws of exponents, to simplify expressions containing rational or variable exponents.

$$\begin{aligned} a(b+c) &= ab+ac \\ (a+b)(c+d) &= ac+ad+bc+bd \\ (a+b)(a-b) &= a^2-b^2 \\ (a+b)^2 &= a^2+2ab+b^2 \end{aligned}$$

Example 3 Self Tutor

Expand and simplify: $x^{-\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})$

$x^{-\frac{1}{2}}(x^{\frac{3}{2}} + 2x^{\frac{1}{2}} - 3x^{-\frac{1}{2}})$
$= x^{-\frac{1}{2}} \times x^{\frac{3}{2}} + x^{-\frac{1}{2}} \times 2x^{\frac{1}{2}} - x^{-\frac{1}{2}} \times 3x^{-\frac{1}{2}}$ {each term is multiplied by $x^{-\frac{1}{2}}$}
$= x^1 + 2x^0 - 3x^{-1}$ {adding indices}
$= x + 2 - \dfrac{3}{x}$

EXERCISE 2B

1 Simplify:

a $x^{\frac{1}{2}} \times x^{-\frac{1}{2}}$ **b** $x^{\frac{3}{2}} \times x^{-\frac{1}{2}}$ **c** $x^2 \times x^{-\frac{3}{2}}$

2 Expand and simplify:

a $x^2(x^3+2x^2+1)$ **b** $2^x(2^x+1)$ **c** $x^{\frac{1}{2}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})$

d $7^x(7^x+2)$ **e** $3^x(2-3^{-x})$ **f** $x^{\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}})$

g $2^{-x}(2^x+5)$ **h** $5^{-x}(5^{2x}+5^x)$ **i** $x^{-\frac{1}{2}}(x^2+x+x^{\frac{1}{2}})$

j $3^x(3^x+5+3^{-x})$ **k** $x^{-\frac{1}{2}}(2x^2-x+5x^{\frac{1}{2}})$ **l** $2^{2x}(2^x-3-2^{-2x})$

Example 4 Self Tutor

Expand and simplify:

a $(2^x+3)(2^x+1)$ **b** $(7^x+7^{-x})^2$

a $(2^x+3)(2^x+1)$
$= 2^x\times 2^x+2^x+3\times 2^x+3$
$= 2^{2x}+4\times 2^x+3$

b $(7^x+7^{-x})^2$
$= (7^x)^2+2\times 7^x\times 7^{-x}+(7^{-x})^2$
$= 7^{2x}+2\times 7^0+7^{-2x}$
$= 7^{2x}+2+7^{-2x}$

3 Expand and simplify:

a $(2^x-1)(2^x+3)$ **b** $(3^x+2)(3^x+5)$ **c** $(5^x-2)(5^x-4)$

d $(2^x+3)^2$ **e** $(3^x-1)^2$ **f** $(4^x+7)^2$

g $(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-2)$ **h** $(2^x+3)(2^x-3)$ **i** $(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})$

j $\left(x+\frac{2}{x}\right)^2$ **k** $(7^x-7^{-x})^2$ **l** $(5-2^{-x})^2$

Example 5 Self Tutor

Factorise:

a $2^{n+3}+2^n$ **b** $2^{n+3}+8$ **c** $2^{3n}+2^{2n}$

a $2^{n+3}+2^n$
$= 2^n2^3+2^n$
$= 2^n(2^3+1)$
$= 2^n\times 9$

b $2^{n+3}+8$
$= 2^n2^3+8$
$= 8(2^n)+8$
$= 8(2^n+1)$

c $2^{3n}+2^{2n}$
$= 2^{2n}2^n+2^{2n}$
$= 2^{2n}(2^n+1)$

4 Factorise:

a $5^{2x}+5^x$ **b** $3^{n+2}+3^n$ **c** 7^n+7^{3n}

d $5^{n+1}-5$ **e** $6^{n+2}-6$ **f** $4^{n+2}-16$

g $2^{2n}-2^{n+3}$ **h** $2^{n+1}+2^{n-1}$ **i** $4^{n+1}+2^{2n-1}$

Example 6 Self Tutor

Factorise:

a $4^x - 9$ **b** $9^x + 4(3^x) + 4$

a $4^x - 9$
$= (2^x)^2 - 3^2$ {compare $a^2 - b^2 = (a+b)(a-b)$}
$= (2^x + 3)(2^x - 3)$

b $9^x + 4(3^x) + 4$
$= (3^x)^2 + 4(3^x) + 4$ {compare $a^2 + 4a + 4$}
$= (3^x + 2)^2$ {as $a^2 + 4a + 4 = (a+2)^2$}

5 Factorise:

a $9^x - 4$ **b** $4^x - 25$ **c** $16 - 9^x$
d $25 - 4^x$ **e** $9^x - 4^x$ **f** $4^x + 6(2^x) + 9$
g $9^x + 10(3^x) + 25$ **h** $4^x - 14(2^x) + 49$ **i** $25^x - 4(5^x) + 4$

6 Factorise:

a $(2^x)^2 - 2^x - 2$ **b** $(3^x)^2 + 3^x - 6$ **c** $4^x - 7(2^x) + 12$
d $4^x + 9(2^x) + 18$ **e** $4^x - 2^x - 20$ **f** $9^x + 9(3^x) + 14$
g $9^x + 4(3^x) - 5$ **h** $25^x + 5^x - 2$ **i** $49^x - 7^{x+1} + 12$

Example 7 Self Tutor

Simplify:

a $\frac{6^n}{3^n}$ **b** $\frac{4^n}{6^n}$

a $\frac{6^n}{3^n}$ **or** $\frac{6^n}{3^n}$
$= \frac{2^n \cancel{3^n}}{\cancel{3^n}_1}$ $= \left(\frac{6}{3}\right)^n$
$= 2^n$ $= 2^n$

b $\frac{4^n}{6^n}$ **or** $\frac{4^n}{6^n}$
$= \frac{\cancel{2^n} 2^n}{\cancel{2^n} 3^n}$ $= \left(\frac{4}{6}\right)^n$
$= \frac{2^n}{3^n}$ $= \left(\frac{2}{3}\right)^n$

7 Simplify:

a $\frac{12^n}{6^n}$ **b** $\frac{20^a}{2^a}$ **c** $\frac{6^b}{2^b}$ **d** $\frac{4^n}{20^n}$ **e** $\frac{35^x}{7^x}$
f $\frac{6^a}{8^a}$ **g** $\frac{24^k}{9^k}$ **h** $\frac{5^{n+1}}{5^n}$ **i** $\frac{5^{n+1}}{5}$

Example 8 Self Tutor

Simplify:

a $\frac{3^n + 6^n}{3^n}$ **b** $\frac{2^{m+2} - 2^m}{2^m}$ **c** $\frac{2^{m+3} + 2^m}{9}$

a $\dfrac{3^n + 6^n}{3^n}$

$= \dfrac{3^n + 2^n 3^n}{3^n}$

$= \dfrac{3^n(1 + 2^n)}{3^n}$

$= 1 + 2^n$

b $\dfrac{2^{m+2} - 2^m}{2^m}$

$= \dfrac{2^m 2^2 - 2^m}{2^m}$

$= \dfrac{2^m(4 - 1)}{2^m}$

$= 3$

c $\dfrac{2^{m+3} + 2^m}{9}$

$= \dfrac{2^m 2^3 + 2^m}{9}$

$= \dfrac{2^m(8 + 1)}{9}$

$= 2^m$

8 Simplify:

a $\dfrac{6^m + 2^m}{2^m}$ **b** $\dfrac{2^n + 12^n}{2^n}$ **c** $\dfrac{8^n + 4^n}{2^n}$

d $\dfrac{12^x - 3^x}{3^x}$ **e** $\dfrac{6^n + 12^n}{1 + 2^n}$ **f** $\dfrac{5^{n+1} - 5^n}{4}$

g $\dfrac{5^{n+1} - 5^n}{5^n}$ **h** $\dfrac{4^n - 2^n}{2^n}$ **i** $\dfrac{2^n - 2^{n-1}}{2^n}$

9 Simplify:

a $2^n(n + 1) + 2^n(n - 1)$ **b** $3^n\left(\dfrac{n-1}{6}\right) - 3^n\left(\dfrac{n+1}{6}\right)$

C EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

For example: $2^x = 8$ and $30 \times 3^x = 7$ are both exponential equations.

There are a number of methods we can use to solve exponential equations. These include graphing, using technology, and by using **logarithms**, which we will study in **Chapter 3**. However, in some cases we can solve the equation algebraically.

If both sides of an exponential equation are written as powers with the same base numbers, we can **equate indices**.

So, if $\boldsymbol{a^x = a^k}$ then $\boldsymbol{x = k}$.

For example, if $2^x = 8$ then $2^x = 2^3$. Thus $x = 3$, and this is the only solution.

Example 9 Self Tutor

Solve for x:

a $2^x = 16$ **b** $3^{x+2} = \frac{1}{27}$

a $2^x = 16$

$\therefore\ 2^x = 2^4$

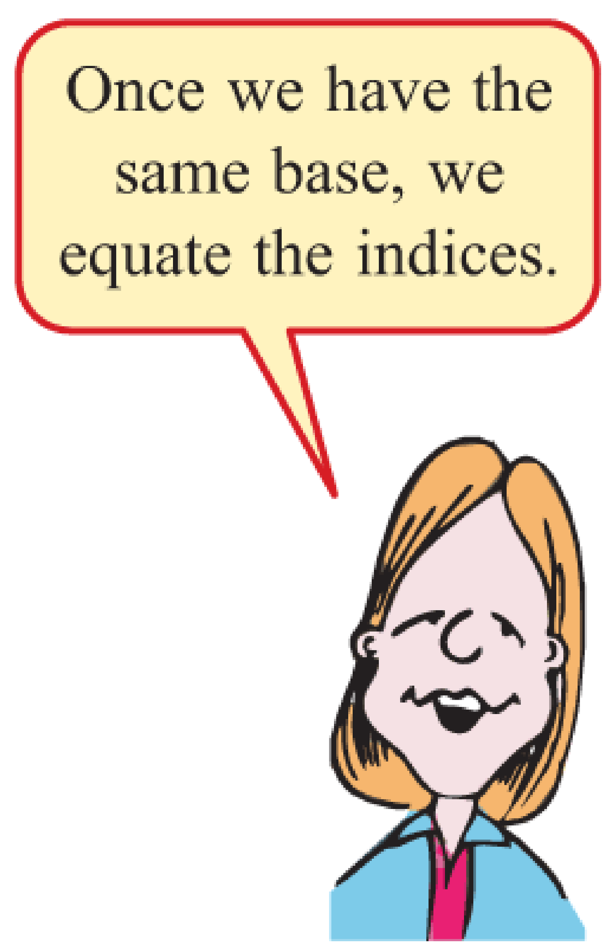
$\therefore\ x = 4$

b $3^{x+2} = \frac{1}{27}$

$\therefore\ 3^{x+2} = 3^{-3}$

$\therefore\ x + 2 = -3$

$\therefore\ x = -5$



EXERCISE 2C

1 Solve for x:

a $2^x = 32$ **b** $5^x = 25$ **c** $3^x = 81$ **d** $7^x = 1$

e $3^x = \frac{1}{3}$ **f** $2^x = \sqrt{2}$ **g** $5^x = \frac{1}{125}$ **h** $4^{x+1} = 64$

i $2^{x-2} = \frac{1}{32}$ **j** $3^{x+1} = \frac{1}{27}$ **k** $7^{x+1} = 343$ **l** $5^{1-2x} = \frac{1}{\sqrt{5}}$

Example 10 Self Tutor

Solve for x:

a $4^x = 8$ **b** $9^{x-2} = \frac{1}{3}$

a

$$4^x = 8$$
$$\therefore (2^2)^x = 2^3$$
$$\therefore 2^{2x} = 2^3$$
$$\therefore 2x = 3$$
$$\therefore x = \frac{3}{2}$$

b

$$9^{x-2} = \frac{1}{3}$$
$$\therefore (3^2)^{x-2} = 3^{-1}$$
$$\therefore 3^{2(x-2)} = 3^{-1}$$
$$\therefore 2(x-2) = -1$$
$$\therefore 2x - 4 = -1$$
$$\therefore 2x = 3$$
$$\therefore x = \frac{3}{2}$$

2 Solve for x:

a $8^x = 32$ **b** $4^x = \frac{1}{8}$ **c** $9^x = \frac{1}{27}$ **d** $25^x = \frac{1}{5}$

e $27^x = \frac{1}{9}$ **f** $16^x = \frac{1}{32}$ **g** $4^{x+2} = 128$ **h** $25^{1-x} = \frac{1}{125}$

i $4^{4x-1} = \frac{1}{2}$ **j** $9^{x-3} = 27$ **k** $\left(\frac{1}{2}\right)^{x+1} = 8$ **l** $\left(\frac{1}{3}\right)^{x+2} = \sqrt{27}$

m $81^x = 27^{-x}$ **n** $\left(\frac{1}{4}\right)^{1-x} = 32$ **o** $\left(\frac{1}{7}\right)^x = \sqrt[3]{49}$ **p** $\left(\frac{1}{3}\right)^{x+1} = 243$

3 Solve for x, if possible:

a $4^{2x+1} = 8^{1-x}$ **b** $9^{2-x} = \left(\frac{1}{3}\right)^{2x+1}$ **c** $2^x \times 8^{1-x} = \frac{1}{4}$

d $3^{x+2} \times 9^x = 27$ **e** $\left(\frac{1}{2}\right)^{x-1} \times 8^x = 4^{-x}$ **f** $\left(\frac{1}{5}\right)^{x^2} \times 25^x = \frac{1}{125}$

4 Solve for x:

a $3 \times 2^x = 24$ **b** $7 \times 2^x = 28$ **c** $4 \times 3^{x+2} = 12$

d $12 \times 3^{-x} = \frac{4}{3}$ **e** $4 \times \left(\frac{1}{3}\right)^x = 36$ **f** $5 \times \left(\frac{1}{2}\right)^x = 20$

Example 11 Self Tutor

Solve for x: $4^x + 2^x - 20 = 0$

$$4^x + 2^x - 20 = 0$$
$$\therefore (2^x)^2 + 2^x - 20 = 0 \quad \{\text{compare } a^2 + a - 20 = 0\}$$
$$\therefore (2^x - 4)(2^x + 5) = 0 \quad \{a^2 + a - 20 = (a-4)(a+5)\}$$
$$\therefore 2^x = 4 \text{ or } 2^x = -5$$
$$\therefore 2^x = 2^2 \quad \{2^x \text{ cannot be negative}\}$$
$$\therefore x = 2$$

5 Solve for x:

a $4^x - 6(2^x) + 8 = 0$ **b** $4^x - 2^x - 2 = 0$ **c** $9^x - 12(3^x) + 27 = 0$

d $9^x = 3^x + 6$ **e** $25^x - 23(5^x) - 50 = 0$ **f** $49^x + 1 = 2(7^x)$

g $3^x - 1 = 6(3^{-x})$ **h** $2(4^x) - 5(2^x) + 2 = 0$

i $4(9^x) - 35(3^x) = 9$ **j** $4^{x+1} + 2 = 9(2^x)$

k $3^{2x-1} = 3^x + 18$ **l** $4^x + 2^{x+\frac{1}{2}} = 4$

Check your answers using technology.

GRAPHICS CALCULATOR INSTRUCTIONS

6 Solve simultaneously: $4^x = 8^y$ and $9^y = \dfrac{243}{3^x}$.

D EXPONENTIAL FUNCTIONS

We have already seen how to evaluate a^n for any $n \in \mathbb{Q}$.

But how do we evaluate a^n when $n \in \mathbb{R}$, so n is real but not necessarily rational?

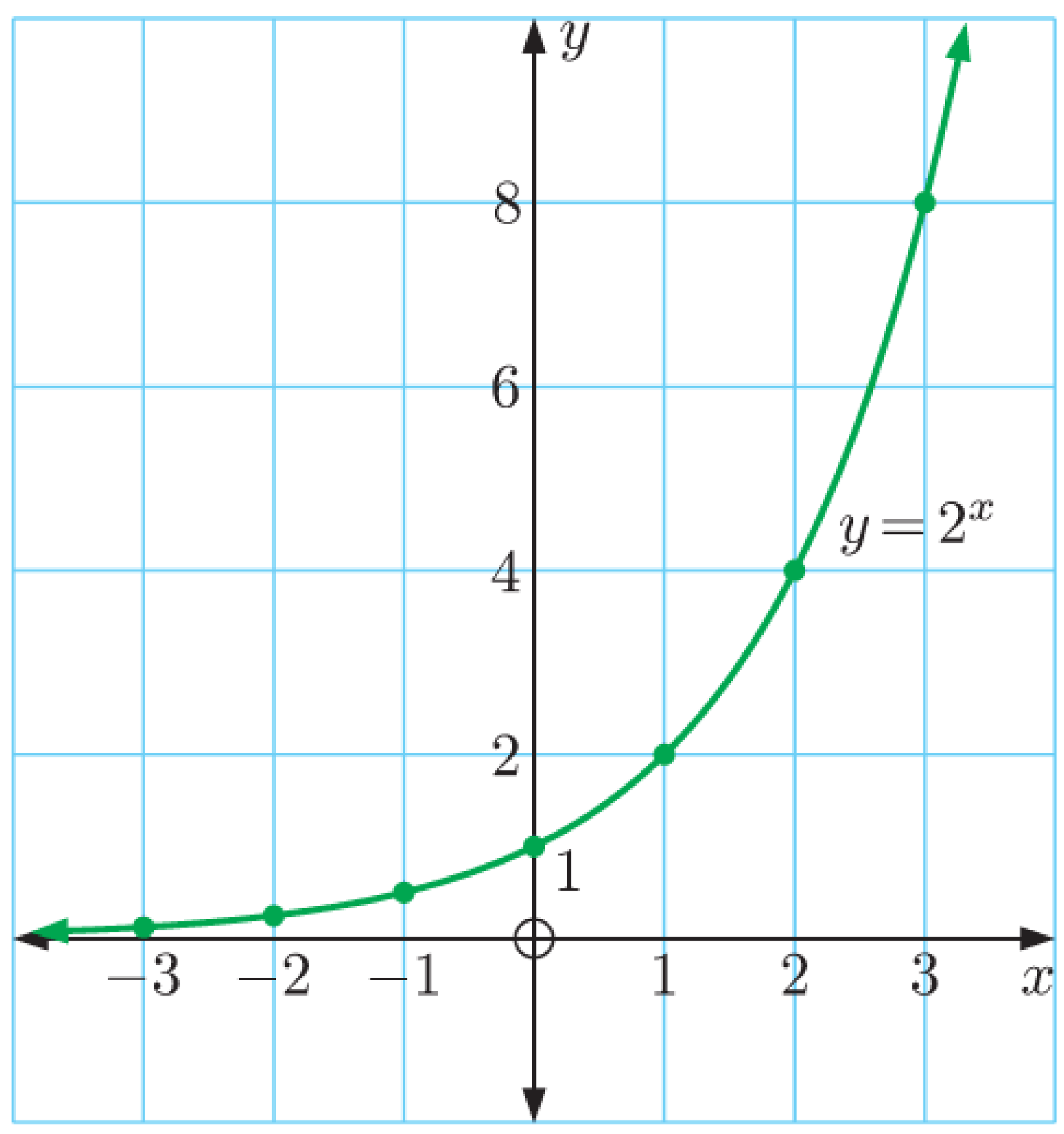
To answer this question, we can study the graphs of exponential functions.

The most simple **exponential function** has the form $y = a^x$ where $a > 0$, $a \neq 1$.

For example, $y = 2^x$ is an exponential function.

We construct a table of values from which we graph the function:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



As x becomes large and negative, the graph of $y = 2^x$ approaches the x-axis from above. However, it never touches the x-axis, since 2^x becomes very small but never zero.

So, as $x \to -\infty$, $y \to 0^+$.

$y = 0$ is therefore a **horizontal asymptote**.

Plotting $y = a^x$ for $x \in \mathbb{Q}$ suggests a smooth, continuous curve. This allows us to complete the curve for all $x \in \mathbb{R}$, giving meaning to a^x for irrational values of x.

INVESTIGATION 1 GRAPHS OF EXPONENTIAL FUNCTIONS

In this Investigation we examine the graphs of various families of exponential functions. You can use the **graphing package** or your calculator.

GRAPHING PACKAGE

What to do:

1 **a** State the transformation which maps $y = a^x$ to $y = a^x + k$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 2^x + 1$, and $y = 2^x - 2$ on the same set of axes.

2 **a** State the transformation which maps $y = a^x$ to $y = a^{x-h}$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 2^{x-1}$, $y = 2^{x+2}$, and $y = 2^{x-3}$ on the same set of axes.

3 **a** State the transformation which maps $y = a^x$ to $y = p \times a^x$, $p > 0$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Check your predictions by graphing $y = 2^x$, $y = 3 \times 2^x$, and $y = \frac{1}{2} \times 2^x$ on the same set of axes.

4 **a** State the transformation which maps $y = a^x$ to $y = -a^x$.

b Predict what the graph of $y = -2^x$ will look like, and check your answer using technology.

5 **a** State the transformation which maps $y = a^x$ to $y = a^{qx}$, $q > 0$.

b Predict the effect, if any, this transformation will have on:

i the shape of the graph **ii** the position of the graph

iii the horizontal asymptote.

c Notice that $2^{2x} = (2^2)^x = 4^x$ and $2^{3x} = (2^3)^x = 8^x$.
Check your predictions by graphing $y = 2^x$, $y = 4^x$, and $y = 8^x$ on the same set of axes.

6 **a** State the transformation which maps $y = a^x$ to $y = a^{-x}$.

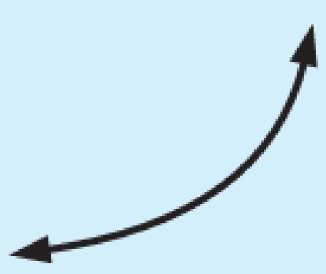
b Notice that $2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$.

Predict what the graph of $y = \left(\frac{1}{2}\right)^x$ will look like, and check your answer using technology.

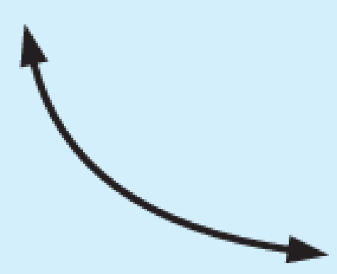
From your **Investigation** you should have discovered that:

For the general exponential function $\boldsymbol{y = p \times a^{x-h} + k}$ where $a > 0$, $a \neq 1$, $p \neq 0$:

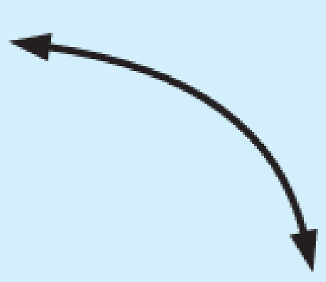
- a controls how steeply the graph increases or decreases.
- h controls horizontal translation.
- k controls vertical translation.
- The equation of the horizontal asymptote is $y = k$.
- If $p > 0$, $a > 1$ the function is increasing.



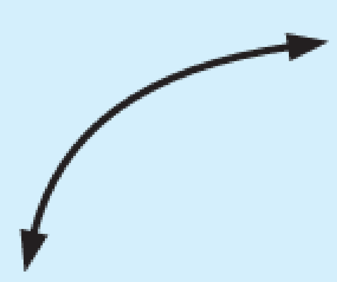
- If $p > 0$, $0 < a < 1$ the function is decreasing.



- If $p < 0$, $a > 1$ the function is decreasing.

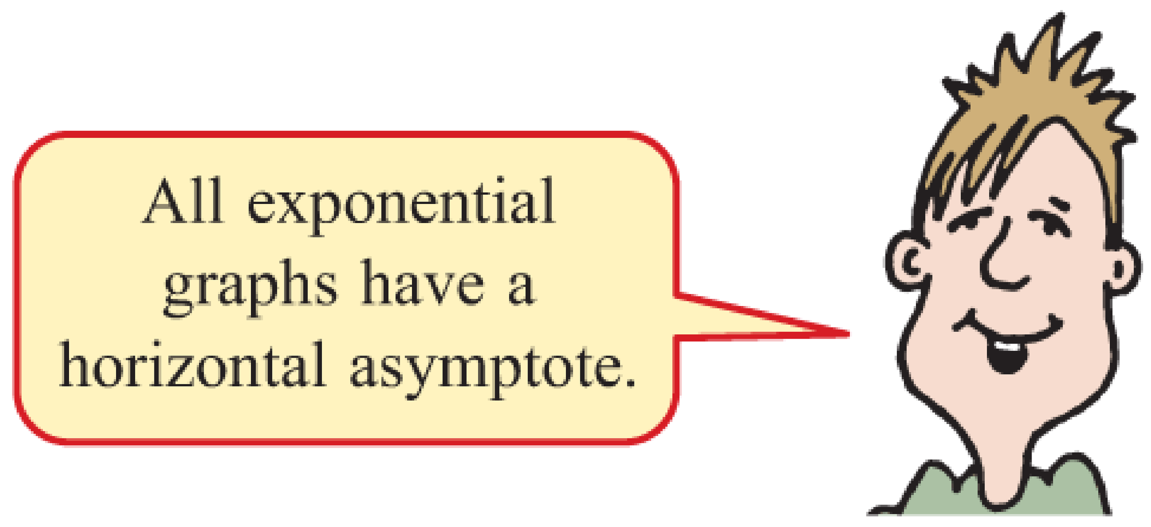


- If $p < 0$, $0 < a < 1$ the function is increasing.



We can sketch the graphs of exponential functions using:

- the horizontal asymptote
- the y-intercept
- two other points.



Example 12 — Self Tutor

Sketch the graph of $y = 2^{-x} - 3$.
Hence state the domain and range of $f(x) = 2^{-x} - 3$.

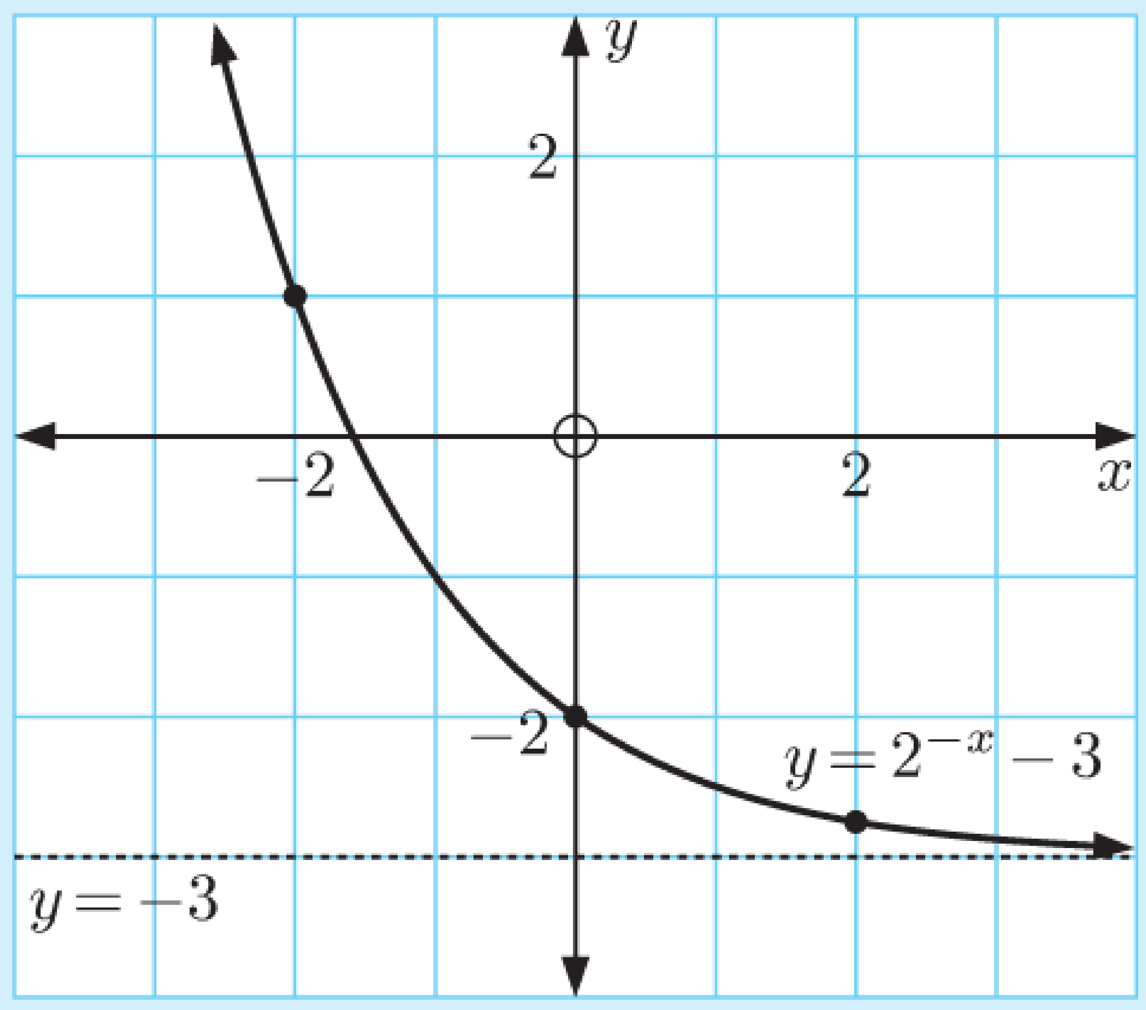
For $y = 2^{-x} - 3$,
the horizontal asymptote is $y = -3$.

When $x = 0$, $y = 2^0 - 3$
$= 1 - 3$
$= -2$

$\therefore$ the y-intercept is -2.

When $x = 2$, $y = 2^{-2} - 3$
$= \frac{1}{4} - 3$
$= -2\frac{3}{4}$

When $x = -2$, $y = 2^2 - 3 = 1$



The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > -3\}$.

EXERCISE 2D

1 Consider the graph of $y = 2^x$ alongside.

a Use the graph to estimate the value of:

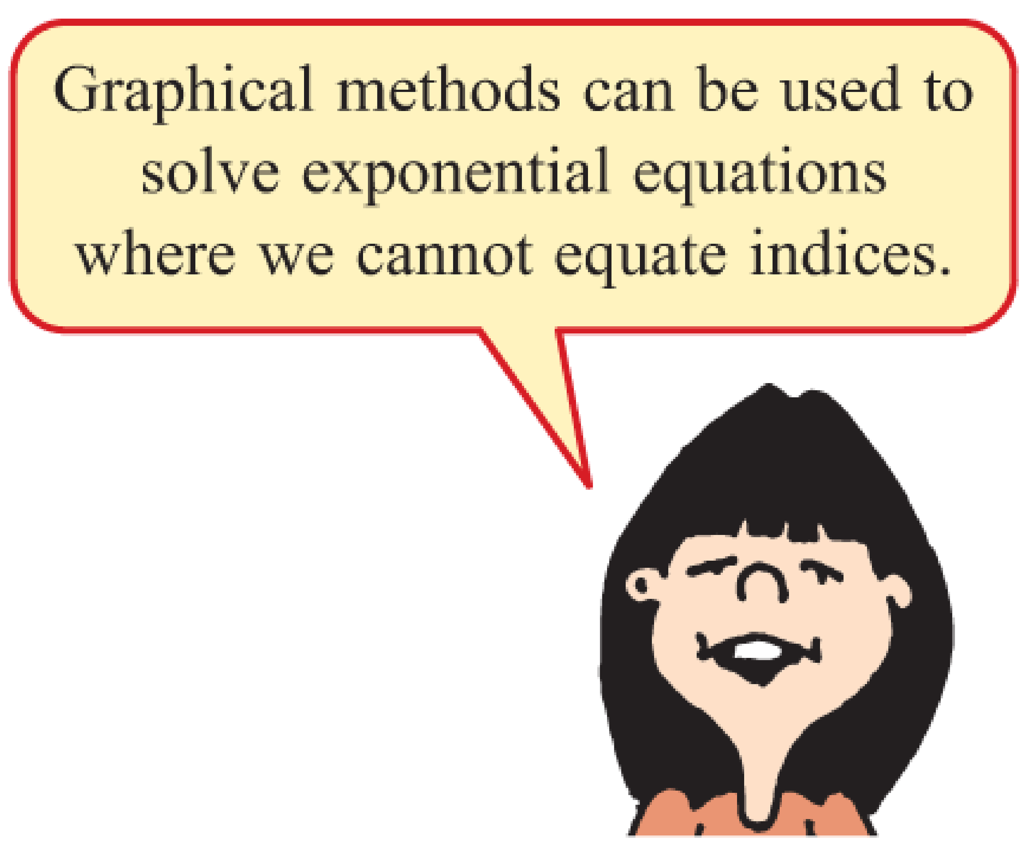
i $2^{\frac{1}{2}}$ or $\sqrt{2}$ **ii** $2^{0.8}$

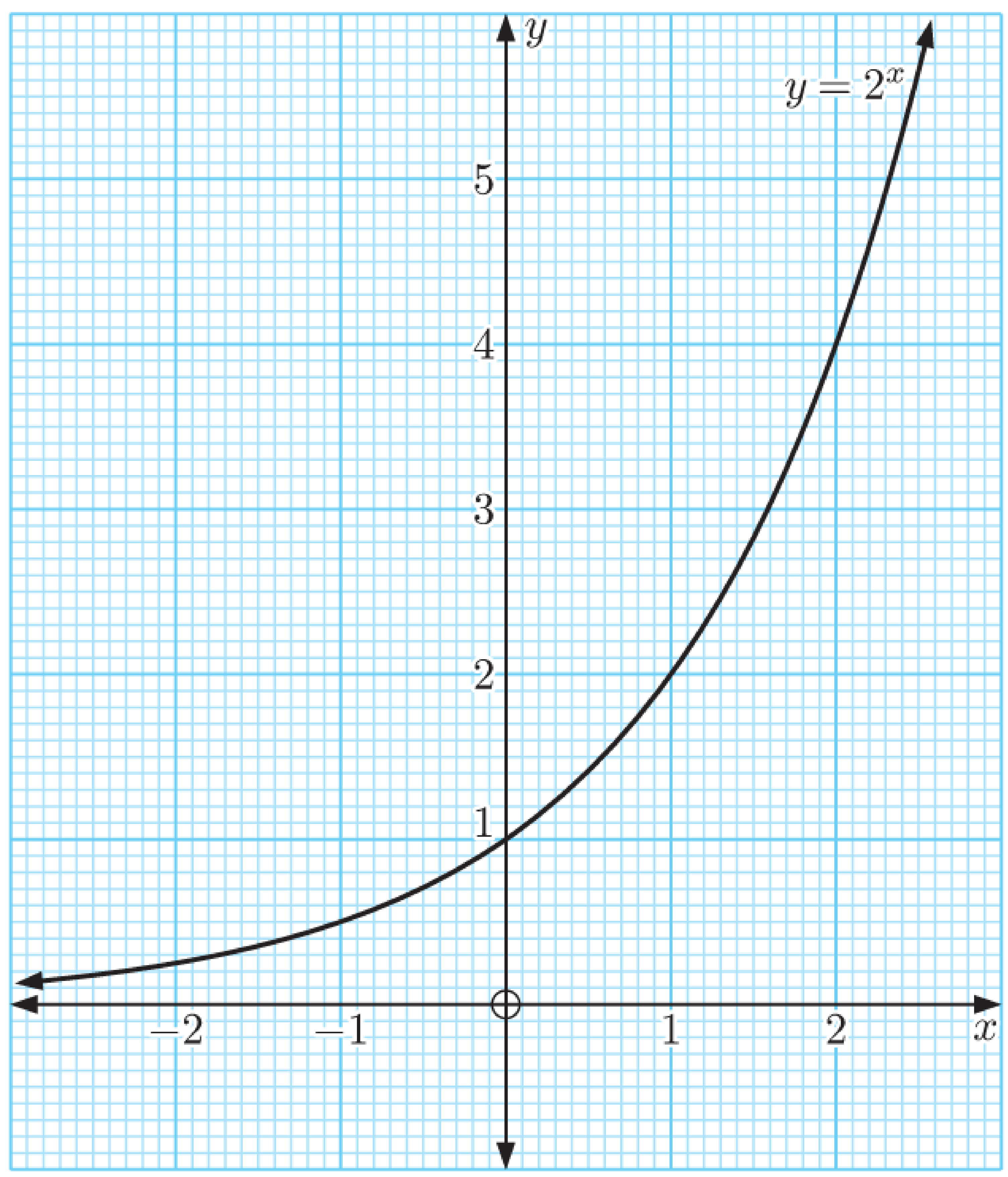
iii $2^{1.5}$ **iv** $2^{-\sqrt{2}}$

b Use the graph to estimate the solution to:

i $2^x = 3$ **ii** $2^x = 0.6$

c Use the graph to explain why $2^x = 0$ has no solutions.



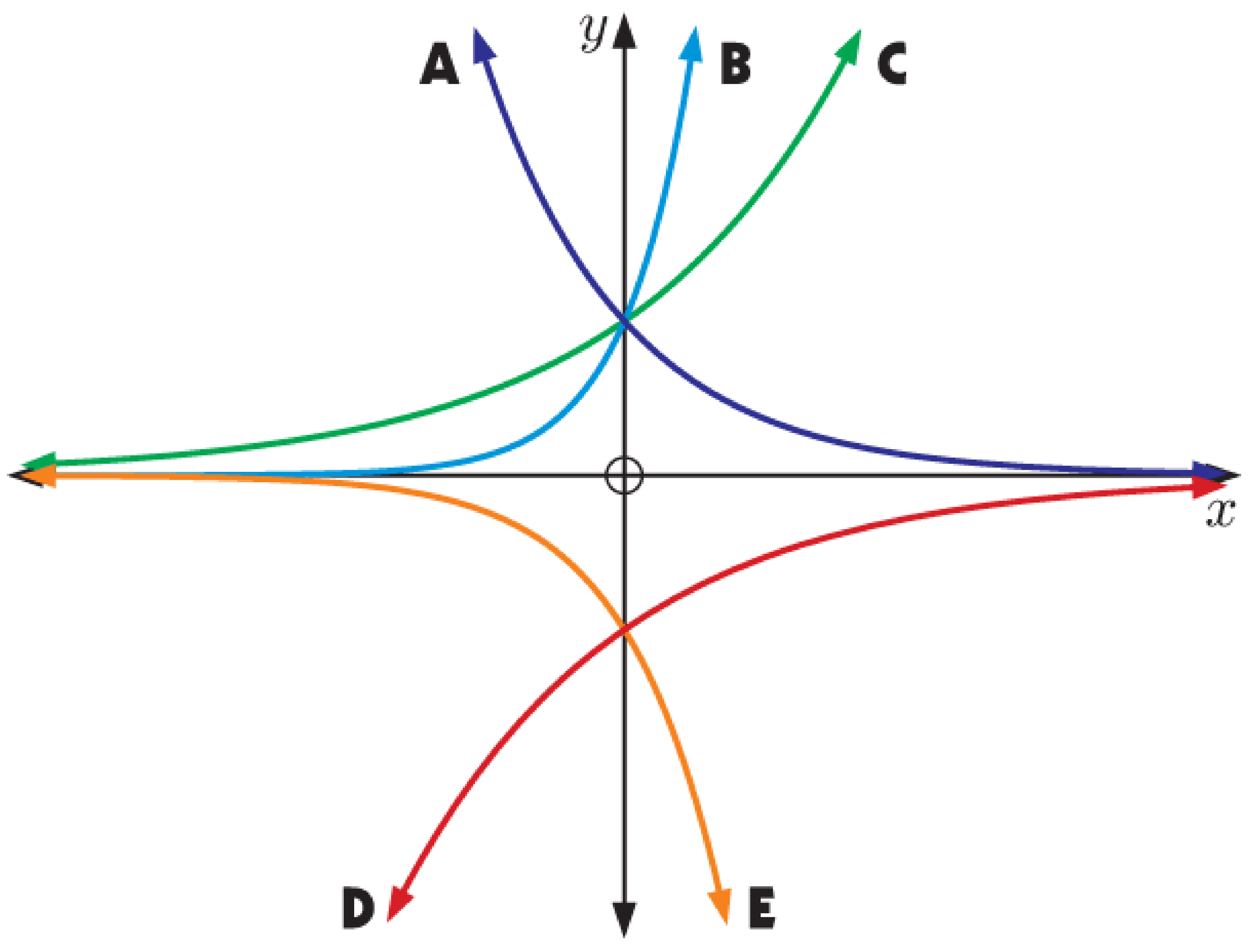


2 Match each function with its graph:

a $y = 2^x$ **b** $y = 10^x$

c $y = -5^x$ **d** $y = \left(\frac{1}{3}\right)^x$

e $y = -\left(\frac{1}{2}\right)^x$



3 Use a transformation to help sketch each pair of functions on the same set of axes:

GRAPHING PACKAGE

a $y = 2^x$ and $y = 2^x - 2$ **b** $y = 2^x$ and $y = 2^{-x}$

c $y = 2^x$ and $y = 2^{x-2}$ **d** $y = 2^x$ and $y = 2(2^x)$

4 Draw freehand sketches of the following pairs of graphs:

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = 3^x + 1$

c $y = 3^x$ and $y = -3^x$ **d** $y = 3^x$ and $y = 3^{x-1}$

5 State the equation of the horizontal asymptote of:

a $y = 3^{-x}$ **b** $y = 2^x - 1$ **c** $y = 3 - 2^{-x}$

d $y = 4 \times 2^x + 2$ **e** $y = 5 \times 3^{x+2}$ **f** $y = -2 \times 3^{1-x} - 4$

6 Consider the exponential function $f(x) = 3^x - 2$.

a Find: **i** $f(0)$ **ii** $f(2)$ **iii** $f(-2)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

7 Consider the function $g(x) = 3 \times \left(\frac{1}{2}\right)^x + 4$.

a Find: **i** $g(0)$ **ii** $g(2)$ **iii** $g(-2)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

8 Consider the function $h(x) = -2^{x-3} + 1$.

a Find: **i** $h(0)$ **ii** $h(3)$ **iii** $h(6)$

b State the equation of the horizontal asymptote.

c Sketch the graph of the function.

d State the domain and range of the function.

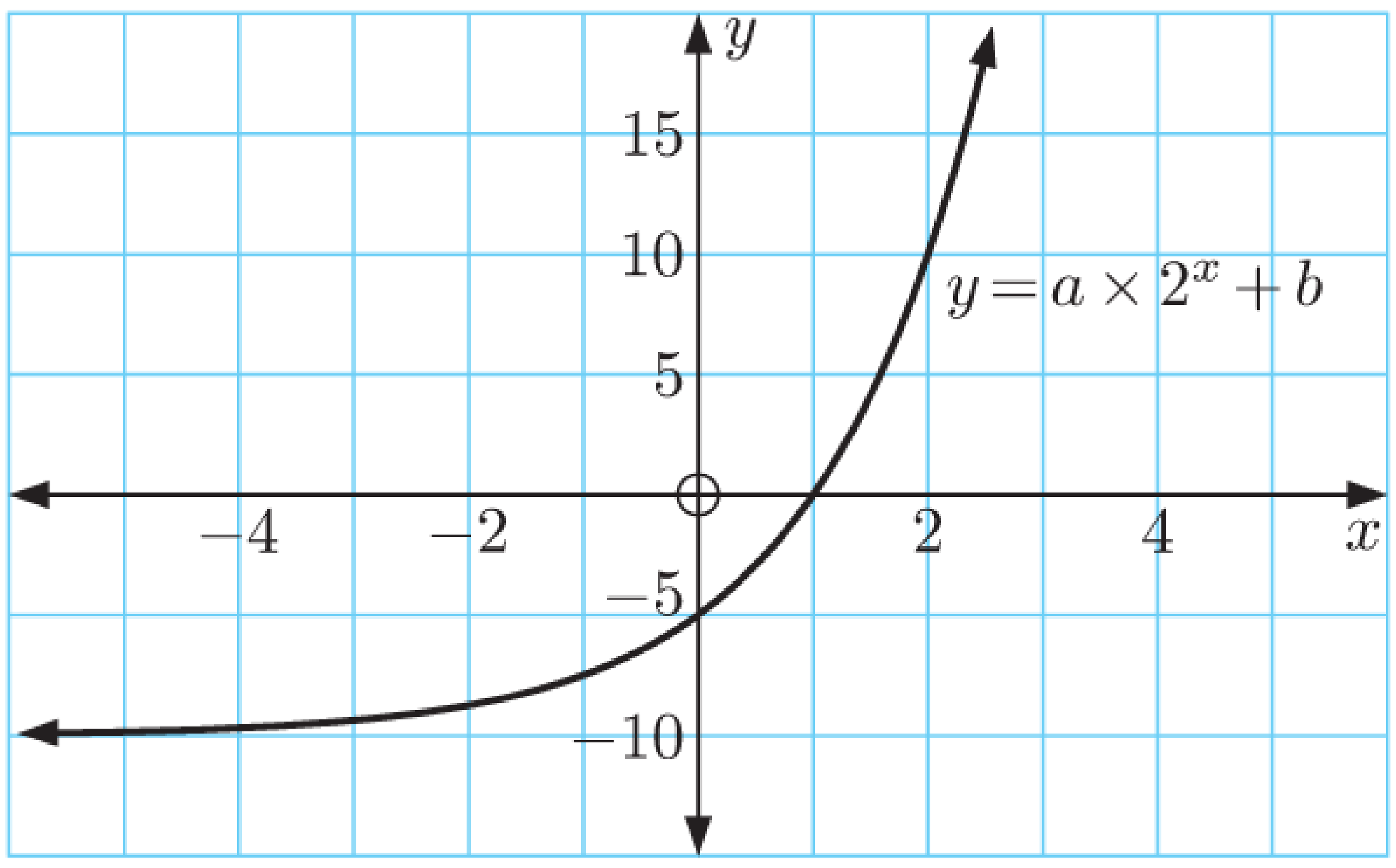
9 For each of the functions below:

- **i** Sketch the graph of the function.
- **ii** State the domain and range.
- **iii** Use your calculator to find the value of y when $x = \sqrt{2}$.
- **iv** Discuss the behaviour of y as $x \to \pm\infty$.
- **v** Determine the horizontal asymptote.

a $y = 2^x + 1$ **b** $y = 2 - 2^x$ **c** $y = 2^{-x} + 3$ **d** $y = 3 - 2^{-x}$

10 The graph alongside shows the curve $y = a \times 2^x + b$, where a and b are constants.

- **a** Find the values of a and b.
- **b** Find y when $x = 6$.



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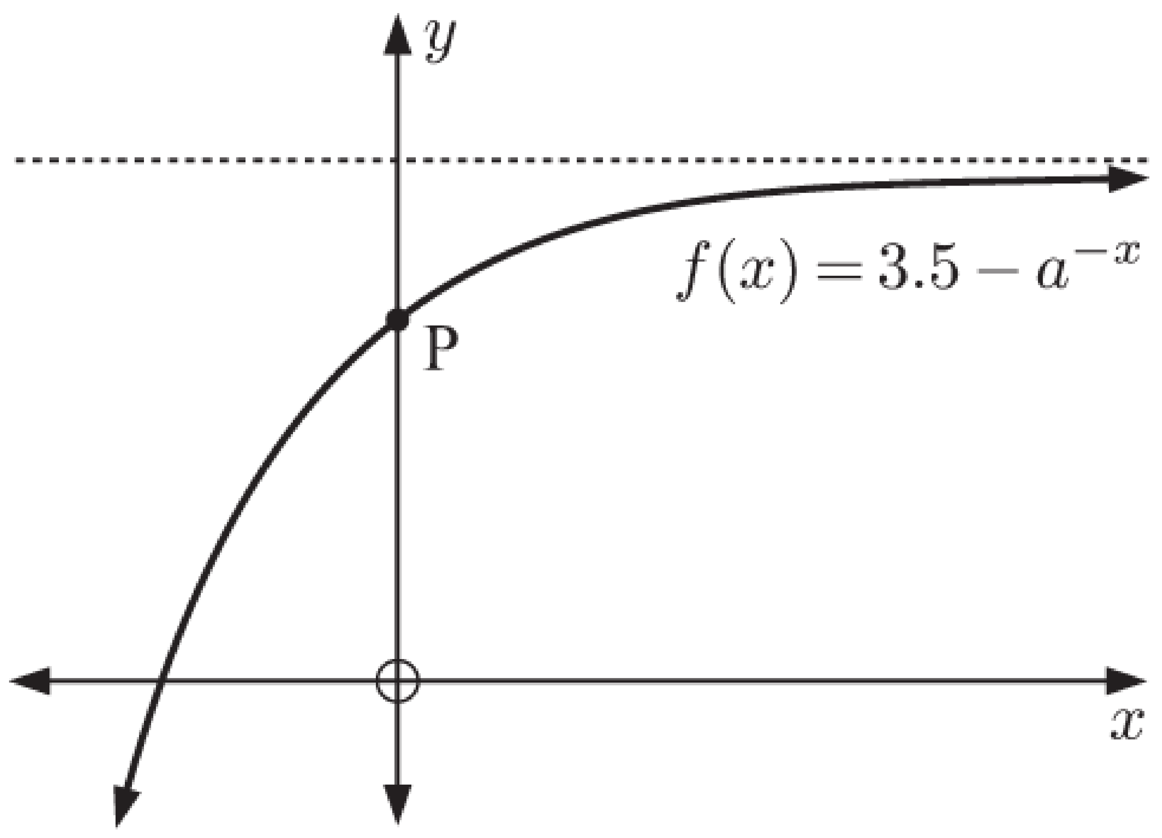


This graph shows the function $f(x) = 3.5 - a^{-x}$, where a is a positive constant.
The point $(-1, 2)$ lies on the graph.

- **a** Write down the coordinates of P.
- **b** Find the value of a.
- **c** Find the equation of the horizontal asymptote.

12 Find the domain and range of:

a $y = 2^{x^2+1}$ **b** $y = \dfrac{1}{3^x - 1}$ **c** $y = \sqrt{5^x - 5}$

13 Let $f(x) = 3^x - 9$ and $g(x) = \sqrt{x}$.

- **a** Find $(f \circ g)(x)$, and state its domain and range.
- **b** Find $(g \circ f)(x)$, and state its domain and range.
- **c** Solve:
  - **i** $(f \circ g)(x) = 0$ **ii** $(g \circ f)(x) = 3\sqrt{2}$

14 Suppose $f(x) = 2^x - 3$ and $g(x) = 1 + 2^{-x}$.

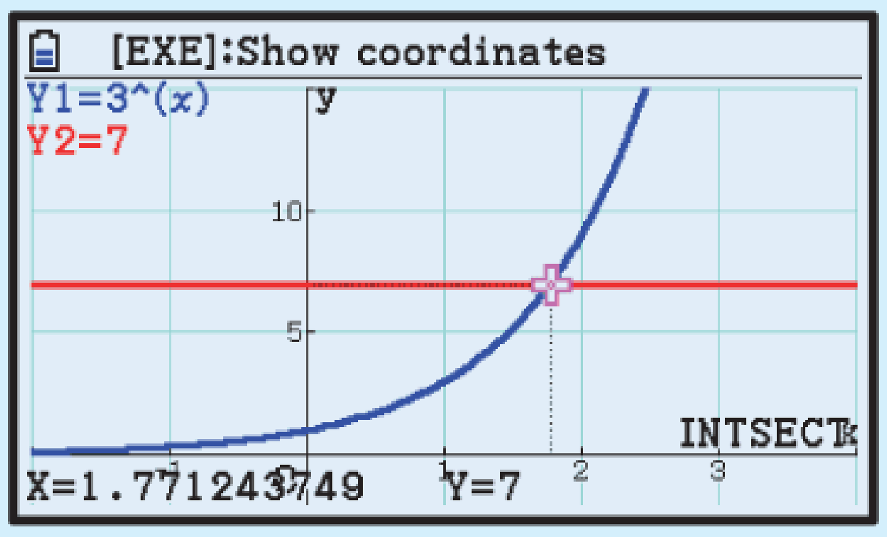
- **a** For each function, find the:
  - **i** horizontal asymptote **ii** range **iii** y-intercept.
- **b** Graph the functions on the same set of axes.
- **c** Find the exact y-coordinate of the point where the graphs intersect.

Example 13 **Self Tutor**

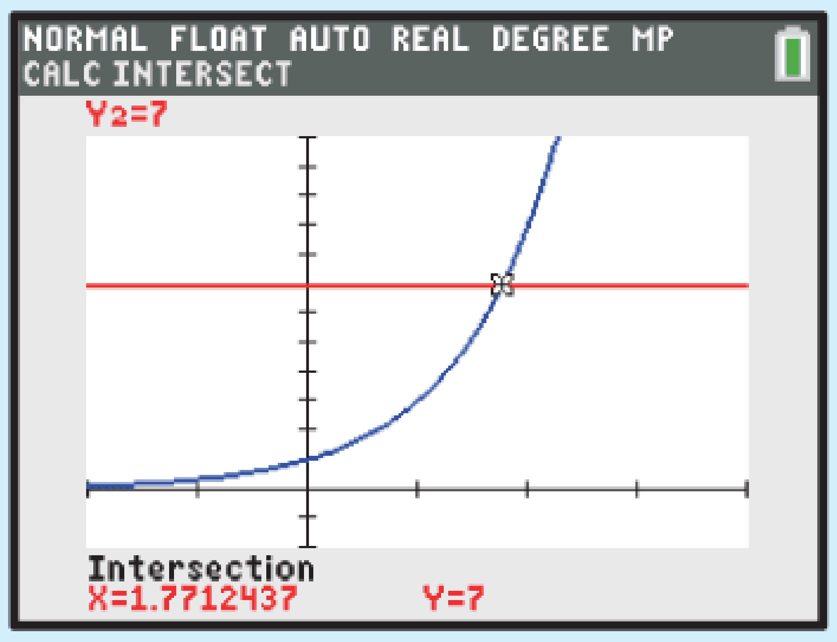
Use technology to solve the equation $3^x = 7$.

We graph $Y_1 = 3^X$ and $Y_2 = 7$ on the same set of axes, and find their point of intersection.

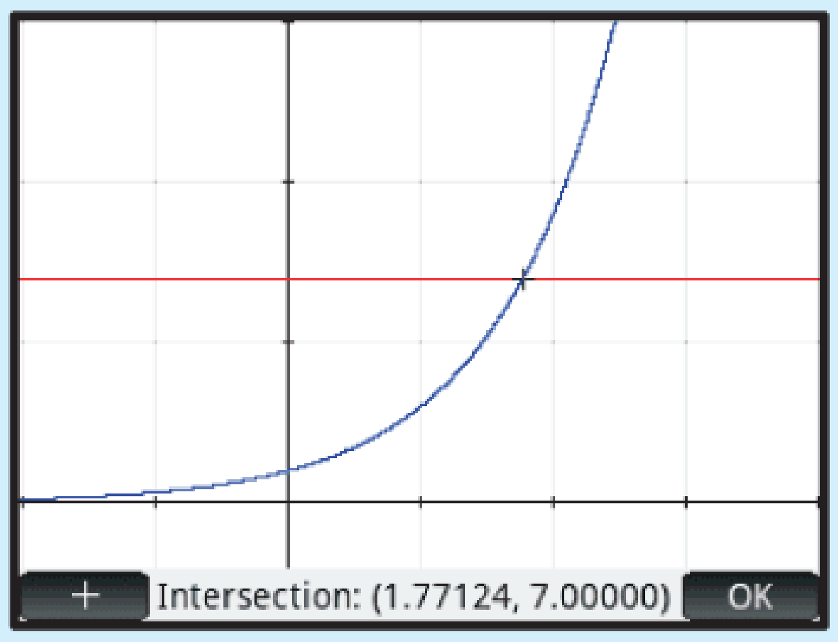
Casio fx-CG50



TI-84 Plus CE



HP Prime



The solution is $x \approx 1.77$.

15 Use technology to solve:

a $2^x = 11$ **b** $3^x = 15$ **c** $4^x + 5 = 10$

d $3^{x+2} = 4$ **e** $5 \times 2^x = 18$ **f** $3^{-x} = 0.9$

g $2 \times 3^{x-2} = 168$ **h** $26 \times (0.95)^x = 2$ **i** $2000 \times (1.03)^x = 5000$

DISCUSSION

For the exponential function $y = a^x$, why do we choose to specify $a > 0$?

What would the graph of $y = (-2)^x$ look like? What is its domain and range?

E GROWTH AND DECAY

In this Section we will examine situations where quantities are either increasing or decreasing exponentially. These situations are known as **growth** and **decay** modelling, and occur frequently in the world around us.

Populations of animals, people, and bacteria usually *grow* in an exponential way.

Radioactive substances, cooling, and items that depreciate in value, usually *decay* exponentially.

For the exponential function $y = p \times a^{x-h} + k$ where $a, p > 0$, $a \neq 1$, we see:

- growth if $a > 1$
- decay if $a < 1$.

GROWTH

Consider a population of 100 mice which under favourable conditions is increasing by 20% each week.

To increase a quantity by 20%, we multiply it by 1.2.

If P_n is the population after n weeks, then:

$P_0 = 100$ {the *original* population}

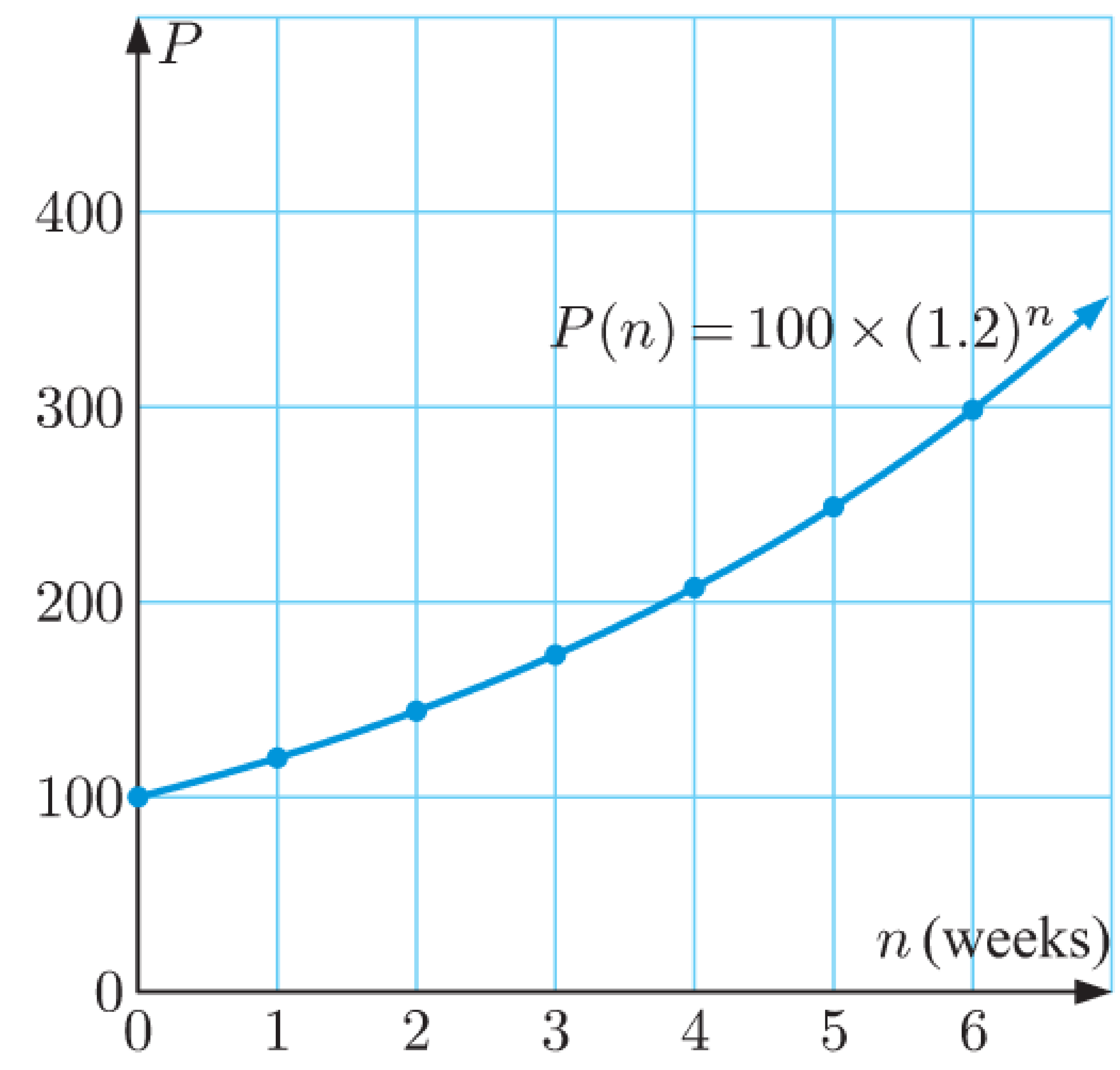
$P_1 = P_0 \times 1.2 = 100 \times 1.2$

$P_2 = P_1 \times 1.2 = 100 \times (1.2)^2$

$P_3 = P_2 \times 1.2 = 100 \times (1.2)^3$, and so on.

From this pattern we see that $P_n = 100 \times (1.2)^n$, $n \in \mathbb{Z}$, which is a geometric sequence.

However, while the population of mice must always be an integer, we expect that the population will grow continuously throughout the year, rather than in big, discrete jumps. We therefore expect it will be well approximated by the corresponding exponential function $P(n) = 100 \times (1.2)^n$, $n \in \mathbb{R}$.



Example 14 **Self Tutor**

A scientist is modelling a grasshopper plague. The area affected by the grasshoppers is given by $A(n) = 1000 \times (1.15)^n$ hectares, where n is the number of weeks after the initial observation.

a Find the original affected area.

b Find the affected area after: **i** 5 weeks **ii** 10 weeks.

c Draw the graph of the affected area over time.

d Use your graph or technology to find how long it will take for the affected area to reach 8000 hectares.

---

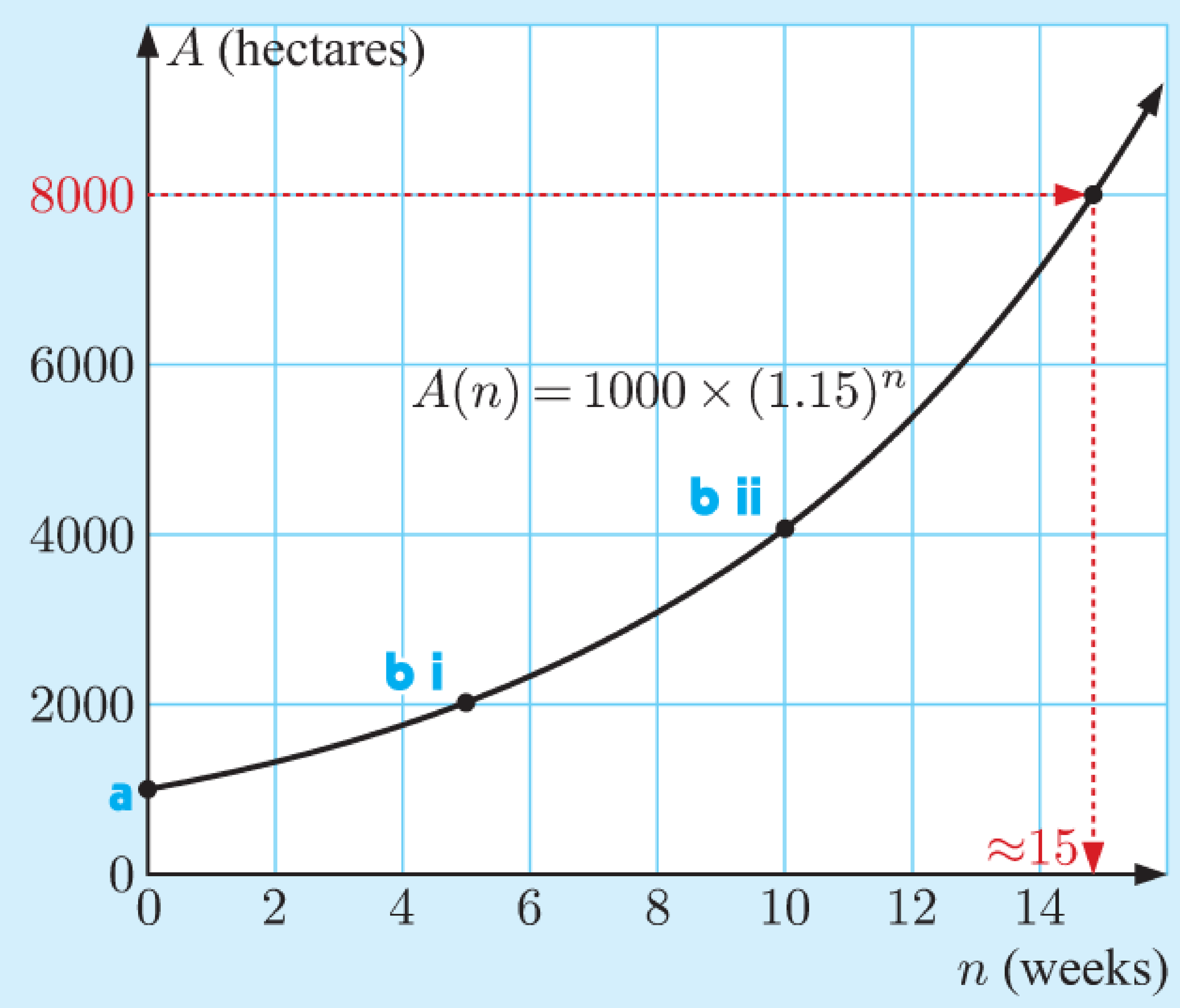
a $A(0) = 1000 \times 1.15^0 = 1000$

$\therefore$ the original affected area was 1000 hectares.

b **i** $A(5) = 1000 \times 1.15^5 \approx 2010$
The affected area is about 2010 hectares.

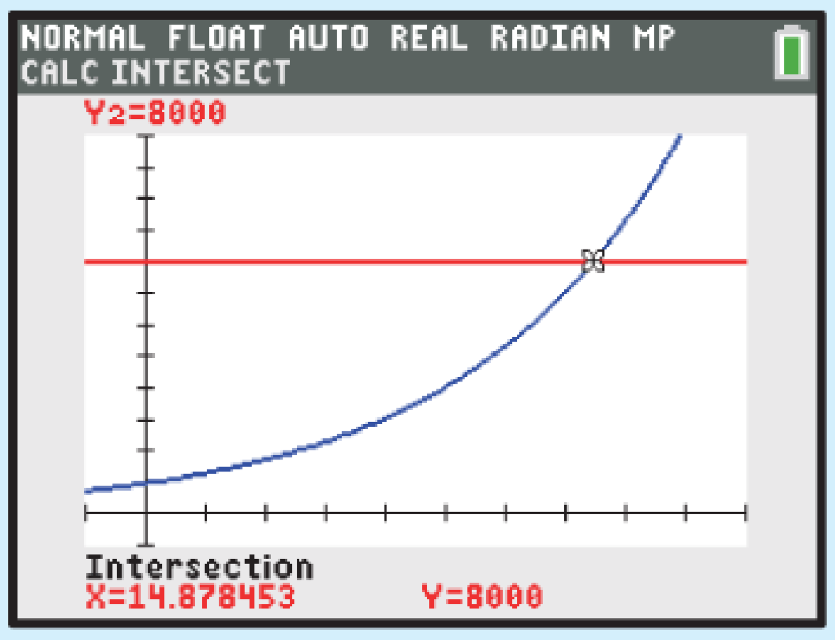
ii $A(10) = 1000 \times 1.15^{10} \approx 4050$
The affected area is about 4050 hectares.

c



d From the graph in **c**, it appears that it would take about 15 weeks for the affected area to reach 8000 hectares.

or Using technology, the solution is ≈ 14.9 weeks.



EXERCISE 2E.1

1 The weight W of bacteria in a culture t hours after establishment is given by $W(t) = 100 \times (1.07)^t$ grams.

a Find the initial weight.

b Find the weight after:

i 4 hours **ii** 10 hours **iii** 24 hours.

c Sketch the graph of the bacteria weight over time using the results of **a** and **b** only.

GRAPHING PACKAGE

Use technology to graph $Y_1 = 100 \times (1.07)^X$ and hence check your answers.

2 A breeding program to ensure the survival of pygmy possums is established with an initial population of 50 (25 pairs). From a previous program, the expected population P in n years' time is given by $P(n) = P_0 \times (1.23)^n$.

a What is the value of P_0?

b What is the expected population after:

i 2 years **ii** 5 years **iii** 10 years?

c Sketch the graph of the population over time using **a** and **b** only.

d Hence estimate the time needed for the population to reach 500.

e Use technology to graph $Y_1 = 50 \times (1.23)^X$. Hence check your answer to **d**.

3 A flu virus spreads in a school. The number of people N infected after t days is given by $N = 4 \times 1.332^t$, $t \geqslant 0$.

a Find the number of people who were initially infected.

b Calculate the number of people who were infected after 16 days.

c There are 1200 people in the school. Estimate the time it will take for everybody in the school to catch the flu.

4 In 1998, 200 bears were introduced to a large island off Alaska where previously there were no bears. The population increased exponentially according to $B(t) = B_0 \times a^t$, where $a > 0$ is a constant and t is the time in years since the introduction.

a Find B_0.

b In 2000 there were 242 bears. Find a, and interpret your answer.

c Find the expected bear population in 2018.

d Find the expected percentage increase in population from 2008 to 2018.

e How long will it take for the population to reach 2000?

5 The speed V of a chemical reaction is given by $V(t) = V_0 \times 2^{0.05t}$ where t is the temperature in °C.

a Find the reaction speed at: **i** 0°C **ii** 20°C.

b Find the percentage increase in reaction speed at 20°C compared with 0°C.

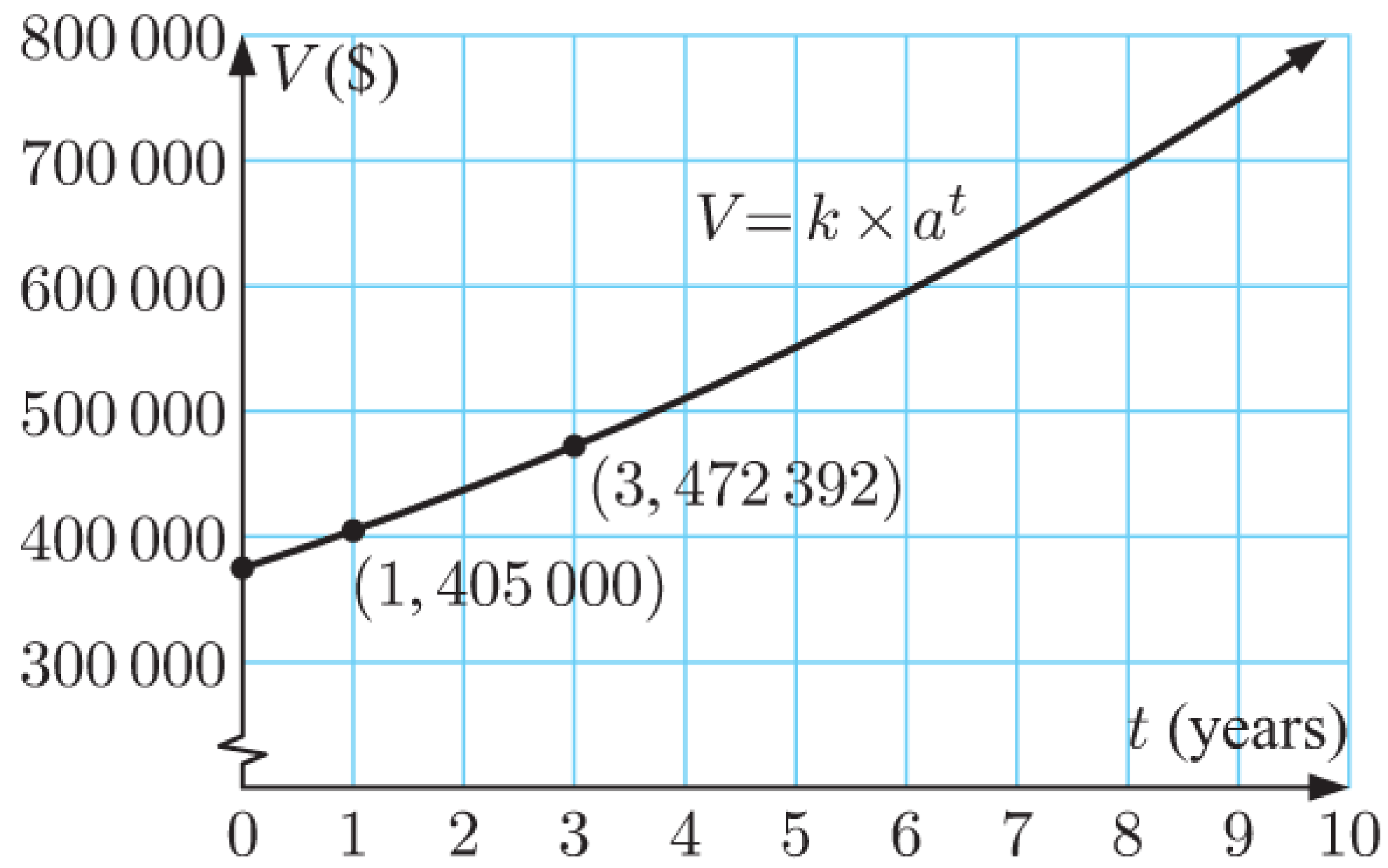
c Find $\left(\dfrac{V(50) - V(20)}{V(20)}\right) \times 100\%$ and explain what this calculation means.

6 Kayla deposited £5000 into an account. The amount in the account increases by 10% each year.

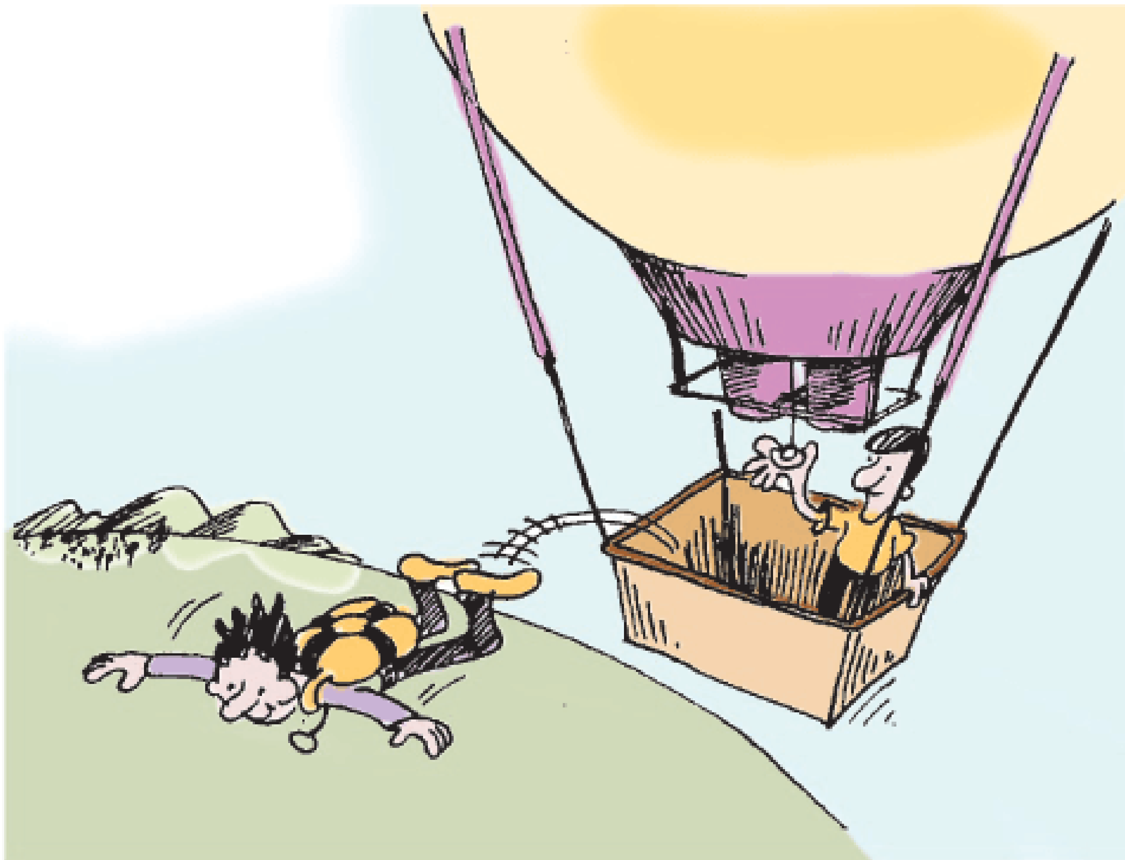
- **a** Write a formula for the amount $A(t)$ in the account after t years.
- **b** Find the amount in the account after: **i** 2 years **ii** 5 years.
- **c** Sketch the graph of $A(t)$.
- **d** How long will it take for the amount in the account to reach £8000?

7 The expected value of a house in t years' time is given by the exponential function $V = k \times a^t$ dollars, where $t \geqslant 0$.
The function is graphed alongside.

- **a** Find a and k, and interpret these values.
- **b** How long will it take for the house's value to reach \$550 000?



8 A parachutist jumps from the basket of a stationary hot air balloon. His speed of descent is given by $V = c - 60 \times 2^{kt}$ m s^{-1} where c and k are constants, and t is the time in seconds.



- **a** Explain why $c = 60$.
- **b** After 5 seconds, the parachutist has speed 30 m s^{-1}. Find k.
- **c** Find the speed of the parachutist after 12 seconds.
- **d** Sketch the graph of V against t.
- **e** Describe how the speed of the parachutist varies over time.

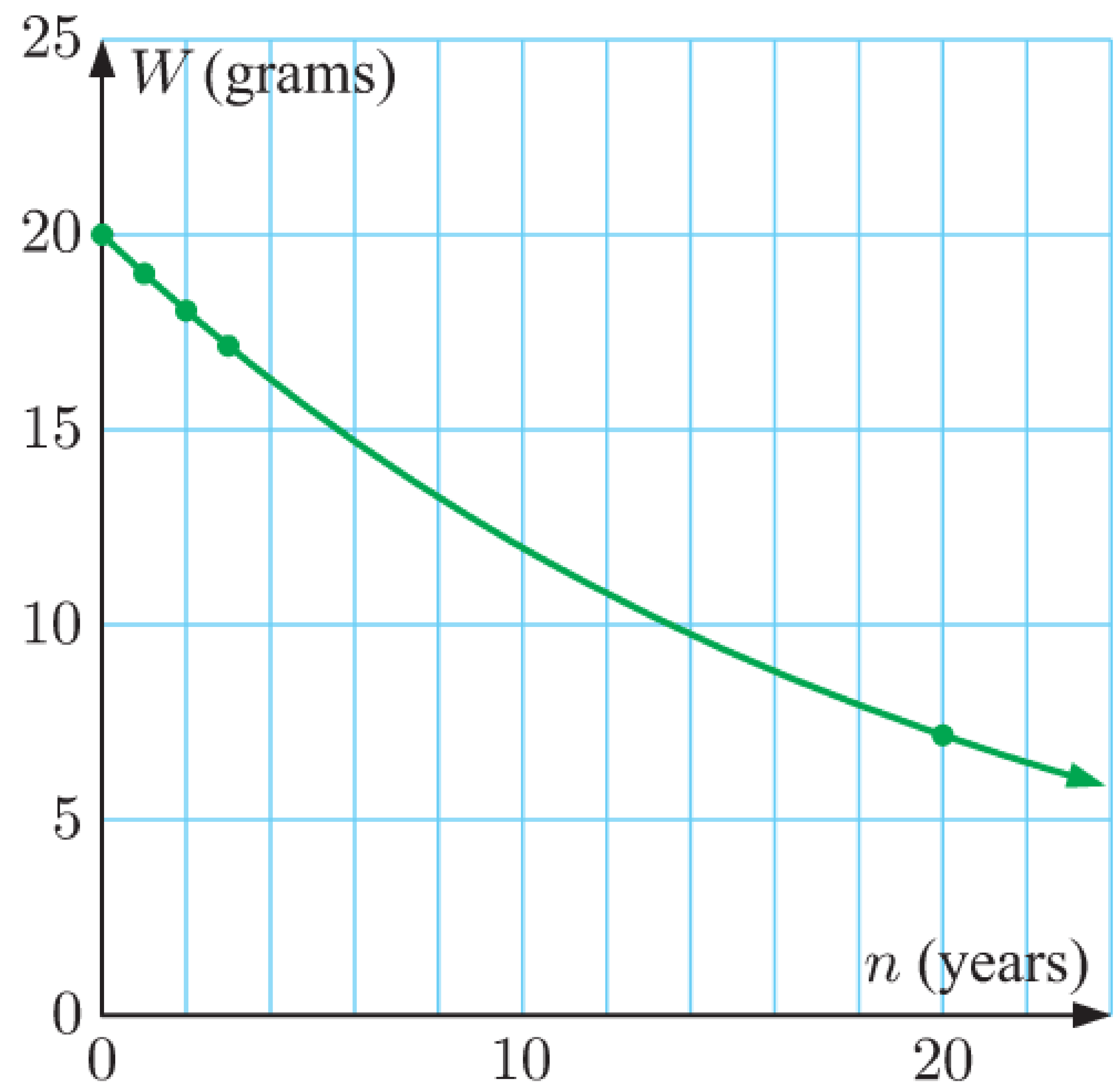
9 The number of microorganisms in a culture doubles every 6 hours. How long will it take for the number of microorganisms to increase by 30%?

DECAY

Consider a radioactive substance with original weight 20 grams. It *decays* or reduces by 5% each year. The multiplier for this is 95% or 0.95.

If W_n is the weight after n years, then:

$$W_0 = 20 \text{ grams}$$
$$W_1 = W_0 \times 0.95 = 20 \times 0.95 \text{ grams}$$
$$W_2 = W_1 \times 0.95 = 20 \times (0.95)^2 \text{ grams}$$
$$W_3 = W_2 \times 0.95 = 20 \times (0.95)^3 \text{ grams}$$
$$\vdots$$
$$W_{20} = 20 \times (0.95)^{20} \approx 7.2 \text{ grams}$$
$$\vdots$$



From this pattern we see that $W_n = 20 \times (0.95)^n$, $n \in \mathbb{Z}$, which is again a geometric sequence.

However, we know that radioactive decay is a continuous process, so the weight remaining will actually be given by the smooth exponential curve $W(n) = 20 \times (0.95)^n$, $n \in \mathbb{R}$.

Example 15 **Self Tutor**

When a diesel-electric generator is switched off, the current dies away according to the formula $I(t) = 24 \times (0.25)^t$ amps, where t is the time in seconds after the power is cut.

a Find $I(t)$ when $t = 0$, 1, 2, and 3.

b What current flowed in the generator at the instant it was switched off?

c Plot the graph of $I(t)$ for $t \geqslant 0$ using the information above.

d Use your graph or technology to find how long it takes for the current to reach 4 amps.

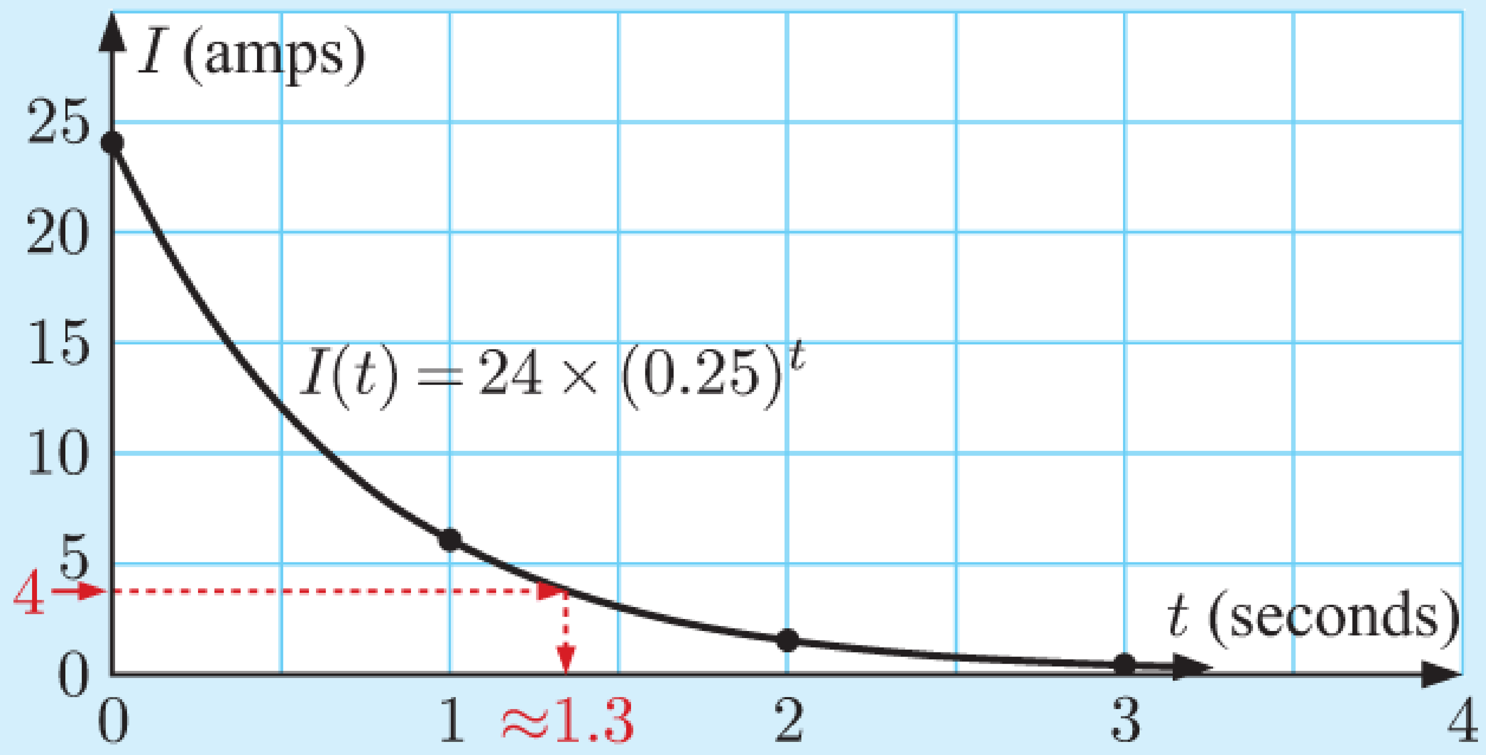
a $I(t) = 24 \times (0.25)^t$ amps

$I(0)$	$I(1)$	$I(2)$	$I(3)$
$= 24 \times (0.25)^0$	$= 24 \times (0.25)^1$	$= 24 \times (0.25)^2$	$= 24 \times (0.25)^3$
$= 24$ amps	$= 6$ amps	$= 1.5$ amps	$= 0.375$ amps

b $I(0) = 24$

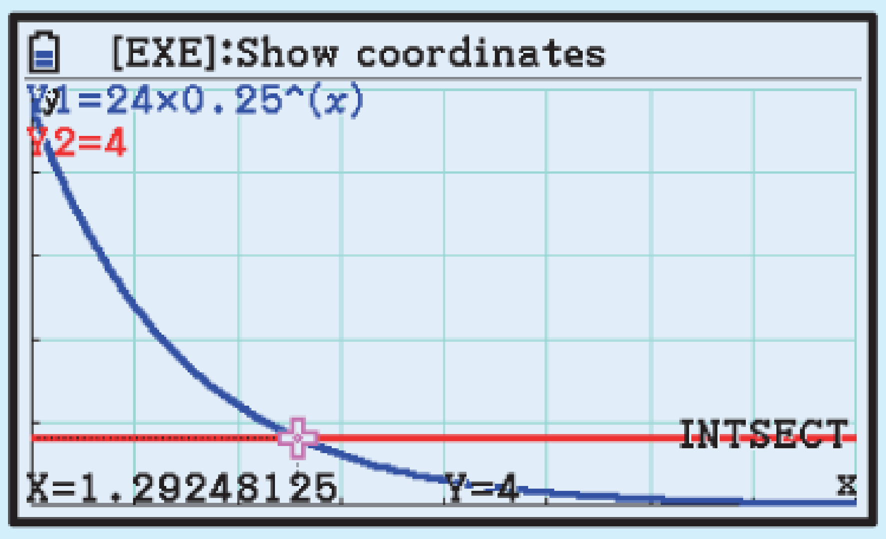
When the generator was switched off, 24 amps of current flowed in the circuit.

c

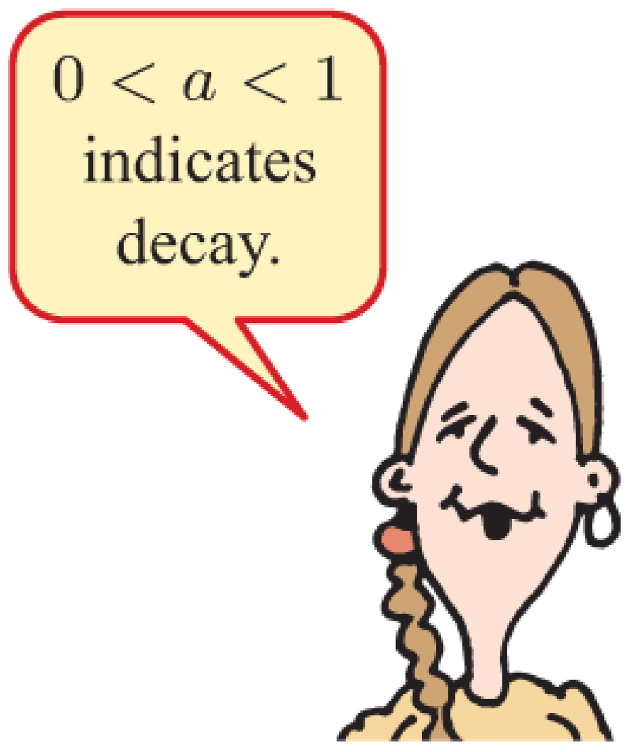


d From the graph above, the time to reach 4 amps is about 1.3 seconds.

or Using technology, the solution is ≈ 1.29 seconds.



EXERCISE 2E.2



1 The weight of a radioactive substance t years after being set aside is given by $W(t) = 250 \times (0.998)^t$ grams.

a How much radioactive substance was initially set aside?

b Determine the weight of the substance after:

i 400 years **ii** 800 years **iii** 1200 years.

c Sketch the graph of $W(t)$ for $t \geqslant 0$ using **a** and **b** only.

d Use your graph or graphics calculator to find how long it takes for the substance to decay to 125 grams.

2 The temperature T of a liquid which has been placed in a refrigerator is given by $T(t) = 100 \times (0.986)^t$ °C where t is the time in minutes.

a Find the initial temperature of the liquid.

b Find the temperature after:

i 15 minutes **ii** 20 minutes **iii** 78 minutes.

c Sketch the graph of $T(t)$ for $t \geqslant 0$ using **a** and **b** only.

3 The weight W of radioactive substance remaining after t years is given by $W(t) = 1000 \times 2^{-0.03t}$ grams.

a Find the initial weight of the radioactive substance.

b Find the weight remaining after:

i 10 years **ii** 100 years **iii** 1000 years.

c Graph the weight remaining over time using **a** and **b** only.

d Use your graph or graphics calculator to find the time when 10 grams of the substance remains.

e Write an expression for the amount of substance that has decayed after t years.

4 An initial count of orangutans in a forest found that the forest contained 400 orangutans. Since then, the destruction of their habitat has caused the population to fall by 8% each year.

a Write a formula for the population P of orangutans t years after the initial count.

b Find the population of orangutans after:

i 1 year **ii** 5 years.

c Sketch the graph of the population over time.

d How long will it take for the population to fall to 200?

5 The intensity of light L diminishes below the surface of the sea according to the formula $L = L_0 \times (0.95)^d$ units, where d is the depth in metres measured from the surface of the sea.

a If the intensity of light at the surface is 10 units, find the value of L_0.

b Find the intensity of light 25 m below the surface.

c A light intensity of 4 units is considered adequate for divers to be able to see clearly. Calculate the depth corresponding to this intensity of light.

d Calculate the range of depths for which the light intensity is between 1 and 3 units.

6 The value of a car after t years is $V = 24\,000 \times r^t$ dollars, $t \geqslant 0$.

a Write down the value of the car when it was first purchased.

b The value of the car after 2 years was \$17 340. Find the value of r.

c How long will it take for the value of the car to reduce to \$8000? Give your answer to the nearest year.

7 The interior of a freezer has temperature $-10°$C. When a packet of peas is placed in the freezer, its temperature after t minutes is given by $T(t) = -10 + 32 \times 2^{-0.2t}$ °C.

a What was the temperature of the packet of peas:

i when placed in the freezer **ii** after 5 minutes **iii** after 10 minutes?

b Sketch the graph of $T(t)$.

c How long does it take for the temperature of the packet of peas to fall to 0°C?

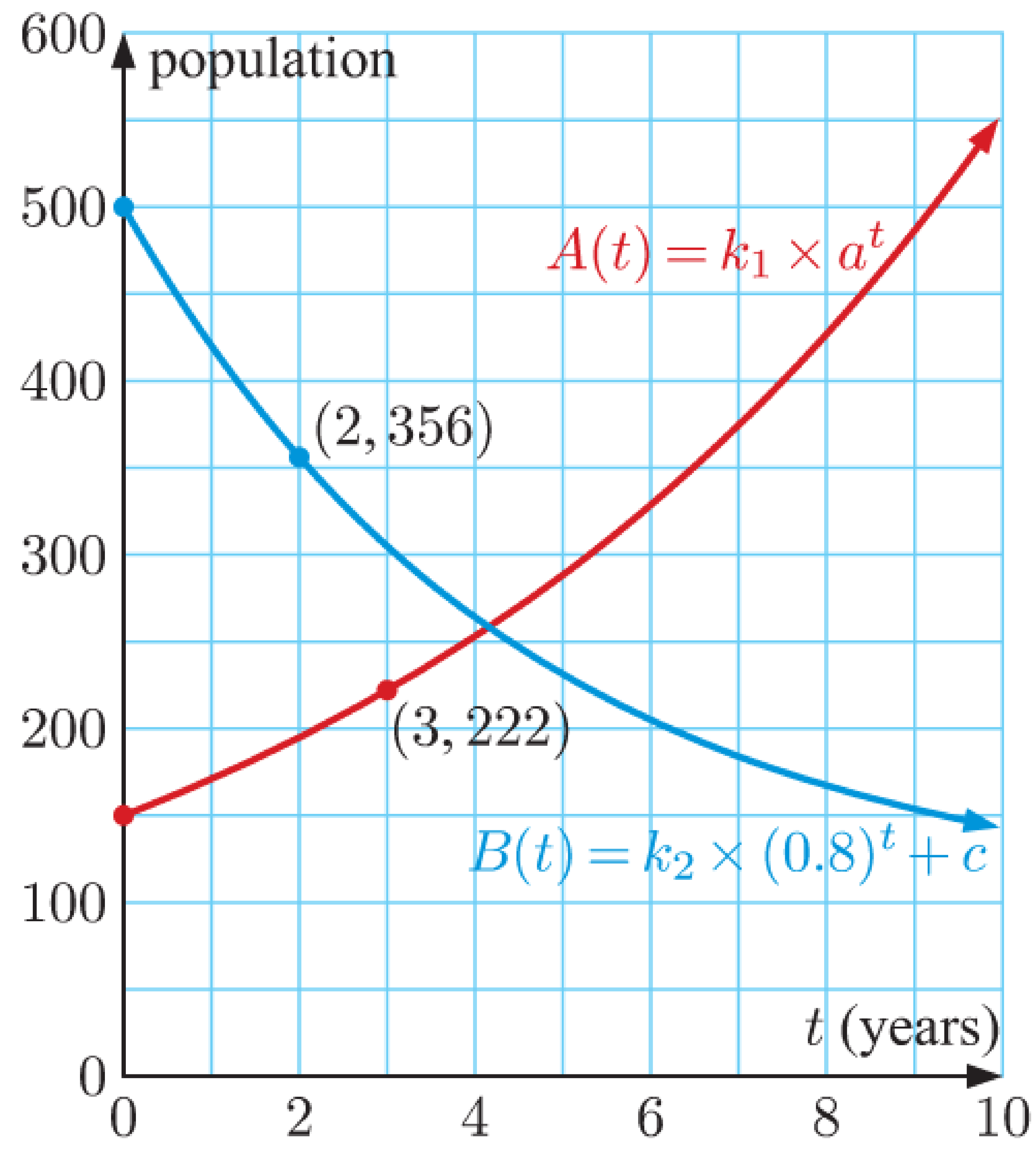
d Will the temperature of the packet of peas ever reach $-10°$C? Explain your answer.

8 The weight W_t of a radioactive uranium-235 sample remaining after t years is given by the formula $W_t = W_0 \times 2^{-0.0002t}$ grams, $t \geqslant 0$.

a Find the original weight. **b** Find the percentage weight loss after 1000 years.

c How long will it take until $\frac{1}{512}$ of the sample remains?

9 When scientists first observe a population of endangered marsupials, they notice two distinct groups. Group A is smaller in number, but appear to be larger and stronger individuals. Group B are more numerous, but smaller animals. The number of animals in each group over time are given by $A(t)$ and $B(t)$ respectively.



a Use the graph to find the exponential function for each animal group.

b Determine the time at which:

i there are the same number of group A and group B animals

ii there are 50 less group A than group B animals

iii there are twice as many group B animals compared to group A animals.

10 The **half-life** of a radioactive substance is the time it takes for the substance's weight to fall to half of its original value.
The radioactive isotope fermium-253 has a half-life of 3 days. The weight of fermium-253 detected t days after an explosion is $W(t) = 10 \times a^t$ mg.

a Interpret the value 10 in this model.

b Calculate the value of a, correct to 4 decimal places, and interpret this value.

c Find the weight of fermium-253 after 2 days.

d How long will it take for the weight of fermium-253 to fall to:

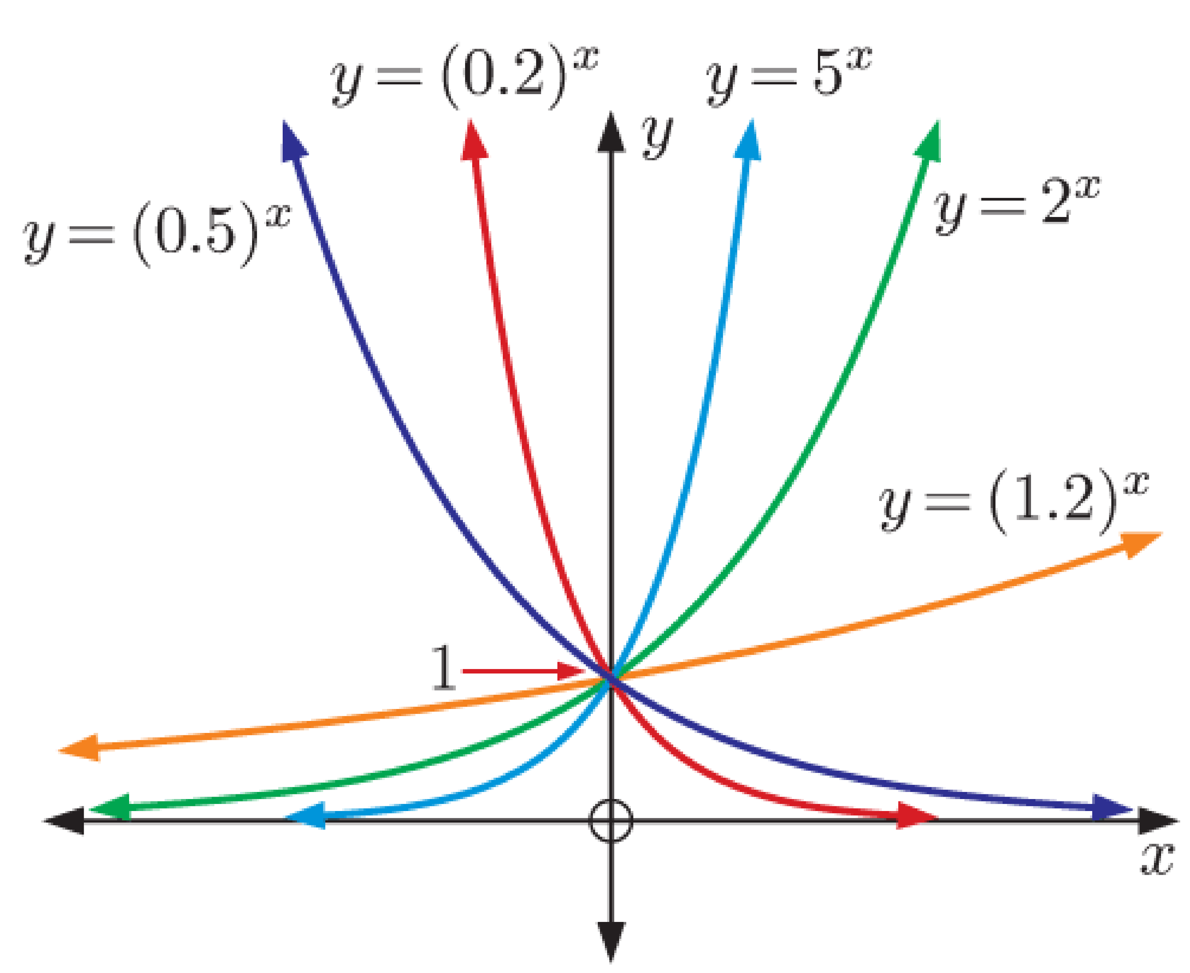
i 3 mg **ii** 1.25 mg?

11 The half-life of nitrogen-13 is 10 minutes. How long will it take for the mass of nitrogen-13 to fall to 10% of its original value?

F THE NATURAL EXPONENTIAL

We have seen that the simplest exponential functions have the form $f(x) = a^x$ where $a > 0$, $a \neq 1$.

Graphs of some of these functions are shown alongside.



We can see that for all positive values of the base a, the graph is always positive.

Hence $a^x > 0$ for all $a > 0$.

There are an infinite number of possible choices for the base number.

However, where exponential data is examined in science, engineering, and finance, the base $e \approx 2.7183$ is commonly used.

e is a special number in mathematics. It is irrational like π, and just as π is the ratio of a circle's circumference to its diameter, e also has a physical meaning. We explore this meaning in the following **Investigation**.

INVESTIGATION 2 CONTINUOUS COMPOUND INTEREST

A discrete formula for calculating the amount to which an investment grows under compound interest is $u_n = u_0(1+i)^n$ where:

u_n is the final amount, u_0 is the initial amount,
i is the interest rate per compounding period,
n is the number of periods, or times the interest is compounded.

We will investigate the final value of an investment for various values of n, and allow n to become extremely large.

What to do:

1 Suppose \$1000 is invested for one year at a fixed rate of 6% per annum. Use your calculator to find the final amount or *maturing value* if the interest is paid:

a annually $(n = 1,\ i = 6\% = 0.06)$ **b** quarterly $(n = 4,\ i = \frac{6\%}{4} = 0.015)$

c monthly **d** daily **e** by the second **f** by the millisecond.

Comment on your answers.

2 If r is the percentage rate per year, t is the number of years, and N is the number of interest payments per year, then $i = \frac{r}{N}$ and $n = Nt$.

If we let $a = \frac{N}{r}$, show that the growth formula becomes $u_n = u_0\left[\left(1+\frac{1}{a}\right)^a\right]^{rt}$.

3 For continuous compound growth, the number of interest payments per year N gets very large.

a Explain why a gets very large as N gets very large.

b Copy and complete the table, giving your answers as accurately as technology permits.

a	$\left(1+\frac{1}{a}\right)^a$
10	
100	
1000	
10 000	
100 000	
1 000 000	
10 000 000	

4 You should have found that for very large values of a, $\left(1+\frac{1}{a}\right)^a \approx 2.718\,281\,828\,459....$

Use the e^x key of your calculator to find the value of e^1. What do you notice?

5 For continuous growth, $u_n = u_0 e^{rt}$ where u_0 is the initial amount, r is the annual percentage rate, and t is the number of years.

Use this formula to find the final amount if \$1000 is invested for 4 years at a fixed rate of 6% per annum, where the interest is paid continuously.

From **Investigation 2** we observe that:

If interest is paid *continuously* or *instantaneously* then the formula for calculating a compounding amount $u_n = u_0(1+i)^n$ can be replaced by $u_n = u_0 e^{rt}$, where r is the percentage rate per annum and t is the number of years.

HISTORICAL NOTE

The natural exponential e was first described in 1683 by Swiss mathematician **Jacob Bernoulli**. He discovered the number while studying compound interest, just as we did in **Investigation 2**.

The natural exponential was first called e by Swiss mathematician and physicist **Leonhard Euler** in a letter to the German mathematician **Christian Goldbach** in 1731. The number was then published with this notation in 1736.

In 1748, Euler evaluated e correct to 18 decimal places.

Leonhard Euler

Euler also discovered some patterns in **continued fraction** expansions of e. He wrote that

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\dots}}}}} \quad \text{and} \quad e-1 = 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}}}}$$

One may think that e was chosen because it was the first letter of Euler's name or for the word exponential, but it is likely that it was just the next vowel available since he had already used a in his work.

EXERCISE 2F

GRAPHING PACKAGE

1 Sketch, on the same set of axes, the graphs of $y = 2^x$, $y = e^x$, and $y = 3^x$. Comment on any observations.

2 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = e^{-x}$. What is the geometric connection between these two graphs?

3 For the general exponential function $y = pe^{qx}$, what is the y-intercept?

4 Consider $y = 2e^x$.

a Explain why y can never be negative.

b Find y if: **i** $x = -20$ **ii** $x = 20$.

5 Find, to 3 significant figures, the value of:

a e^2 **b** e^3 **c** $e^{0.7}$ **d** $\sqrt{e}$ **e** e^{-1}

6 Write the following as powers of e:

a $\sqrt{e}$ **b** $\frac{1}{\sqrt{e}}$ **c** $\frac{1}{e^2}$ **d** $e\sqrt{e}$

7 Evaluate, to five significant figures:

a $e^{2.31}$ **b** $e^{-2.31}$ **c** $e^{4.829}$ **d** $e^{-4.829}$

e $50e^{-0.1764}$ **f** $80e^{-0.6342}$ **g** $1000e^{1.2642}$ **h** $0.25e^{-3.6742}$

8 Expand and simplify:

a $(e^x+1)^2$ **b** $(1+e^x)(1-e^x)$ **c** $e^x(e^{-x}-3)$

9 Factorise:

a $e^{2x}+e^x$ **b** $e^{2x}-16$ **c** $e^{2x}-8e^x+12$

10 **a** On the same set of axes, sketch and clearly label the graphs of:

$f: x \mapsto e^x, \quad g: x \mapsto e^{x-2}, \quad h: x \mapsto e^x+3$

b State the domain and range of each function.

11 **a** On the same set of axes, sketch and clearly label the graphs of:

$f: x \mapsto e^x, \quad g: x \mapsto -e^x, \quad h: x \mapsto 10-e^x$

b State the domain and range of each function.

c Describe the behaviour of each function as $x \to \pm\infty$.

12 Let $f(x)=e^x-1$ and $g(x)=\frac{1}{x}$.

a Find $(f \circ g)(x)$, and state its domain and range.

b Find $(g \circ f)(x)$, and state its domain and range.

13 The weight of bacteria in a culture is given by $W(t)=2e^{\frac{t}{2}}$ grams where t is the time in hours after the culture was set to grow.

a Find the weight of the culture:

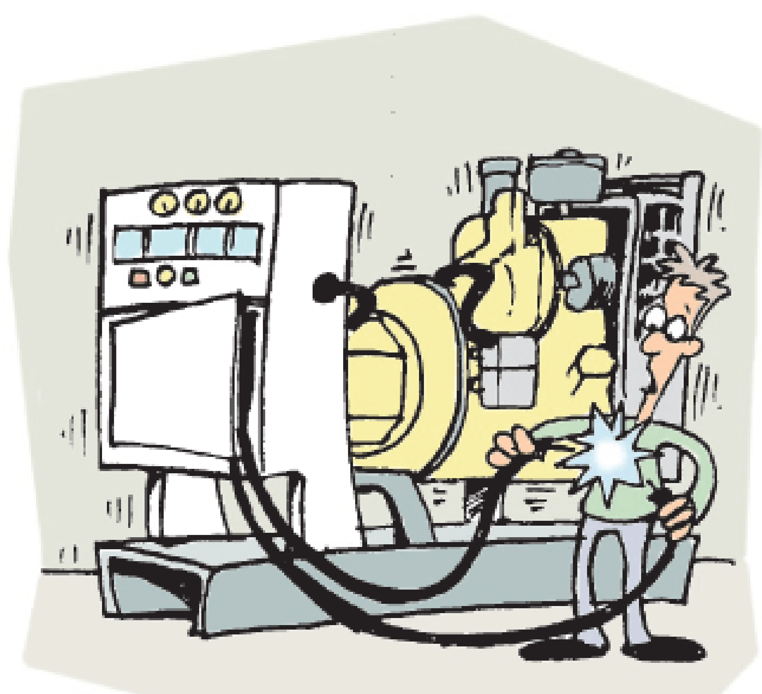
i initially **ii** after 30 minutes **iii** after $1\frac{1}{2}$ hours **iv** after 6 hours.

b Hence sketch the graph of $W(t)=2e^{\frac{t}{2}}$.

14 Solve for x:

a $e^x=\sqrt{e}$ **b** $e^{\frac{1}{2}x}=\frac{1}{e^2}$ **c** $e^{2x}+e^x=2$

15 The current flowing in an electrical circuit t seconds after it is switched off is given by $I(t)=75e^{-0.15t}$ amps.



a What current is still flowing in the circuit after:

i 1 second **ii** 10 seconds?

b Use your graphics calculator to help sketch the graph of $I(t)=75e^{-0.15t}$.

c How long will it take for the current to fall to 1 amp?

16 The population P of trout in a lake is given by $P(t)=\frac{800}{1+ke^{-0.5t}}$, where t is the time in months.

a Given that there were initially 20 trout in the lake, find the value of k.

b Find the population after 6 months.

c Use technology to sketch the graph of $P(t)$.

d Describe what happens to the population as t increases.

e How long will it take for the population to reach 600?

17 Consider the function $f(x)=e^x$.

a On the same set of axes, sketch $y=f(x)$, $y=x$, and $y=f^{-1}(x)$.

b State the domain and range of f^{-1}.

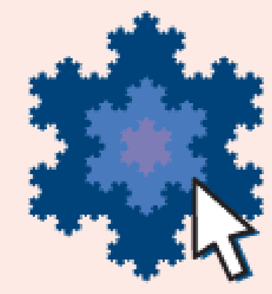
18 It can be shown that $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\times3}x^3 + \frac{1}{2\times3\times4}x^4 + ... = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$ which is an infinite polynomial expansion.

Check this statement by using the first 20 terms of the series to find an approximation for e^1.

ACTIVITY

Click on the icon to run a card game for exponential functions.

CARD GAME



REVIEW SET 2A

1 Evaluate:

a $8^{\frac{2}{3}}$ **b** $27^{-\frac{2}{3}}$ **c** $81^{-\frac{1}{4}}$

2 Solve for x:

a $2^{x-3} = \frac{1}{32}$ **b** $9^x = 27^{2-2x}$ **c** $e^{2x} = \frac{1}{\sqrt{e}}$

3 Expand and simplify:

a $e^x(e^{-x} + e^x)$ **b** $(2^x + 5)^2$ **c** $(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 7)$

4 Solve for x:

a $6 \times 2^x = 192$ **b** $9^{x-1} \times \left(\frac{1}{27}\right)^x = \sqrt{3}$ **c** $4^x - 32 = 4(2^x)$

5 The point $(1, \sqrt{8})$ lies on the graph of $y = 2^{kx}$. Find the value of k.

6 Consider the graph of $y = 3^x$ alongside.

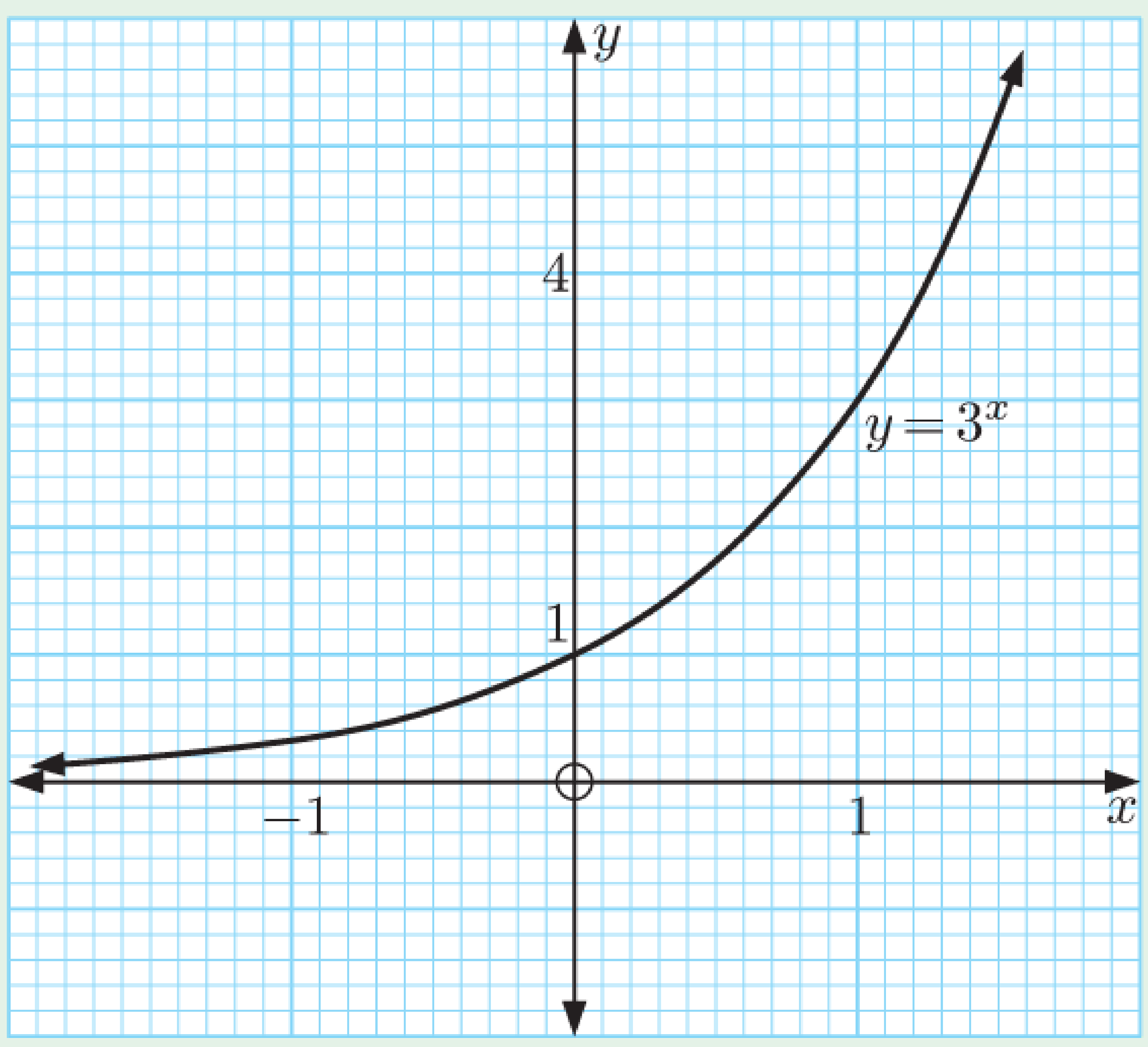
a Use the graph to estimate the value of:

i $3^{0.7}$ **ii** $3^{-0.5}$

b Use the graph to estimate the solution to:

i $3^x = 5$ **ii** $3^x = \frac{1}{2}$

iii $6 \times 3^x = 20$



7 If $f(x) = 3 \times 2^x$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-2)$

8 On the same set of axes, draw the graphs of $y = 2^x$ and $y = 2^x - 4$. Include on your graph the y-intercept and the equation of the horizontal asymptote of each function.

9 Consider $y = 3^x - 5$.

a Find y when $x = 0, \pm1, \pm2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 3^x - 5$.

d State the equation of any asymptote.

10 Consider $y = 3 - 2^{-x}$.

a Find y when $x = 0, \pm1, \pm2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 3 - 2^{-x}$.

d State the equation of any asymptote.

11 Let $f(x) = 2^x$ and $g(x) = 3 - x^2$.

a Find $(f \circ g)(x)$, and state its domain and range.

b Find $(g \circ f)(x)$, and state its domain and range.

c Solve for x:

i $(f \circ g)(x) = 2$ **ii** $(g \circ f)(x) = -13$

12 **a** On the same set of axes, sketch and clearly label the graphs of:

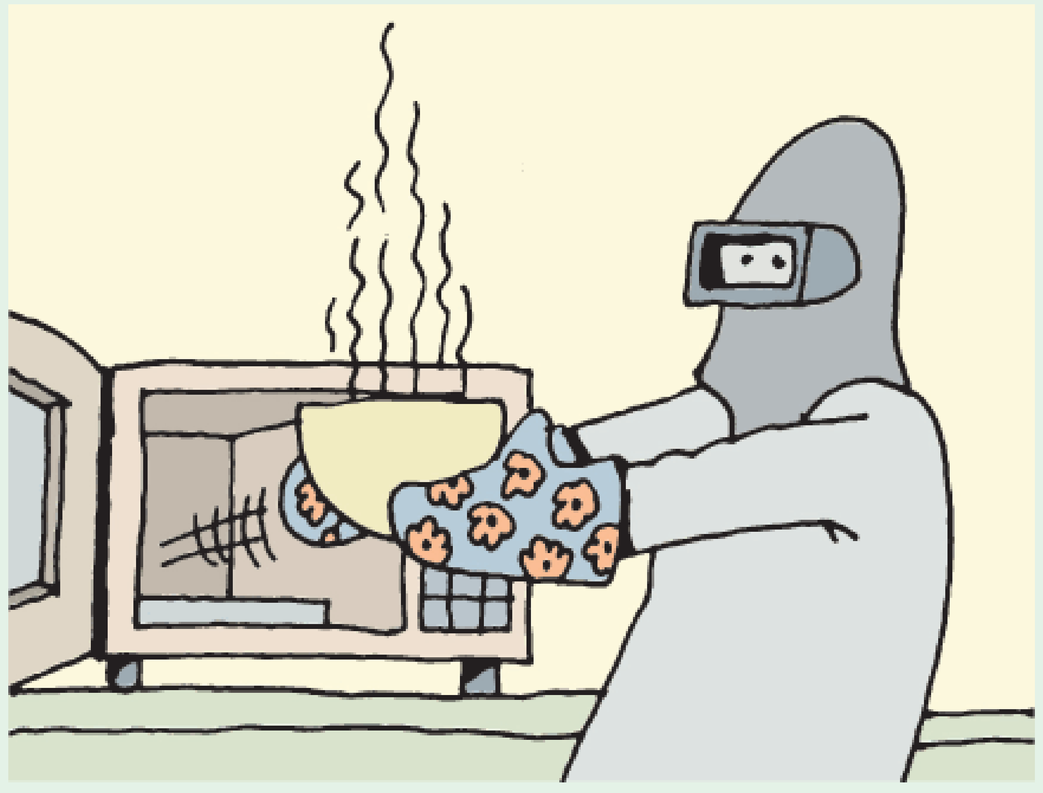
$f : x \mapsto e^x$, $g : x \mapsto e^{x-1}$, $h : x \mapsto 3 - e^x$

b State the domain and range of each function in **a**.

c Describe the behaviour of each function as $x \to \pm\infty$.

13 A plant doubles in size every 5 days. How often does it treble in size?

14 The temperature of a dish t minutes after it is removed from the microwave, is given by $T(t) = 80 \times (0.913)^t$ °C.



a Find the initial temperature of the dish.

b Find the temperature after:

i 12 minutes **ii** 24 minutes **iii** 36 minutes.

c Draw the graph of T against t for $t \geqslant 0$, using **a** and **b** or technology.

d Hence find the time taken for the temperature of the dish to fall to 25°C.

REVIEW SET 2B

1 Evaluate, correct to 3 significant figures:

a $3^{\frac{5}{4}}$ **b** $27^{-\frac{1}{5}}$ **c** $\sqrt[4]{100}$

2 Expand and simplify:

a $(3 - e^x)^2$ **b** $x^{-\frac{1}{2}}(x^{\frac{3}{2}} - 2x^{\frac{1}{2}} - x^{-\frac{1}{2}})$ **c** $2^{-x}(2^{2x} + 2^x)$

3 Factorise:

a $3^{x+2} - 3^x$ **b** $4^x - 2^x - 12$ **c** $e^{2x} + 2e^x - 15$

4 Solve for x:

a $3 \times \left(\frac{1}{7}\right)^{x+1} = 1029$ **b** $9^x - 10(3^x) + 9 = 0$ **c** $2(4^{x+1}) + 1 = 6(2^x)$

5 Suppose $f(x) = 2^{-x} + 1$.

a Find $f\left(\frac{1}{2}\right)$. **b** Find a such that $f(a) = 3$.

6 Consider $y = 2e^{-x} + 1$.

a Find y when $x = 0, \pm 1, \pm 2$. **b** Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 2e^{-x} + 1$. **d** State the equation of any asymptote.

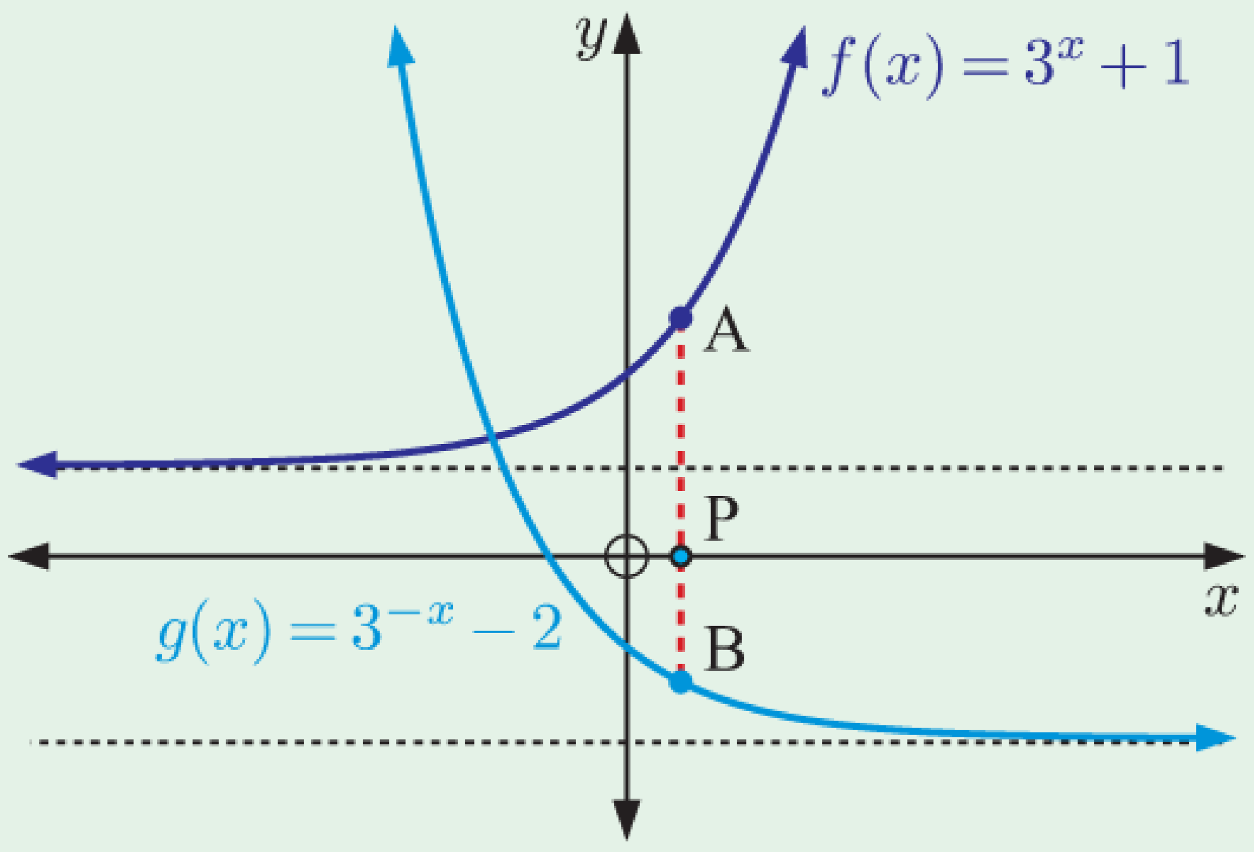
7 Answer the **Opening Problem** on page **42**.

8 Find the domain and range of $f(x) = 3^{\sqrt{x+1}}$.

9 The exponential functions $f(x) = 3^x + 1$ and $g(x) = 3^{-x} - 2$ are graphed alongside.

a Find the y-intercept of each function.

b Given that the vertical line segment [AB] has length 4 units, find the exact length of [PB]. Give your answer in the form $a + b\sqrt{5}$ units, where $a, b \in \mathbb{Q}$.



10 Let $f(x) = 3^x$.

a Write down the value of:

i $f(4)$ **ii** $f(-1)$

b Find the value of k such that $f(x+2) = k\,f(x)$, $k \in \mathbb{Z}$.

11 Suppose $y = a^x$. Express in terms of y:

a a^{2x} **b** a^{-x} **c** $\dfrac{1}{\sqrt{a^x}}$

12 The weight of a radioactive substance after t years is given by $W = 1500 \times (0.993)^t$ grams.

a Find the original amount of radioactive material.

b Find the amount of radioactive material remaining after:

i 400 years **ii** 800 years.

c Sketch the graph of W against t for $t \geqslant 0$.

d Hence find the time taken for the weight to reduce to 100 grams.

13 A phycologist investigates an algal bloom in a lake. Initially it covers 10 square metres of water. Each day after it was discovered, the area covered increases by 15%.

a Write a formula for the area $A(t)$ covered after t days.

b Find the area covered after:

i 2 days **ii** 5 days.

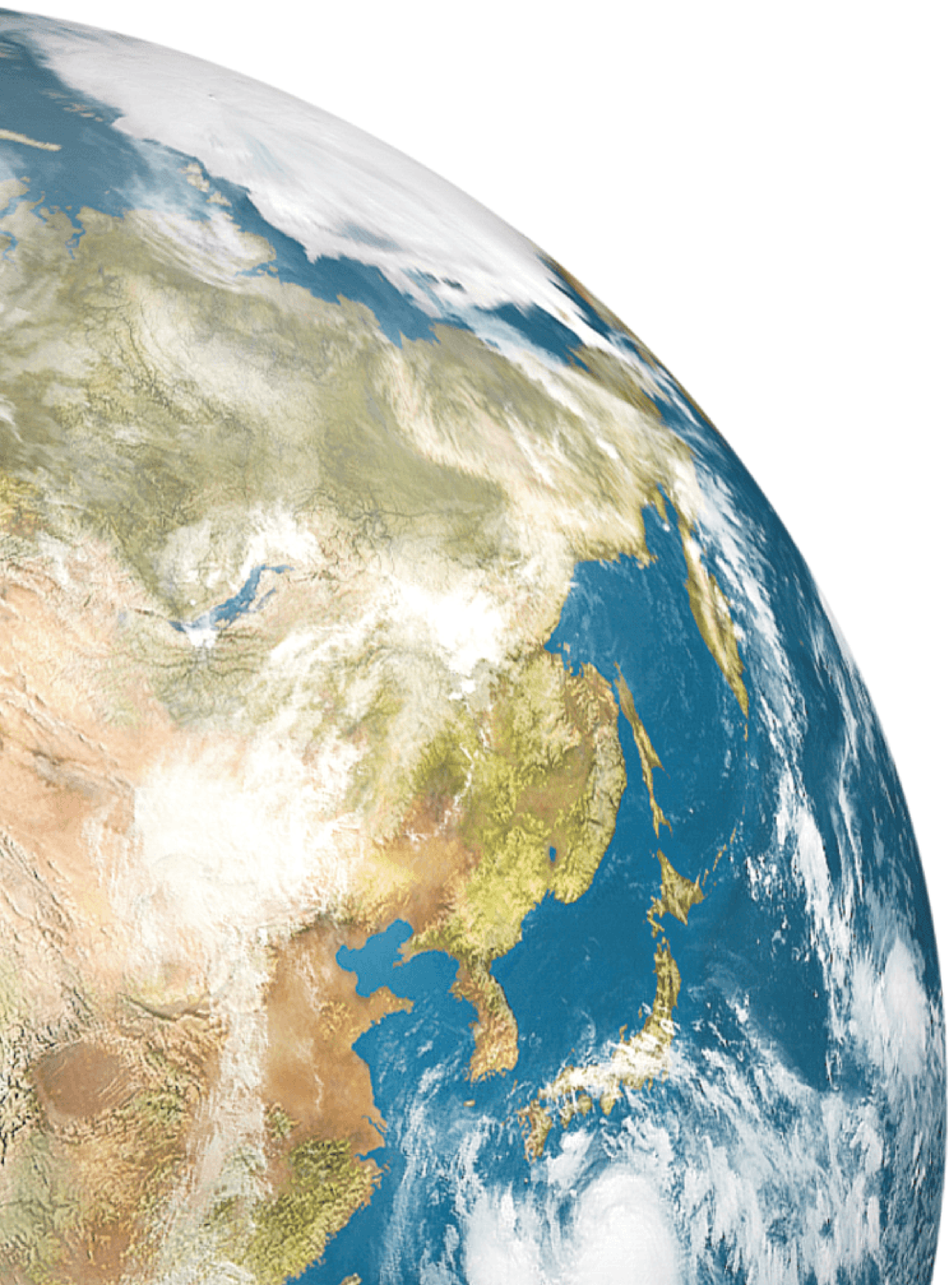
c Sketch the graph of $A(t)$.

d How long will it take for the affected area to reach 300 m^2?

Chapter 3

Logarithms

Contents:



OPENING PROBLEM

In a plentiful springtime, a population of 1000 mice will double every week.

The population after t weeks is given by the exponential function $P(t) = 1000 \times 2^t$ mice.

Things to think about:

a What does the graph of the population over time look like?

b How long will it take for the population to reach 20 000 mice?

c Can we write a function for t in terms of P, which determines the time at which the population P is reached?

In the previous Chapter we solved exponential equations by writing both sides with the same base, and by using graphs.

In this Chapter we study a more formal solution to exponential equations in which we use the **inverse** of the exponential function. We call this a **logarithm**.

A LOGARITHMS IN BASE 10

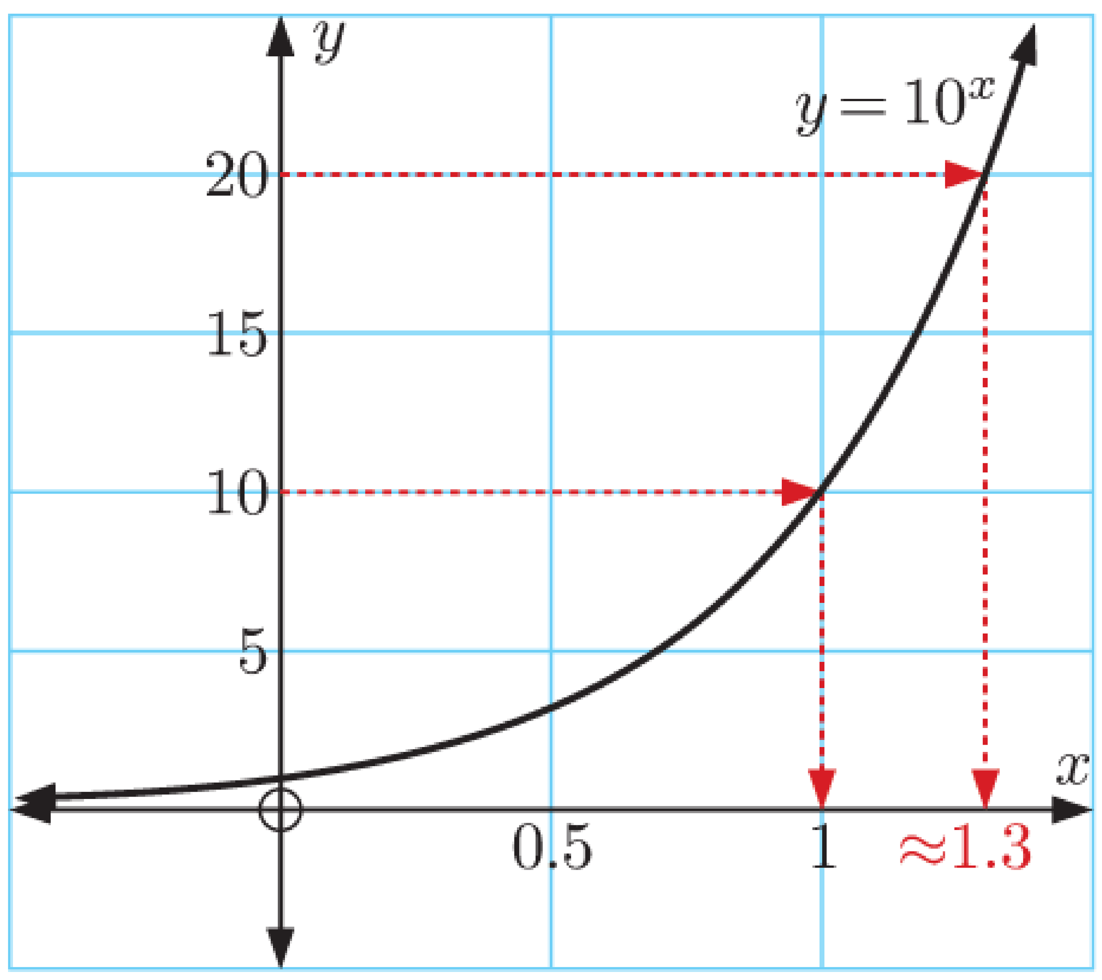
Consider the graph of $y = 10^x$ shown.

Notice that the range of the function is $\{y \mid y > 0\}$. This means that every positive number y can be written in the form 10^x.

For example:

- When $y = 10$, $x = 1$, so $10 = 10^1$.
- When $y = 20$, $x \approx 1.3$, so $20 \approx 10^{1.3}$.

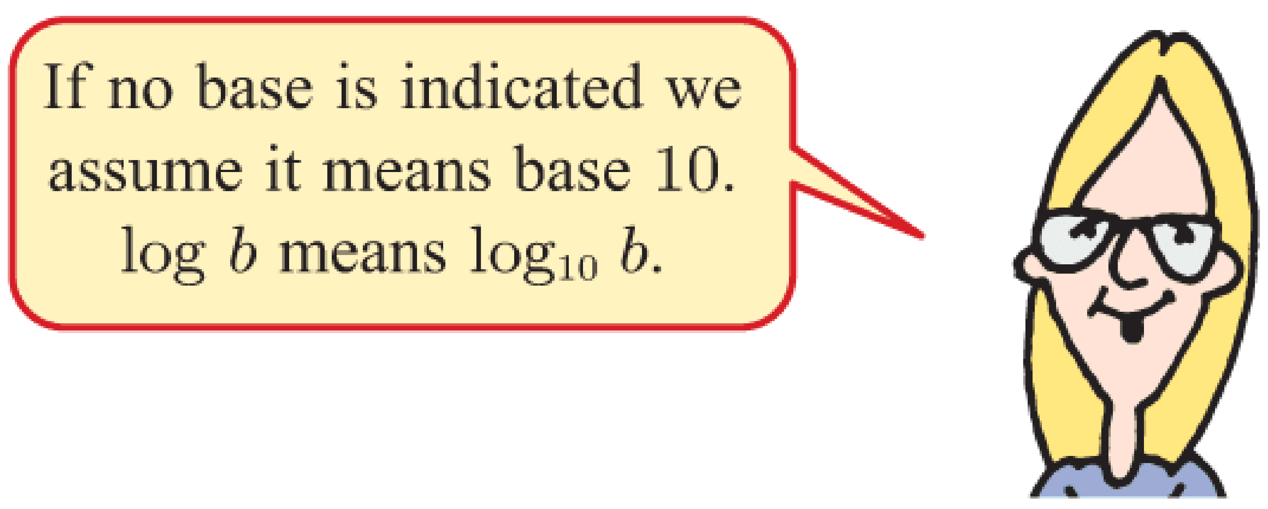
When we write a positive number y in the form 10^x, we say that x is the **logarithm in base 10**, of y.



The **logarithm in base 10** of a positive number is the power that 10 must be raised to in order to obtain that number.

For example:

- The logarithm in base 10 of 1000 is 3, since $1000 = 10^3$.
  We write $\log_{10} 1000 = 3$ or simply $\log 1000 = 3$.
- $\log(0.01) = -2$ since $0.01 = 10^{-2}$.



By observing that $\log 1000 = \log(10^3) = 3$ and $\log(0.01) = \log(10^{-2}) = -2$, we conclude that $\log 10^x = x$ for any $x \in \mathbb{R}$.

Example 1 Self Tutor

Find: **a** $\log 100$ **b** $\log \sqrt[4]{10}$

a $\log 100 = \log(10^2) = 2$

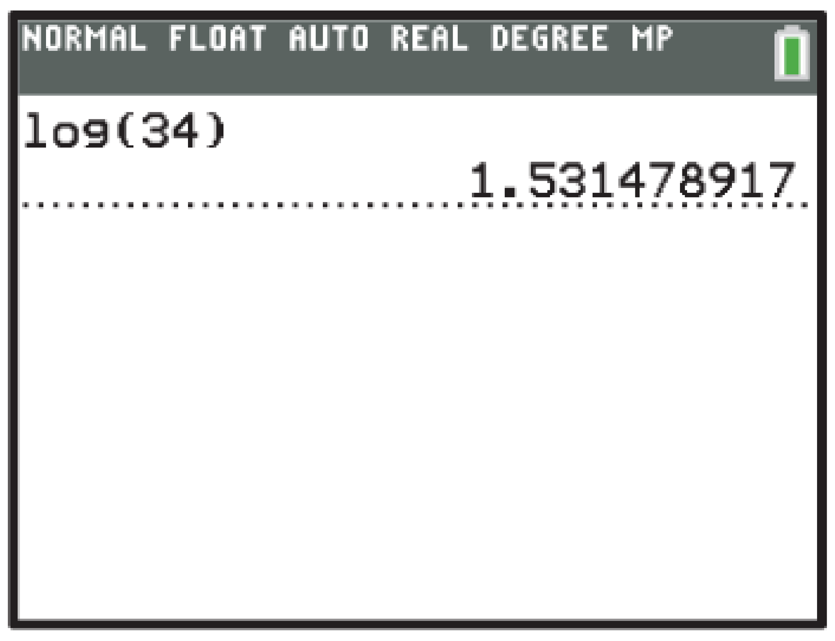
b $\log \sqrt[4]{10} = \log(10^{\frac{1}{4}}) = \frac{1}{4}$

The logarithms in **Example 1** can be found by hand because it is easy to write 100 and $\sqrt[4]{10}$ as powers of 10. The logarithms of most values, however, can only be found using a calculator.

For example, $\log 34 \approx 1.53$

so $34 \approx 10^{1.53}$

GRAPHICS CALCULATOR INSTRUCTIONS



Logarithms allow us to write any number as a power of 10. In particular:

$\boldsymbol{x = 10^{\log x}}$ for any $x > 0$.

Example 2 Self Tutor

Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

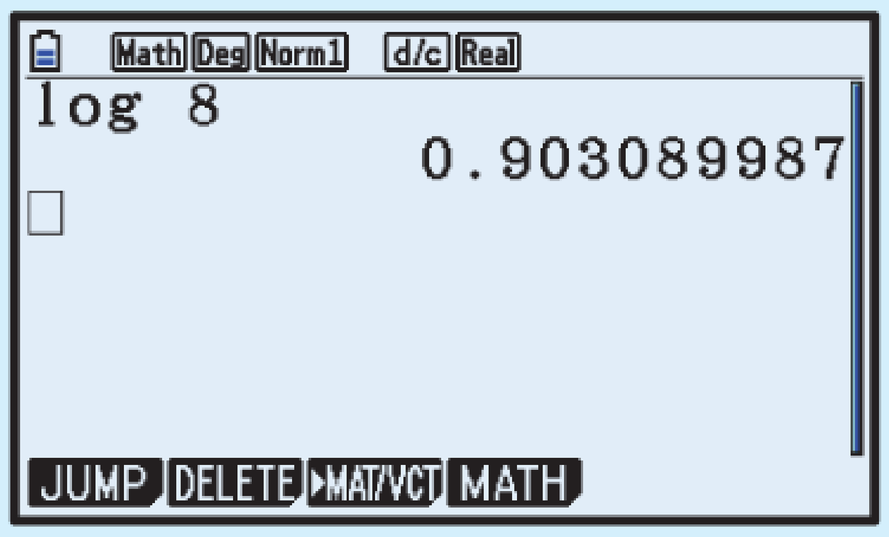
a 8 **b** 800 **c** 0.08

a $8 = 10^{\log 8}$
$\approx 10^{0.9031}$

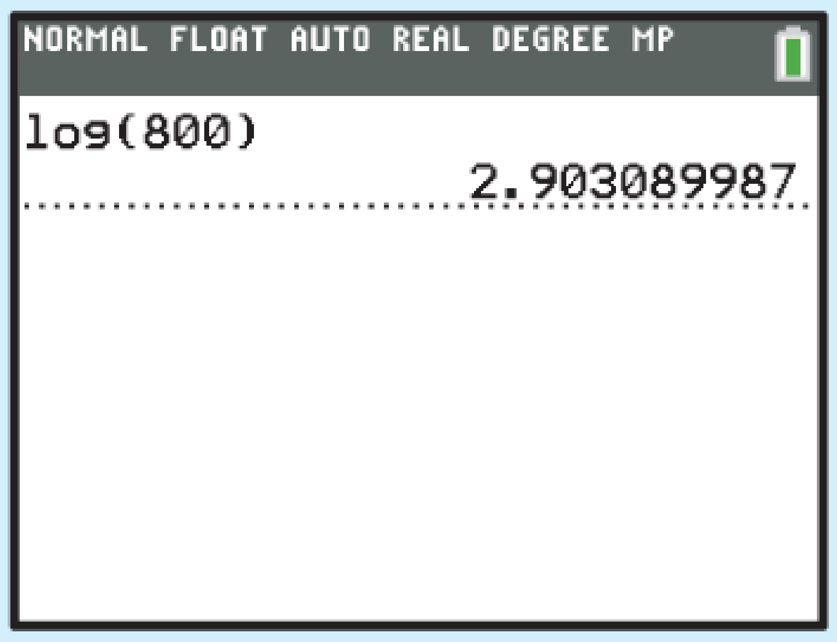
b $800 = 10^{\log 800}$
$\approx 10^{2.9031}$

c $0.08 = 10^{\log 0.08}$
$\approx 10^{-1.0969}$

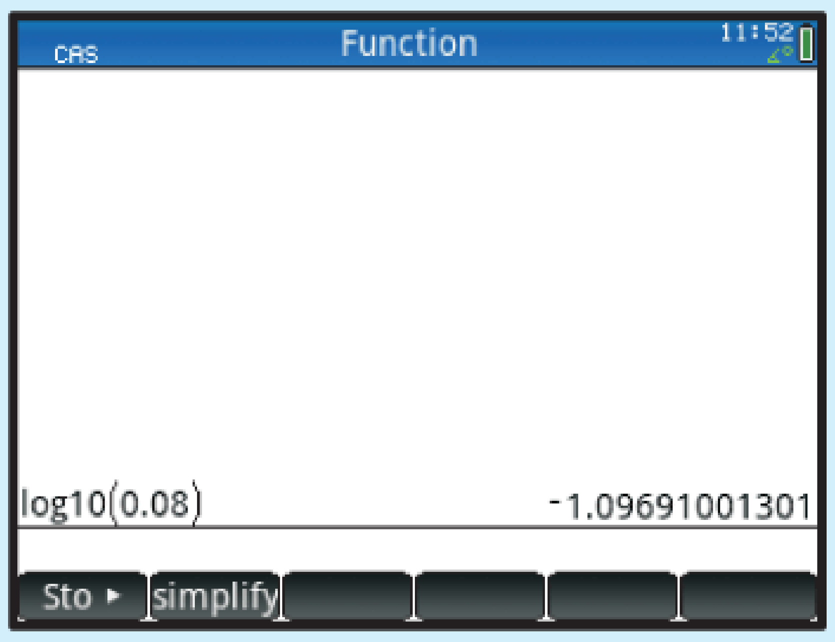
Casio fx-CG50



TI-84 Plus CE



HP Prime



EXERCISE 3A

1 Without using a calculator, find:

a $\log 10\,000$ **b** $\log 0.001$ **c** $\log 10$ **d** $\log 1$

e $\log \sqrt{10}$ **f** $\log \sqrt[3]{10}$ **g** $\log\left(\frac{1}{\sqrt[4]{10}}\right)$ **h** $\log\left(10\sqrt{10}\right)$

i $\log \sqrt[3]{100}$ **j** $\log\left(\frac{100}{\sqrt{10}}\right)$ **k** $\log\left(10 \times \sqrt[3]{10}\right)$ **l** $\log\left(1000\sqrt{10}\right)$

Check your answers using your calculator.

2 Simplify:

a $\log(10^n)$ **b** $\log(10^a \times 100)$ **c** $\log\left(\frac{10}{10^m}\right)$ **d** $\log\left(\frac{10^a}{10^b}\right)$

3 **a** Explain why $\log 237$ must lie between 2 and 3.

b Use your calculator to evaluate $\log 237$ correct to 2 decimal places.

4 **a** Between which two consecutive whole numbers does $\log(0.6)$ lie?
b Check your answer by evaluating $\log(0.6)$ correct to 2 decimal places.

5 Use your calculator to evaluate, correct to 2 decimal places:

a $\log 76$ **b** $\log 114$ **c** $\log 3$ **d** $\log 831$
e $\log(0.4)$ **f** $\log 3247$ **g** $\log(0.008)$ **h** $\log(-7)$

6 For what values of x is $\log x$:

a positive **b** zero **c** negative **d** undefined?

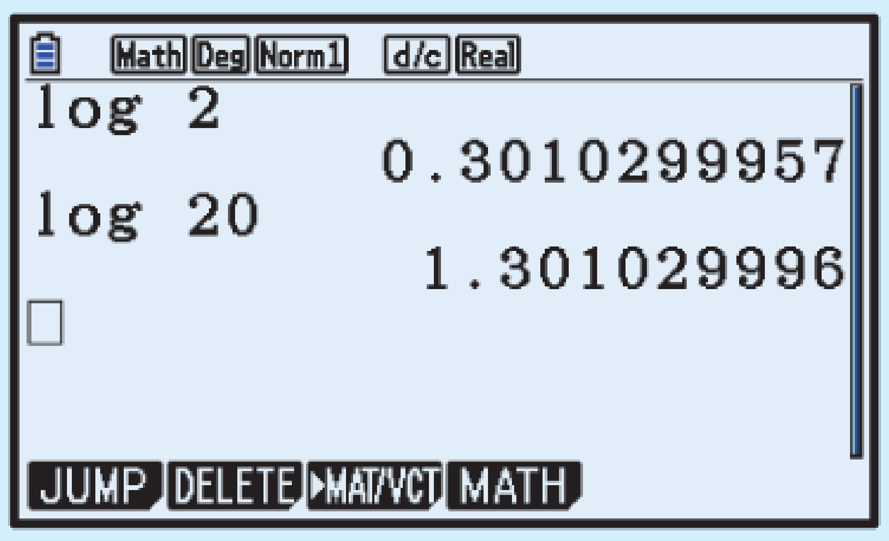
7 Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006
f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

Example 3 **Self Tutor**

a Use your calculator to find: **i** $\log 2$ **ii** $\log 20$
b Explain why $\log 20 = \log 2 + 1$.

a



i $\log 2 \approx 0.3010$
ii $\log 20 \approx 1.3010$

b
$$\begin{aligned}\log 20 &= \log(2 \times 10)\\ &= \log(10^{\log 2} \times 10^1) \quad \{x = 10^{\log x}\}\\ &= \log(10^{\log 2+1}) \quad \{\text{adding indices}\}\\ &= \log 2 + 1\end{aligned}$$

8 **a** Use your calculator to find: **i** $\log 3$ **ii** $\log 300$
b Explain why $\log 300 = \log 3 + 2$.

9 **a** Use your calculator to find: **i** $\log 5$ **ii** $\log(0.05)$
b Explain why $\log(0.05) = \log 5 - 2$.

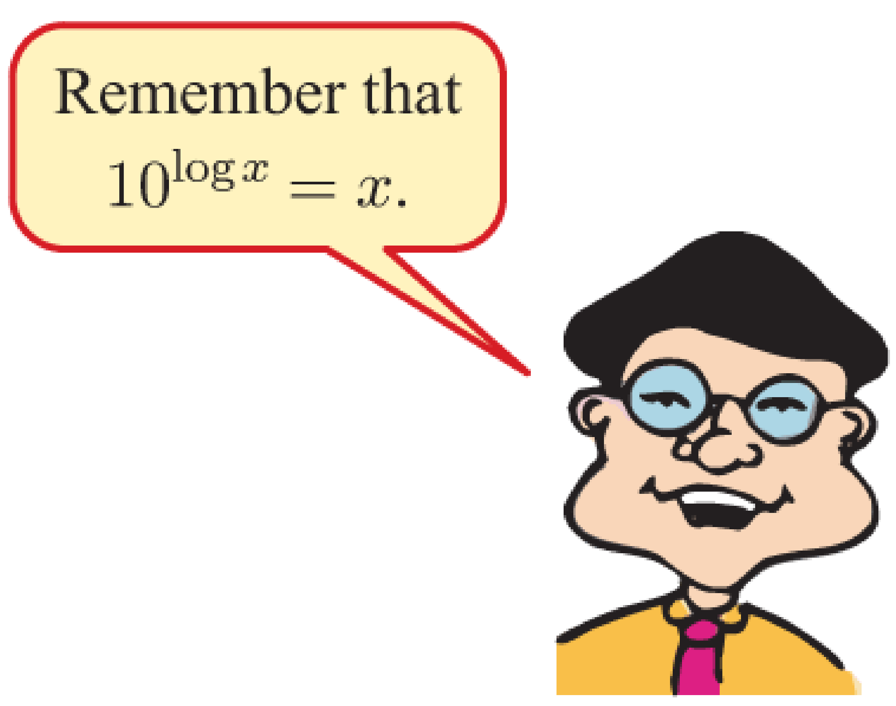
Example 4 **Self Tutor**

Find x such that:

a $\log x = 3$ **b** $\log x \approx -0.271$

a
$$\begin{aligned}\log x &= 3\\ \therefore\ 10^{\log x} &= 10^3\\ \therefore\ x &= 1000\end{aligned}$$

b
$$\begin{aligned}\log x &\approx -0.271\\ \therefore\ 10^{\log x} &\approx 10^{-0.271}\\ \therefore\ x &\approx 0.536\end{aligned}$$



10 Find x such that:

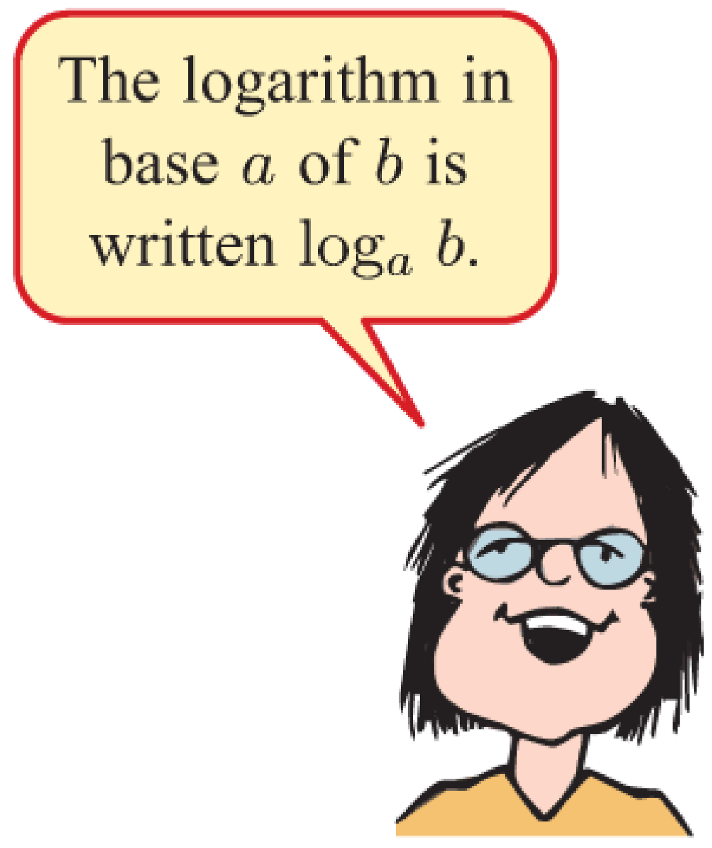
a $\log x = 2$ **b** $\log x = 1$ **c** $\log x = 0$ **d** $\log x = -1$
e $\log x = \frac{1}{2}$ **f** $\log x = -\frac{1}{2}$ **g** $\log x = 4$ **h** $\log x = -5$
i $\log x \approx 0.8351$ **j** $\log x \approx 2.1457$ **k** $\log x \approx -1.378$ **l** $\log x \approx -3.1997$

B LOGARITHMS IN BASE a

In the previous Section we defined the logarithm in base 10 of a number as the power that 10 must be raised to in order to obtain that number.

We can use the same principle to define logarithms in other bases:

> The **logarithm in base a of b** is the power that a must be raised to in order to obtain b.



For example, to find $\log_2 8$, we ask "What power must 2 be raised to in order to obtain 8?". We know that $2^3 = 8$, so $\log_2 8 = 3$.

$a^x = b$ and $x = \log_a b$ are *equivalent* statements.

> For any $b > 0$, $a^x = b \Leftrightarrow x = \log_a b$

Example 5 — Self Tutor

a Write an equivalent exponential statement for $\log_{10} 1000 = 3$.

b Write an equivalent logarithmic statement for $3^4 = 81$.

a From $\log_{10} 1000 = 3$ we deduce that $10^3 = 1000$.

b From $3^4 = 81$ we deduce that $\log_3 81 = 4$.

EXERCISE 3B

1 Write an equivalent exponential statement for:

a $\log_{10} 100 = 2$ **b** $\log_{10} 10\,000 = 4$ **c** $\log_{10}(0.1) = -1$

d $\log_{10} \sqrt{10} = \frac{1}{2}$ **e** $\log_2 8 = 3$ **f** $\log_3 9 = 2$

g $\log_2\left(\frac{1}{4}\right) = -2$ **h** $\log_3 \sqrt{27} = 1.5$ **i** $\log_5\left(\frac{1}{\sqrt{5}}\right) = -\frac{1}{2}$

2 Write an equivalent logarithmic statement for:

a $4^3 = 64$ **b** $5^2 = 25$ **c** $7^2 = 49$ **d** $2^6 = 64$

e $2^{-3} = \frac{1}{8}$ **f** $10^{-2} = 0.01$ **g** $2^{-1} = \frac{1}{2}$ **h** $3^{-3} = \frac{1}{27}$

Example 6 — Self Tutor

Find: **a** $\log_2 16$ **b** $\log_5(0.2)$ **c** $\log_{10} \sqrt[5]{100}$ **d** $\log_2\left(\frac{1}{\sqrt{2}}\right)$

a
$$\begin{aligned}&\log_2 16\\&= \log_2(2^4)\\&= 4\end{aligned}$$

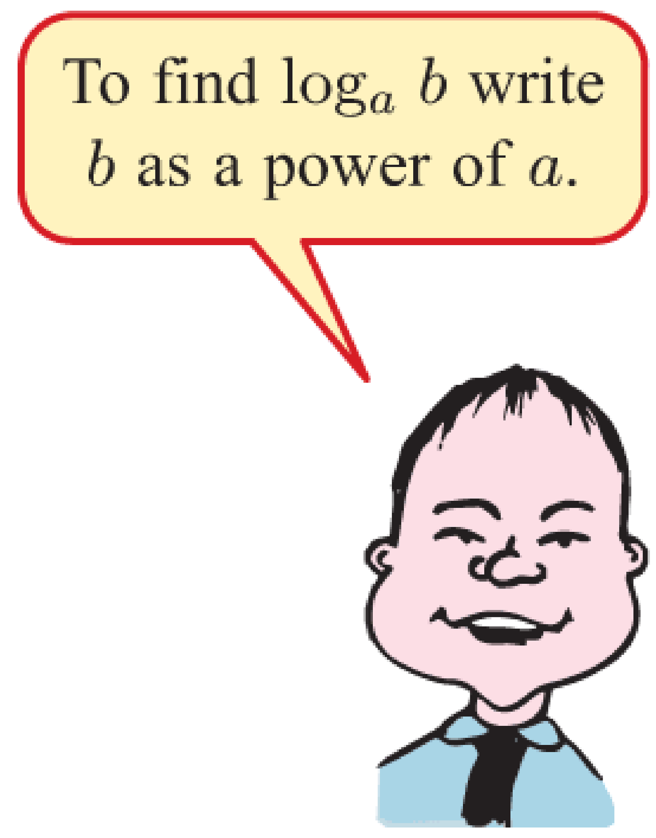
b
$$\begin{aligned}&\log_5(0.2)\\&= \log_5\left(\tfrac{1}{5}\right)\\&= \log_5(5^{-1})\\&= -1\end{aligned}$$

c
$$\begin{aligned}&\log_{10} \sqrt[5]{100}\\&= \log_{10}((10^2)^{\frac{1}{5}})\\&= \log_{10}(10^{\frac{2}{5}})\\&= \tfrac{2}{5}\end{aligned}$$

d
$$\begin{aligned}&\log_2\left(\tfrac{1}{\sqrt{2}}\right)\\&= \log_2(2^{-\frac{1}{2}})\\&= -\tfrac{1}{2}\end{aligned}$$

3 Without using a calculator, find:

a $\log_{10} 100\,000$	**b** $\log_{10}(0.01)$	**c** $\log_3 \sqrt{3}$
d $\log_2 4$	**e** $\log_2 64$	**f** $\log_2 128$
g $\log_5 25$	**h** $\log_5 125$	**i** $\log_2(0.125)$
j $\log_9 3$	**k** $\log_4 16$	**l** $\log_{36} 6$
m $\log_3 243$	**n** $\log_2 \sqrt[3]{2}$	**o** $\log_8 2$
p $\log_6\left(6\sqrt{6}\right)$	**q** $\log_4 1$	**r** $\log_9 9$
s $\log_3\left(\frac{1}{3}\right)$	**t** $\log_{10} \sqrt[4]{1000}$	**u** $\log_7\left(\frac{1}{\sqrt{7}}\right)$
v $\log_5\left(25\sqrt{5}\right)$	**w** $\log_3\left(\frac{1}{\sqrt{27}}\right)$	**x** $\log_4\left(\frac{1}{2\sqrt{2}}\right)$



Check your answers using technology.

GRAPHICS CALCULATOR INSTRUCTIONS

4 Simplify:

a $\log_x(x^2)$	**b** $\log_t\left(\frac{1}{t}\right)$	**c** $\log_x \sqrt{x}$
d $\log_m(m^3)$	**e** $\log_k \sqrt[4]{k}$	**f** $\log_x(x\sqrt{x})$
g $\log_a\left(\frac{1}{a^2}\right)$	**h** $\log_a\left(\frac{1}{\sqrt{a}}\right)$	**i** $\log_m \sqrt{m^5}$

Example 7 — **Self Tutor**

Solve for x: $\log_3 x = 5$

$$\begin{aligned} \log_3 x &= 5 \\ \therefore \quad x &= 3^5 \\ \therefore \quad x &= 243 \end{aligned}$$

5 Solve for x:

a $\log_2 x = 3$	**b** $\log_4 x = \frac{1}{2}$	**c** $\log_5 x = -3$
d $\log_x 81 = 4$	**e** $\log_2(x-6) = 3$	**f** $\log_2(\log_3 x) = -1$

6 Suppose $\log_a b = x$. Find, in terms of x, the value of $\log_b a$.

7 If $y = \log_2 \sqrt{5x-1}$, write x in terms of y.

HISTORICAL NOTE

Acharya Virasena was an 8th century Indian mathematician. Among other areas, he worked with the concept of *ardhaccheda*, which is how many times a number of the form 2^n can be divided by 2. The result is the integer n, and is the logarithm of the number 2^n in base 2.

In 1544, the German **Michael Stifel** published *Arithmetica Integra* which contains a table expressing many other integers as powers of 2. In effect, he had created an early version of a logarithmic table.

C LAWS OF LOGARITHMS

INVESTIGATION 1 DISCOVERING THE LAWS OF LOGARITHMS

What to do:

1 **a** Use your calculator to find:

i $\log 2 + \log 3$ **ii** $\log 3 + \log 7$ **iii** $\log 4 + \log 20$

iv $\log 6$ **v** $\log 21$ **vi** $\log 80$

b From your answers, suggest a possible simplification for $\log m + \log n$.

2 **a** Use your calculator to find:

i $\log 6 - \log 2$ **ii** $\log 12 - \log 3$ **iii** $\log 3 - \log 5$

iv $\log 3$ **v** $\log 4$ **vi** $\log(0.6)$

b From your answers, suggest a possible simplification for $\log m - \log n$.

3 **a** Use your calculator to find:

i $3\log 2$ **ii** $2\log 5$ **iii** $-4\log 3$

iv $\log(2^3)$ **v** $\log(5^2)$ **vi** $\log(3^{-4})$

b From your answers, suggest a possible simplification for $m\log b$.

From the **Investigation**, you should have discovered the three important **laws of logarithms**:

- $\boldsymbol{\log m + \log n = \log(mn)}$ for $m,\ n > 0$
- $\boldsymbol{\log m - \log n = \log\left(\frac{m}{n}\right)}$ for $m,\ n > 0$
- $\boldsymbol{m\log b = \log(b^m)}$ for $b > 0$

More generally, in any base a where $a \neq 1$, $a > 0$, we have these **laws of logarithms**:

- $\boldsymbol{\log_a m + \log_a n = \log_a(mn)}$ for $m,\ n > 0$
- $\boldsymbol{\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)}$ for $m,\ n > 0$
- $\boldsymbol{m\log_a b = \log_a(b^m)}$ for $b > 0$

Proof:

- $$\begin{aligned}&\log_a(mn)\\ &= \log_a\left(a^{\log_a m} \times a^{\log_a n}\right)\\ &= \log_a\left(a^{\log_a m + \log_a n}\right)\\ &= \log_a m + \log_a n\end{aligned}$$

- $$\begin{aligned}&\log_a\left(\frac{m}{n}\right)\\ &= \log_a\left(\frac{a^{\log_a m}}{a^{\log_a n}}\right)\\ &= \log_a\left(a^{\log_a m - \log_a n}\right)\\ &= \log_a m - \log_a n\end{aligned}$$

- $$\begin{aligned}&\log_a(b^m)\\ &= \log_a\left(\left(a^{\log_a b}\right)^m\right)\\ &= \log_a\left(a^{m\log_a b}\right)\\ &= m\log_a b\end{aligned}$$

Example 8 Self Tutor

Use the laws of logarithms to write as a single logarithm or as an integer:

a $\log 5 + \log 3$ **b** $\log_3 24 - \log_3 8$ **c** $\log_2 5 - 1$

a $\log 5 + \log 3$
$= \log(5 \times 3)$
$= \log 15$

b $\log_3 24 - \log_3 8$
$= \log_3\left(\frac{24}{8}\right)$
$= \log_3 3$
$= 1$

c $\log_2 5 - 1$
$= \log_2 5 - \log_2(2^1)$
$= \log_2\left(\frac{5}{2}\right)$

EXERCISE 3C

1 Write as a single logarithm or as an integer:

a $\log 8 + \log 2$ **b** $\log 4 + \log 5$ **c** $\log 40 - \log 5$
d $\log p - \log m$ **e** $\log_4 8 - \log_4 2$ **f** $\log 5 + \log(0.4)$
g $\log 250 + \log 4$ **h** $\log_5 100 - \log_5 4$ **i** $\log 2 + \log 3 + \log 4$
j $\log 5 + \log 4 - \log 2$ **k** $\log_3 6 - \log_3 2 - \log_3 3$ **l** $\log\left(\frac{4}{3}\right) + \log 3 + \log 7$

2 Write as a single logarithm:

a $\log 7 + 2$ **b** $\log 4 - 1$ **c** $1 + \log_2 3$
d $\log_3 5 - 2$ **e** $2 + \log 2$ **f** $\log 50 - 4$
g $t + \log w$ **h** $\log_m 40 - 2$ **i** $3 - \log_5 50$

Example 9 Self Tutor

Simplify by writing as a single logarithm or as a rational number:

a $2\log 7 - 3\log 2$ **b** $2\log 3 + 3$ **c** $\dfrac{\log 8}{\log 4}$

a $2\log 7 - 3\log 2$
$= \log(7^2) - \log(2^3)$
$= \log 49 - \log 8$
$= \log\left(\frac{49}{8}\right)$

b $2\log 3 + 3$
$= \log(3^2) + \log(10^3)$
$= \log 9 + \log 1000$
$= \log 9000$

c $\dfrac{\log 8}{\log 4} = \dfrac{\log(2^3)}{\log(2^2)}$
$= \dfrac{3\log 2}{2\log 2}$
$= \frac{3}{2}$

3 Write as a single logarithm or integer:

a $5\log 2 + \log 3$ **b** $2\log 3 + 3\log 2$ **c** $3\log 4 - \log 8$
d $2\log_3 5 - 3\log_3 2$ **e** $\frac{1}{2}\log_6 4 + \log_6 3$ **f** $\frac{1}{3}\log\left(\frac{1}{8}\right)$
g $3 - \log 2 - 2\log 5$ **h** $1 - 3\log 2 + \log 20$ **i** $2 - \frac{1}{2}\log_n 4 - \log_n 5$

4 Simplify without using a calculator:

a $\dfrac{\log 4}{\log 2}$ **b** $\dfrac{\log_5 27}{\log_5 9}$ **c** $\dfrac{\log 8}{\log 2}$
d $\dfrac{\log 3}{\log 9}$ **e** $\dfrac{\log_3 25}{\log_3(0.2)}$ **f** $\dfrac{\log_4 8}{\log_4(0.25)}$

Example 10 — Self Tutor

Show that:

a $\log\left(\frac{1}{9}\right) = -2\log 3$ **b** $\log 500 = 3 - \log 2$

a $\log\left(\frac{1}{9}\right)$
$= \log(3^{-2})$
$= -2\log 3$

b $\log 500$
$= \log\left(\frac{1000}{2}\right)$
$= \log 1000 - \log 2$
$= \log(10^3) - \log 2$
$= 3 - \log 2$

5 Show that:

a $\log 9 = 2\log 3$ **b** $\log\sqrt{2} = \frac{1}{2}\log 2$ **c** $\log\left(\frac{1}{8}\right) = -3\log 2$

d $\log\left(\frac{1}{5}\right) = -\log 5$ **e** $\log 5 = 1 - \log 2$ **f** $\log 5000 = 4 - \log 2$

6 The number $a \times 10^k$ where $1 \leqslant a < 10$, $k \in \mathbb{Z}$ is written in standard form. Show that $\log(a \times 10^k) = \log a + k$.

7 Suppose $p = \log_b 2$, $q = \log_b 3$, and $r = \log_b 5$. Write in terms of p, q, and r:

a $\log_b 6$ **b** $\log_b 45$ **c** $\log_b 108$

d $\log_b\left(\frac{5\sqrt{3}}{2}\right)$ **e** $\log_b\left(\frac{5}{32}\right)$ **f** $\log_b\left(\frac{2}{9}\right)$

8 Suppose $\log_2 P = x$, $\log_2 Q = y$, and $\log_2 R = z$. Write in terms of x, y, and z:

a $\log_2(PR)$ **b** $\log_2(RQ^2)$ **c** $\log_2\left(\frac{PR}{Q}\right)$

d $\log_2\left(P^2\sqrt{Q}\right)$ **e** $\log_2\left(\frac{Q^3}{\sqrt{R}}\right)$ **f** $\log_2\left(\frac{R^2\sqrt{Q}}{P^3}\right)$

9 If $\log_t M = 1.29$ and $\log_t N^2 = 1.72$, find:

a $\log_t N$ **b** $\log_t(MN)$ **c** $\log_t\left(\frac{N^2}{\sqrt{M}}\right)$

10 Suppose $\log_a(x+2) = \log_a x + 2$ and $a > 1$. Find x in terms of a.

11 In **factorial notation** we use $n!$ to denote the product $1 \times 2 \times 3 \times .... \times n$.

a Write $\log(8!) - \log(7!) + \log(6!) - \log(5!) + \log(4!) - \log(3!) + \log(2!) - \log(1!)$ as a single logarithm.

b Write $\log_2(6!)$ in the form $a + \log_2 b$, where $a, b \in \mathbb{Z}$ and b is as small as possible.

12 Write $\log x^4 + \log\left(\frac{x^4}{y}\right) + \log\left(\frac{x^4}{y^2}\right) + ... + \log\left(\frac{x^4}{y^9}\right)$ in the form $\log\left(\frac{x^m}{y^n}\right)$.

13 Evaluate the infinite series $\log\sqrt{3} - \log\sqrt[4]{3} + \log\sqrt[8]{3} - \log\sqrt[16]{3} + ...$

14 Suppose $x^2 + y^2 = 52xy$ where $0 < y < x$. Show that $\log\left(\frac{x-y}{5}\right) = \frac{1}{2}(\log x + \log 2y)$.

D NATURAL LOGARITHMS

The logarithm in base e is called the **natural logarithm**.

We use $\ln x$ to represent $\log_e x$, and call $\ln x$ the natural logarithm of x.

$$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x.$$

Example 11 Self Tutor

Find: **a** $\ln e^3$ **b** $\ln \sqrt{e}$ **c** $e^{2\ln 5}$

a $\ln e^3 = 3$

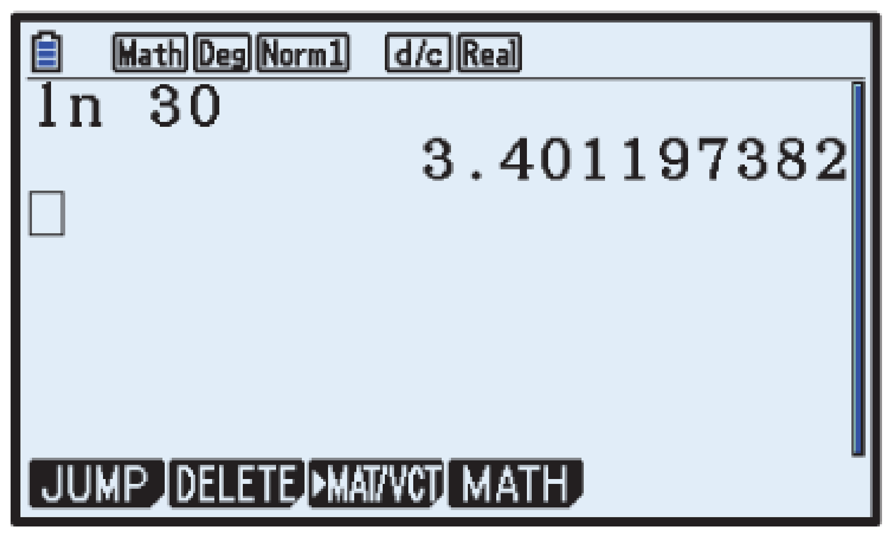
b $\ln \sqrt{e} = \ln(e^{\frac{1}{2}})$
$= \frac{1}{2}$

c $e^{2\ln 5} = (e^{\ln 5})^2$
$= 5^2$
$= 25$

As with base 10 logarithms, we can use our calculator to find natural logarithms.

For example, $\ln 30 \approx 3.40$, which means that $30 \approx e^{3.40}$.

GRAPHICS CALCULATOR INSTRUCTIONS



Example 12 Self Tutor

Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 50 **b** 0.005

a $50 = e^{\ln 50}$ $\{x = e^{\ln x}\}$
$\approx e^{3.9120}$

b $0.005 = e^{\ln 0.005}$
$\approx e^{-5.2983}$

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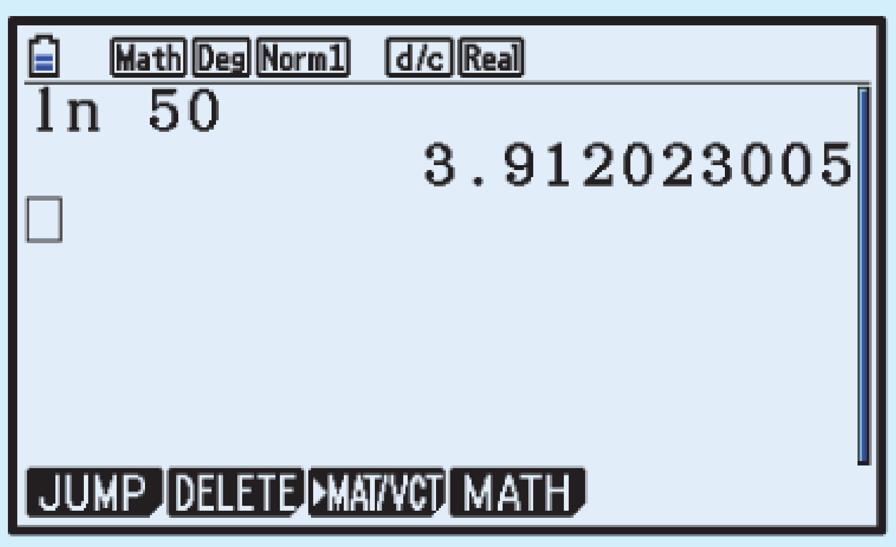

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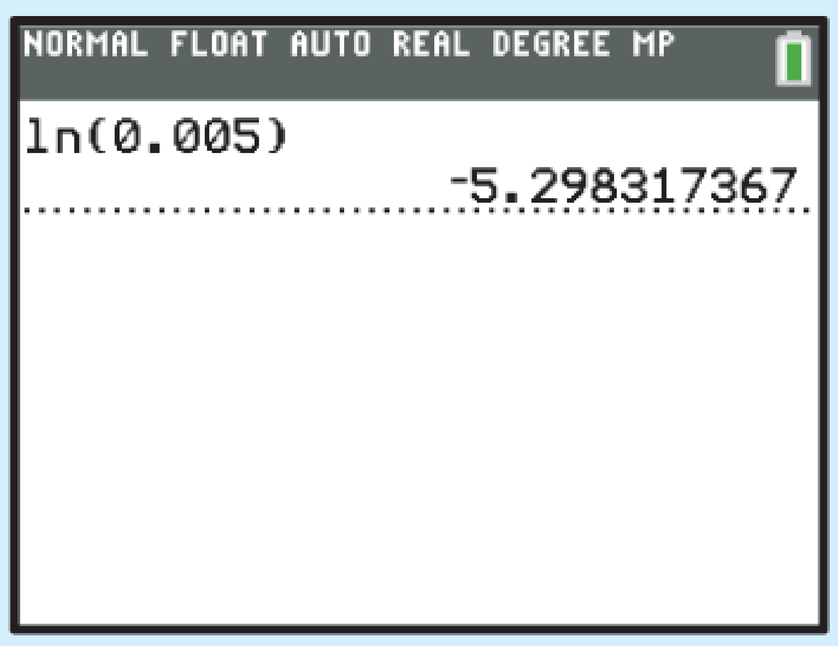


LAWS OF NATURAL LOGARITHMS

The laws for natural logarithms are the laws for logarithms written in base e:

- $\ln m + \ln n = \ln(mn)$ for $m, n > 0$
- $\ln m - \ln n = \ln\left(\frac{m}{n}\right)$ for $m, n > 0$
- $m \ln b = \ln(b^m)$ for $b > 0$

EXERCISE 3D

1 Without using a calculator find:

a $\ln(e^2)$ b $\ln(e^4)$ c $\ln\left((\sqrt{e})^3\right)$ d $\ln 1$

e $\ln\left(\frac{1}{e}\right)$ f $\ln \sqrt[3]{e}$ g $\ln\left(\frac{1}{e^2}\right)$ h $\ln\left(\frac{1}{\sqrt{e}}\right)$

Check your answers using a calculator.

2 Simplify:

a $e^{\ln 3}$ b $e^{2\ln 3}$ c $e^{-\ln 5}$ d $e^{-2\ln 2}$

e $\ln e^a$ f $\ln(e \times e^a)$ g $\ln(e^a \times e^b)$ h $\ln\left((e^a)^b\right)$

3 Use your calculator to find, correct to 3 decimal places:

a $\ln 12$ b $\ln 68$ c $\ln(1.4)$ d $\ln(0.7)$ e $\ln 500$

4 Explain why $\ln(-2)$ and $\ln 0$ cannot be found.

5 Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 6 b 60 c 6000 d 0.6 e 0.006

f 15 g 1500 h 1.5 i 0.15 j 0.000 15

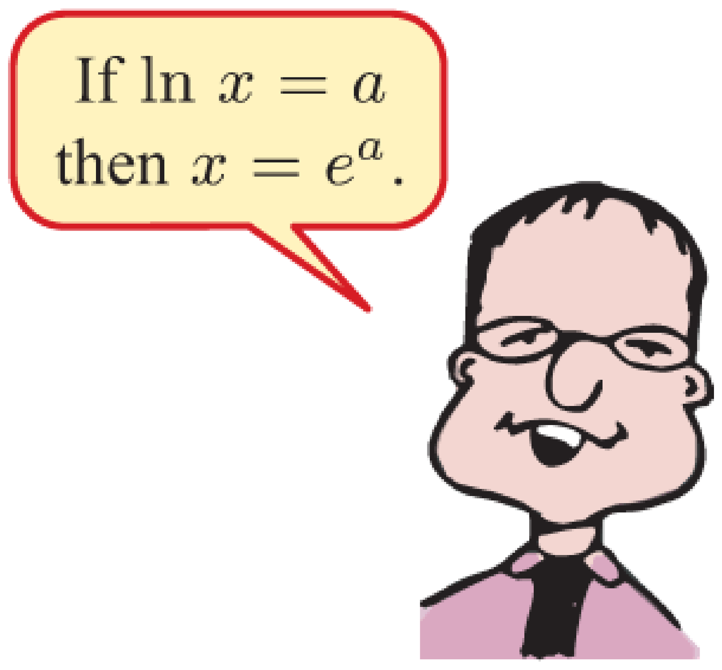
Example 13 **Self Tutor**

Find x if:

a $\ln x = 2.17$ b $\ln x = -0.384$

a $\ln x = 2.17$
$\therefore\ x = e^{2.17}$
$\therefore\ x \approx 8.76$

b $\ln x = -0.384$
$\therefore\ x = e^{-0.384}$
$\therefore\ x \approx 0.681$



6 Find x if:

a $\ln x = 3$ b $\ln x = 1$ c $\ln x = 0$ d $\ln x = -1$

e $\ln x = -5$ f $\ln x \approx 0.835$ g $\ln x \approx 2.145$ h $\ln x \approx -3.2971$

7 a Write in simplest form:

i $\ln(e^x)$ ii $e^{\ln x}$

b What does this tell us about the functions $y = e^x$ and $y = \ln x$?

Example 14 **Self Tutor**

Use the laws of logarithms to write as a single logarithm:

a $\ln 5 + \ln 3$ b $\ln 24 - \ln 8$ c $\ln 5 - 1$

a $\ln 5 + \ln 3$
$= \ln(5 \times 3)$
$= \ln 15$

b $\ln 24 - \ln 8$
$= \ln\left(\frac{24}{8}\right)$
$= \ln 3$

c $\ln 5 - 1$
$= \ln 5 - \ln(e^1)$
$= \ln\left(\frac{5}{e}\right)$

8 Write as a single logarithm or integer:

a $\ln 15+\ln 3$ **b** $\ln 15-\ln 3$ **c** $\ln 20-\ln 5$
d $\ln 4+\ln 6$ **e** $\ln 5+\ln(0.2)$ **f** $\ln 2+\ln 3+\ln 5$
g $1+\ln 4$ **h** $\ln 6-1$ **i** $\ln 5+\ln 8-\ln 2$
j $2+\ln 4$ **k** $\ln 20-2$ **l** $\ln 12-\ln 4-\ln 3$

Example 15 **Self Tutor**

Use the laws of logarithms to simplify:

a $2\ln 7-3\ln 2$ **b** $2\ln 3+3$

a
$$\begin{aligned}&2\ln 7-3\ln 2\\ &=\ln(7^2)-\ln(2^3)\\ &=\ln 49-\ln 8\\ &=\ln\left(\tfrac{49}{8}\right)\end{aligned}$$

b
$$\begin{aligned}&2\ln 3+3\\ &=\ln(3^2)+\ln(e^3)\\ &=\ln 9+\ln(e^3)\\ &=\ln(9e^3)\end{aligned}$$

9 Write in the form $\ln a,\ \ a\in\mathbb{R}$:

a $5\ln 3+\ln 4$ **b** $3\ln 2+2\ln 5$ **c** $3\ln 2-\ln 8$
d $3\ln 4-2\ln 2$ **e** $\frac{1}{3}\ln 8+\ln 3$ **f** $\frac{1}{3}\ln\left(\frac{1}{27}\right)$
g $-\ln 2$ **h** $-\ln\left(\frac{1}{2}\right)$ **i** $-2\ln\left(\frac{1}{4}\right)$
j $4\ln 2+2$ **k** $\frac{1}{2}\ln 9-1$ **l** $-3\ln 2+\frac{1}{2}$

10 Show that:

a $\ln 27=3\ln 3$ **b** $\ln\sqrt{3}=\frac{1}{2}\ln 3$ **c** $\ln\left(\frac{1}{16}\right)=-4\ln 2$
d $\ln\left(\frac{1}{6}\right)=-\ln 6$ **e** $\ln\left(\frac{1}{\sqrt{2}}\right)=-\frac{1}{2}\ln 2$ **f** $\ln\left(\frac{e}{5}\right)=1-\ln 5$
g $\ln(6e)=\ln 6+1$ **h** $\ln\sqrt[3]{5}=\frac{1}{3}\ln 5$ **i** $\ln\left(\frac{1}{\sqrt[5]{2}}\right)=-\frac{1}{5}\ln 2$
j $\ln\left(\frac{e^2}{8}\right)=2-3\ln 2$ **k** $\ln\left(\frac{\sqrt{3}}{e^4}\right)=\frac{1}{2}\ln 3-4$ **l** $\ln\left(\frac{1}{16\times\sqrt[3]{e}}\right)=-4\ln 2-\frac{1}{3}$

11 Find x and y given that $\ln\left(\frac{x}{y}\right)=6$ and $\ln(x^3y^4)=4$.

THEORY OF KNOWLEDGE

It is easy to take modern technology, such as the electronic calculator, for granted. Until electronic computers became affordable in the 1980s, a "calculator" was a *profession*, literally someone who would spend their time performing calculations by hand. They used mechanical calculators and techniques such as logarithms. They often worked in banks, but sometimes for astronomers and other scientists.

The logarithm was invented by **John Napier** (1550 - 1617) and first published in 1614 in a Latin book which translates as a *Description of the Wonderful Canon of Logarithms*. John Napier was the 8th Lord of Merchiston, which is now part of Edinburgh, Scotland. Napier wrote a number of other books on many subjects including religion and mathematics. One of his other inventions was a device for performing long multiplication which is now called "Napier's Bones". Other calculators, such as slide rules, used logarithms as part of their design. He also popularised the use of the decimal point in mathematical notation.

Logarithms were an extremely important development, and they had an immediate effect on the seventeenth century scientific community. **Johannes Kepler** used logarithms to assist with his calculations. This helped him develop his laws of planetary motion. Without logarithms these calculations would have taken many years. Kepler published a letter congratulating and acknowledging Napier. Kepler's laws gave **Sir Isaac Newton** important evidence to support his theory of universal gravitation. 200 years later, **Laplace** said that logarithms "by shortening the labours, doubled the life of the astronomer".

Johannes Kepler

1 Can anyone claim to have *invented* logarithms?

2 Can we consider the process of mathematical discovery as an *evolution* of ideas?

3 Has modern computing effectively doubled the life of a mathematician?

Many areas of mathematics have been developed over centuries as several mathematicians have worked in a particular area, or taken the knowledge from one area and applied it to another field. Sometimes the process is held up because a method for solving a particular class of problem has not yet been found. In other cases, pure mathematicians have published research papers on seemingly useless mathematical ideas, which have then become vital in applications much later.

In *Everybody Counts*: *A report to the nation on the future of Mathematical Education* by the National Academy of Sciences (National Academy Press, 1989), there is an excellent section on the Nature of Mathematics. It includes:

> "Even the most esoteric and abstract parts of mathematics - number theory and logic, for example - are now used routinely in applications (for example, in computer science and cryptography). Fifty years ago, the leading British mathematician G.H. Hardy could boast that number theory was the most pure and least useful part of mathematics. Today, Hardy's mathematics is studied as an essential prerequisite to many applications, including control of automated systems, data transmission from remote satellites, protection of financial records, and efficient algorithms for computation."

4 Should we only study the mathematics required to enter our chosen profession?

5 Why should we explore mathematics for its own sake, rather than to address the needs of science?

E LOGARITHMIC EQUATIONS

We can use the laws of logarithms to write equations in a different form. This can be particularly useful if an unknown appears as an exponent.

Since the logarithmic function is one-to-one, we can take the logarithm of both sides of an equation without changing the solution. However, we can only do this if both sides are positive.

Example 16 Self Tutor

Write as a logarithmic equation (in base 10):

a $y = a^2b$ **b** $P = \frac{20}{\sqrt{n}}$

a
$$y = a^2b$$
$$\therefore \log y = \log(a^2b)$$
$$\therefore \log y = \log(a^2) + \log b$$
$$\therefore \log y = 2\log a + \log b$$

b
$$P = \left(\frac{20}{\sqrt{n}}\right)$$
$$\therefore \log P = \log\left(\frac{20}{n^{\frac{1}{2}}}\right)$$
$$\therefore \log P = \log 20 - \log(n^{\frac{1}{2}})$$
$$\therefore \log P = \log 20 - \tfrac{1}{2}\log n$$

Example 17 Self Tutor

Write without logarithms:

a $\log A = \log b + 2\log c$ **b** $\log_2 M = 3\log_2 a - 2$

a
$$\log A = \log b + 2\log c$$
$$\therefore \log A = \log b + \log(c^2)$$
$$\therefore \log A = \log(bc^2)$$
$$\therefore A = bc^2$$

b
$$\log_2 M = 3\log_2 a - 2$$
$$\therefore \log_2 M = \log_2(a^3) - \log_2(2^2)$$
$$\therefore \log_2 M = \log_2\left(\frac{a^3}{4}\right)$$
$$\therefore M = \frac{a^3}{4}$$

EXERCISE 3E

1 Write as a logarithmic equation (in base 10), with no powers, assuming all variables are positive:

a $y = 2^x$ **b** $y = 20b^3$ **c** $M = ad^4$ **d** $T = 5\sqrt{d}$

e $R = b\sqrt{l}$ **f** $Q = \frac{a}{b^n}$ **g** $y = ab^x$ **h** $F = \frac{20}{\sqrt{n}}$

i $L = \frac{ab}{c}$ **j** $N = \sqrt{\frac{a}{b}}$ **k** $S = 200 \times 2^t$ **l** $y = \frac{a^m}{b^n}$

2 Write without logarithms:

a $\log D = \log e + \log 2$ **b** $\log_a F = \log_a 5 - \log_a t$ **c** $\log P = \frac{1}{2}\log x$

d $\log_n M = 2\log_n b + \log_n c$ **e** $\log B = 3\log m - 2\log n$ **f** $\log N = -\frac{1}{3}\log p$

g $\log P = 3\log x + 1$ **h** $\log_a Q = 2 - \log_a x$

3 Write without logarithms:

a $\ln D = \ln x + 1$
b $\ln F = -\ln p + 2$
c $\ln P = \frac{1}{2}\ln x$
d $\ln M = 2\ln y + 3$
e $\ln B = 3\ln t - 1$
f $\ln N = -\frac{1}{3}\ln g$
g $\ln Q \approx 3\ln x + 2.159$
h $\ln D \approx 0.4\ln n - 0.6582$

4 **a** Write $y = 3 \times 2^x$ as a logarithmic equation in base 2.

b Hence write x in terms of y.

c Find the value of x when: **i** $y = 3$ **ii** $y = 12$ **iii** $y = 30$

5 Solve for x:

a $\log_3 27 + \log_3\left(\frac{1}{3}\right) = \log_3 x$
b $\log_5 x = \log_5 8 - \log_5(6 - x)$
c $\log_5 125 - \log_5 \sqrt{5} = \log_5 x$
d $\log_{20} x = 1 + \log_{20} 10$
e $\log x + \log(x + 1) = \log 30$
f $\log(x + 2) - \log(x - 2) = \log 5$
g $\log 24 = \log 3 + x\log 2$
h $x\log_2 3 + \log_2 36 = 2$

6 Solve simultaneously for x and y:

a $\begin{cases} \log_x y = 2 \\ \log_{y-2} x = 1 \end{cases}$

b $\begin{cases} \log_x y = 3 \\ \log_{y+1}(x+1) = \frac{1}{2} \end{cases}$

7 Let $x = \log_2 7$.

a Write the equation without logarithms.

b Take the logarithm in base 10 of both sides of your equation from **a**. Hence show that $\log_2 7 = \dfrac{\log 7}{\log 2}$, and calculate this number.

8 Consider the exponential equation $a^x = b$ where $a, b > 0$.

a Explain why $x = \log_a b$.

b Take the logarithm in base 10 of both sides of $a^x = b$.

c Hence show that $x = \log_a b = \dfrac{\log b}{\log a}$.

F THE CHANGE OF BASE RULE

In the previous Exercise you should have proven the base 10 case of the **change of base rule**:

$$\log_b a = \frac{\log_c a}{\log_c b} \quad \text{for } a, b, c > 0 \text{ and } b, c \neq 1.$$

Proof: If $\log_b a = x$, then $b^x = a$

$\therefore \log_c b^x = \log_c a$ {taking logarithms in base c}

$\therefore x\log_c b = \log_c a$ {power law of logarithms}

$\therefore x = \dfrac{\log_c a}{\log_c b}$

$\therefore \log_b a = \dfrac{\log_c a}{\log_c b}$

We need the change of base rule to evaluate logarithms in bases other than 10 or e.

Example 18 Self Tutor

Find $\log_2 9$ by:

a changing to base 10 **b** changing to base e.

a $\log_2 9 = \dfrac{\log_{10} 9}{\log_{10} 2}$
≈ 3.17

b $\log_2 9 = \dfrac{\ln 9}{\ln 2}$
≈ 3.17

EXERCISE 3F

1 Use the change of base rule with base 10 to calculate:

a $\log_3 7$ **b** $\log_2 40$ **c** $\log_5 180$

d $\log_{\frac{1}{2}} 1250$ **e** $\log_3(0.067)$ **f** $\log_{0.4}(0.006\,984)$

Check your results using the change of base rule with base e.

2 Simplify $\log_m n \times \log_n(m^2)$.

3 Without using technology, show that $2^{\frac{4}{\log_5 4} + \frac{3}{\log_7 8}} = 175$.

Hint: Use the change of base rule with base 2.

4 Solve for x:

a $\log_4(x^3) + \log_2 \sqrt{x} = 8$ **b** $\log_{\frac{1}{9}} x = \log_9 5$

c $\log_{16}(x^5) = \log_{64} 125 - \log_4 \sqrt{x}$ **d** $\log_3(x^3) - 4\log_9 x - 5\log_{27} \sqrt{x} = \log_9 4$

e $\log_x 4 + \log_2 x = 3$

5 Given $x = \log_3(y^2)$, express $\log_y 81$ in terms of x.

6 Given $m = \log_4 3$, express $\log_2 24$ in terms of m.

7 **a** By evaluating each expression when $x = 9$, show that $\log_9(\log_3 x)$ and $\log_3(\log_9 x)$ are not always equal.

b Find the value of x such that $\log_9(\log_3 x) = \log_3(\log_9 x)$.

c Find, in terms of a, the solution to $\log_{a^2}(\log_a x) = \log_a(\log_{a^2} x)$.

d Show that when $\log_{a^k}(\log_a x) = \log_a(\log_{a^k} x)$, $k \neq 1$, both sides of the equation have the value $\log_a k^{\frac{1}{k-1}}$.

G SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS

In **Chapter 2** we found solutions to simple exponential equations where we could make equal bases and then equate exponents. However, it is not always easy to make the bases the same. In these situations we can use **logarithms**.

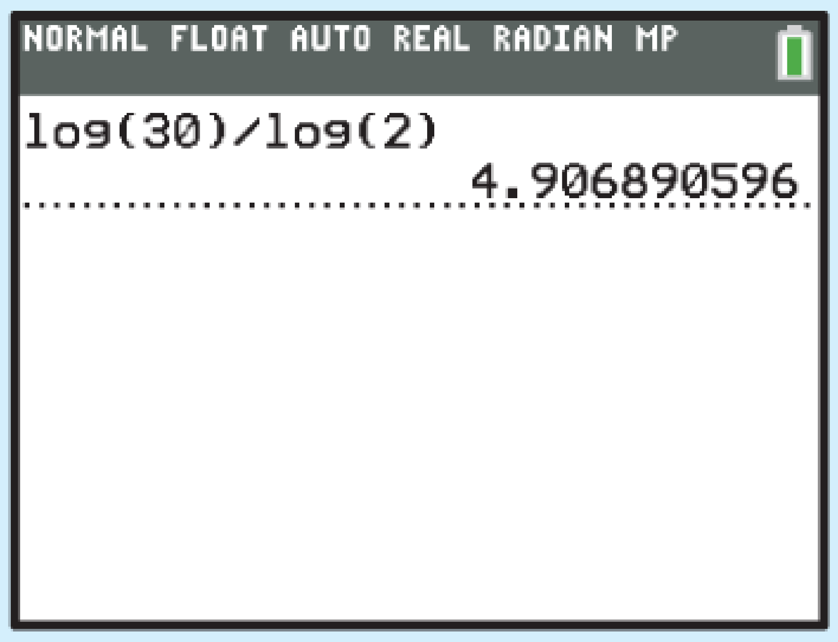
Example 19 — Self Tutor

a Solve the equation $2^x = 30$ exactly.

b Use your calculator to evaluate the solution correct to 2 decimal places.

a
$$2^x = 30$$
$$\therefore \log(2^x) = \log 30 \quad \{\text{taking the logarithm of each side}\}$$
$$\therefore x\log 2 = \log 30 \quad \{\log(b^m) = m\log b\}$$
$$\therefore x = \frac{\log 30}{\log 2}$$

b $\dfrac{\log 30}{\log 2} \approx 4.91$, so the solution is $x \approx 4.91$.



EXERCISE 3G

1 Consider the equation $3^x = 40$.

- **a** Explain why the solution to this equation lies between $x = 3$ and $x = 4$.
- **b** Find the solution exactly.
- **c** Use your calculator to evaluate the solution correct to 2 decimal places.

2 Solve for x: **i** exactly **ii** correct to 2 decimal places.

- **a** $2^x = 10$
- **b** $3^x = 20$
- **c** $4^x = 50$
- **d** $\left(\frac{1}{2}\right)^x = 0.0625$
- **e** $\left(\frac{3}{4}\right)^x = 0.1$
- **f** $10^x = 0.000\,015$

3 Solve for x, correct to 3 significant figures:

- **a** $5^x = 40$
- **b** $3^x = 2^{x+3}$
- **c** $2^{x+4} = 5^{2-x}$

Example 20 — Self Tutor

Find x exactly:

a $e^x = 30$ **b** $3e^{\frac{x}{2}} = 21$

a
$$e^x = 30$$
$$\therefore x = \ln 30$$

b
$$3e^{\frac{x}{2}} = 21$$
$$\therefore e^{\frac{x}{2}} = 7$$
$$\therefore \frac{x}{2} = \ln 7$$
$$\therefore x = 2\ln 7$$

4 Solve for x, giving an exact answer:

- **a** $e^x = 10$
- **b** $e^x = 1000$
- **c** $2e^x = 0.3$
- **d** $e^{\frac{x}{2}} = 5$
- **e** $e^{2x} = 18$
- **f** $e^{-\frac{x}{2}} = 1$

5 Solve for x, giving an exact answer:

a $3 \times 2^x = 75$ **b** $7 \times (1.5)^x = 20$ **c** $5 \times (0.8)^x = 3$

d $4 \times 2^{-x} = 0.12$ **e** $300 \times 5^{0.1x} = 1000$ **f** $32 \times e^{-0.25x} = 4$

6 Solve for x exactly:

a $25^x - 3 \times 5^x = 0$ **b** $8 \times 9^x - 3^x = 0$ **c** $2^x - 2 \times 4^x = 0$

7 Solve $3^{2x} = \frac{1}{16}$, writing your answer in terms of $\ln 2$ and $\ln 3$.

8 Solve $10^{2x} = 4^{x+3}$, writing your answer in terms of $\ln 2$ and $\ln 5$.

Example 21 Self Tutor

Find exactly the points of intersection of $y = e^x - 3$ and $y = 1 - 3e^{-x}$.
Check your solution using technology.

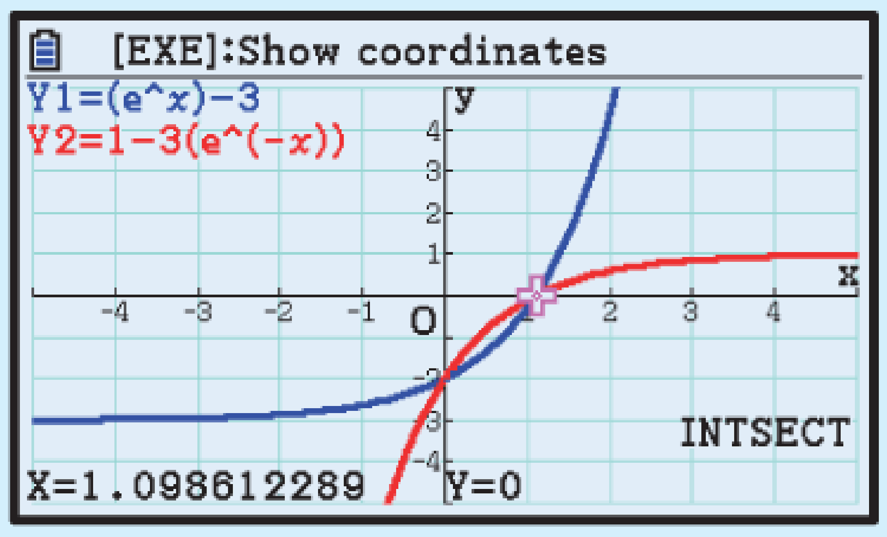
The functions meet where

$$e^x - 3 = 1 - 3e^{-x}$$
$$\therefore \quad e^x - 4 + 3e^{-x} = 0$$
$$\therefore \quad e^{2x} - 4e^x + 3 = 0 \qquad \{\text{multiplying each term by } e^x\}$$
$$\therefore \quad (e^x - 1)(e^x - 3) = 0$$
$$\therefore \quad e^x = 1 \text{ or } 3$$
$$\therefore \quad x = \ln 1 \text{ or } \ln 3$$
$$\therefore \quad x = 0 \text{ or } \ln 3$$

When $x = 0$, $y = e^0 - 3 = -2$
When $x = \ln 3$, $y = e^{\ln 3} - 3 = 0$

$\therefore$ the functions meet at $(0, -2)$ and at $(\ln 3, 0)$.

GRAPHING PACKAGE



9 Solve for x:

a $e^{2x} = 2e^x$ **b** $e^x = e^{-x}$ **c** $e^{2x} - 5e^x + 6 = 0$

d $e^x + 2 = 3e^{-x}$ **e** $1 + 12e^{-x} = e^x$ **f** $e^x + e^{-x} = 3$

10 Find algebraically the point(s) of intersection of:

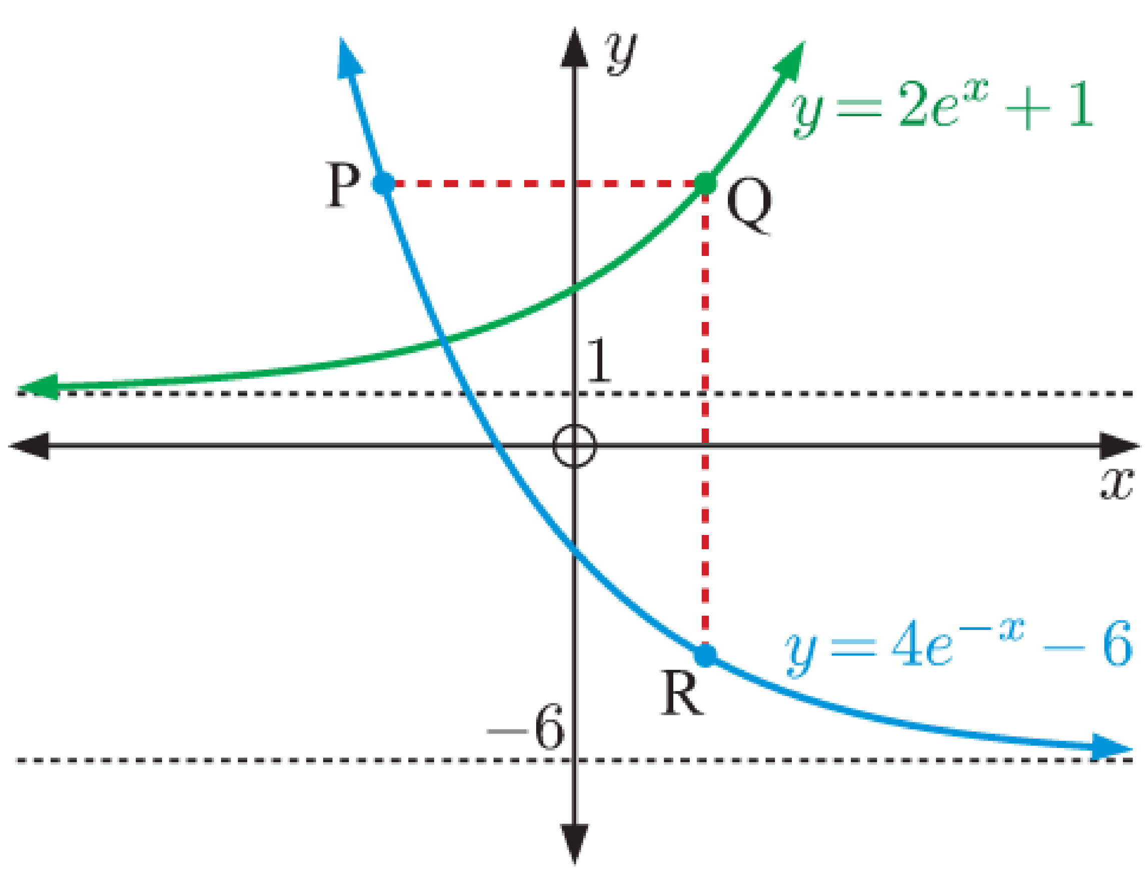
a $y = e^x$ and $y = e^{2x} - 6$ **b** $y = 2e^x + 1$ and $y = 7 - e^x$

c $y = 3 - e^x$ and $y = 5e^{-x} - 3$

Check your answers using technology.

11 **a** Find the exact coordinates of the intersection point of the graphs shown.

b Given that the vertical line segment [QR] has length 9 units, find the length of the horizontal line segment [PQ]. Give your answer in the form $\ln k$ units, where $k \in \mathbb{Q}$.



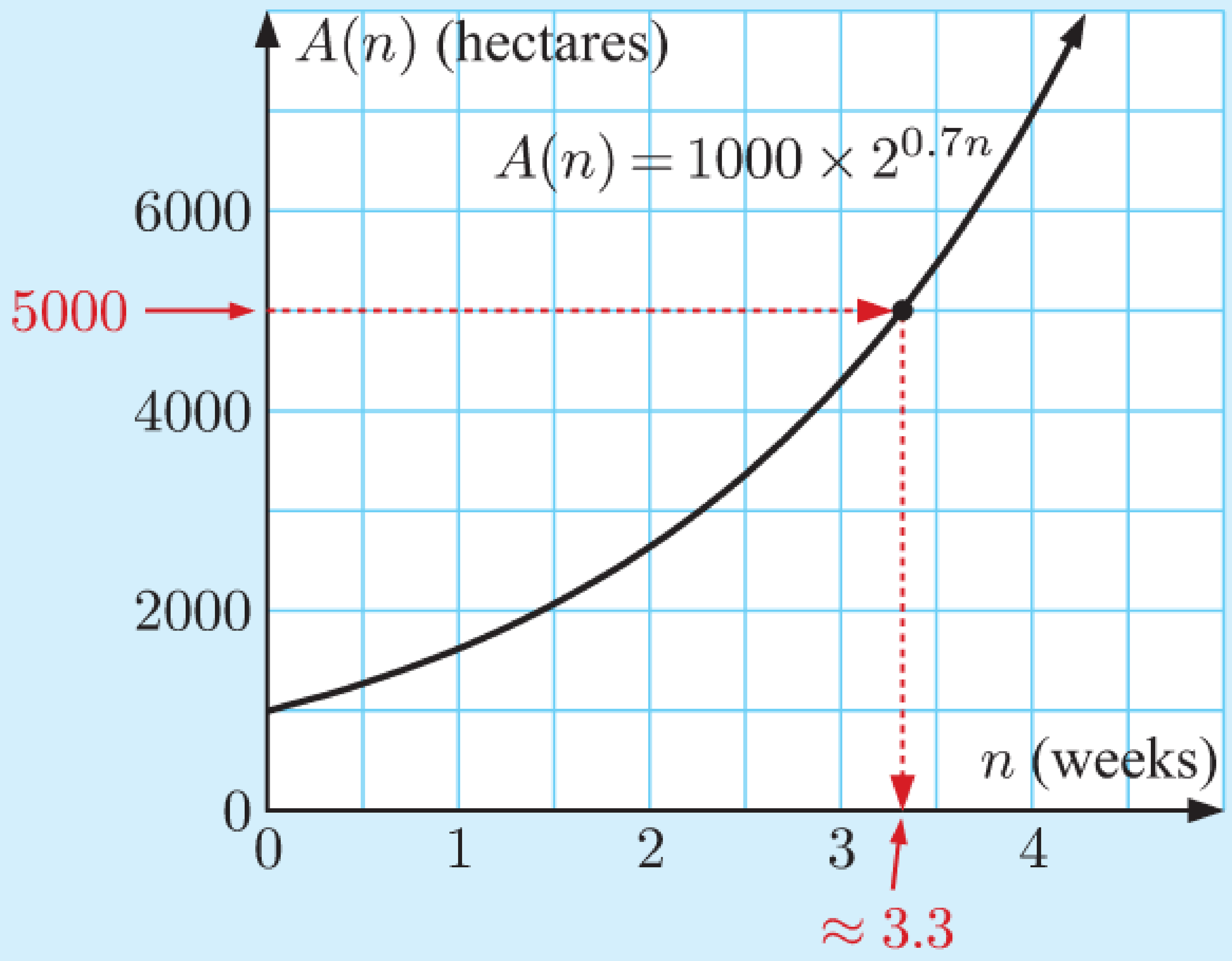
Example 22 **Self Tutor**

A farmer monitoring an insect plague finds that the area affected by the insects is given by $A(n) = 1000 \times 2^{0.7n}$ hectares, where n is the number of weeks after the initial observation.

a Use technology to help sketch the graph of $A(n)$. Hence estimate the time taken for the affected area to reach 5000 hectares.

b Check your answer to **a** using logarithms.

a From the graph, it appears that it will take about 3.3 weeks for the affected area to reach 5000 hectares.



b When $A(n) = 5000$,

$$1000 \times 2^{0.7n} = 5000$$
$$\therefore\ 2^{0.7n} = 5$$
$$\therefore\ \log(2^{0.7n}) = \log 5$$
$$\therefore\ 0.7n \log 2 = \log 5$$
$$\therefore\ n = \frac{\log 5}{0.7 \times \log 2} \approx 3.32$$

$\therefore$ it will take about 3 weeks and 2 days.

12 The population of turtles in an isolated colony is $P(t) = 852 \times (1.07)^t$, where t is the time in years after the colony was first recorded. How long will it take for the population to reach:

a 1000 turtles **b** 1500 turtles?

13 The weight of bacteria in a culture t hours after establishment is given by $W(t) = 20 \times 2^{0.15t}$ grams. Find, using logarithms, the time for the weight of the culture to reach:

a 30 grams **b** 100 grams.

14 A biologist is modelling an infestation of fire ants. He determines that the area affected by the ants is given by $A(n) = 2000 \times e^{0.57n}$ hectares, where n is the number of weeks after the initial observation.

a Use technology to help sketch the graph of $A(n)$.

b Hence estimate the time taken for the infested area to reach 10 000 hectares.

c Check your answer to **b** using logarithms.

15 A house is expected to increase in value at an average rate of 7.5% p.a. If the house is worth £360 000 now, when would you expect it to be worth £550 000?

16 Thabo has $10 000 to invest in an account that pays 4.8% p.a. compounded annually. How long will it take for his investment to grow to $15 000?

17 Dien invests $15 000 at 8.4% p.a. compounded *monthly*. He will withdraw his money when it reaches $25 000, at which time he plans to travel. The formula $t_n = t_0 \times r^n$ can be used to model the investment, where n is the time in months.

a Explain why $r = 1.007$.

b After how many months will Dien withdraw the money?

18 The mass M_t of radioactive substance remaining after t years is given by $M_t = 1000 \times e^{-0.04t}$ grams. Find the time taken for the mass to:

- **a** halve
- **b** reach 25 grams
- **c** reach 1% of its original value.

19 The current I flowing in a transistor radio t seconds after it is switched off, is given by $I = I_0 \times 2^{-0.02t}$ amps. Show that it takes $\frac{50}{\log 2}$ seconds for the current to drop to 10% of its original value.

20 A sky diver jumps from an aeroplane. His speed of descent is given by $V(t) = 50(1 - e^{-0.2t})$ m s^{-1}, where t is the time in seconds.

- **a** Show that it will take $5 \ln 5$ seconds for the sky diver's speed to reach 40 m s^{-1}.
- **b** Write an expression for the time taken for his speed to reach v m s^{-1}.

21 Answer the **Opening Problem** on page **68**.

22 The weight of radioactive substance remaining after t years is given by $W = 1000 \times 2^{-0.04t}$ grams.

- **a** Sketch the graph of W against t.
- **b** Write a function for t in terms of W.
- **c** Hence find the time required for the weight to reach:
  - **i** 20 grams
  - **ii** 0.001 grams.

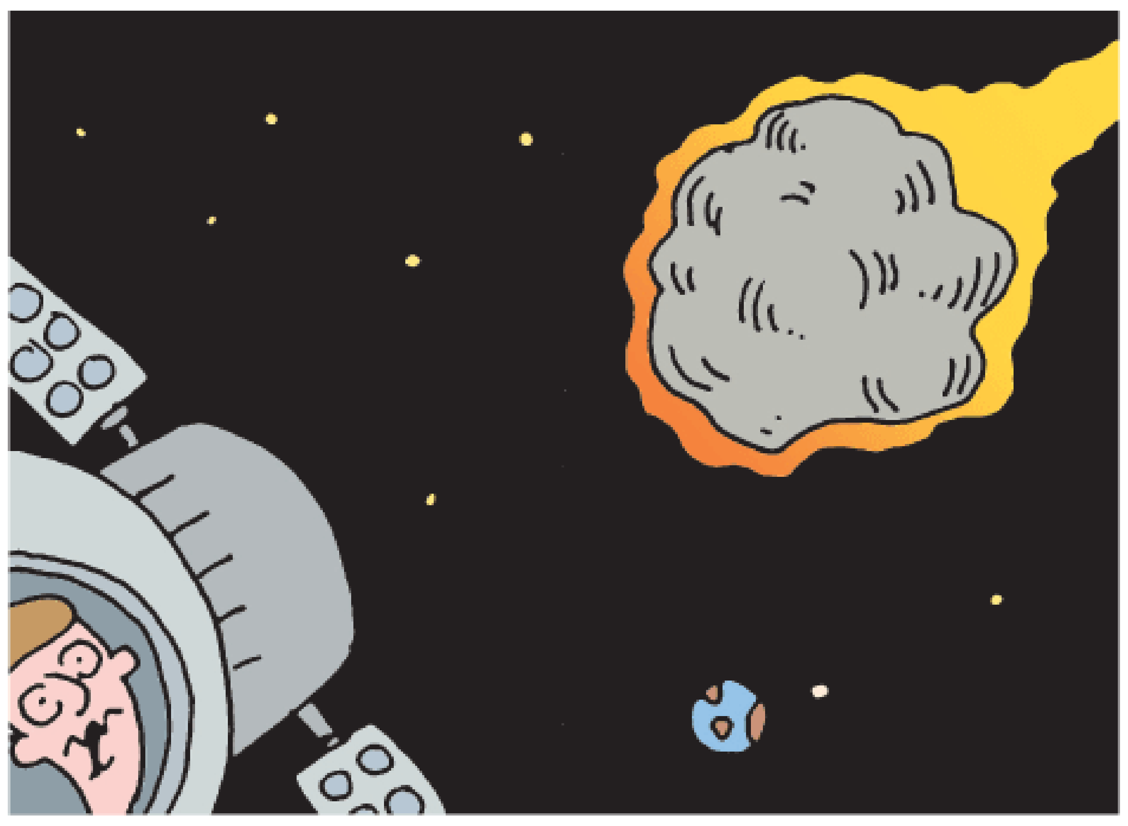
23 The temperature of a liquid t minutes after it is placed in a refrigerator, is given by $T = 4 + 96 \times e^{-0.03t}$ °C.

- **a** Sketch the graph of T against t.
- **b** Write a function for t in terms of T.
- **c** Find the time required for the temperature to reach:
  - **i** 25°C
  - **ii** 5°C.

24 A meteor hurtling through the atmosphere has speed of descent given by

$$V(t) = 650(4 + 2 \times e^{-0.1t}) \quad \text{m s}^{-1}$$

where t is the time in seconds after the meteor is sighted.



- **a** Is the meteor's speed increasing or decreasing?
- **b** Find the speed of the meteor:
  - **i** when it was first sighted
  - **ii** after 2 minutes.
- **c** How long will it take for the meteor's speed to reach 3000 m s^{-1}?

INVESTIGATION 2 THE "RULE OF 72"

The "rule of 72" is used to estimate the time a quantity takes to double in value, given the rate at which the quantity grows.

Click on the icon to view this Investigation.

RULE OF 72

H LOGARITHMIC FUNCTIONS

We have seen that $\log_a a^x = a^{\log_a x} = x$.

Letting $f(x) = \log_a x$ and $g(x) = a^x$, we have $f \circ g = g \circ f = x$.

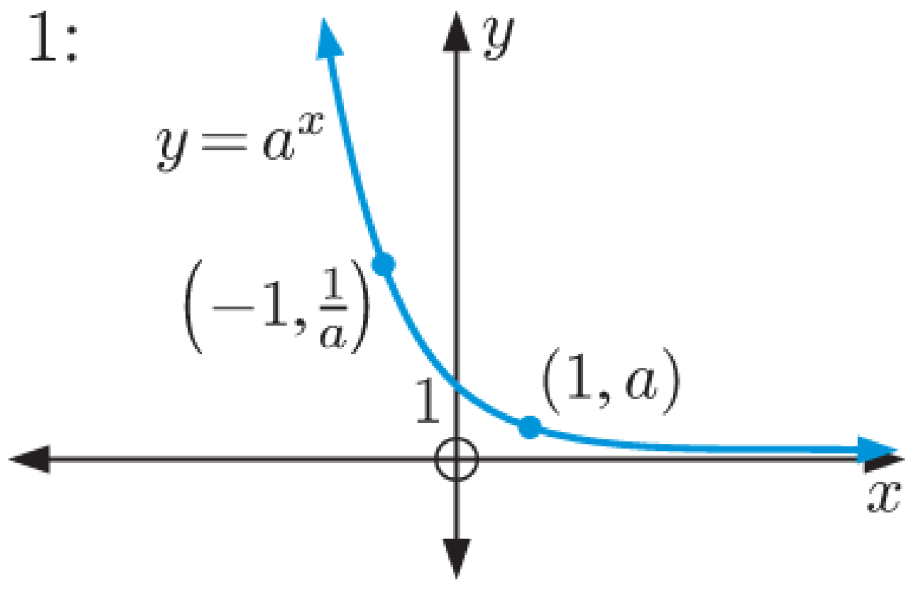
We can therefore say that the logarithmic function $\log_a x$ is the **inverse** of the exponential function a^x.

Algebraically, this has the effect that the logarithmic and exponential functions "undo" one another.

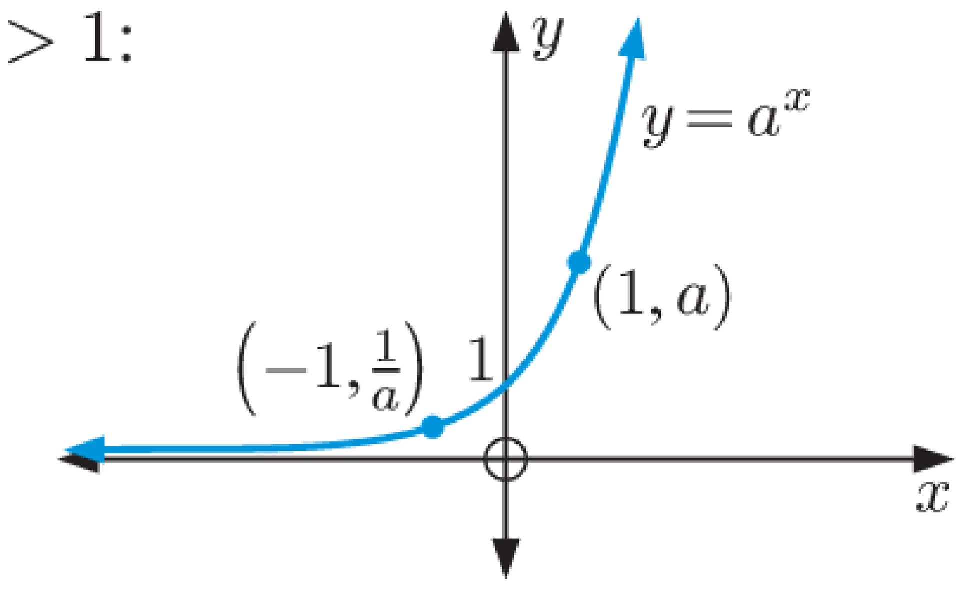
Geometrically, it means that the graph of $y = \log_a x$, $a > 0$, $a \neq 1$ is the *reflection* of the graph of $y = a^x$ in the line $y = x$.

We have seen previously the shape of the exponential function $y = a^x$ where $a > 0$, $a \neq 1$.

For $0 < a < 1$:



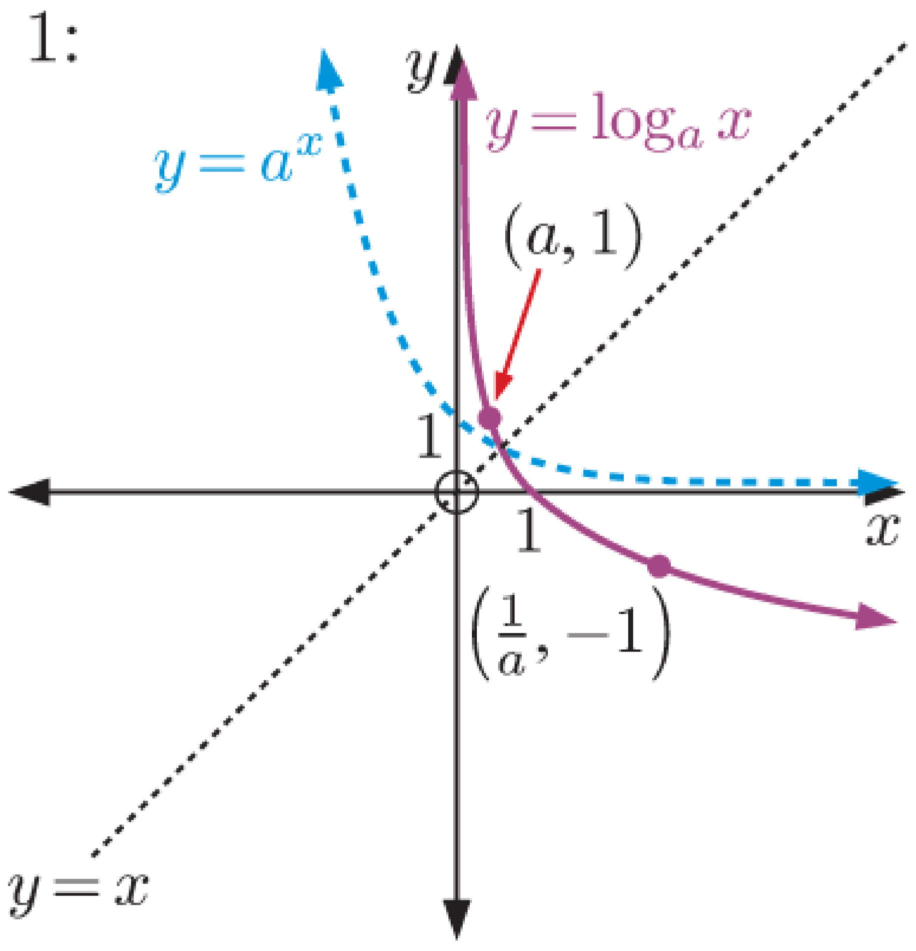
For $a > 1$:



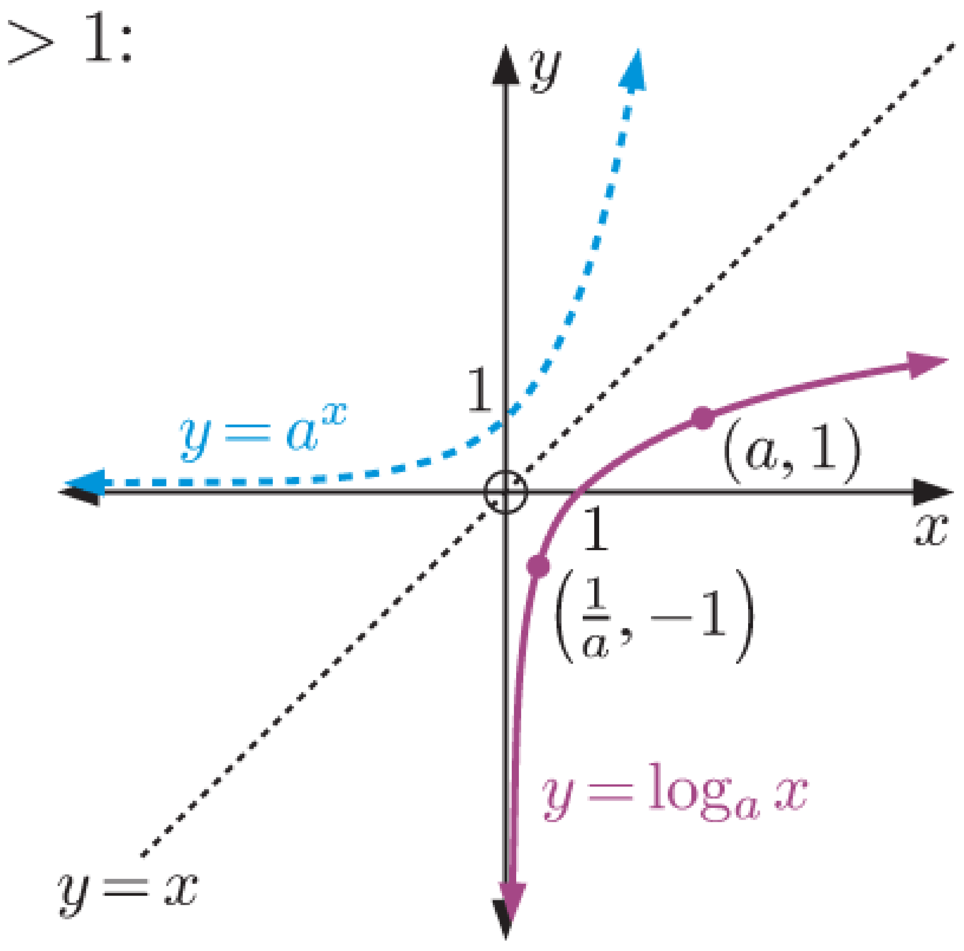
The horizontal asymptote for all of these functions is the x-axis $y = 0$.

By reflecting these graphs in the line $y = x$, we obtain the graphs for $y = \log_a x$.

For $0 < a < 1$:



For $a > 1$:



The **vertical asymptote** of $y = \log_a x$ is the y-axis $x = 0$.

For $0 < a < 1$: as $x \to \infty$, $y \to -\infty$
as $x \to 0^+$, $y \to \infty$

For $a > 1$: as $x \to \infty$, $y \to \infty$
as $x \to 0^+$, $y \to -\infty$

PROPERTIES OF $y = \log_a x$

Since we can only find logarithms of positive numbers, the domain of $y = \log_a x$ is $\{x \mid x > 0\}$.

We can compare the functions $y = a^x$ and $y = \log_a x$ as follows:

Function	$y = a^x$	$y = \log_a x$
Domain	$x \in \mathbb{R}$	$x > 0$
Range	$y > 0$	$y \in \mathbb{R}$
Asymptote	horizontal $y = 0$	vertical $x = 0$

TRANSFORMATIONS OF LOGARITHMIC FUNCTIONS

LOGARITHMIC GRAPHS

Click on the icon to explore the graphs of functions of the form $y = p\ln(x-h)+k$.

Example 23 — Self Tutor

Consider the function $f(x) = \log_2(x-1)+1$.

- **a** State the transformation which maps $y = \log_2 x$ to $y = f(x)$.
- **b** Find the domain and range of f.
- **c** Find any asymptotes and axes intercepts.
- **d** Sketch the graph of $y = f(x)$.
- **e** Find the inverse function f^{-1}.

a $f(x)$ is a translation of $y = \log_2 x$ by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

b $x-1>0$ when $x>1$

So, the domain is $x>1$ and the range is $y \in \mathbb{R}$.

c As $x \to 1^+$, $y \to -\infty$, so the vertical asymptote is $x = 1$.

As $x \to \infty$, $y \to \infty$, so there is no horizontal asymptote.

When $x = 0$, y is undefined, so there is no y-intercept.

When $y = 0$, $\log_2(x-1) = -1$

$\therefore\ x - 1 = 2^{-1}$

$\therefore\ x = \frac{3}{2}$ So, the x-intercept is $\frac{3}{2}$.

GRAPHICS CALCULATOR INSTRUCTIONS

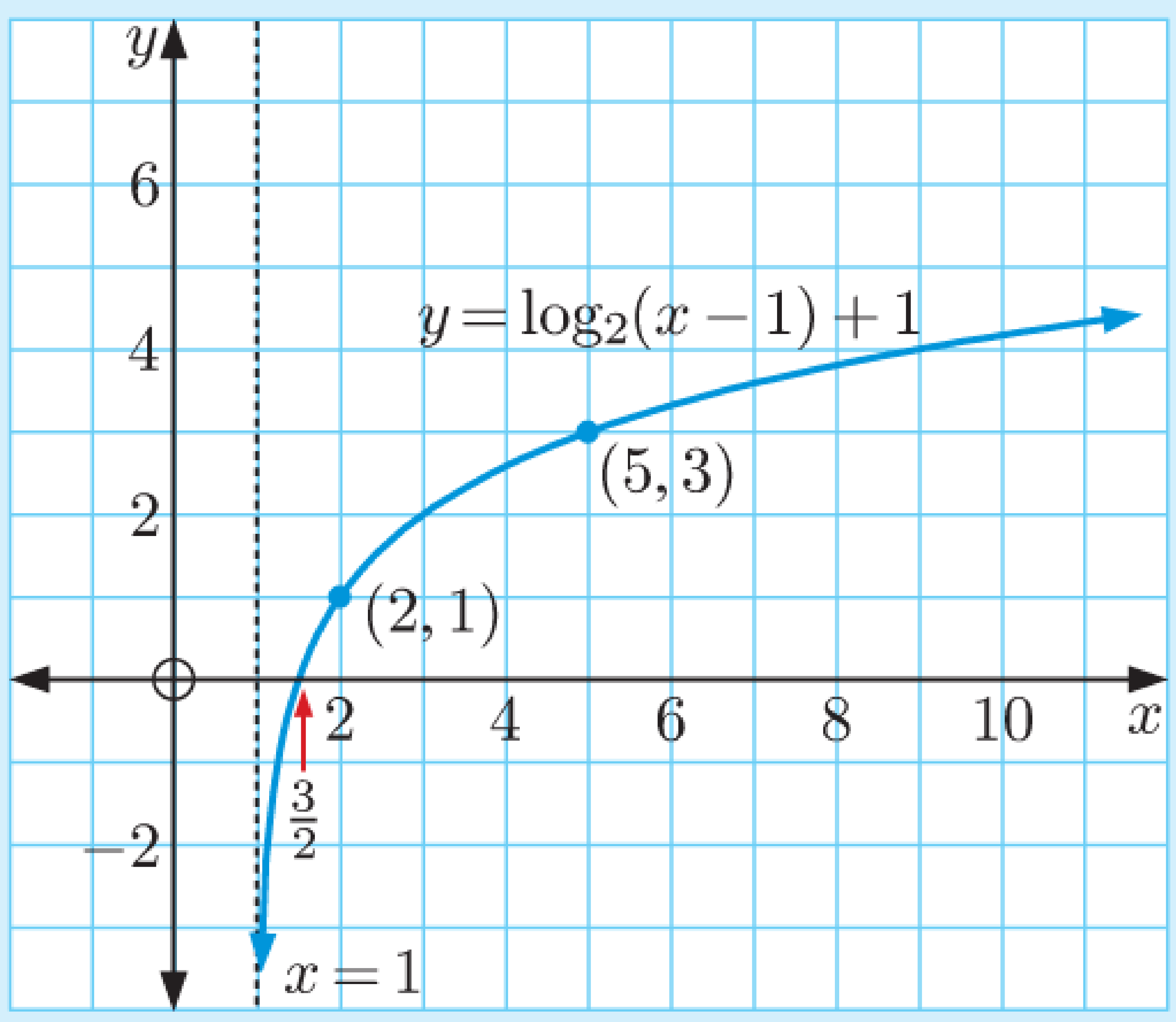
d When $x = 2$, $y = \log_2(2-1)+1 = 1$

When $x = 5$, $y = \log_2(5-1)+1$

$= \log_2 4 + 1$

$= 2+1$

$= 3$



e f is defined by $y = \log_2(x-1)+1$

$\therefore\ f^{-1}$ is defined by $x = \log_2(y-1)+1$

$\therefore\ x - 1 = \log_2(y-1)$

$\therefore\ y - 1 = 2^{x-1}$

$\therefore\ y = 2^{x-1}+1$

$\therefore\ f^{-1}(x) = 2^{x-1}+1$

EXERCISE 3H

1 For each of the following functions f:

- **i** Find the domain and range.
- **ii** Find any asymptotes and axes intercepts.
- **iii** Sketch the graph of $y = f(x)$, showing all important features.
- **iv** Solve $f(x) = -1$ algebraically and check the solution on your graph.
- **v** Find the inverse function f^{-1}.

a $f: x \mapsto \log_2 x - 2$ **b** $f: x \mapsto \log_3(x+1)$ **c** $f: x \mapsto 1 - \log_3(x+1)$

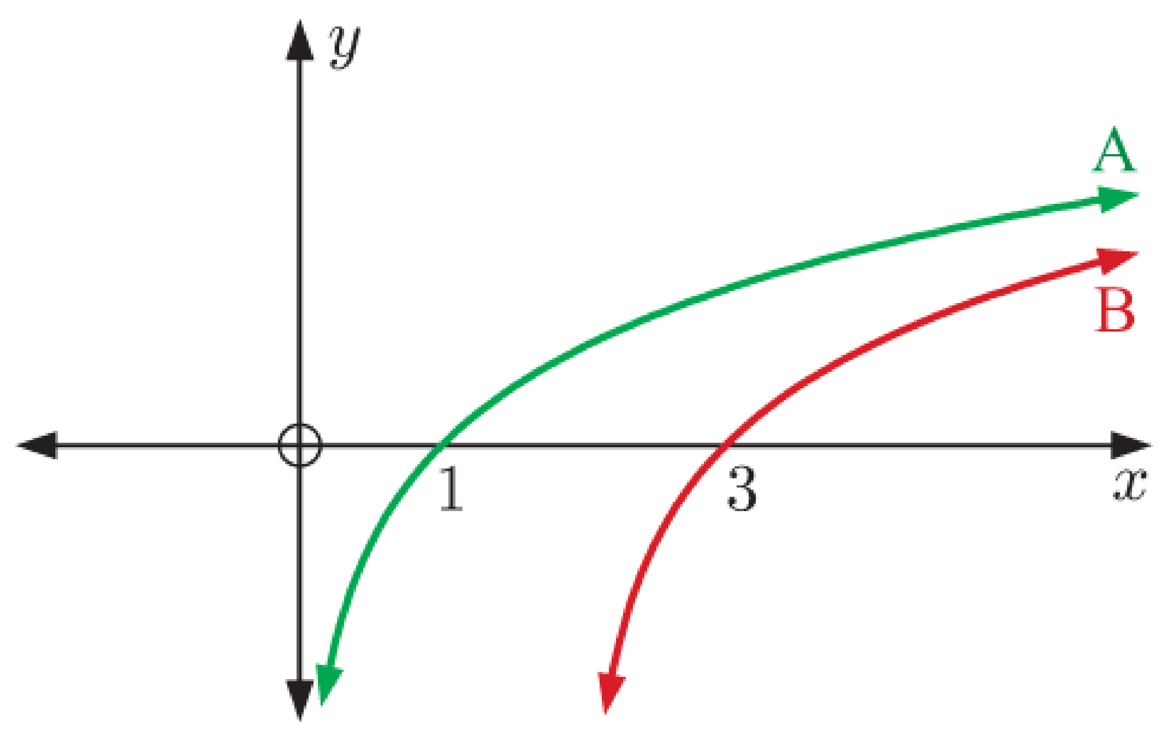
d $f: x \mapsto \log_5(x-2) - 2$ **e** $f: x \mapsto 1 - \log_5(x-2)$ **f** $f: x \mapsto 1 - 2\log_2 x$

2 For each of the functions f:

- **i** State the transformation which maps $y = \ln x$ to $y = f(x)$.
- **ii** State the domain and range.
- **iii** Find any asymptotes and intercepts.
- **iv** Sketch the graph of $y = f(x)$, showing all important features.
- **v** Find the inverse function f^{-1}.

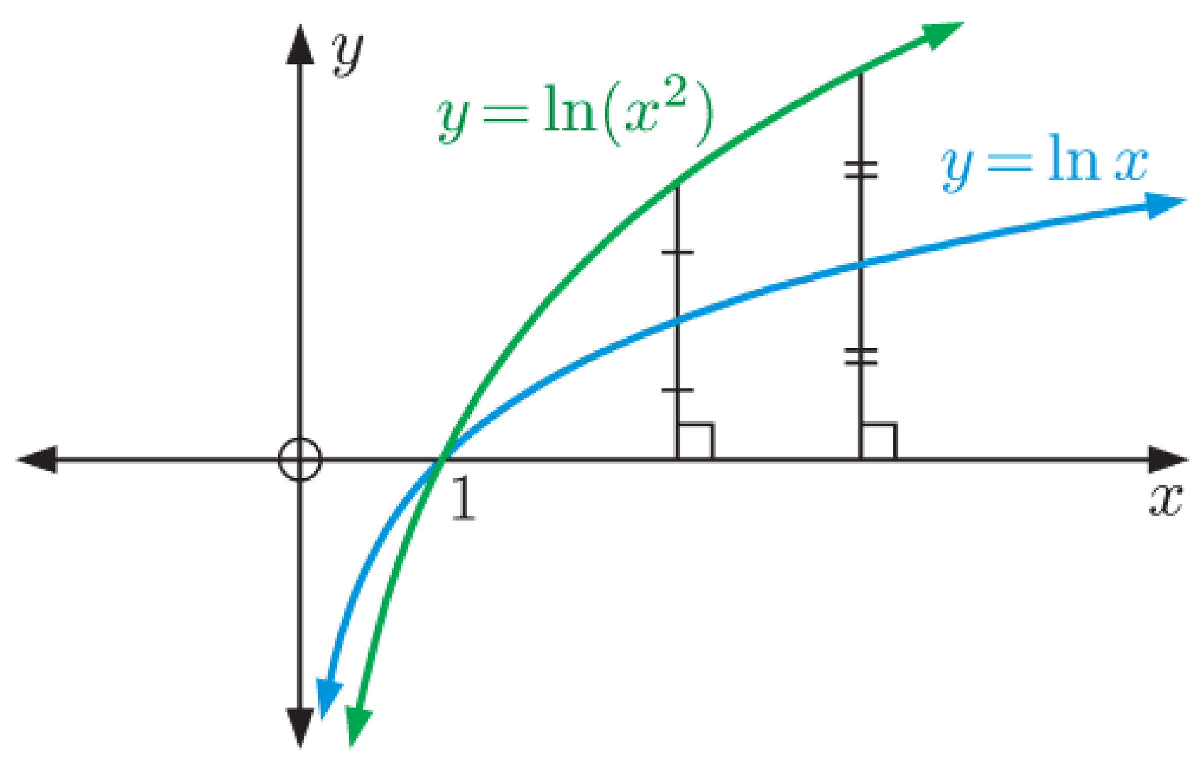
a $f(x) = \ln x - 4$ **b** $f(x) = \ln(x-1) + 2$ **c** $f(x) = 3\ln x - 1$

3 Consider the curves A and B. One of them is the graph of $y = \ln x$ and the other is the graph of $y = \ln(x-2)$.



- **a** Identify which curve is which, giving evidence for your answer.
- **b** Copy the graphs onto a new set of axes, and then draw the graph of $y = \ln(x+2)$.
- **c** Find the equation of the vertical asymptote for each graph.

4 Kelly said that in order to graph $y = \ln(x^2)$, $x > 0$, you could first graph $y = \ln x$ and then double the distance of each point on the graph from the x-axis. Is Kelly correct? Explain your answer.



5 Draw, on the same set of axes, the graphs of:

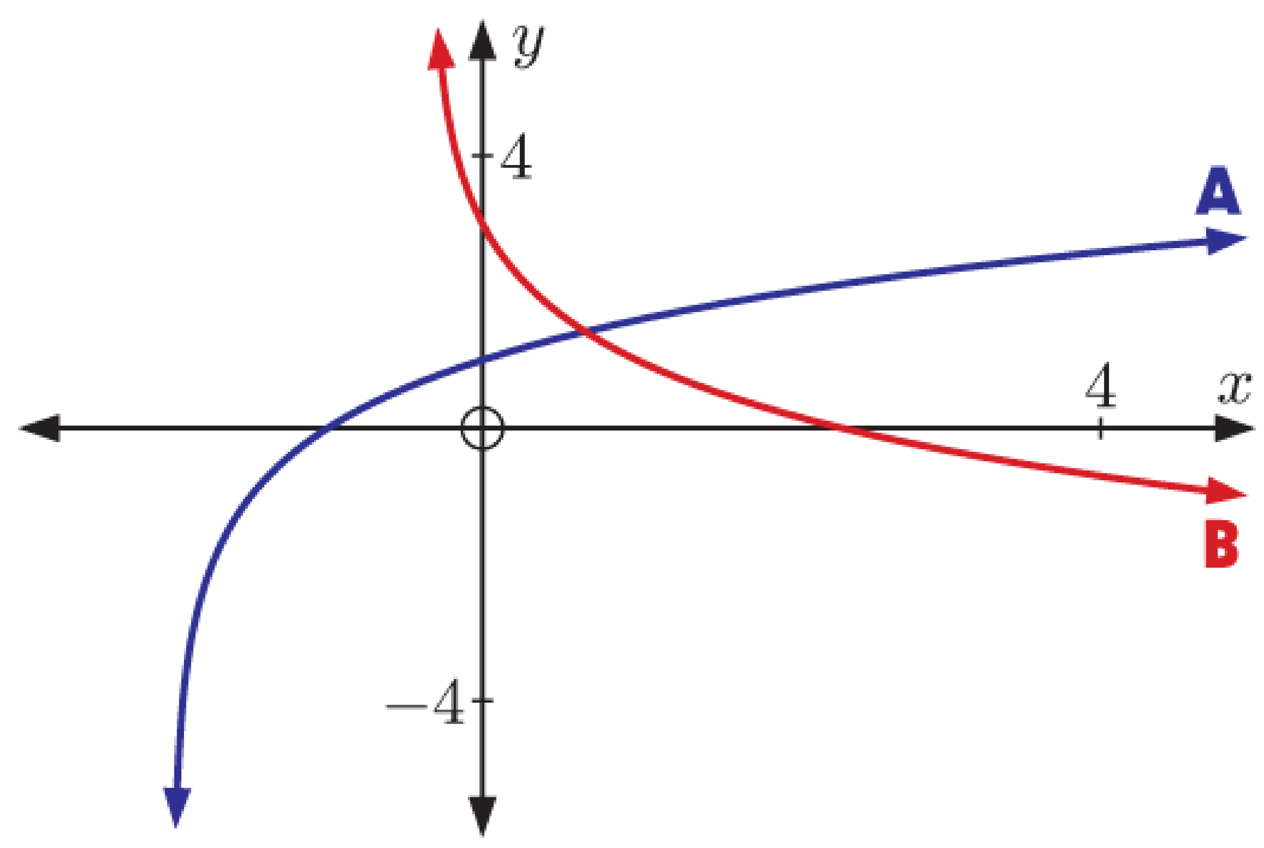
a $y = \ln x$ and $y = \ln(x^3)$ **b** $y = \ln x$ and $y = \ln\left(\frac{1}{x}\right)$

c $y = \ln x$ and $y = \ln(x + e)$ **d** $y = \ln x$ and $y = \ln(x-2) - 3$

e $y = 2\ln x$ and $y = \ln(x^2) + 2$

6 Describe a transformation which maps the graph of $y = \log_2 x$ to the graph of $y = \log_5 x$.

7 The logarithmic functions $y = 3 - \log_2(3x+1)$ and $y = \log_2(x+2)$ are graphed alongside.



- **a** Identify which curve is which, giving evidence for your answer.
- **b** Find the axes intercepts and asymptotes of each graph.
- **c** Find the exact coordinates of the point where the graphs intersect.

8 For each of the following functions, find the inverse function f^{-1}, and state the domain and range of f^{-1}:

a $f(x) = 3^x$ **b** $f(x) = 2^{x+1}$ **c** $f(x) = e^{2x}$ **d** $f(x) = 5^x - 3$

9 Suppose $f(x) = be^x$ and $g(x) = \ln(bx)$. Find:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

c the value of x, in terms of b, for which $(f \circ g)(x) = (g \circ f)(x)$.

10 Suppose $f(x) = e^{x+2}$ and $g(x) = \ln x - 3$.

a Find $(f \circ g)(x)$, and state its domain and range.

b Find $(g \circ f)(x)$, and state its domain and range.

11 Let $f(x) = \log_2(1 - 3x)$ and $g(x) = 2x + 1$.

a Find the domain and range of $f(x)$.

b Solve for x:

i $(f \circ g)(x) = 5$ **ii** $(g \circ f)(x) = 0$

c Find $f^{-1}(x)$, and state its domain and range.

12 Given $f : x \mapsto e^{2x}$ and $g : x \mapsto 2x - 1$, find:

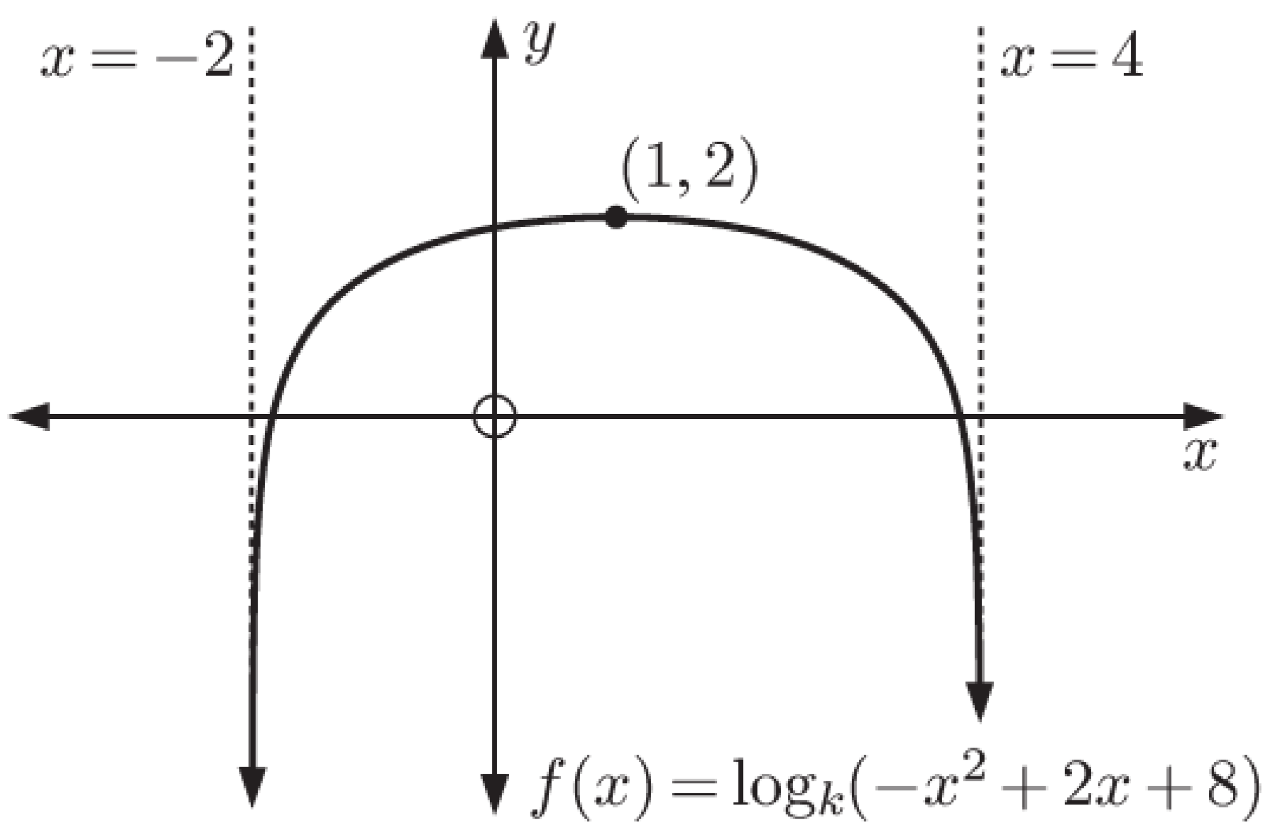
a $(f^{-1} \circ g)(x)$ **b** $(g \circ f)^{-1}(x)$

13 The function $f(x) = \log_k(-x^2 + 2x + 8)$ is graphed alongside.

a Find k.

b Find the axes intercepts of $f(x)$.

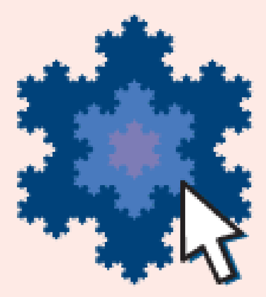
c Let $g(x) = f(x)$ on the restricted domain $-2 < x \leqslant 1$. Find $g^{-1}(x)$, and state its domain and range.



ACTIVITY

Click on the icon to obtain a card game for logarithmic functions.

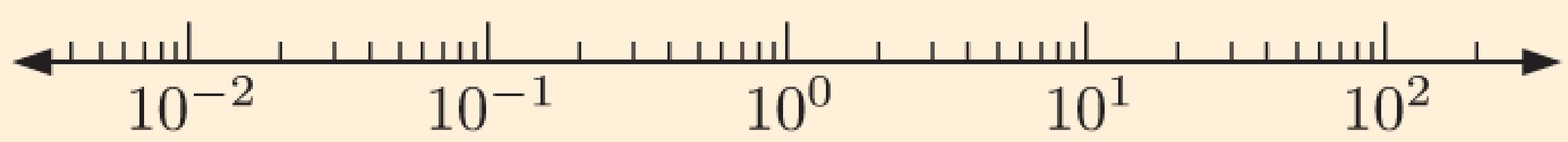
CARD GAME



INVESTIGATION 3 LOGARITHMIC SCALES

In a **logarithmic scale**, equally spaced major tick marks correspond to integer *powers* of a base number. We often call these **orders of magnitude**.

For example, in the logarithmic scale alongside, each major tick mark represents a power of 10.



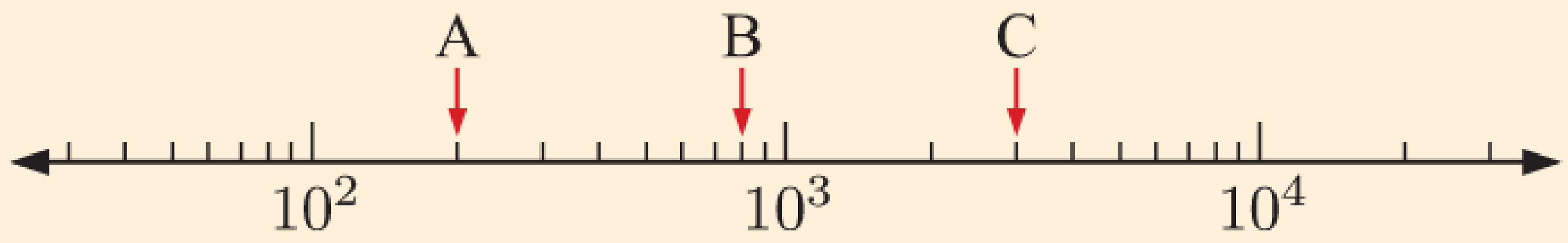
The minor tick marks correspond to integer *multiples* of each power of 10. So the minor tick marks between 10^1 and 10^2 represent 20, 30, 40, ..., and so on.

Logarithmic scales are useful when we want to represent both very large and very small numbers on the same number line. They allow us to compare real world quantities or events which are many orders of magnitude apart.

In this Investigation, we will explore the use of logarithmic scales in a variety of contexts.

What to do:

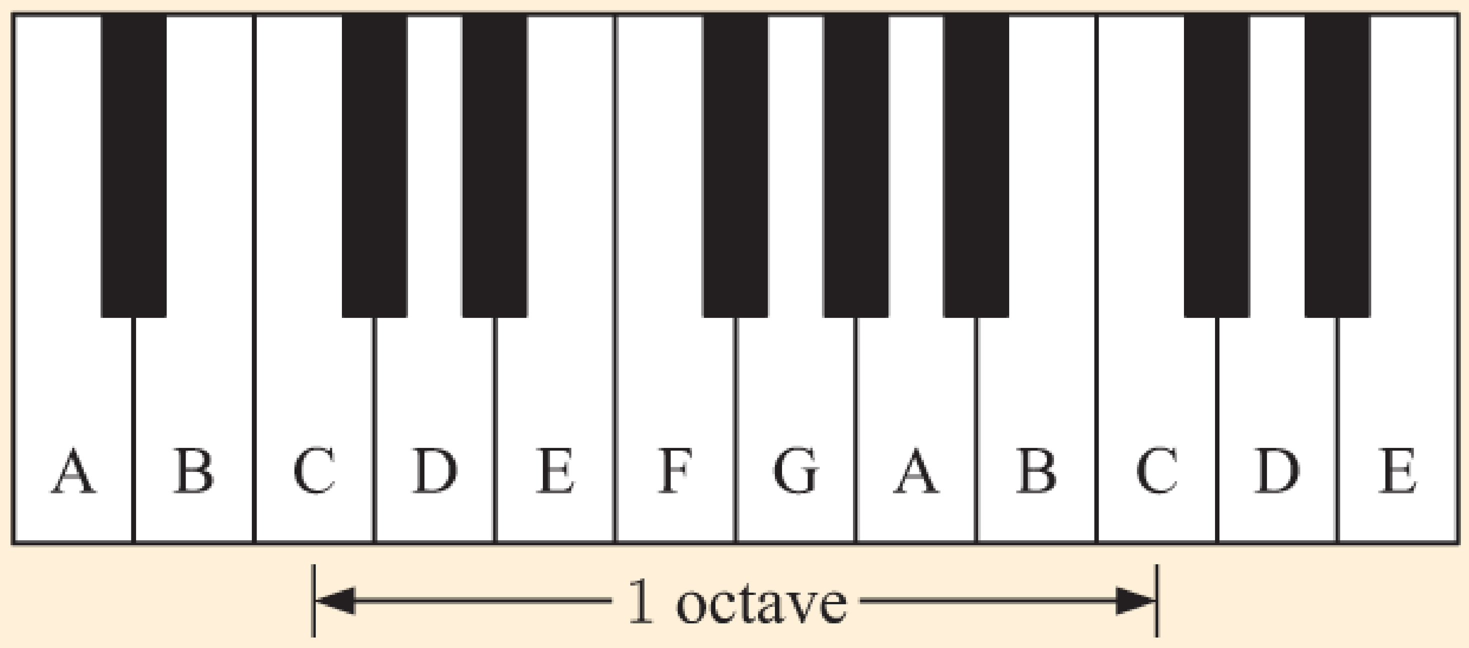
1 **a** For the logarithmic scale alongside, state the values of the points A, B, and C.



b Explain why the minor tick marks in a logarithmic scale are not equally spaced.

c Where is the value 0 on a logarithmic scale? Explain your answer.

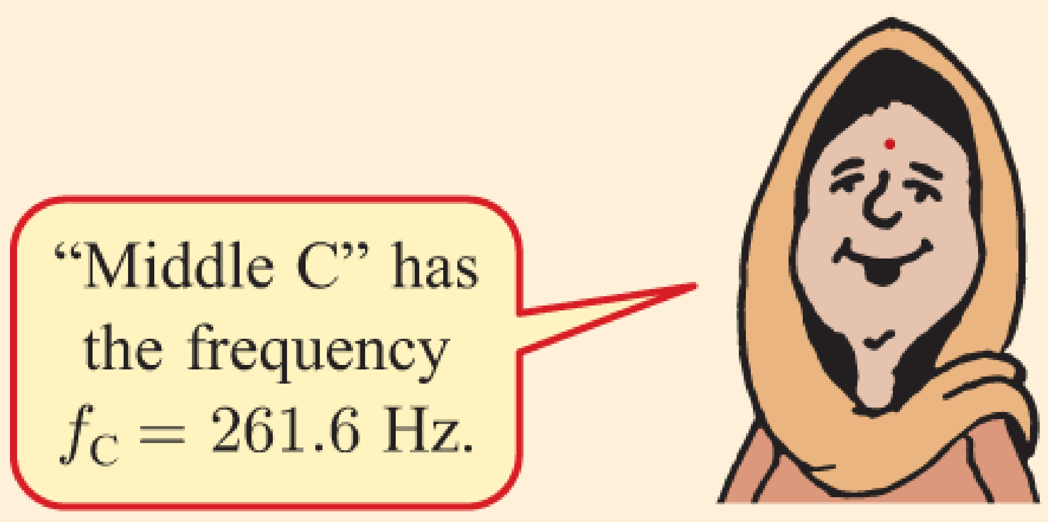
2 Musical notes are named according to the frequency of their sound waves. They are labelled with letters of the alphabet. A note which has *twice* the frequency of another is said to be one **octave** higher than it. So, one C is an octave below the next C.



a How many orders of magnitude apart are the frequencies of two notes separated by 3 octaves?

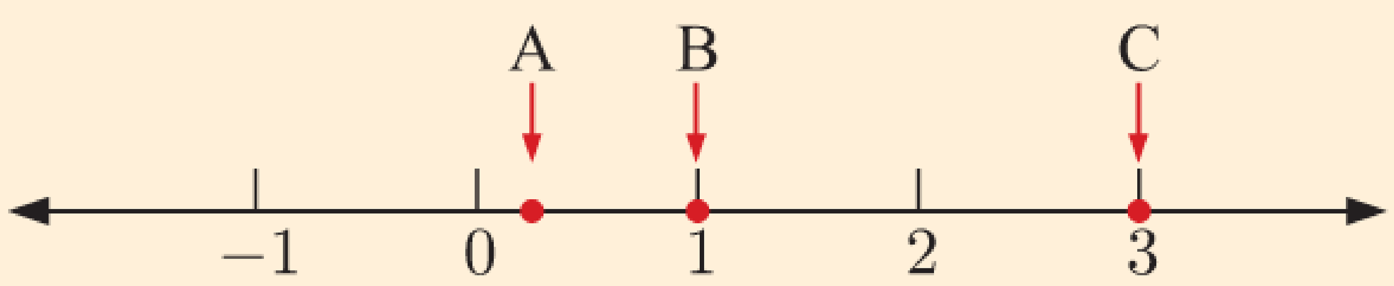
b Write an expression for the frequency of a musical note f, in terms of the number of octaves n *above* middle C.

c There are 12 different notes in an octave. They are equally spaced on the logarithmic scale. Find the ratio of frequencies between two adjacent notes.



3 In some situations, the logarithm is already applied to values placed on the number line. In these cases, the major tick marks represent the *exponents* rather than the numbers themselves.

For example, suppose the scale alongside is logarithmic with base 10. The major tick mark "2" represents the value 10^2, the major tick mark "3" represents the value 10^3, and so on.



a How many times larger is the value at C than the value at B?

b Estimate the position on the scale representing the value:

i 10 times smaller than A **ii** twice as large as B.

4 Earthquakes can range from microscopic tremors to huge natural disasters. The magnitude of earthquakes is measured on the **Richter scale** which relates to the energy released by the earthquake. For this logarithmic scale, the logarithm is part of the formula. It is calculated as $M = \log\left(\frac{I}{I_0}\right)$, where I is the earthquake intensity and I_0 is a reference intensity level.

a What does it mean for a tremor to have magnitude:

i 0 **ii** 1?

b Explain why an earthquake of magnitude 6 is *not* twice as intense as a magnitude 3 earthquake.

c Find the magnitude of an earthquake which has half the intensity of a magnitude 4 earthquake.

5 The *acidity* of a solution is determined by the concentration of hydronium ions (H_3O^+).
The higher the concentration of H_3O^+, the more acidic it is. The opposite of acidic is *alkaline*.

a In extremely acidic solutions, the concentration of H_3O^+ is typically more than 10^{-3} units. In very alkaline solutions, it is usually less than 10^{-12} units.
Explain why a logarithmic scale would be useful in describing the acidity of a solution.

b In chemistry, the **pH** scale is used to measure acidity. The pH of a solution is given by $\text{pH} = -\log C$, where C is the concentration of H_3O^+. Find:

i the pH of a solution with H_3O^+ concentration 0.000 234 units

ii the H_3O^+ concentration in a solution with pH 7.

c Is it possible for a solution to have a negative pH? Explain what this means in terms of the concentration of H_3O^+.

6 Research the use of **decibels** in acoustics as a unit of measurement for loudness of sound. Compare the use of decibels to the scales in questions **4** and **5**.

REVIEW SET 3A

1 Find:

a $\log\sqrt{10}$ **b** $\log\left(\frac{1}{\sqrt[3]{10}}\right)$ **c** $\log(10^a \times 10^{b+1})$

2 Find:

a $\log_4 64$ **b** $\log_2 256$ **c** $\log_2(0.25)$ **d** $\log_{25} 5$

e $\log_8 1$ **f** $\log_{81} 3$ **g** $\log_9\left(\frac{1}{9}\right)$ **h** $\log_k \sqrt{k}$

3 Use your calculator to evaluate, correct to 3 decimal places:

a $\log 27$ **b** $\log(0.58)$ **c** $\log 400$ **d** $\ln 40$

4 If $y = \log_3\sqrt{2-x}$, write x in terms of y.

5 Simplify:

a $4\ln 2 + 2\ln 3$ **b** $\frac{1}{2}\ln 9 - \ln 2$ **c** $2\ln 5 - 1$ **d** $\frac{1}{4}\ln 81$

6 Write as a single logarithm:

a $\log 16 + 2\log 3$ **b** $\log_2 16 - 2\log_2 3$ **c** $2 + \log_4 5$

7 Suppose $A = \log_5 2$ and $B = \log_5 3$. Write in terms of A and B:

a $\log_5 36$ **b** $\log_5 54$ **c** $\log_5\left(8\sqrt{3}\right)$

d $\log_5\left(\sqrt{6}\right)$ **e** $\log_5(20.25)$ **f** $\log_5\left(\frac{8}{9}\right)$

8 Write as a logarithmic equation:

a $M = ab^n$ **b** $T = \frac{5}{\sqrt{l}}$ **c** $G = \frac{a^2 b}{c}$

9 Solve for x:

a $3^x = 300$ **b** $30 \times 5^{1-x} = 0.15$ **c** $3^{x+2} = 2^{1-x}$

10 Solve exactly for x:

a $e^{2x} = 3e^x$ **b** $e^{2x} - 7e^x + 12 = 0$

11 Write without logarithms:

a $\ln P = 1.5 \ln Q + \ln T$ **b** $\ln M = 1.2 - 0.5 \ln N$

12 Solve for x:

a $3e^x - 5 = -2e^{-x}$ **b** $2 \ln x - 3 \ln\left(\frac{1}{x}\right) = 10$

13 Find x if:

a $\log_2 x = -3$ **b** $\log_5 x \approx 2.743$ **c** $\log_3 x \approx -3.145$

14 Solve for x: **i** exactly **ii** rounded to 2 decimal places.

a $2^x = 50$ **b** $7^x = 4$ **c** $(0.6)^x = 0.01$

15 Suppose $\log_a b = x$. Find, in terms of x, the value of $\log_a\left(\frac{1}{b}\right)$.

16 Write $\dfrac{8}{\log_5 9}$ in the form $a \log_3 b$.

17 Show that the solution to $16^x - 5 \times 8^x = 0$ is $x = \log_2 5$.

18 Solve for x, giving exact answers:

a $\ln x = 5$ **b** $3 \ln x + 2 = 0$ **c** $e^x = 400$

d $e^{2x+1} = 11$ **e** $25e^{\frac{x}{2}} = 750$

19 Consider $f(x) = e^{3x-4} + 1$.

a Show that $f^{-1}(x) = \dfrac{\ln(x-1)+4}{3}$.

b Calculate $f^{-1}(8) - f^{-1}(3)$. Give your answer in the form $a \ln b$, where $a, b \in \mathbb{Q}^+$.

20 Solve simultaneously: $x = 16y$ and $\log_y x - \log_x y = \frac{8}{3}$.

21 Consider the function $g : x \mapsto \log_3(x+2) - 2$.

a State the transformation which maps $y = \log_3 x$ to $y = g(x)$.

b Find the domain and range.

c Find any asymptotes and axes intercepts for the graph of the function.

d Find the inverse function g^{-1}.

e Sketch the graphs of g, g^{-1}, and $y = x$ on the same set of axes.

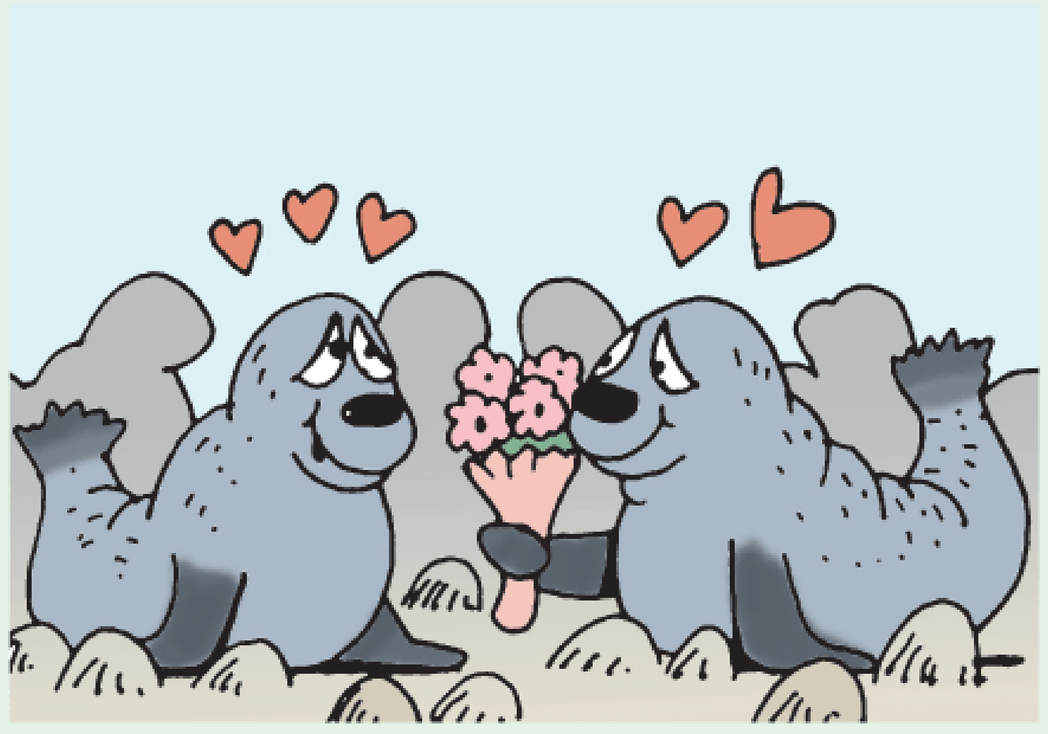
22 The weight of a radioactive isotope remaining after t weeks is given by $W_t = 8000 \times e^{-\frac{t}{20}}$ grams. Find the time for the weight to:

a halve **b** reach 1000 g **c** reach 0.1% of its original value.

23 A population of seals is given by $P(t) = 80 \times (1.15)^t$ where t is the time in years, $t \geqslant 0$.

a Find the time required for the population to double in size.

b Find the percentage increase in population during the first 4 years.



24 For each of the following functions:

i State the domain and range.
ii Find any asymptotes and axes intercepts.
iii Sketch the graph of the function, showing all important features.

a $f(x) = \log_2(x+4) - 1$ **b** $f(x) = \ln x + 2$

25 Draw, on the same set of axes, the graphs of:

a $y = \ln x$ and $y = \ln(x-3)$ **b** $y = \ln x$ and $y = \frac{1}{2}\ln x$

26 Consider $f(x) = e^x$ and $g(x) = \ln(x+4)$, $x > -4$. Find:

a $(f \circ g)(5)$ **b** $(g \circ f)(0)$

REVIEW SET 3B

1 Without using a calculator, find the base 10 logarithms of:

a $\sqrt{1000}$ **b** $\dfrac{10}{\sqrt[3]{10}}$ **c** $\dfrac{10^a}{10^{-b}}$

2 Find:

a $\log_2 128$ **b** $\log_3\left(\frac{1}{27}\right)$ **c** $\log_5\left(\frac{1}{\sqrt{5}}\right)$

3 Write in the form 10^x, giving x correct to 4 decimal places:

a 32 **b** 0.0013 **c** 8.963×10^{-5}

4 Find:

a $\ln(e\sqrt{e})$ **b** $\ln\left(\dfrac{1}{e^3}\right)$ **c** $\ln(e^{2x})$ **d** $\ln\left(\dfrac{e}{e^x}\right)$

5 Simplify:

a $\dfrac{\log_2 25}{\log_2 125}$ **b** $\dfrac{\log 64}{\log 32}$ **c** $\dfrac{\log_5 81}{\log_5 \sqrt{3}}$

6 Simplify:

a $e^{4\ln x}$ **b** $\ln(e^5)$ **c** $\ln(\sqrt{e})$

d $10^{\log x + \log 3}$ **e** $\ln\left(\dfrac{1}{e^x}\right)$ **f** $\dfrac{\log(x^2)}{\log_3 9}$

7 Write in the form e^x, where x is correct to 4 decimal places:

a 20 **b** 3000 **c** 0.075

8 Solve for x: **i** exactly **ii** rounded to 2 decimal places.

a $5^x = 7$ **b** $2^x = 0.1$

9 Write as a single logarithm:

a $\ln 60 - \ln 20$ **b** $\ln 4 + \ln 1$ **c** $\ln 200 - \ln 8 + \ln 5$

10 Solve for x, giving exact answers:

a $e^{2x} = 70$ **b** $3 \times (1.3)^x = 11$ **c** $5 \times 2^{0.3x} = 16$

11 What is the only value of x for which $\log x = \ln x$?

12 Show that the equation $\log_3(x-k)+\log_3(x+2)=1$ has a real solution for every real value of k.

13 Write as a logarithmic equation:

a $P=3\times b^x$ **b** $m=\dfrac{n^3}{p^2}$

14 Show that $\log_3 7\times 2\log_7 x=2\log_3 x$.

15 Solve for x:

a $\log_2(x^2)+\log_8\sqrt{x}=3$ **b** $\log_{27}\left(\dfrac{1}{x}\right)+\log_3(x^4)=\log_3 10$

16 Write the following equations without logarithms:

a $\log T=2\log x-\log y$ **b** $\log_2 K=\log_2 n+\frac{1}{2}\log_2 t$

17 Write in the form $a\ln k$ where a and k are positive whole numbers and k is prime:

a $\ln 32$ **b** $\ln 125$ **c** $\ln 729$

18 Copy and complete:

	$y=\log_2 x$	$y=\ln(x+5)$
Domain		
Range		

19 **a** Factorise 4^x-2^x-20 in the form $(2^x+a)(2^x-b)$ where $a, b\in\mathbb{Z}^+$.

b Hence find the exact solution of $2^x(2^x-1)=20$.

c Suppose $p=\log_5 2$.

i Write the solution to **b** in terms of p.

ii Find the solution to $8^x=5^{1-x}$ in terms of p only.

20 Consider $g:x\mapsto 2e^x-5$.

a Find the inverse function g^{-1}.

b Sketch the graphs of g and g^{-1} on the same set of axes.

c State the domain and range of g and g^{-1}.

d State the asymptotes and intercepts of g and g^{-1}.

21 The temperature of a mug of water t minutes after it has been poured from a kettle is given by $T=60e^{-0.1t}+20$ °C.
Show that it will take $10\ln 3$ minutes for the temperature of the water to fall to 40°C.

22 The weight of a radioactive isotope after t years is given by $W(t)=2500\times 3^{-\frac{t}{3000}}$ grams.

a Find the initial weight of the isotope.

b Find the time taken for the isotope to reduce to 30% of its original weight.

c Find the percentage of weight lost after 1500 years.

23 Solve for x, giving an exact answer:

a $5^{\frac{x}{2}} = 9$ **b** $e^x = 30$ **c** $e^{1-3x} = 2$

24 Draw, on the same set of axes, the graphs of:

a $y = \ln x$ and $y = \ln(x + 2)$ **b** $y = \ln x$ and $y = \ln(ex)$

25 *Hick's law* models the time taken for a person to make a selection from a number of possible options.

For a particular person, Hick's law determines that the time taken to choose between n equally probable choices is $T = 2\ln(n + 1)$ seconds.

a Sketch the graph of T against n for $0 \leqslant n \leqslant 50$.

b How long will it take this person to choose between:

i 5 possible choices **ii** 15 possible choices?

c If the number of possible choices increases from 20 to 40, how much longer will the person take to make a selection?

26 Let $f(x) = \log_3 x - 2$ and $g(x) = 3 - \sqrt{x}$.

a Find the following functions, and state the domain and range of each function:

i $f^{-1}(x)$ **ii** $(f \circ g)(x)$ **iii** $(g \circ f)(x)$

b Solve for x:

i $(f \circ g)(x) = -2$ **ii** $(g \circ f)(x) = 0$

c Find $(g \circ f)^{-1}(x)$, and state its domain and range.

13 $3\sin x - 5\cos x \approx \sqrt{34}\cos(x+3.68)$

14 **a** $2\sin x + \sqrt{3}\cos x \approx \sqrt{7}\sin(x+0.714)$

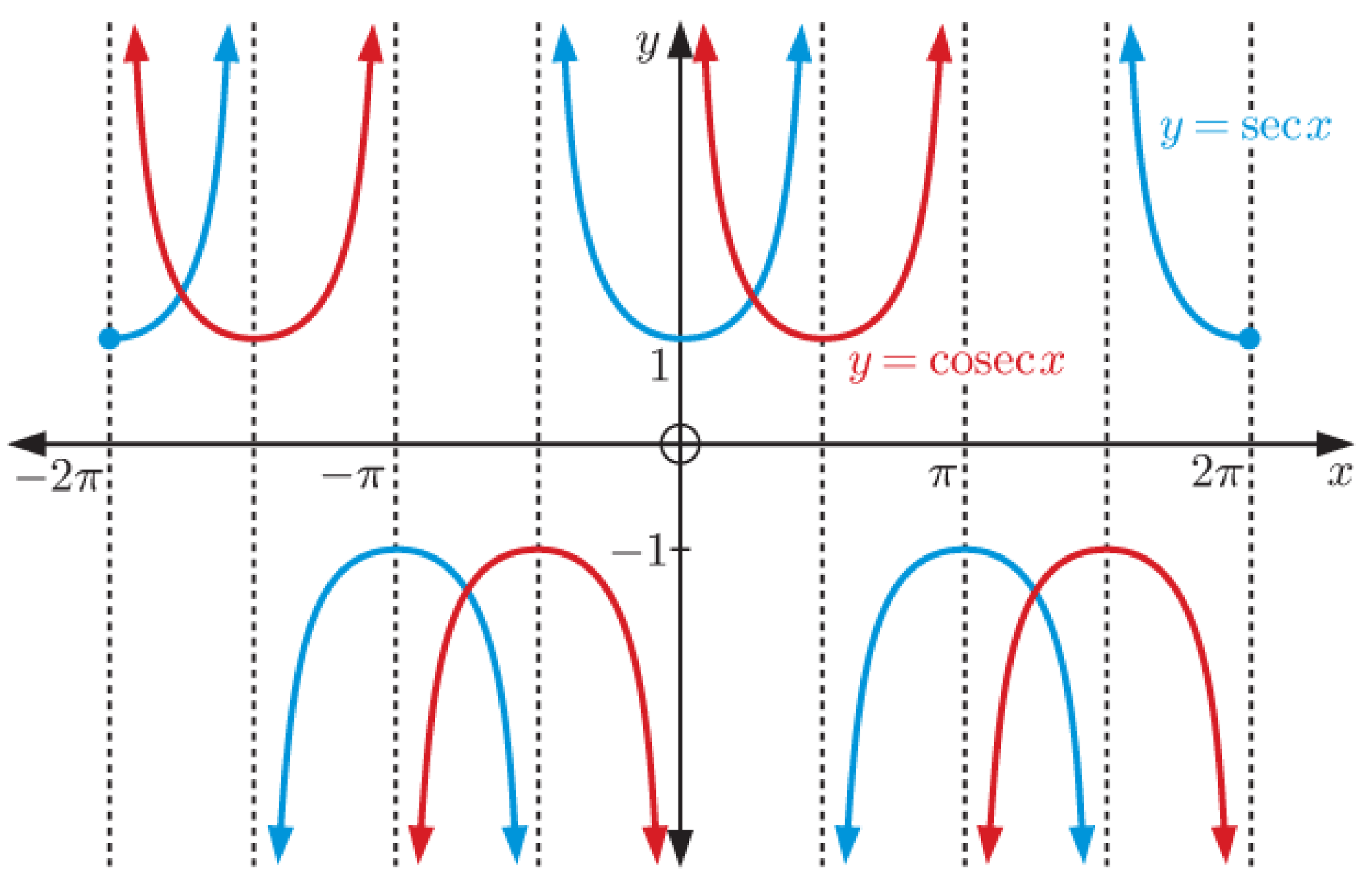
b **i** $A = \sqrt{7}$ **ii** $b \approx 2.43$

15 $\frac{\pi}{4}$

REVIEW SET 1B

1 $\sin x = -\frac{2\sqrt{2}}{3}$, $\tan x = 2\sqrt{2}$, $\operatorname{cosec} x = -\frac{3}{2\sqrt{2}}$,
$\sec x = -3$, $\cot x = \frac{1}{2\sqrt{2}}$

2 **a**



b translation $\frac{\pi}{2}$ units right

3 $x = -\frac{5\pi}{6}$ or $\frac{\pi}{6}$ **4** **a** $x = \frac{\sqrt{3}}{2}$ **b** $x = 2 + \frac{1}{\sqrt{3}}$

5 **a** $\sec x$ **b** $\sin x$ **c** $\cos x$

6 **a** $\cos\theta$ **b** $-\sin\theta$ **c** $5\cos^2\theta$ **d** $-\cos\theta$
e $\operatorname{cosec}\theta$ **f** $\sin 2\theta$

7 **a** $\frac{120}{169}$ **b** $\frac{119}{169}$ **c** $\frac{120}{119}$

10 **a** $x = -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}$, or $\frac{\pi}{2}$ **b** $\theta = \frac{\pi}{3}$

11 $\sin\left(\theta + \frac{\pi}{6}\right) = \frac{3\sqrt{3}-\sqrt{7}}{8}$ **12** $\tan\theta = \frac{9}{19}$

13 $3\sin x + 4\cos x \approx 5\sin(x+0.927)$ **14** 1.5 m

15 **b** $y = 2\sec 2x$ has range $\{y \mid y \leqslant -2 \text{ or } y \geqslant 2\}$

$\therefore \frac{1}{1+\sqrt{2}\sin x} + \frac{1}{1-\sqrt{2}\sin x} = 1$ has no solutions.

EXERCISE 2A

1 **a** $2^{\frac{1}{5}}$ **b** $2^{-\frac{1}{5}}$ **c** $2^{\frac{3}{2}}$ **d** $2^{\frac{5}{2}}$ **e** $2^{-\frac{1}{3}}$
f $2^{\frac{4}{3}}$ **g** $2^{\frac{3}{2}}$ **h** $2^{\frac{3}{2}}$ **i** $2^{-\frac{4}{3}}$ **j** $2^{-\frac{3}{2}}$

2 **a** $3^{\frac{1}{3}}$ **b** $3^{-\frac{1}{3}}$ **c** $3^{\frac{1}{4}}$ **d** $3^{\frac{3}{2}}$ **e** $3^{-\frac{5}{2}}$

3 **a** $7^{\frac{1}{3}}$ **b** $3^{\frac{3}{4}}$ **c** $2^{\frac{4}{5}}$ **d** $2^{\frac{5}{3}}$ **e** $7^{\frac{2}{7}}$
f $7^{-\frac{1}{3}}$ **g** $3^{-\frac{3}{4}}$ **h** $2^{-\frac{4}{5}}$ **i** $2^{-\frac{5}{3}}$ **j** $7^{-\frac{2}{7}}$

4 **a** $x^{\frac{1}{2}}$ **b** $x^{\frac{3}{2}}$ **c** $x^{-\frac{1}{2}}$ **d** $x^{\frac{5}{2}}$ **e** $x^{-\frac{3}{2}}$

5 **a** ≈ 2.28 **b** ≈ 0.435 **c** ≈ 1.68 **d** ≈ 1.93
e ≈ 0.523

6 **a** $\sqrt[3]{5}$ **b** $\frac{1}{\sqrt{3}}$ **c** $9\sqrt{3}$ **d** $m\sqrt{m}$ **e** $x^3\sqrt{x}$

7 **a** 8 **b** 32 **c** 8 **d** 125 **e** 4
f $\frac{1}{2}$ **g** $\frac{1}{27}$ **h** $\frac{1}{16}$ **i** $\frac{1}{81}$ **j** $\frac{1}{25}$

EXERCISE 2B

1 **a** 1 **b** x **c** $x^{\frac{1}{2}}$ or $\sqrt{x}$

2 **a** $x^5 + 2x^4 + x^2$ **b** $2^{2x} + 2^x$ **c** $x+1$
d $7^{2x} + 2(7^x)$ **e** $2(3^x) - 1$ **f** $x^2 + 2x + 3$
g $1 + 5(2^{-x})$ **h** $5^x + 1$ **i** $x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1$
j $3^{2x} + 5(3^x) + 1$ **k** $2x^{\frac{3}{2}} - x^{\frac{1}{2}} + 5$ **l** $2^{3x} - 3(2^{2x}) - 1$

3 **a** $2^{2x} + 2^{x+1} - 3$ **b** $3^{2x} + 7(3^x) + 10$
c $5^{2x} - 6(5^x) + 8$ **d** $2^{2x} + 6(2^x) + 9$
e $3^{2x} - 2(3^x) + 1$ **f** $4^{2x} + 14(4^x) + 49$
g $x - 4$ **h** $4^x - 9$ **i** $x - \frac{1}{x}$ **j** $x^2 + 4 + \frac{4}{x^2}$
k $7^{2x} - 2 + 7^{-2x}$ **l** $25 - 10(2^{-x}) + 2^{-2x}$

4 **a** $5^x(5^x + 1)$ **b** $10(3^n)$ **c** $7^n(1 + 7^{2n})$
d $5(5^n - 1)$ **e** $6(6^{n+1} - 1)$ **f** $16(4^n - 1)$
g $2^n(2^n - 8)$ **h** $\frac{5}{2}(2^n)$ **i** $\frac{9}{2}(2^{2n})$

5 **a** $(3^x + 2)(3^x - 2)$ **b** $(2^x + 5)(2^x - 5)$
c $(4 + 3^x)(4 - 3^x)$ **d** $(5 + 2^x)(5 - 2^x)$
e $(3^x + 2^x)(3^x - 2^x)$ **f** $(2^x + 3)^2$
g $(3^x + 5)^2$ **h** $(2^x - 7)^2$ **i** $(5^x - 2)^2$

6 **a** $(2^x + 1)(2^x - 2)$ **b** $(3^x + 3)(3^x - 2)$
c $(2^x - 3)(2^x - 4)$ **d** $(2^x + 3)(2^x + 6)$
e $(2^x + 4)(2^x - 5)$ **f** $(3^x + 2)(3^x + 7)$
g $(3^x + 5)(3^x - 1)$ **h** $(5^x + 2)(5^x - 1)$
i $(7^x - 4)(7^x - 3)$

7 **a** 2^n **b** 10^a **c** 3^b **d** $\frac{1}{5^n}$ **e** 5^x
f $\left(\frac{3}{4}\right)^a$ **g** $\left(\frac{8}{3}\right)^k$ **h** 5 **i** 5^n

8 **a** $3^m + 1$ **b** $1 + 6^n$ **c** $4^n + 2^n$ **d** $4^x - 1$
e 6^n **f** 5^n **g** 4 **h** $2^n - 1$ **i** $\frac{1}{2}$

9 **a** $n\,2^{n+1}$ **b** -3^{n-1}

EXERCISE 2C

1 **a** $x = 5$ **b** $x = 2$ **c** $x = 4$ **d** $x = 0$
e $x = -1$ **f** $x = \frac{1}{2}$ **g** $x = -3$ **h** $x = 2$
i $x = -3$ **j** $x = -4$ **k** $x = 2$ **l** $x = \frac{3}{4}$

2 **a** $x = \frac{5}{3}$ **b** $x = -\frac{3}{2}$ **c** $x = -\frac{3}{2}$ **d** $x = -\frac{1}{2}$
e $x = -\frac{2}{3}$ **f** $x = -\frac{5}{4}$ **g** $x = \frac{3}{2}$ **h** $x = \frac{5}{2}$
i $x = \frac{1}{8}$ **j** $x = \frac{9}{2}$ **k** $x = -4$ **l** $x = -\frac{7}{2}$
m $x = 0$ **n** $x = \frac{7}{2}$ **o** $x = -\frac{2}{3}$ **p** $x = -6$

3 **a** $x = \frac{1}{7}$ **b** no solution **c** $x = \frac{5}{2}$
d $x = \frac{1}{3}$ **e** $x = -\frac{1}{4}$ **f** $x = -1$ or 3

4 **a** $x = 3$ **b** $x = 2$ **c** $x = -1$ **d** $x = 2$
e $x = -2$ **f** $x = -2$

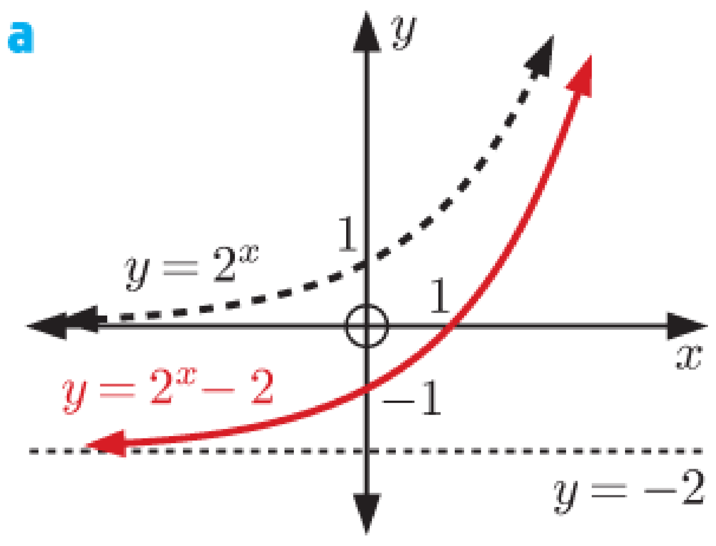
5 **a** $x = 1$ or 2 **b** $x = 1$ **c** $x = 1$ or 2
d $x = 1$ **e** $x = 2$ **f** $x = 0$
g $x = 1$ **h** $x = 1$ or -1 **i** $x = 2$
j $x = -2$ or 1 **k** $x = 2$ **l** $x = \frac{1}{2}$

6 $x = \frac{15}{7}$, $y = \frac{10}{7}$

EXERCISE 2D

1 **a** **i** ≈ 1.4 **ii** ≈ 1.7 **iii** ≈ 2.8 **iv** ≈ 0.4
b **i** $x \approx 1.6$ **ii** $x \approx -0.7$
c $y = 2^x$ has a horizontal asymptote of $y = 0$.

2 **a** C **b** B **c** E **d** A **e** D

3 **a**


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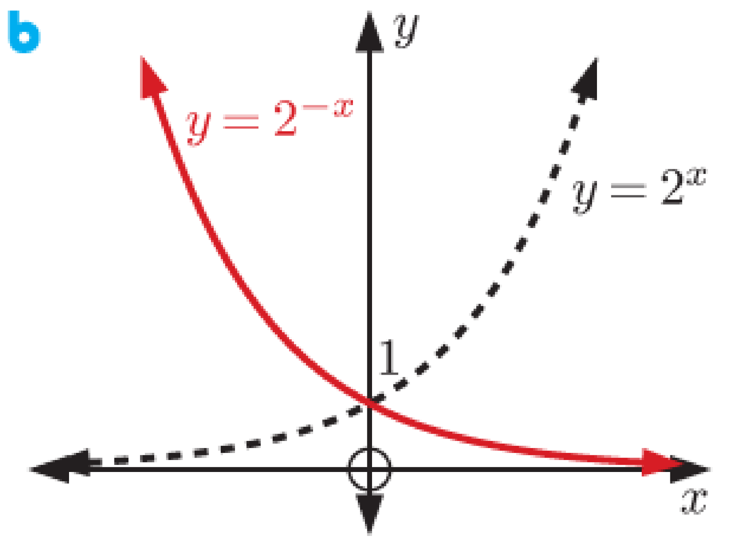

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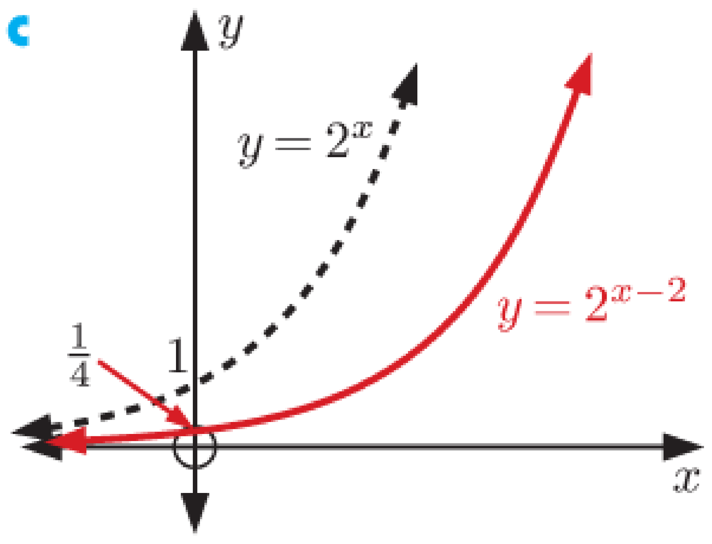
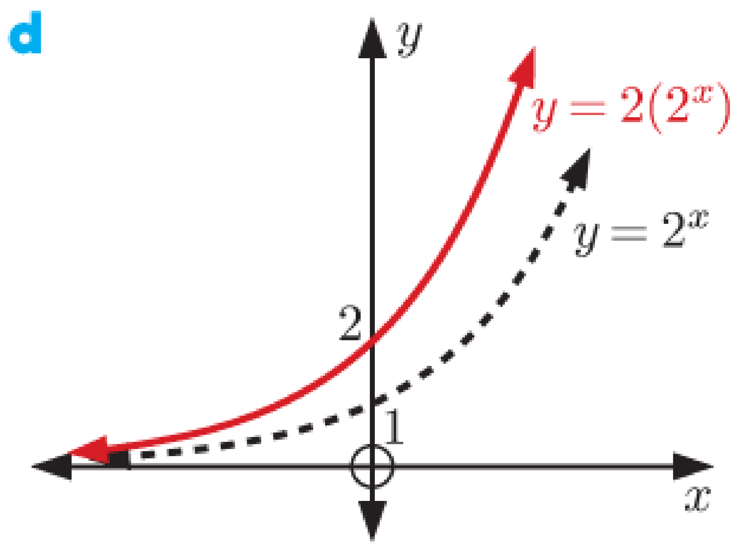
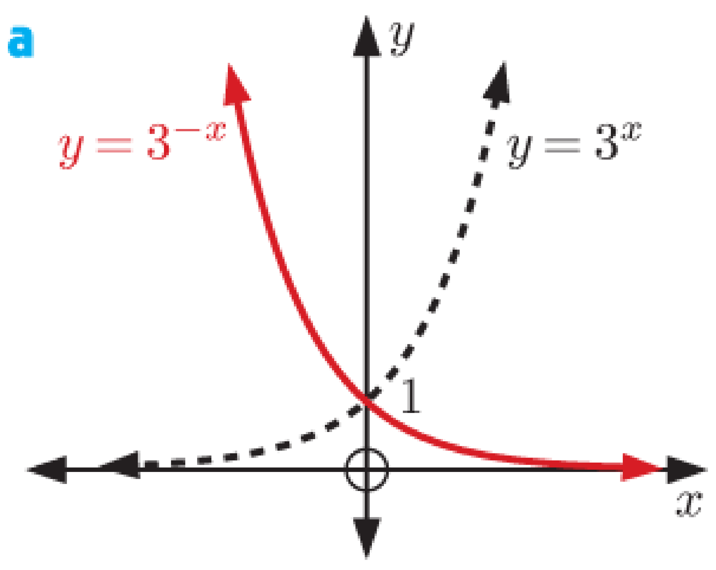
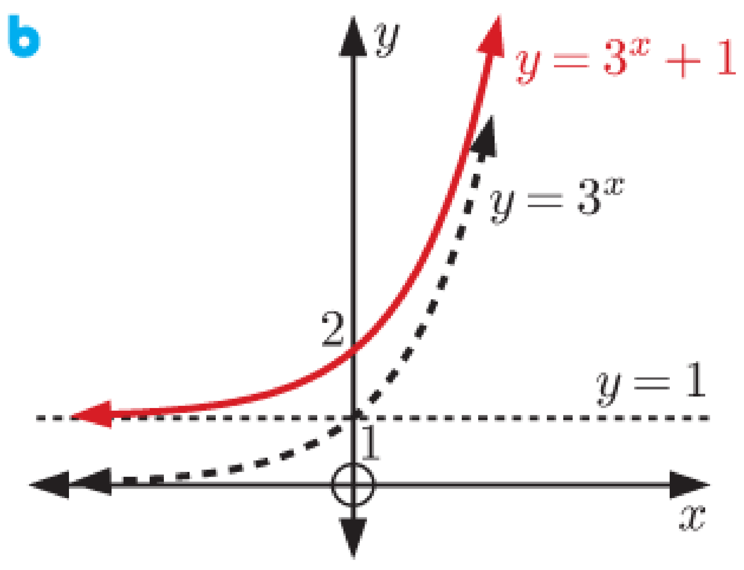
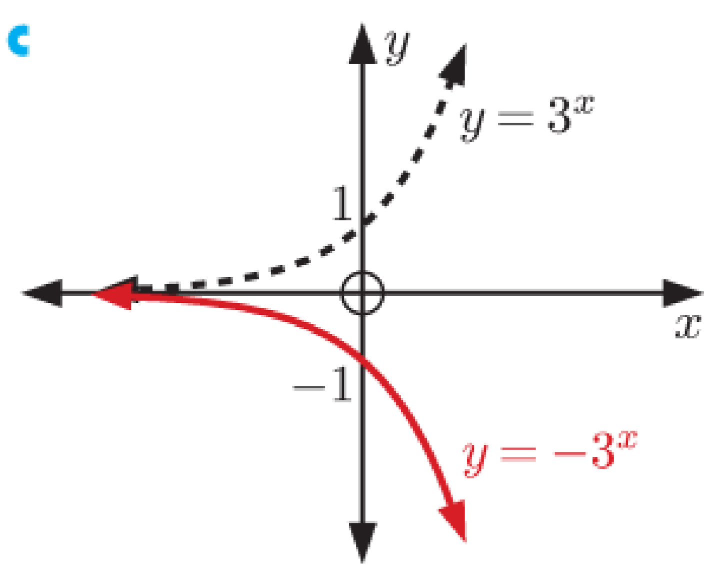
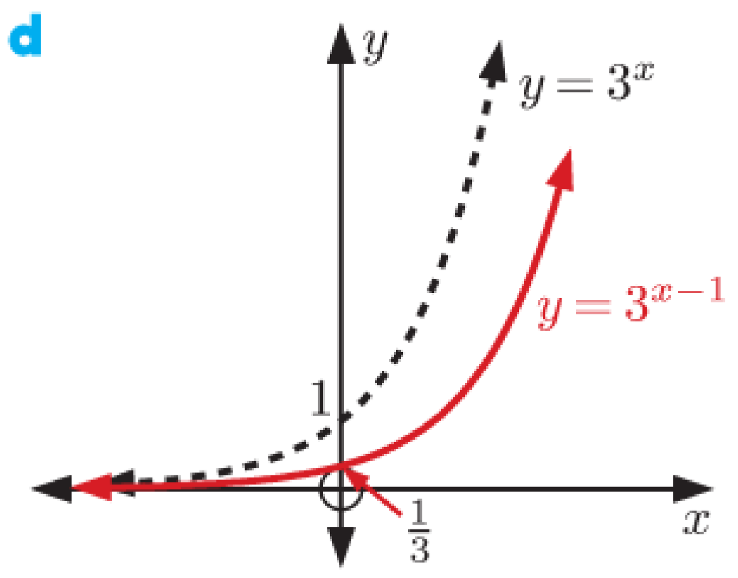


d


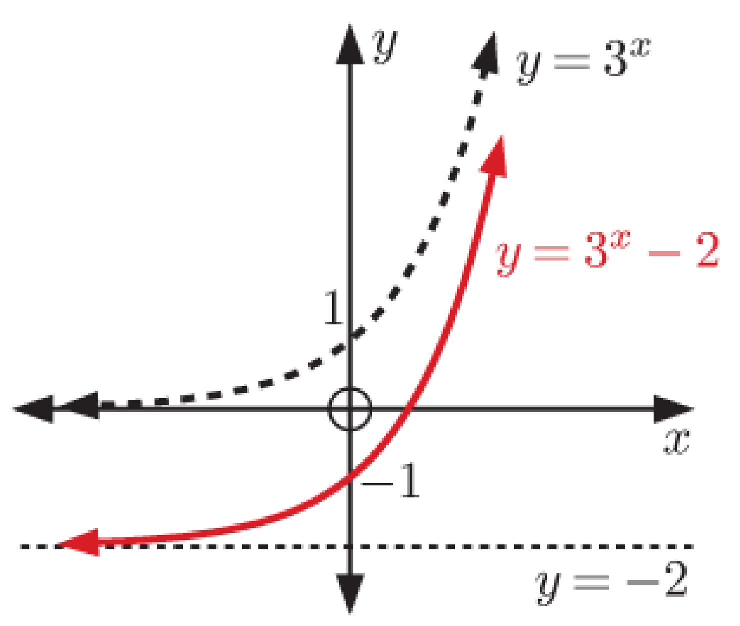
4 **a**


b


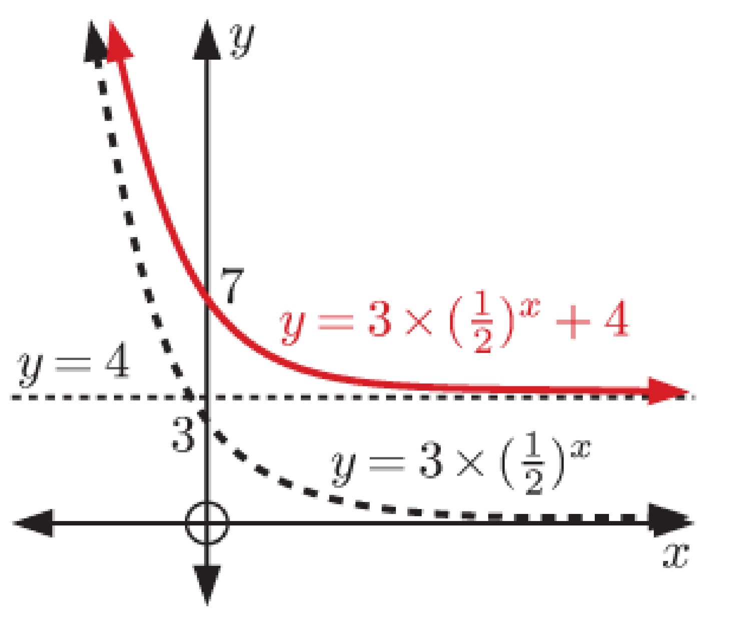
c


d


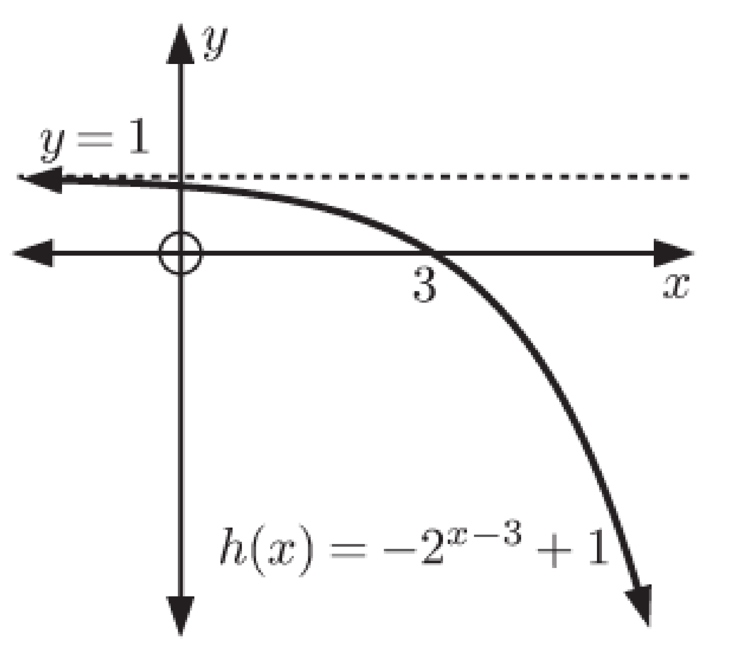
5 **a** $y=0$ **b** $y=-1$ **c** $y=3$ **d** $y=2$
e $y=0$ **f** $y=-4$

6 **a** **i** -1 **ii** 7
iii $-\frac{17}{9}=-1\frac{8}{9}$
b $y=-2$
c


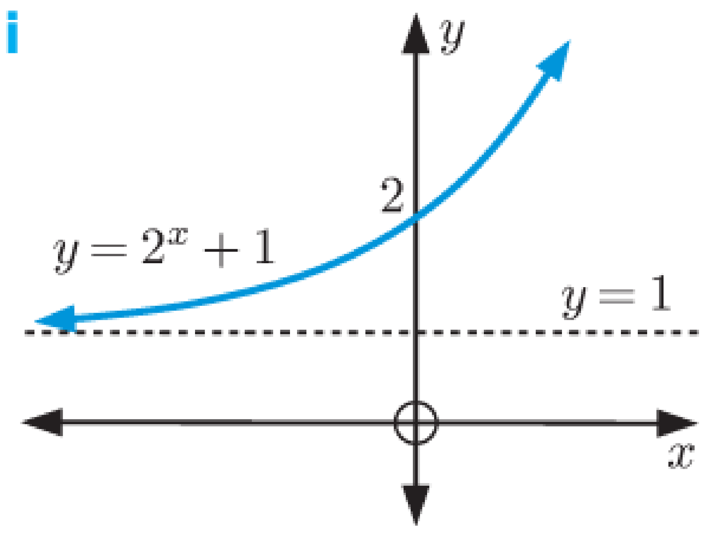
d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > -2\}$

7 **a** **i** 7
ii $\frac{19}{4}=4\frac{3}{4}$
iii 16
b $y=4$
c


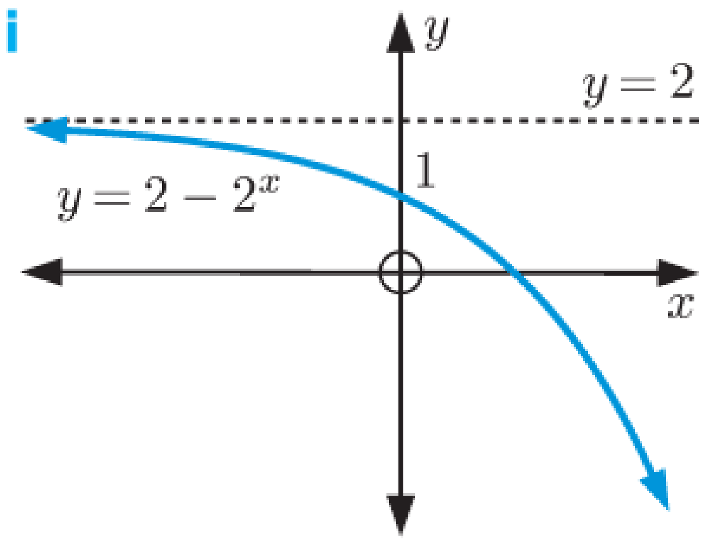
d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 4\}$

8 **a** **i** $\frac{7}{8}$ **ii** 0
iii -7
b $y=1$
c


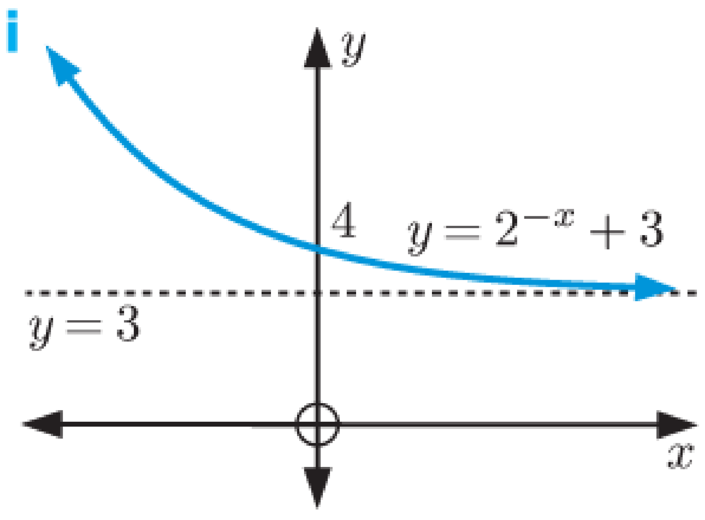
d Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y < 1\}$

9 **a** **i**


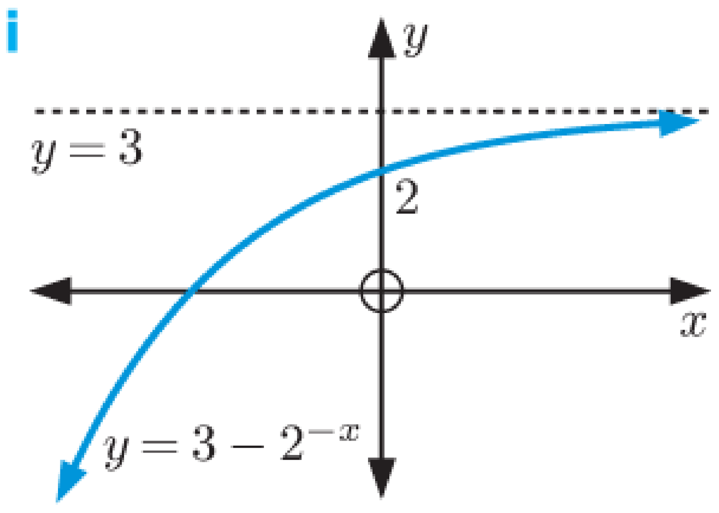
ii Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 1\}$
iii $y \approx 3.67$
iv as $x \to \infty,\ y \to \infty$
as $x \to -\infty,\ y \to 1^+$
v $y=1$

b **i**


ii Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y < 2\}$
iii $y \approx -0.665$
iv as $x \to \infty,\ y \to -\infty$
as $x \to -\infty,\ y \to 2^-$
v $y=2$

c **i**


ii Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 3\}$
iii $y \approx 3.38$
iv as $x \to \infty,\ y \to 3^+$
as $x \to -\infty,\ y \to \infty$
v $y=3$

d **i**


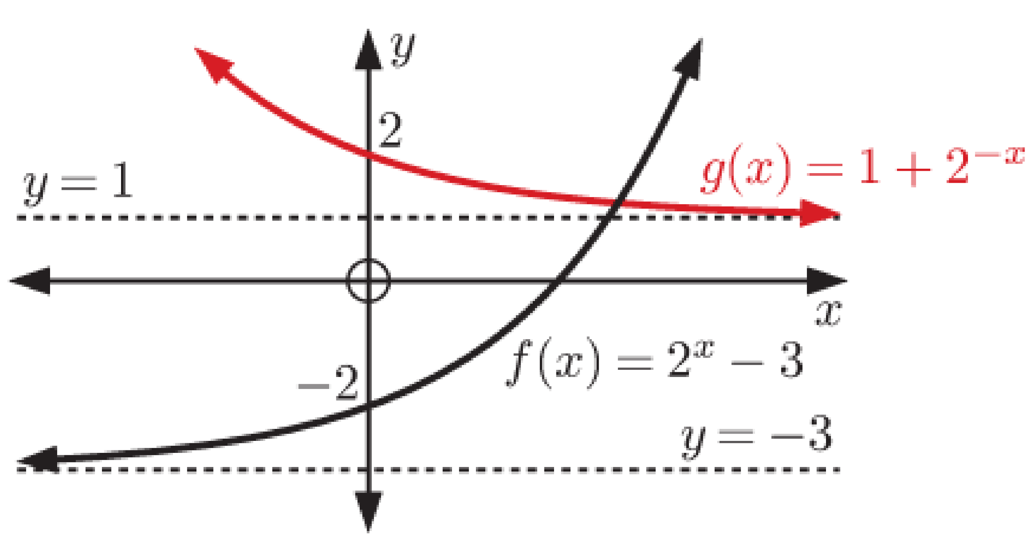
ii Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y < 3\}$
iii $y \approx 2.62$
iv as $x \to \infty,\ y \to 3^-$
as $x \to -\infty,\ y \to -\infty$
v $y=3$

10 **a** $a=5,\ b=-10$ **b** $y=310$

11 **a** $P(0,\ 2.5)$ **b** $a=1.5$ **c** $y=3.5$

12 **a** Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \geqslant 2\}$
b Domain is $\{x \mid x \neq 0\}$, Range is $\{y \mid y > 0,\ y < -1\}$
c Domain is $\{x \mid x \geqslant 1\}$, Range is $\{y \mid y \geqslant 0\}$

13 **a** $(f \circ g)(x) = 3^{\sqrt{x}} - 9$
Domain is $\{x \mid x \geqslant 0\}$, Range is $\{y \mid y \geqslant -8\}$
b $(g \circ f)(x) = \sqrt{3^x - 9}$
Domain is $\{x \mid x \geqslant 2\}$, Range is $\{y \mid y \geqslant 0\}$
c **i** $x=4$ **ii** $x=3$

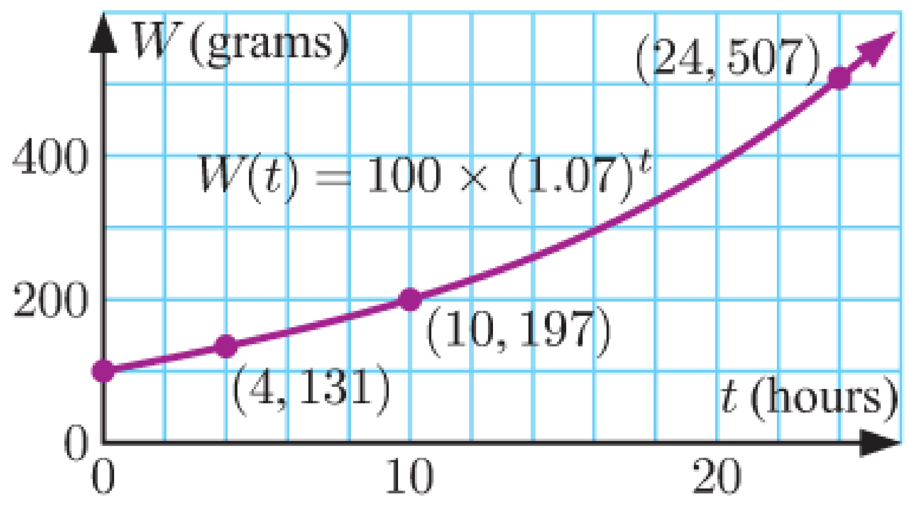
14 **a** **i** $f(x)$: $y=-3$, $g(x)$: $y=1$
ii $f(x)$: Range is $\{y \mid y > -3\}$
$g(x)$: Range is $\{y \mid y > 1\}$
iii $f(x)$: y-intercept -2, $g(x)$: y-intercept 2
b


c $-1+\sqrt{5}$

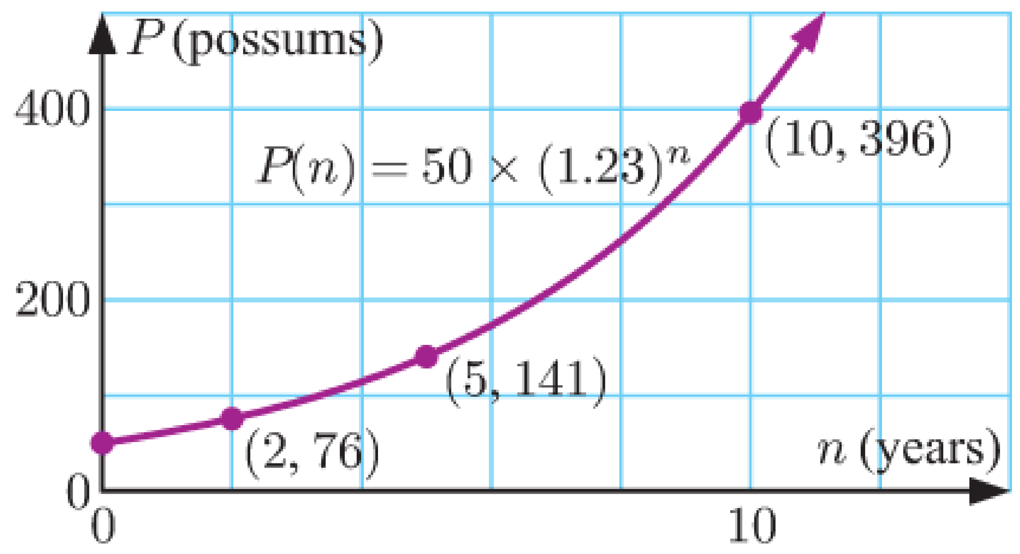
15 **a** $x \approx 3.46$ **b** $x \approx 2.46$ **c** $x \approx 1.16$
d $x \approx -0.738$ **e** $x \approx 1.85$ **f** $x \approx 0.0959$
g $x \approx 6.03$ **h** $x \approx 50.0$ **i** $x \approx 31.0$

EXERCISE 2E.1

1 **a** 100 grams
b **i** ≈ 131 g **ii** ≈ 197 g **iii** ≈ 507 g
c



2 **a** $P_0 = 50$
b **i** ≈ 76 possums **ii** ≈ 141 possums
iii ≈ 396 possums
c



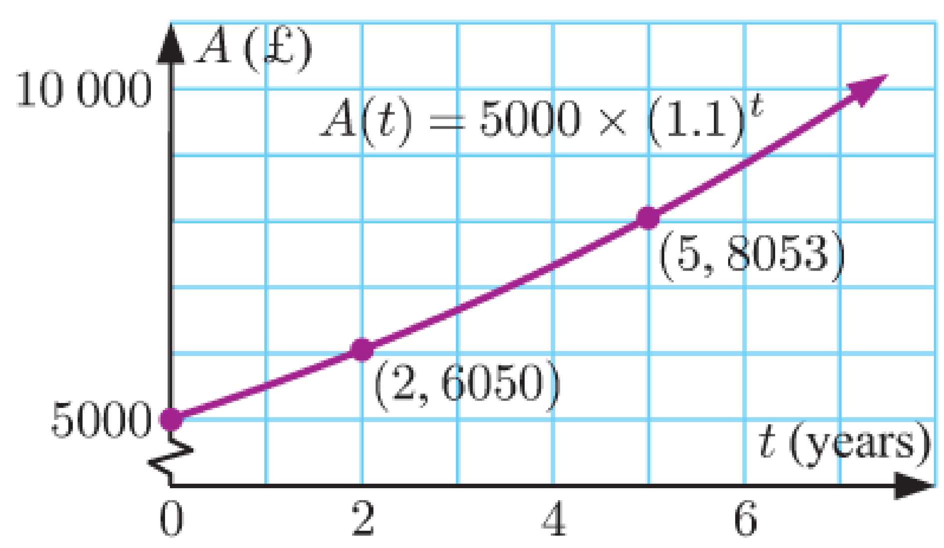
d ≈ 11 years **e** ≈ 11.1 years

3 **a** 4 people **b** ≈ 393 people **c** ≈ 19.9 days

4 **a** $B_0 = 200$
b $a = 1.1$, the bear population is increasing by 10% every year.
c ≈ 1350 bears **d** $\approx 159\%$ increase **e** ≈ 24.2 years

5 **a** **i** V_0 **ii** $2V_0$ **b** 100%
c $\approx 183\%$ increase, it is the percentage increase at 50°C compared with 20°C.

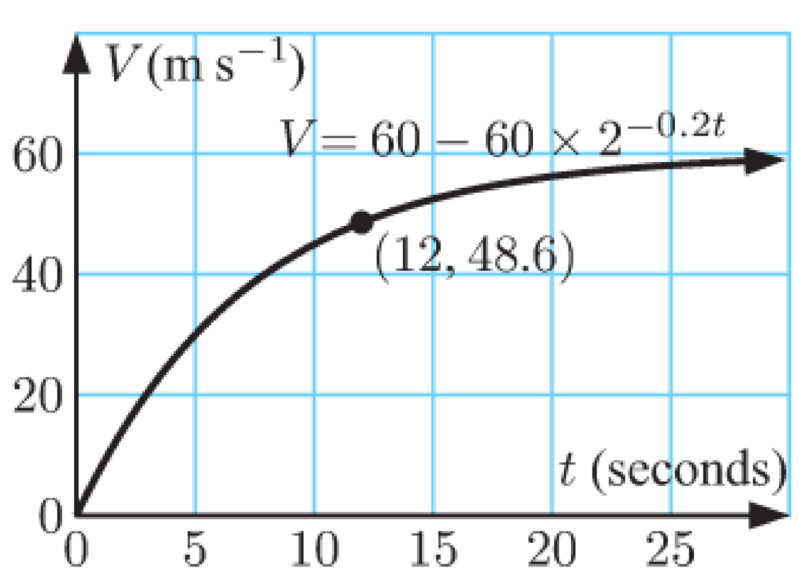
6 **a** $A(t) = 5000 \times (1.1)^t$ **b** **i** £6050 **ii** £8052.55
c



d ≈ 4.93 years

7 **a** $a = 1.08$, the expected value of the house is increasing by 8% per year.
$k = 375\,000$, the original value of the house was $375 000.
b ≈ 4.98 years

8 **a** When $t = 0$, $V = c - 60 = 0$
$\therefore \; c = 60$
b $k = -\frac{1}{5} = -0.2$
c $\approx 48.6 \text{ m s}^{-1}$ **d**

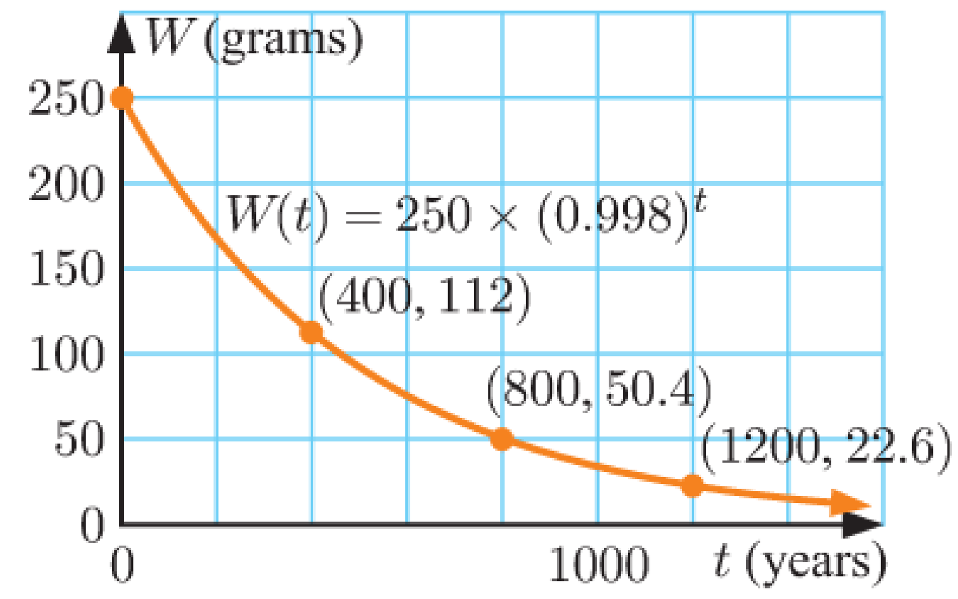


e The parachutist accelerates rapidly until he approaches his terminal velocity of 60 m s^{-1}.

9 ≈ 2.27 hours

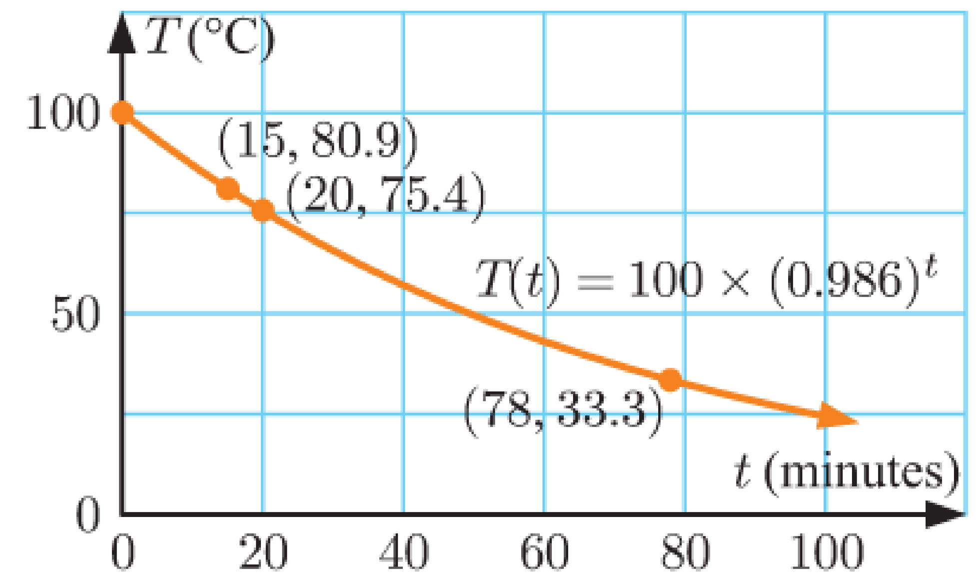
EXERCISE 2E.2

1 **a** 250 g **b** **i** ≈ 112 g **ii** ≈ 50.4 g **iii** ≈ 22.6 g
c

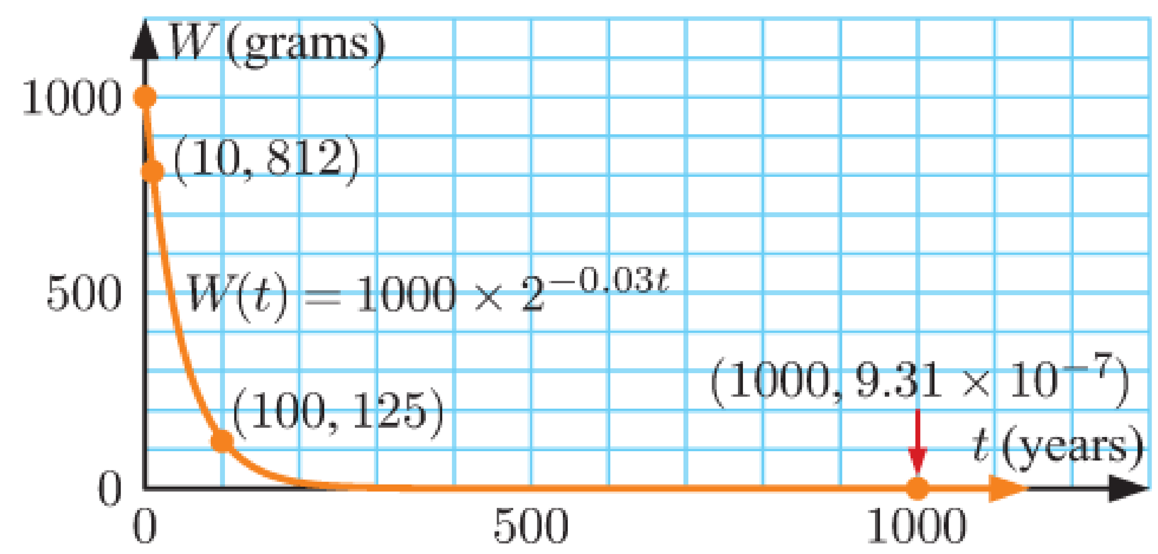


d ≈ 346 years

2 **a** 100°C
b **i** ≈ 80.9°C **ii** ≈ 75.4°C **iii** ≈ 33.3°C
c

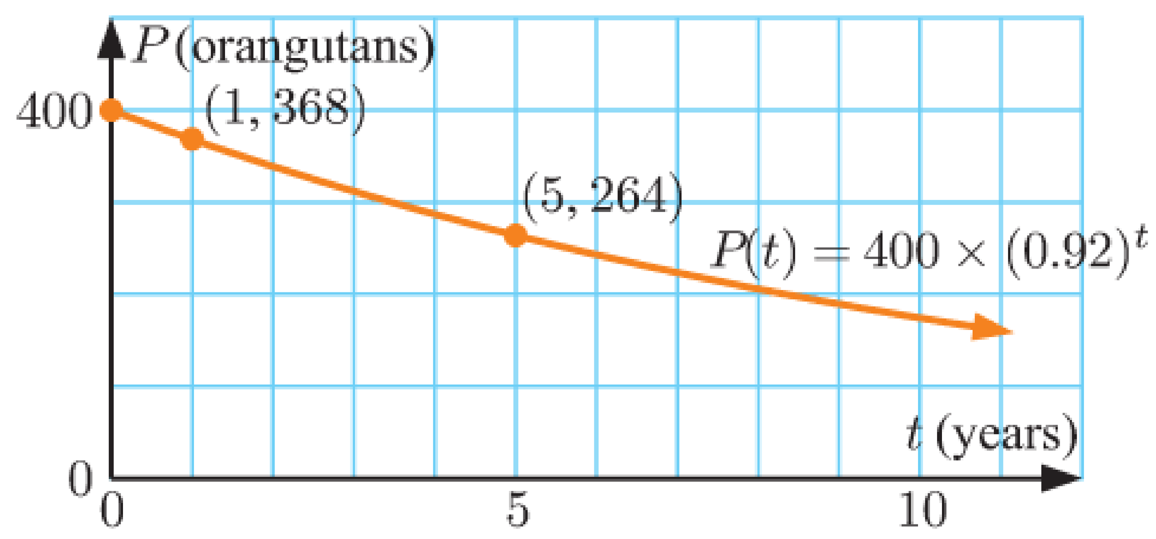


3 **a** 1000 g
b **i** ≈ 812 g **ii** 125 g **iii** $\approx 9.31 \times 10^{-7}$ g
c



d ≈ 221 years **e** $1000(1 - 2^{-0.03t})$ grams

4 **a** $P(t) = 400 \times (0.92)^t$
b **i** 368 orangutans **ii** ≈ 264 orangutans
c

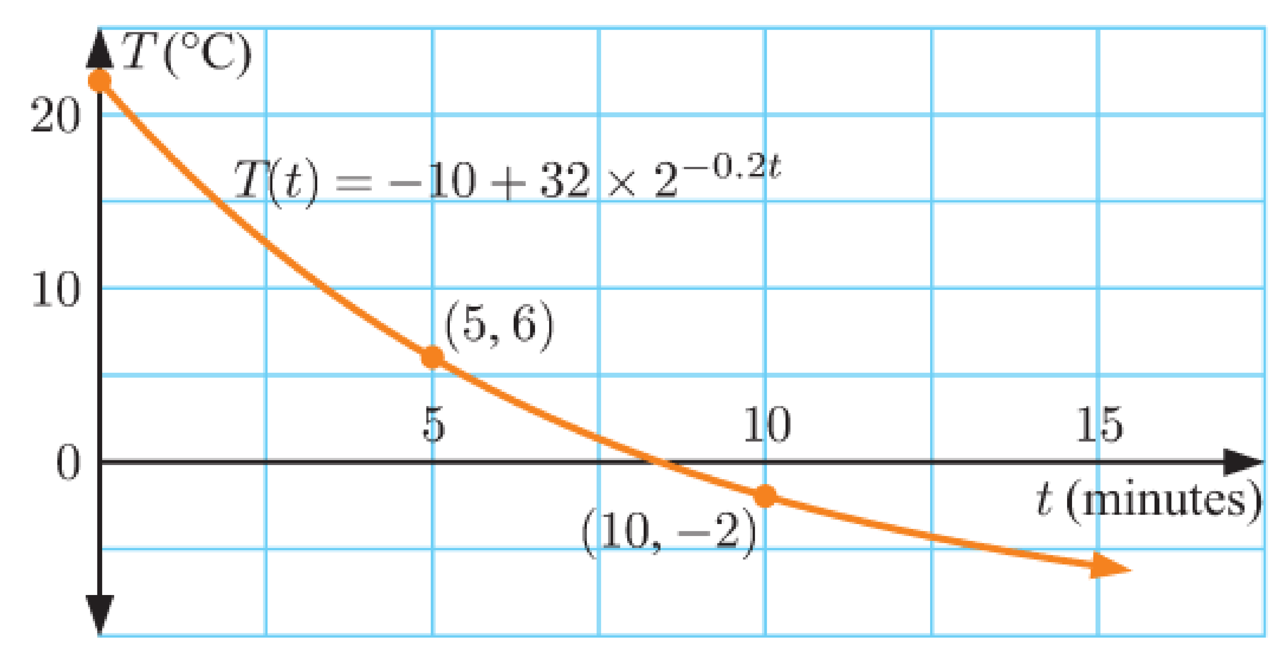


d ≈ 8.31 years, or ≈ 8 years 114 days

5 **a** $L_0 = 10$ units **b** ≈ 2.77 units **c** ≈ 17.9 m
d between ≈ 23.5 m and ≈ 44.9 m

6 **a** $24 000 **b** $r = 0.85$ **c** 7 years

7 **a** **i** 22°C **ii** 6°C **iii** −2°C
b



c ≈ 8.39 min or ≈ 8 min 23 s
d No, as $32 \times 2^{-0.2t} > 0$ for any value of t.

8 **a** W_0 **b** $\approx 12.9\%$ **c** 45 000 years

9 **a** $A(t) = 150 \times (1.48)^{\frac{t}{3}}$, $B(t) = 400 \times (0.8)^t + 100$
b **i** $t \approx 4.16$ years **ii** $t \approx 3.45$ years **iii** $t \approx 1.69$ years

10 **a** The initial weight of the isotope is 10 mg.

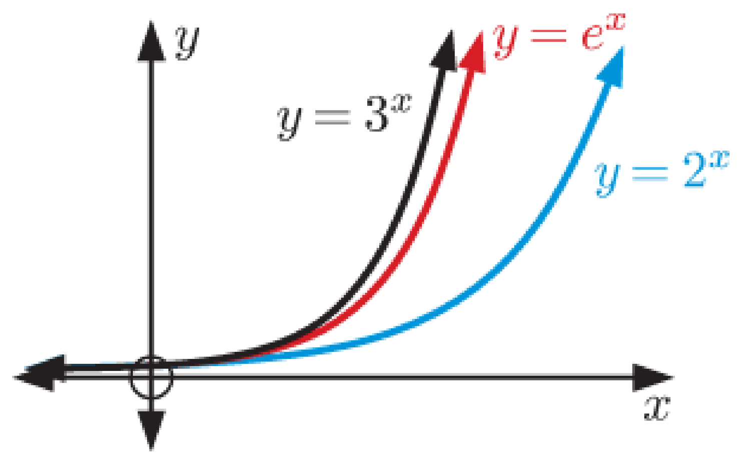
b $a \approx 0.7937$; each day the isotope's weight is decreasing by $\approx 20.63\%$.

c ≈ 6.30 mg **d** **i** ≈ 5.21 days **ii** ≈ 9.00 days

11 ≈ 33.2 minutes or ≈ 33 minutes 13.2 seconds

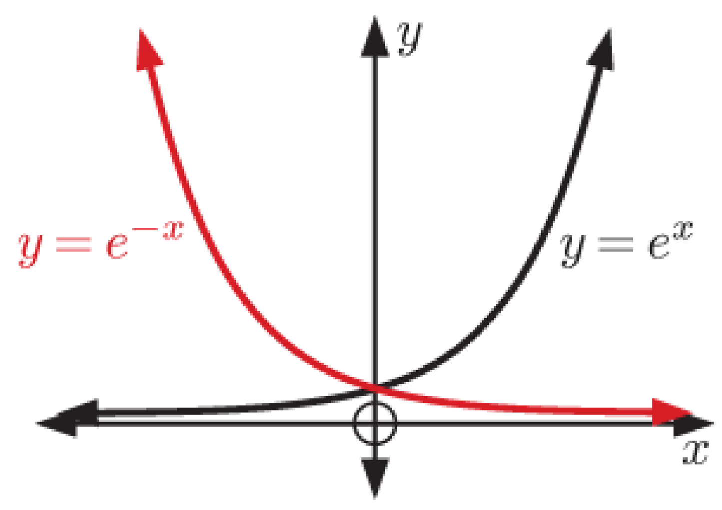
EXERCISE 2F

1



The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.

2



One is the other reflected in the y-axis.

3 p

4 **a** $e^x > 0$ for all x

b **i** $y \approx 4.12 \times 10^{-9}$ **ii** $y \approx 9.70 \times 10^{8}$

5 **a** ≈ 7.39 **b** ≈ 20.1 **c** ≈ 2.01 **d** ≈ 1.65

e ≈ 0.368

6 **a** $e^{\frac{1}{2}}$ **b** $e^{-\frac{1}{2}}$ **c** e^{-2} **d** $e^{\frac{3}{2}}$

7 **a** ≈ 10.074 **b** $\approx 0.099\,261$ **c** ≈ 125.09

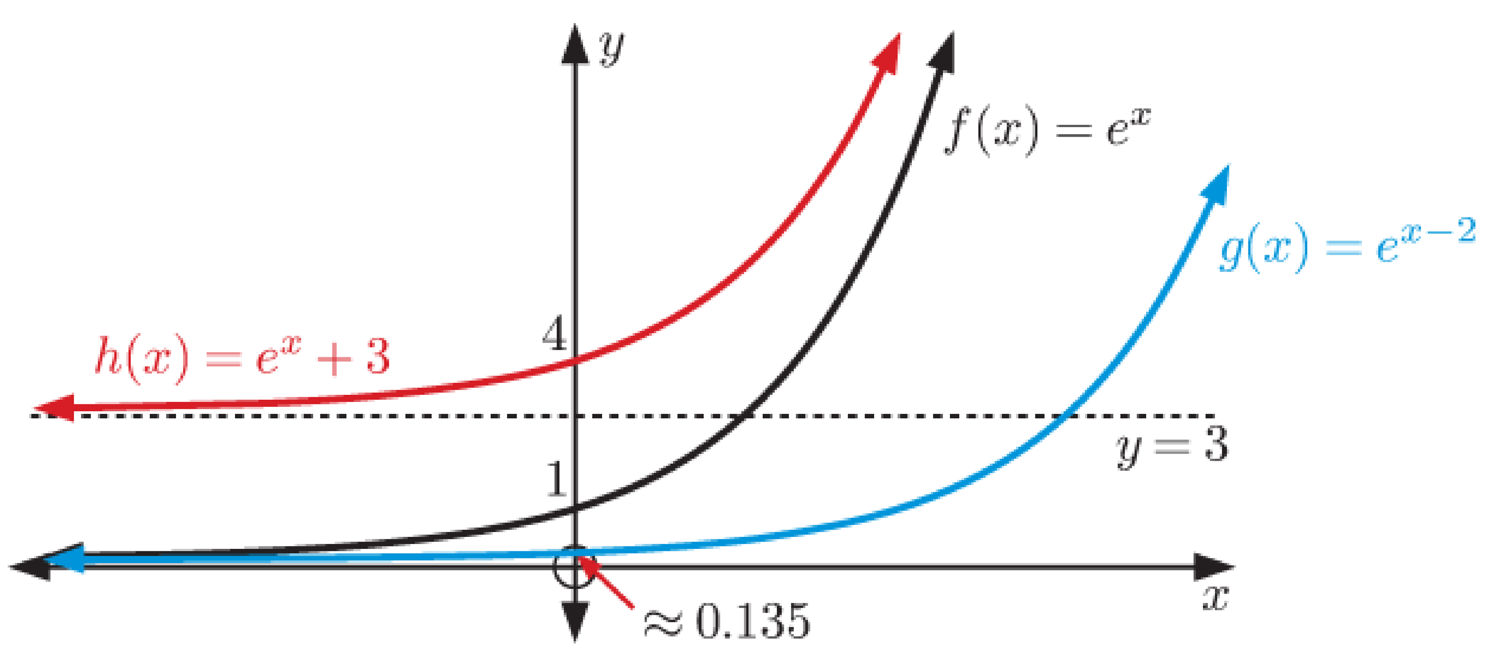
d $\approx 0.007\,994\,5$ **e** ≈ 41.914 **f** ≈ 42.429

g ≈ 3540.3 **h** $\approx 0.006\,342\,4$

8 **a** $e^{2x} + 2e^x + 1$ **b** $1 - e^{2x}$ **c** $1 - 3e^x$

9 **a** $e^x(e^x + 1)$ **b** $(e^x + 4)(e^x - 4)$ **c** $(e^x - 6)(e^x - 2)$

10 **a**

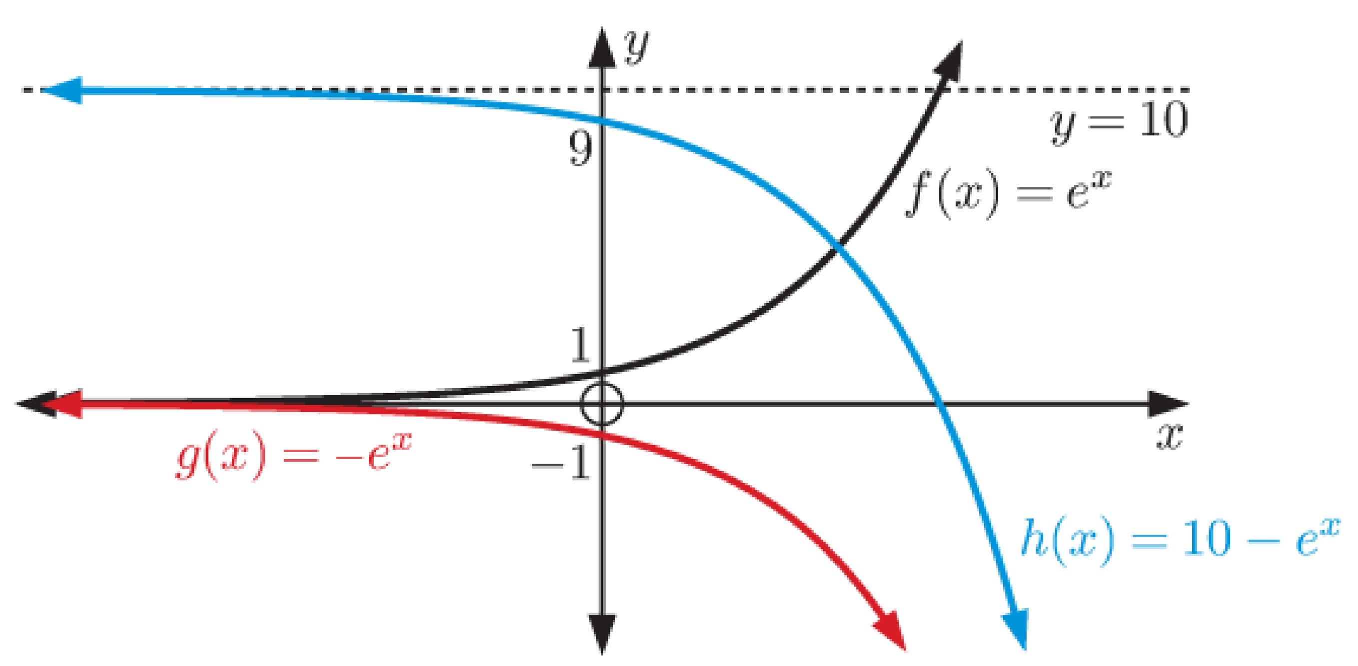


b Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$

Range of f is $\{y \mid y > 0\}$, Range of g is $\{y \mid y > 0\}$,

Range of h is $\{y \mid y > 3\}$

11 **a**



b Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$

Range of f is $\{y \mid y > 0\}$, Range of g is $\{y \mid y < 0\}$,

Range of h is $\{y \mid y < 10\}$

c For f: as $x \to \infty$, $y \to \infty$

as $x \to -\infty$, $y \to 0^+$

For g: as $x \to \infty$, $y \to -\infty$

as $x \to -\infty$, $y \to 0^-$

For h: as $x \to \infty$, $y \to -\infty$

as $x \to -\infty$, $y \to 10^-$

12 **a** $(f \circ g)(x) = e^{\frac{1}{x}} - 1$

Domain is $\{x \mid x \neq 0\}$

Range is $\{y \mid -1 < y < 0 \text{ or } y > 0\}$

b $(g \circ f)(x) = \dfrac{1}{e^x - 1}$

Domain is $\{x \mid x \neq 0\}$

Range is $\{y \mid y < -1 \text{ or } y > 0\}$

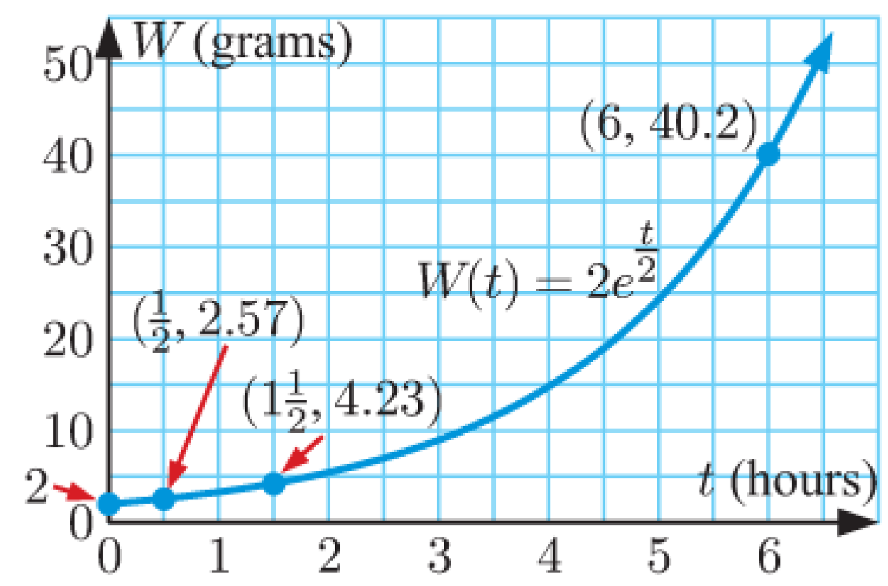
13 **a** **i** 2 g

ii ≈ 2.57 g

iii ≈ 4.23 g

iv ≈ 40.2 g

b

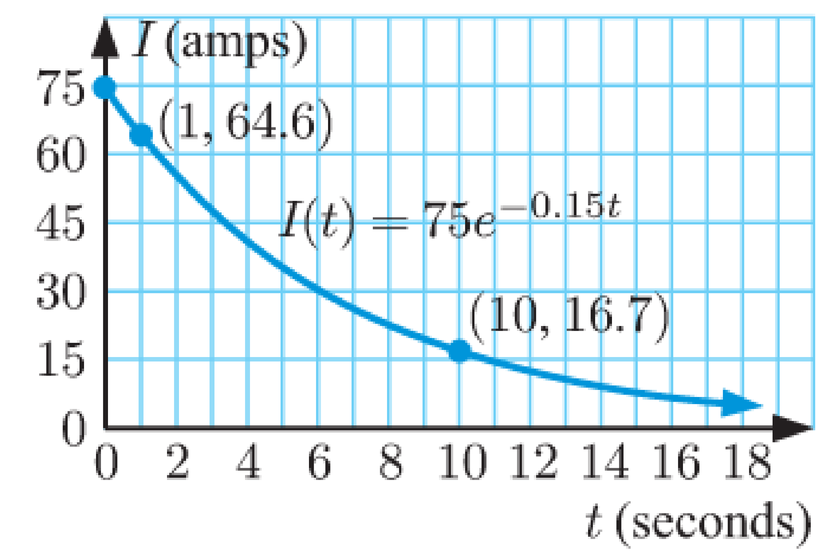


14 **a** $x = \frac{1}{2}$ **b** $x = -4$ **c** $x = 0$

15 **a** **i** ≈ 64.6 amps

ii ≈ 16.7 amps

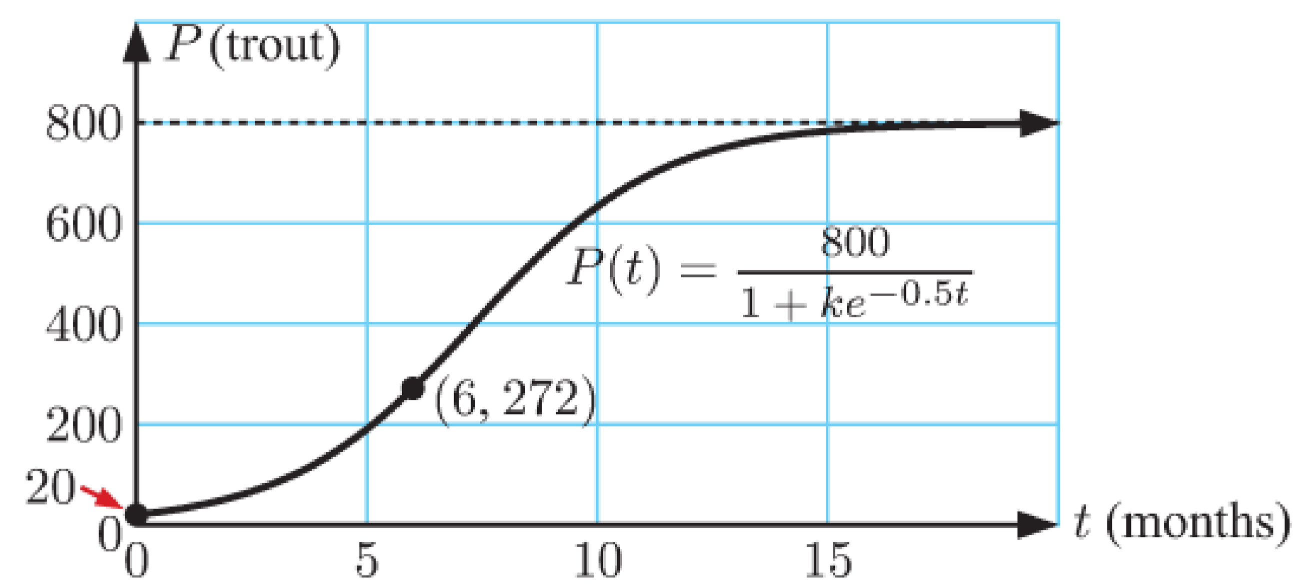
b



c ≈ 28.8 seconds

16 **a** $k = 39$ **b** ≈ 272 trout

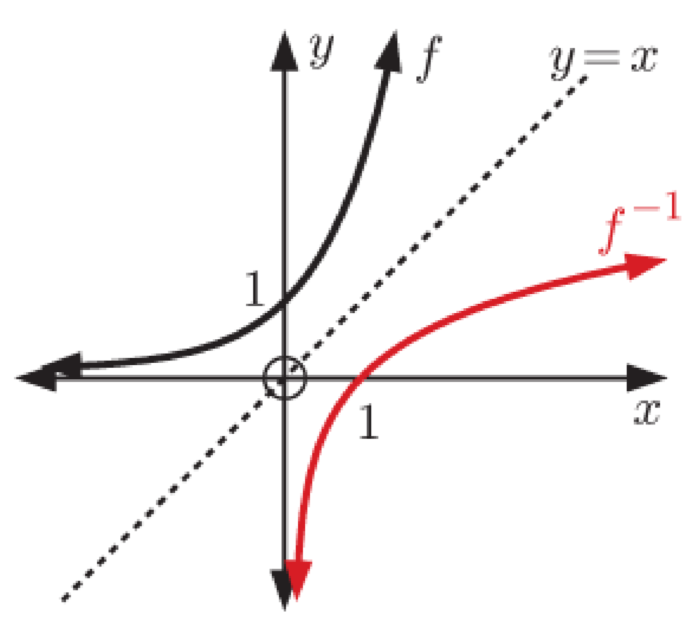
c



d As t increases, the population approaches a limiting value of 800 trout.

e ≈ 9.52 months

17 **a** $f^{-1}(x) = \log_e x$



b Domain of f^{-1} is $\{x \mid x > 0\}$

Range of f^{-1} is $\{y \mid y \in \mathbb{R}\}$

18 $e^1 \approx \displaystyle\sum_{i=0}^{19} \frac{1}{k!} 1^k \approx 2.718\,281\,828$

REVIEW SET 2A

1 **a** 4 **b** $\frac{1}{9}$ **c** $\frac{1}{3}$

2 **a** $x = -2$ **b** $x = \frac{3}{4}$ **c** $x = -\frac{1}{4}$

3 **a** $1 + e^{2x}$ **b** $2^{2x} + 10(2^x) + 25$ **c** $x - 49$

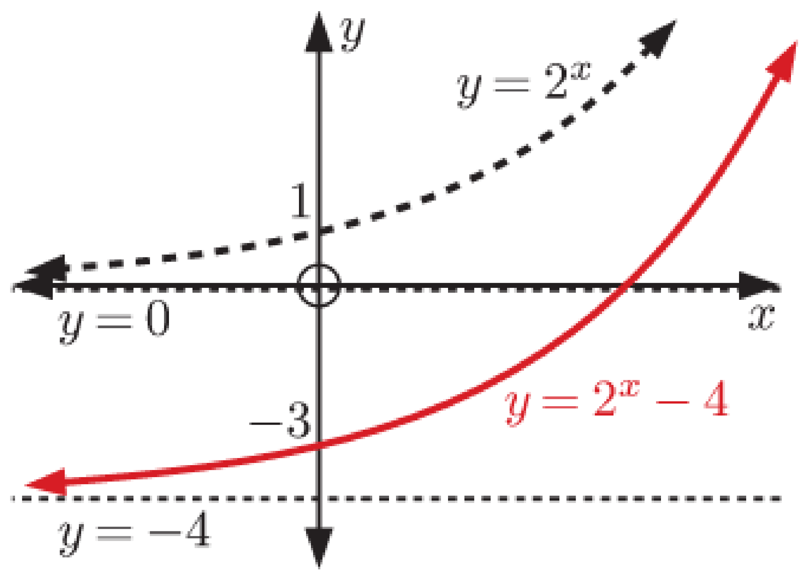
4 **a** $x = 5$ **b** $x = -\frac{5}{2}$ **c** $x = 3$ **5** $k = \frac{3}{2}$

6 **a** **i** ≈ 2.2 **ii** ≈ 0.6

b **i** $x \approx 1.45$ **ii** $x \approx -0.6$ **iii** $x \approx 1.1$

7 **a** 3 **b** 24 **c** $\frac{3}{4}$

8

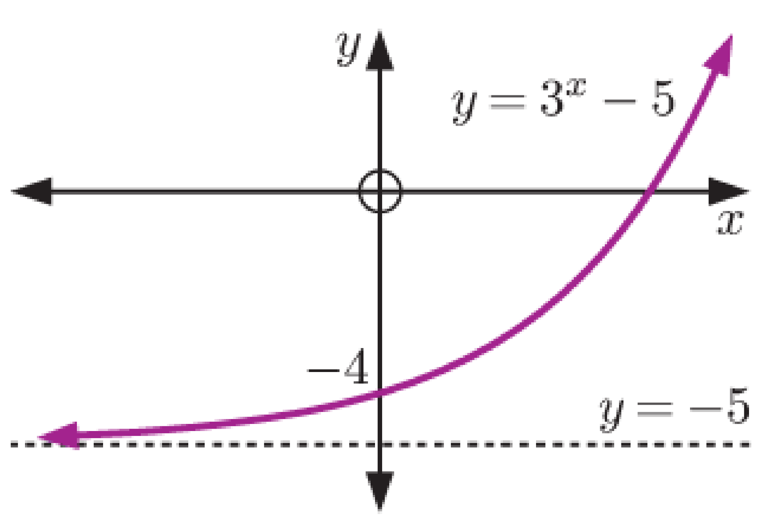


9 **a**

x	-2	-1	0	1	2
y	$-4\frac{8}{9}$	$-4\frac{2}{3}$	-4	-2	4

b as $x \to \infty$, $y \to \infty$
as $x \to -\infty$, $y \to -5^+$

c



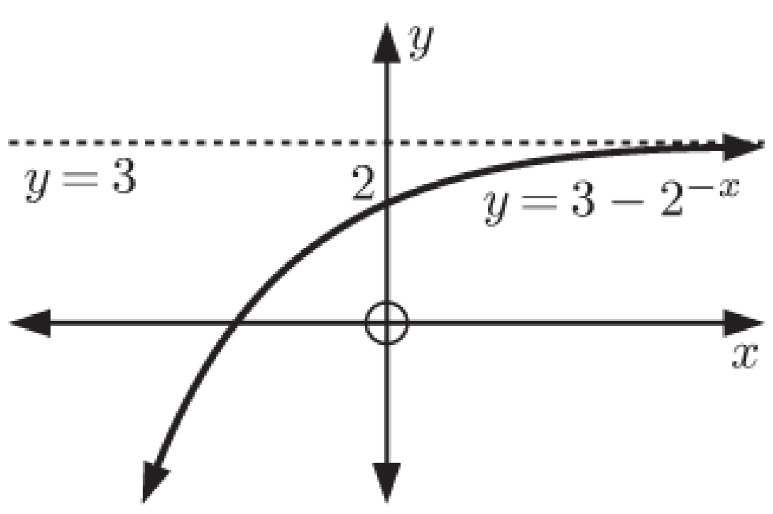
d $y = -5$

10 **a**

x	-2	-1	0	1	2
y	-1	1	2	$2\frac{1}{2}$	$2\frac{3}{4}$

b as $x \to \infty$, $y \to 3^-$
as $x \to -\infty$, $y \to -\infty$

c



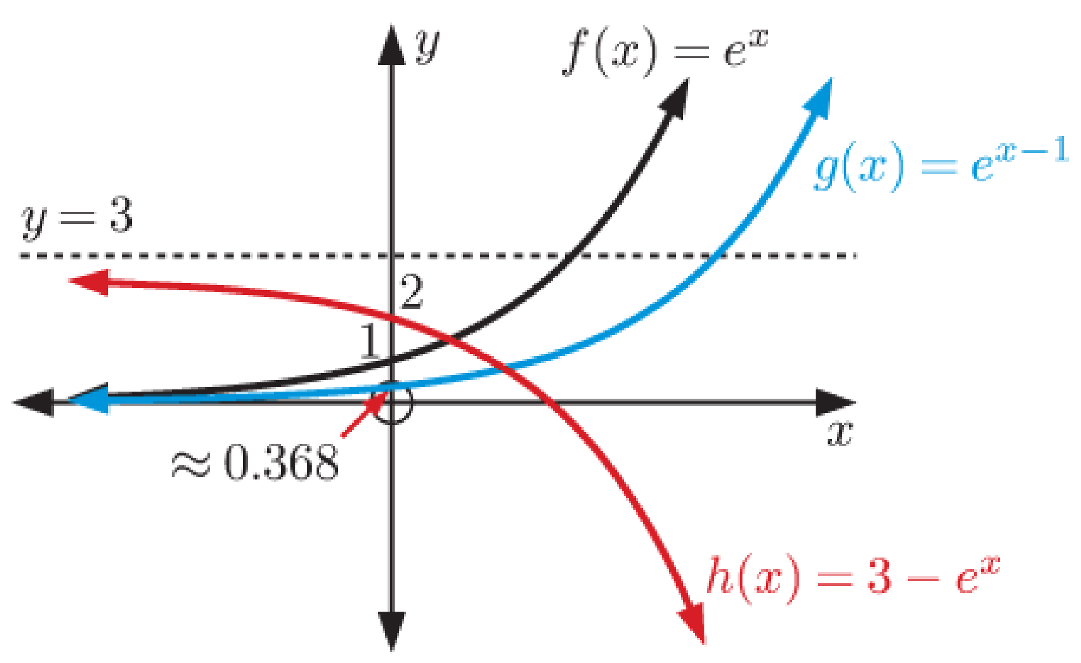
d $y = 3$

11 **a** $(f \circ g)(x) = 2^{3-x^2}$
Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid 0 < y \leqslant 8\}$

b $(g \circ f)(x) = 3 - 2^{2x} = 3 - 4^x$
Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < 3\}$

c **i** $x = \pm\sqrt{2}$ **ii** $x = 2$

12 **a**

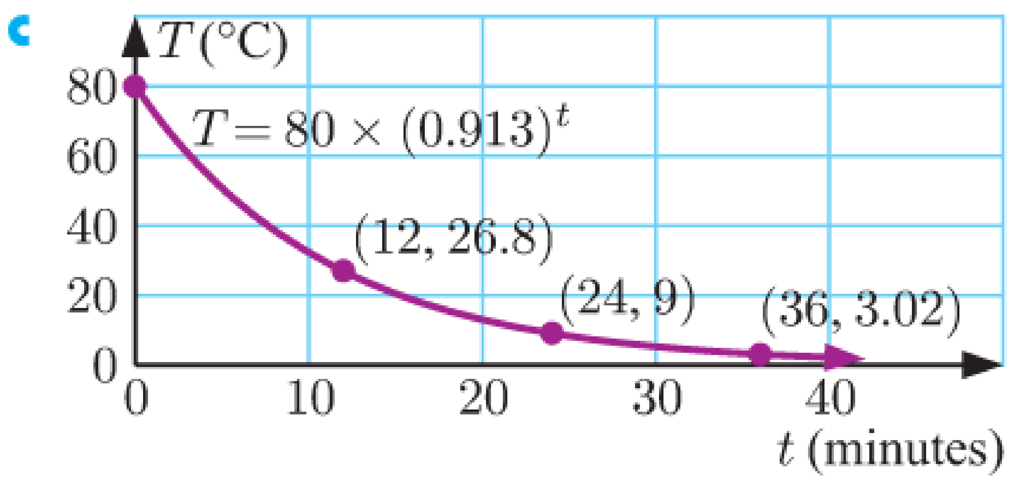


b For $f(x)$: domain is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 0\}$
For $g(x)$: domain is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 0\}$
For $h(x)$: domain is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y < 3\}$

c For $f(x)$: as $x \to \infty$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to 0^+$
For $g(x)$: as $x \to \infty$, $g(x) \to \infty$
as $x \to -\infty$, $g(x) \to 0^+$
For $h(x)$: as $x \to \infty$, $h(x) \to -\infty$
as $x \to -\infty$, $h(x) \to 3^-$

13 about every ≈ 7.92 days

14 **a** 80°C
b **i** ≈ 26.8°C
ii ≈ 9.00°C
iii ≈ 3.02°C
c



d ≈ 12.8 min

REVIEW SET 2B

1 **a** ≈ 3.95 **b** ≈ 0.517 **c** ≈ 3.16

2 **a** $9 - 6e^x + e^{2x}$ **b** $x - 2 - x^{-1}$ **c** $2^x + 1$

3 **a** $8(3^x)$ **b** $(2^x - 4)(2^x + 3)$ **c** $(e^x + 5)(e^x - 3)$

4 **a** $x = -4$ **b** $x = 0$ or 2 **c** $x = -1$ or -2

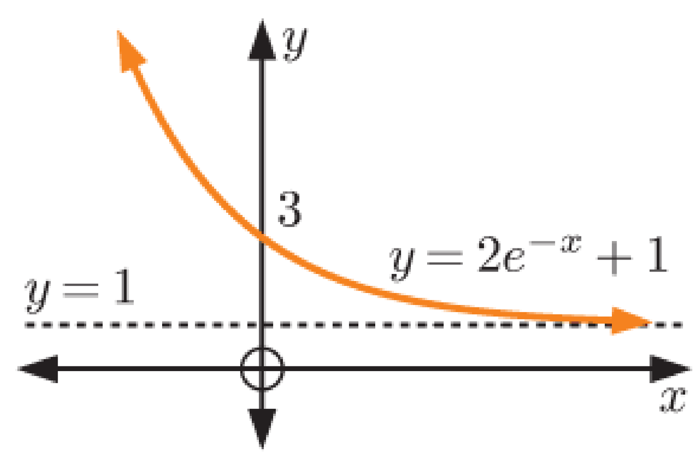
5 **a** $\frac{1}{\sqrt{2}} + 1 \approx 1.71$ **b** $a = -1$

6 **a**

x	-2	-1	0	1	2
y	15.8	6.44	3	1.74	1.27

b as $x \to \infty$, $y \to 1^+$
as $x \to -\infty$, $y \to \infty$

c



d $y = 1$

7 **a** clock: £525, vase: £428
b clock: $V(t) = 500 \times (1.05)^t$
vase: $V(t) = 400 \times (1.07)^t$
c clock $\approx$ £1039.46, vase $\approx$ £1103.61 $\therefore$ the vase
d $500 \times (1.05)^t = 400 \times (1.07)^t$ and solve for t;
$t \approx 11.8$ years

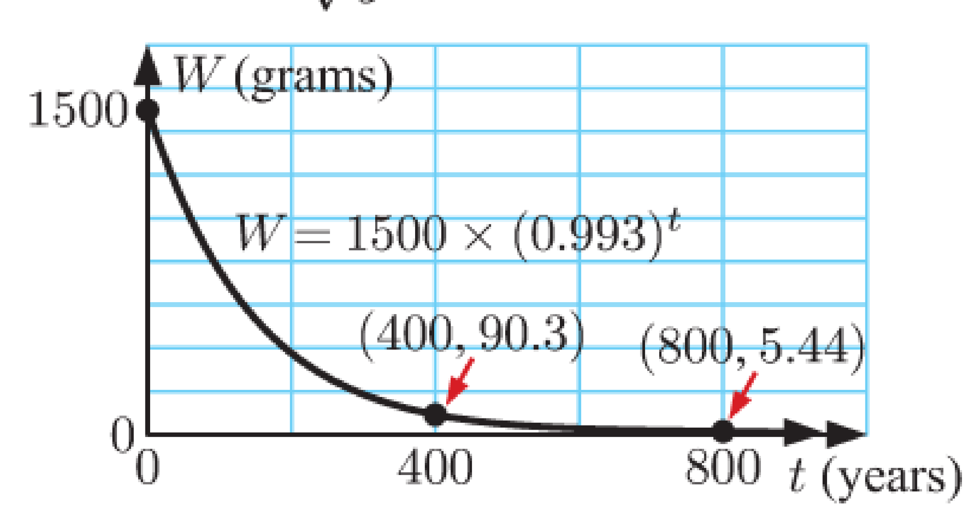
8 Domain is $\{x \mid x \geqslant -1\}$, Range is $\{y \mid y \geqslant 1\}$

9 **a** $f(x)$: y-intercept 2, $g(x)$: y-intercept -1
b $-\frac{5}{2} + \frac{1}{2}\sqrt{5}$ units

10 **a** **i** 81 **ii** $\frac{1}{3}$ **b** $k = 9$

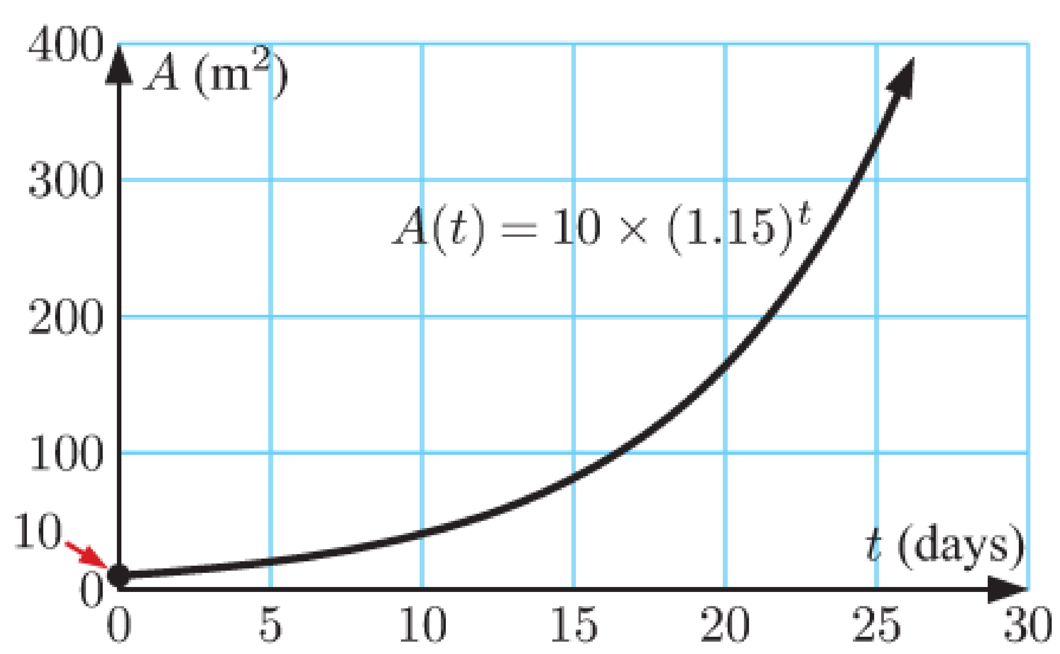
11 **a** y^2 **b** y^{-1} **c** $\frac{1}{\sqrt{y}}$ or $y^{-\frac{1}{2}}$

12 **a** 1500 g
b **i** ≈ 90.3 g
ii ≈ 5.44 g
c



d ≈ 386 years

13 **a** $A(t) = 10 \times (1.15)^t$
b **i** 13.225 m^2 **ii** ≈ 20.1 m^2
c



d ≈ 24.3 days

EXERCISE 3A

1 **a** 4 **b** -3 **c** 1 **d** 0 **e** $\frac{1}{2}$ **f** $\frac{1}{3}$
g $-\frac{1}{4}$ **h** $1\frac{1}{2}$ **i** $\frac{2}{3}$ **j** $1\frac{1}{2}$ **k** $1\frac{1}{3}$ **l** $3\frac{1}{2}$

2 **a** n **b** $a+2$ **c** $1-m$ **d** $a-b$

3 **a** $100 < 237 < 1000$
$\therefore \log 100 < \log 237 < \log 1000$
$\therefore 2 < \log 237 < 3$
b ≈ 2.37

4 **a** $-1 < \log(0.6) < 0$ **b** ≈ -0.22

5 **a** ≈ 1.88 **b** ≈ 2.06 **c** ≈ 0.48 **d** ≈ 2.92
e ≈ -0.40 **f** ≈ 3.51 **g** ≈ -2.10 **h** does not exist

6 **a** $x > 1$ **b** $x = 1$ **c** $0 < x < 1$ **d** $x \leqslant 0$

7 **a** $\approx 10^{0.7782}$ **b** $\approx 10^{1.7782}$ **c** $\approx 10^{3.7782}$
d $\approx 10^{-0.2218}$ **e** $\approx 10^{-2.2218}$ **f** $\approx 10^{1.1761}$
g $\approx 10^{3.1761}$ **h** $\approx 10^{0.1761}$ **i** $\approx 10^{-0.8239}$
j $\approx 10^{-3.8239}$

8 **a** **i** ≈ 0.477 **ii** ≈ 2.477
b $\log 300 = \log(3 \times 10^2) = \log(10^{\log 3} \times 10^2) = ...$

9 **a** **i** ≈ 0.699 **ii** ≈ -1.301
b $\log(0.05) = \log(5 \times 10^{-2}) = \log(10^{\log 5} \times 10^{-2}) = ...$

10 **a** $x = 100$ **b** $x = 10$ **c** $x = 1$
d $x = \frac{1}{10}$ **e** $x = \sqrt{10}$ **f** $x = \frac{1}{\sqrt{10}}$
g $x = 10\,000$ **h** $x = 0.000\,01$ **i** $x \approx 6.84$
j $x \approx 140$ **k** $x \approx 0.0419$ **l** $x \approx 0.000\,631$

EXERCISE 3B

1 **a** $10^2 = 100$ **b** $10^4 = 10\,000$ **c** $10^{-1} = 0.1$
d $10^{\frac{1}{2}} = \sqrt{10}$ **e** $2^3 = 8$ **f** $3^2 = 9$
g $2^{-2} = \frac{1}{4}$ **h** $3^{1.5} = \sqrt{27}$ **i** $5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$

2 **a** $\log_4 64 = 3$ **b** $\log_5 25 = 2$
c $\log_7 49 = 2$ **d** $\log_2 64 = 6$
e $\log_2\left(\frac{1}{8}\right) = -3$ **f** $\log_{10}(0.01) = -2$
g $\log_2\left(\frac{1}{2}\right) = -1$ **h** $\log_3\left(\frac{1}{27}\right) = -3$

3 **a** 5 **b** -2 **c** $\frac{1}{2}$ **d** 2 **e** 6 **f** 7
g 2 **h** 3 **i** -3 **j** $\frac{1}{2}$ **k** 2 **l** $\frac{1}{2}$
m 5 **n** $\frac{1}{3}$ **o** $\frac{1}{3}$ **p** $\frac{3}{2}$ **q** 0 **r** 1
s -1 **t** $\frac{3}{4}$ **u** $-\frac{1}{2}$ **v** $\frac{5}{2}$ **w** $-\frac{3}{2}$ **x** $-\frac{3}{4}$

4 **a** 2 **b** -1 **c** $\frac{1}{2}$ **d** 3 **e** $\frac{1}{4}$ **f** $\frac{3}{2}$
g -2 **h** $-\frac{1}{2}$ **i** $\frac{5}{2}$

5 **a** $x = 8$ **b** $x = 2$ **c** $x = \frac{1}{125}$ **d** $x = 3$
e $x = 14$ **f** $x = \sqrt{3}$

6 $\log_b a = \dfrac{1}{x}$ **7** $x = \dfrac{2^{2y}+1}{5}$

EXERCISE 3C

1 **a** $\log 16$ **b** $\log 20$ **c** $\log 8$ **d** $\log\left(\dfrac{p}{m}\right)$
e 1 **f** $\log 2$ **g** 3 **h** 2
i $\log 24$ **j** 1 **k** 0 **l** $\log 28$

2 **a** $\log 700$ **b** $\log\left(\frac{2}{5}\right)$ **c** $\log_2 6$
d $\log_3\left(\frac{5}{9}\right)$ **e** $\log 200$ **f** $\log(0.005)$
g $\log(10^t \times w)$ **h** $\log_m\left(\dfrac{40}{m^2}\right)$ **i** $\log_5\left(\frac{5}{2}\right)$

3 **a** $\log 96$ **b** $\log 72$ **c** $\log 8$ **d** $\log_3\left(\frac{25}{8}\right)$
e 1 **f** $\log\left(\frac{1}{2}\right)$ **g** $\log 20$ **h** $\log 25$ **i** $\log_n\left(\dfrac{n^2}{10}\right)$

4 **a** 2 **b** $\frac{3}{2}$ **c** 3 **d** $\frac{1}{2}$ **e** -2 **f** $-\frac{3}{2}$

5 For example, for **a**, $\log 9 = \log(3^2) = 2\log 3$.

7 **a** $p+q$ **b** $2q+r$ **c** $2p+3q$ **d** $r+\frac{1}{2}q-p$
e $r-5p$ **f** $p-2q$

8 **a** $x+z$ **b** $z+2y$ **c** $x+z-y$ **d** $2x+\frac{1}{2}y$
e $3y-\frac{1}{2}z$ **f** $2z+\frac{1}{2}y-3x$

9 **a** 0.86 **b** 2.15 **c** 1.075 **10** $x = \dfrac{2}{a^2-1}$

11 **a** $\log 384$ **b** $4+\log_2 45$ **12** $\log\left(\dfrac{x^{40}}{y^{45}}\right)$

13 $\log\sqrt[3]{3}$ **14** **Hint:** Subtract $2xy$ from both sides.

EXERCISE 3D

1 **a** 2 **b** 4 **c** $\frac{3}{2}$ **d** 0 **e** -1
f $\frac{1}{3}$ **g** -2 **h** $-\frac{1}{2}$

2 **a** 3 **b** 9 **c** $\frac{1}{5}$ **d** $\frac{1}{4}$ **e** a
f $1+a$ **g** $a+b$ **h** ab

3 **a** ≈ 2.485 **b** ≈ 4.220 **c** ≈ 0.336
d ≈ -0.357 **e** ≈ 6.215

4 x does not exist such that $e^x = -2$ or 0 since $e^x > 0$ for all $x \in \mathbb{R}$.

5 **a** $\approx e^{1.7918}$ **b** $\approx e^{4.0943}$ **c** $\approx e^{8.6995}$
d $\approx e^{-0.5108}$ **e** $\approx e^{-5.1160}$ **f** $\approx e^{2.7081}$
g $\approx e^{7.3132}$ **h** $\approx e^{0.4055}$ **i** $\approx e^{-1.8971}$
j $\approx e^{-8.8049}$

6 **a** $x \approx 20.1$ **b** $x \approx 2.72$ **c** $x = 1$
d $x \approx 0.368$ **e** $x \approx 0.006\,74$ **f** $x \approx 2.30$
g $x \approx 8.54$ **h** $x \approx 0.0370$

7 **a** **i** x **ii** x **b** They are inverses of each other.

8 **a** $\ln 45$ **b** $\ln 5$ **c** $\ln 4$ **d** $\ln 24$
e $\ln 1 = 0$ **f** $\ln 30$ **g** $\ln(4e)$ **h** $\ln\left(\dfrac{6}{e}\right)$
i $\ln 20$ **j** $\ln(4e^2)$ **k** $\ln\left(\dfrac{20}{e^2}\right)$ **l** $\ln 1 = 0$

9 **a** $\ln 972$ **b** $\ln 200$ **c** $\ln 1 = 0$ **d** $\ln 16$
e $\ln 6$ **f** $\ln\left(\frac{1}{3}\right)$ **g** $\ln\left(\frac{1}{2}\right)$ **h** $\ln 2$
i $\ln 16$ **j** $\ln(16e^2)$ **k** $\ln\left(\dfrac{3}{e}\right)$ **l** $\ln\left(\dfrac{\sqrt{e}}{8}\right)$

10 For example, for **a**, $\ln 27 = \ln(3^3) = 3\ln 3$.

11 $x = e^4$, $y = \dfrac{1}{e^2}$

EXERCISE 3E

1 **a** $\log y = x\log 2$ **b** $\log y = \log 20 + 3\log b$
c $\log M = \log a + 4\log d$ **d** $\log T = \log 5 + \frac{1}{2}\log d$
e $\log R = \log b + \frac{1}{2}\log l$ **f** $\log Q = \log a - n\log b$
g $\log y = \log a + x\log b$ **h** $\log F = \log 20 - \frac{1}{2}\log n$
i $\log L = \log a + \log b - \log c$
j $\log N = \frac{1}{2}\log a - \frac{1}{2}\log b$
k $\log S = \log 200 + t\log 2$ **l** $\log y = m\log a - n\log b$

2 **a** $D = 2e$ **b** $F = \frac{5}{t}$ **c** $P = \sqrt{x}$ **d** $M = b^2c$

e $B = \frac{m^3}{n^2}$ **f** $N = \frac{1}{\sqrt[3]{p}}$ **g** $P = 10x^3$ **h** $Q = \frac{a^2}{x}$

3 **a** $D = ex$ **b** $F = \frac{e^2}{p}$ **c** $P = \sqrt{x}$

d $M = e^3y^2$ **e** $B = \frac{t^3}{e}$ **f** $N = \frac{1}{\sqrt[3]{g}}$

g $Q \approx 8.66x^3$ **h** $D \approx 0.518n^{0.4}$

4 **a** $\log_2 y = \log_2 3 + x$ **b** $x = \log_2\left(\frac{y}{3}\right)$

c **i** $x = 0$ **ii** $x = 2$ **iii** $x \approx 3.32$

5 **a** $x = 9$ **b** $x = 2$ or 4 **c** $x = 25\sqrt{5}$ **d** $x = 200$

e $x = 5$ **f** $x = 3$ **g** $x = 3$ **h** $x = -2$

6 **a** $x = 2,\ y = 4$ **b** $x = 2,\ y = 8$

7 **a** $2^x = 7$ **b** $\log 2^x = \log 7$

$\therefore\ x\log 2 = \log 7$

$\therefore\ x = \log_2 7 = \frac{\log 7}{\log 2} \approx 2.81$

8 **a** Taking the logarithm in base a of both sides, $x = \log_a b$.

b $\log a^x = \log b$

c Using **b**, $x\log a = \log b$

$\therefore\ x = \frac{\log b}{\log a}$

and using part **a**, $x = \log_a b = \frac{\log b}{\log a}$

EXERCISE 3F

1 **a** ≈ 1.77 **b** ≈ 5.32 **c** ≈ 3.23 **d** ≈ -10.3

e ≈ -2.46 **f** ≈ 5.42

2 2

4 **a** $x = 16$ **b** $x = \frac{1}{5}$ **c** $x = \sqrt[3]{5}$ **d** $x = 64$

e $x = 2$ or 4

5 $\frac{8}{x}$ **6** $2m + 3$

7 **a** We get $\log_9 2 \neq 0$. **b** $x = 81$ **c** $x = a^4$

d **Hint:** First show that $\log_a x = k^{\frac{k}{k-1}}$.

EXERCISE 3G

1 **a** $3^3 = 27,\ 3^4 = 81,\ \therefore$ if $3^x = 40$, then $3 < x < 4$

b $x = \frac{\log 40}{\log 3}$ **c** $x \approx 3.36$

2 **a** **i** $x = \frac{1}{\log 2}$ **ii** $x \approx 3.32$

b **i** $x = \frac{\log 20}{\log 3}$ **ii** $x \approx 2.73$

c **i** $x = \frac{\log 50}{\log 4}$ **ii** $x \approx 2.82$

d **i** $x = 4$ **ii** $x = 4$

e **i** $x = -\frac{1}{\log\left(\frac{3}{4}\right)}$ **ii** $x \approx 8.00$

f **i** $x = \log(0.000\,015)$ **ii** $x \approx -4.82$

3 **a** $x \approx 2.29$ **b** $x \approx 5.13$ **c** $x \approx 0.194$

4 **a** $x = \ln 10$ **b** $x = \ln 1000$ **c** $x = \ln(0.15)$

d $x = 2\ln 5$ **e** $x = \frac{1}{2}\ln 18$ **f** $x = 0$

5 **a** $x = \frac{\log 25}{\log 2}$ **b** $x = \frac{\log\left(\frac{20}{7}\right)}{\log 1.5}$ **c** $x = \frac{\log(0.6)}{\log(0.8)}$

d $x = -\frac{\log(0.03)}{\log 2}$ **e** $x = \frac{10\log\left(\frac{10}{3}\right)}{\log 5}$ **f** $x = 4\ln 8$

6 **a** $x = \frac{\log 3}{\log 5}$ **b** $x = -\frac{\log 8}{\log 3}$ **c** $x = -1$

7 $x = -\frac{2\ln 2}{\ln 3}$ **8** $x = \frac{3\ln 2}{\ln 5}$

9 **a** $x = \ln 2$ **b** $x = 0$ **c** $x = \ln 2$ or $\ln 3$ **d** $x = 0$

e $x = \ln 4$ **f** $x = \ln\left(\frac{3+\sqrt{5}}{2}\right)$ or $\ln\left(\frac{3-\sqrt{5}}{2}\right)$

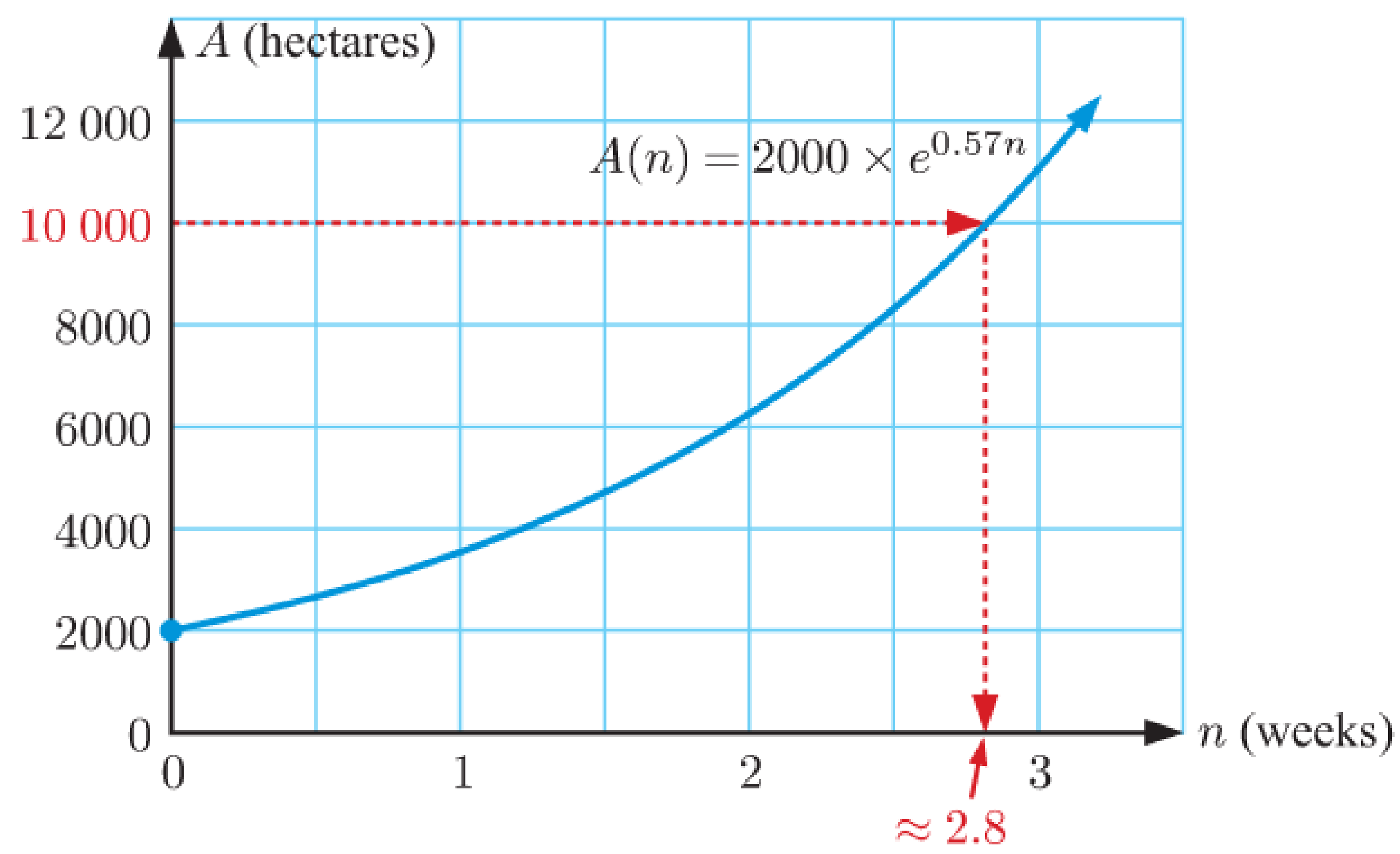
10 **a** $(\ln 3,\ 3)$ **b** $(\ln 2,\ 5)$ **c** $(0,\ 2)$ and $(\ln 5,\ -2)$

11 **a** $(-\ln 2,\ 2)$ **b** $\ln\left(\frac{11}{2}\right)$ units

12 **a** ≈ 2.37 years **b** ≈ 8.36 years

13 **a** ≈ 3.90 hours **b** ≈ 15.5 hours

14 **a, b** see graph below



$\therefore$ approximately 2.8 weeks.

15 In ≈ 5.86 years or ≈ 5 years 10 months. **16** 9 years

17 **a** $\frac{8.4\%}{12} = 0.7\% = 0.007,\ \ r = 1 + 0.007 = 1.007$

b after 74 months

18 **a** ≈ 17.3 years **b** ≈ 92.2 years **c** ≈ 115 years

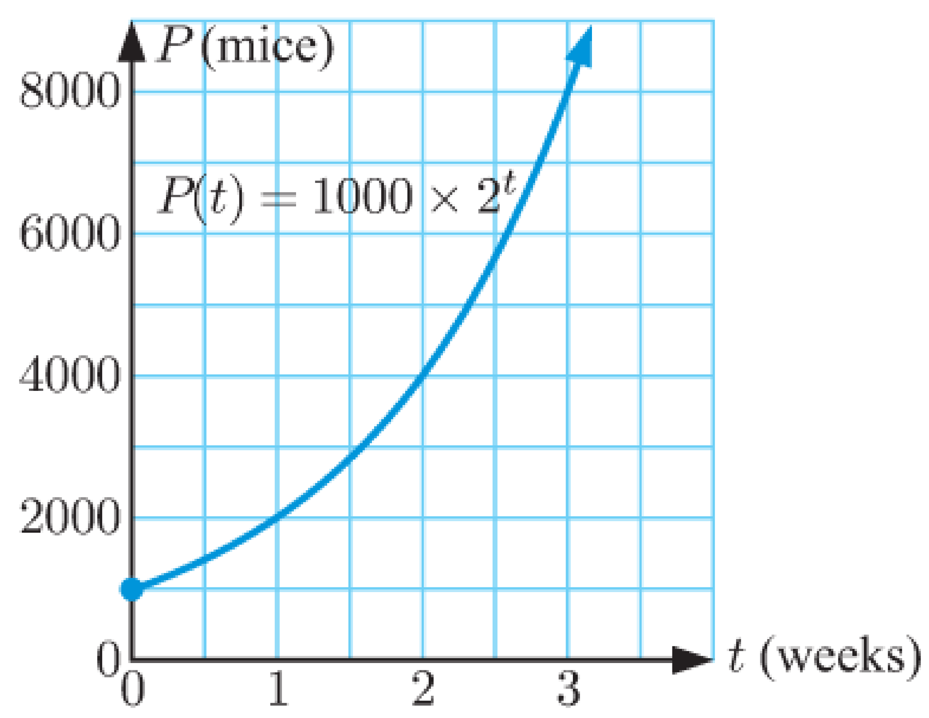
19 **Hint:** $0.1 \times I_0 = I_0 \times 2^{-0.02t}$

$\therefore\ 0.1 = 2^{-0.02t}$ and solve for t using logarithms.

20 **a** **Hint:** Set $V = 40$, solve for t.

b $t = -5\ln\left(1 - \frac{V}{50}\right)$ s

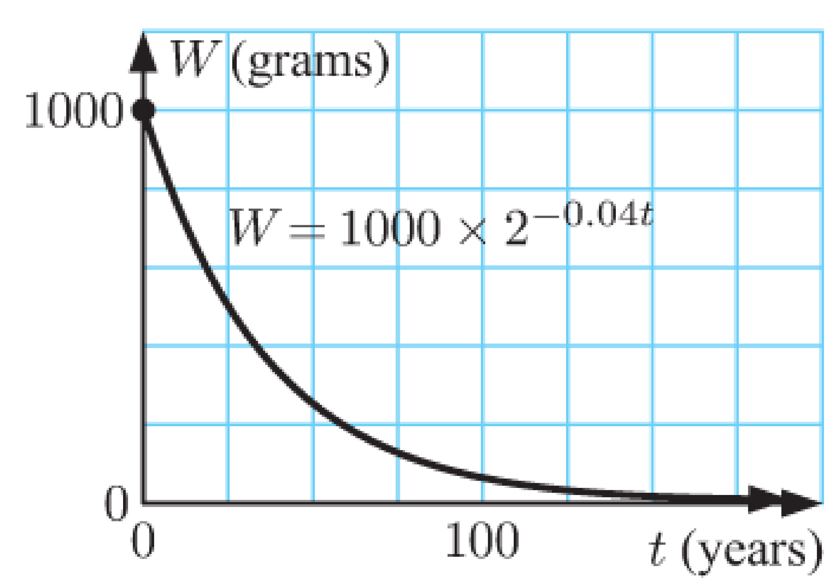
21 **a**



b ≈ 4.32 weeks

c $t = \frac{\log P - 3}{\log 2}$

22 **a**

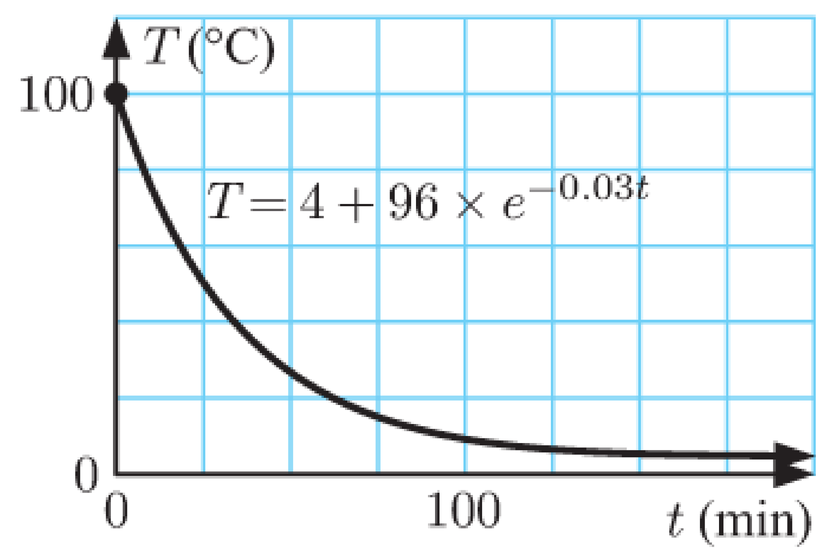


b $t = \frac{3 - \log W}{0.04\log 2}$

c **i** $t \approx 141$ years

ii $t \approx 498$ years

23 **a**



b $t = \dfrac{\ln 96 - \ln(T-4)}{0.03}$

c **i** ≈ 50.7 minutes **ii** ≈ 152 minutes

24 **a** decreasing **b** **i** 3900 m s^{-1} **ii** $\approx 2600 \text{ m s}^{-1}$

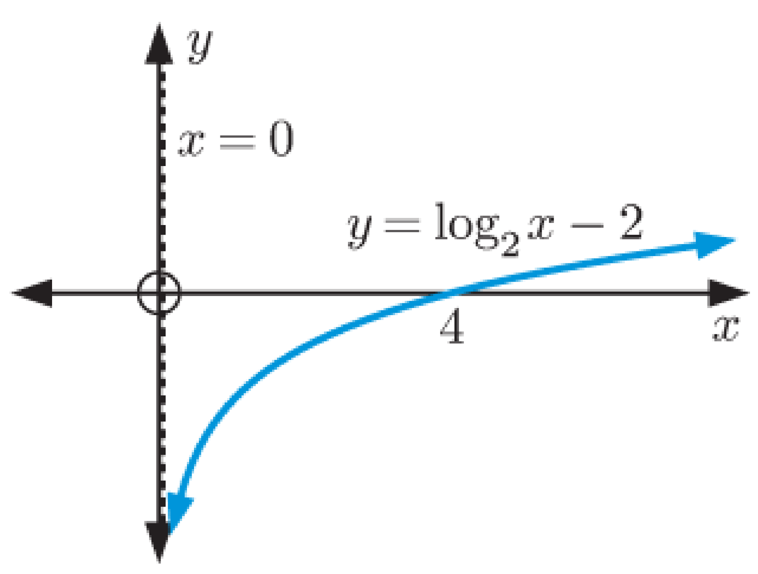
c ≈ 11.8 s

EXERCISE 3H

1 **a** **i** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 0$, x-intercept 4, no y-intercept

iii



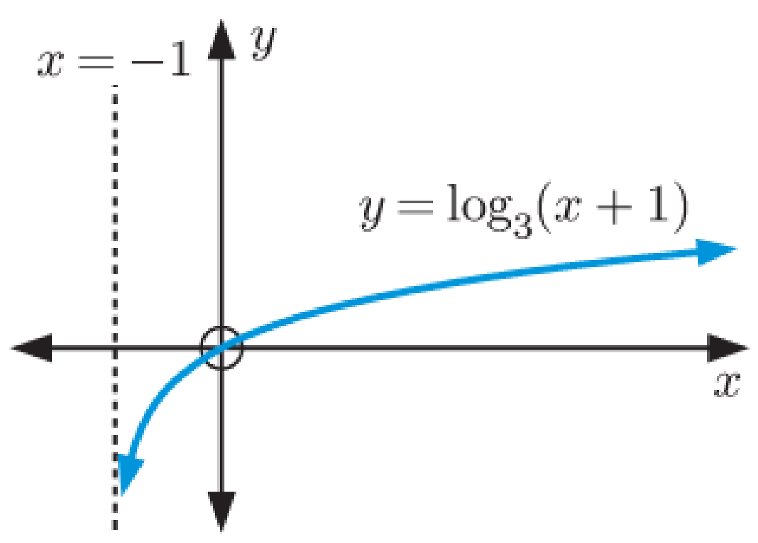
iv $x = 2$

v $f^{-1}(x) = 2^{x+2}$

b **i** Domain is $\{x \mid x > -1\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = -1$, x and y-intercepts 0

iii

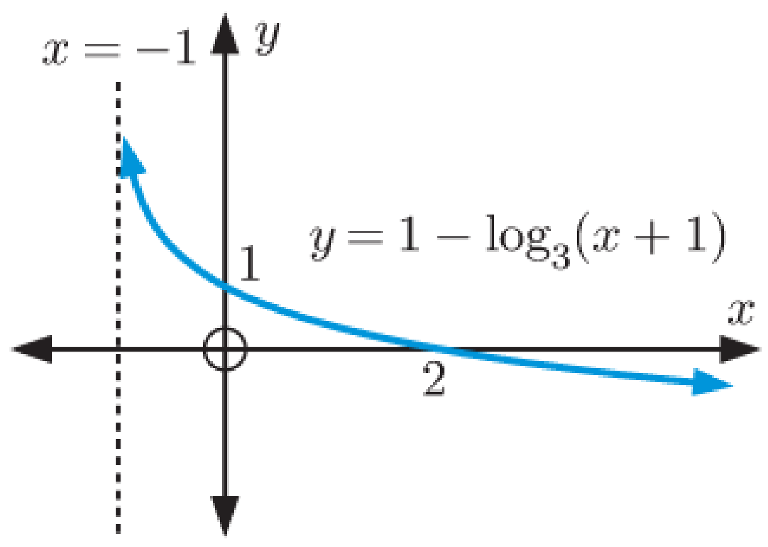


iv $x = -\frac{2}{3}$ **v** $f^{-1}(x) = 3^x - 1$

c **i** Domain is $\{x \mid x > -1\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = -1$, x-intercept 2, y-intercept 1

iii

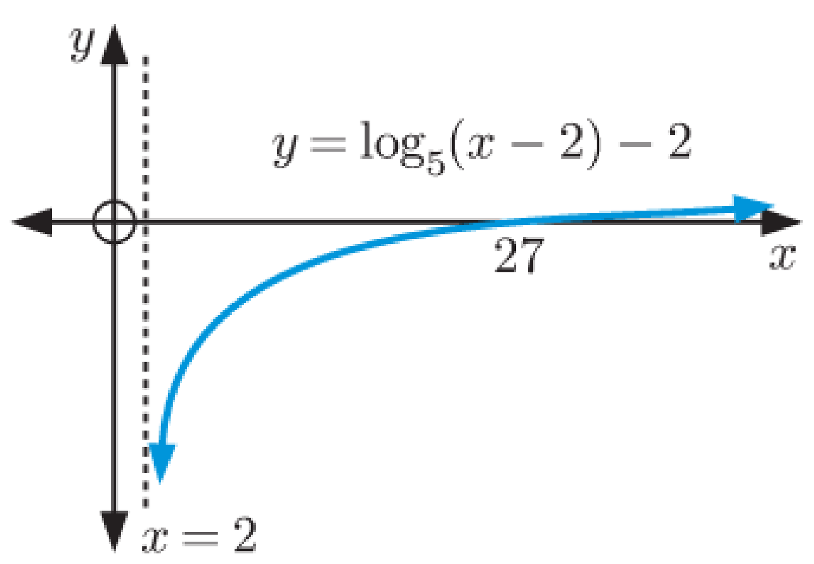


iv $x = 8$ **v** $f^{-1}(x) = 3^{1-x} - 1$

d **i** Domain is $\{x \mid x > 2\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 2$, x-intercept 27, no y-intercept

iii

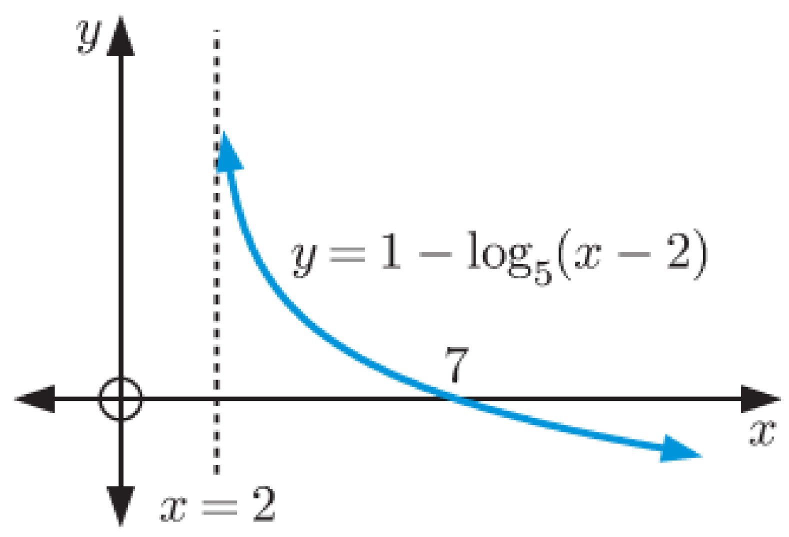


iv $x = 7$ **v** $f^{-1}(x) = 5^{x+2} + 2$

e **i** Domain is $\{x \mid x > 2\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 2$, x-intercept 7, no y-intercept

iii

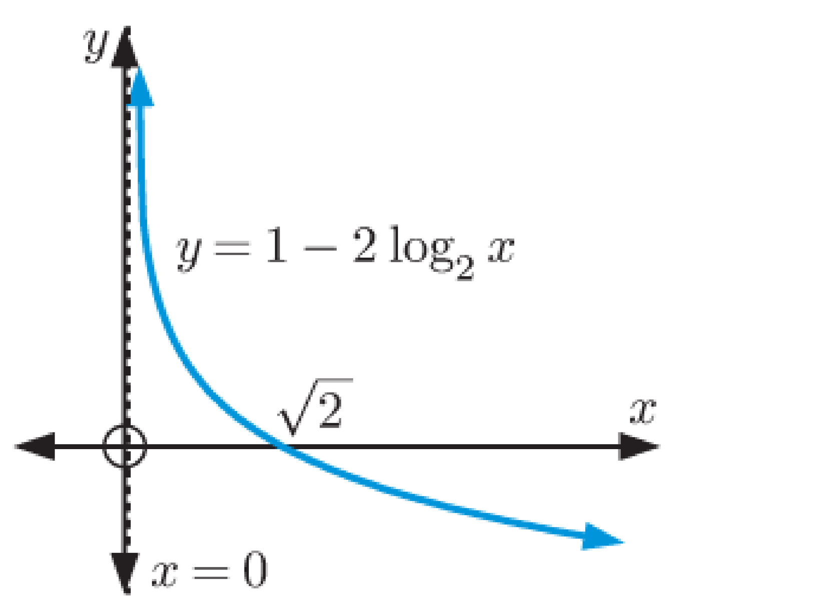


iv $x = 27$ **v** $f^{-1}(x) = 5^{1-x} + 2$

f **i** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

ii vertical asymptote is $x = 0$, x-intercept $\sqrt{2}$, no y-intercept

iii



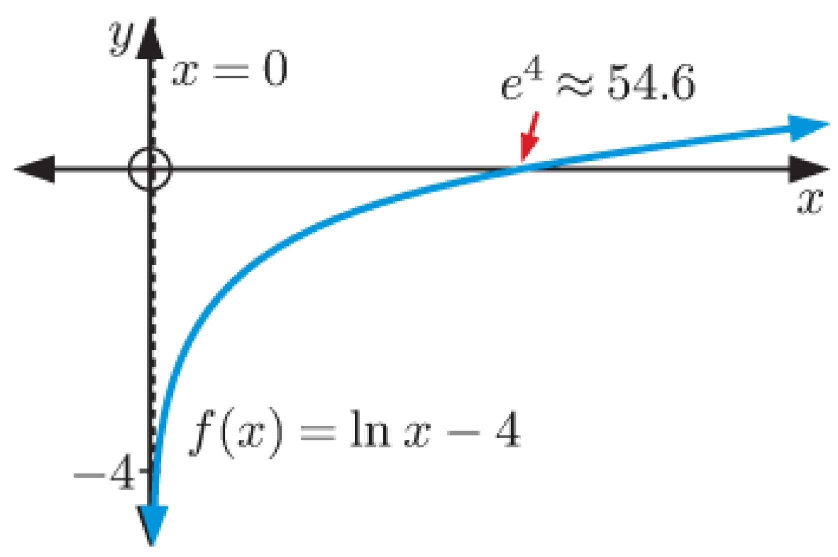
iv $x = 2$ **v** $f^{-1}(x) = 2^{\frac{1-x}{2}}$

2 **a** **i** A translation through $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$.

ii Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii vertical asymptote is $x = 0$, x-intercept e^4, no y-intercept

iv



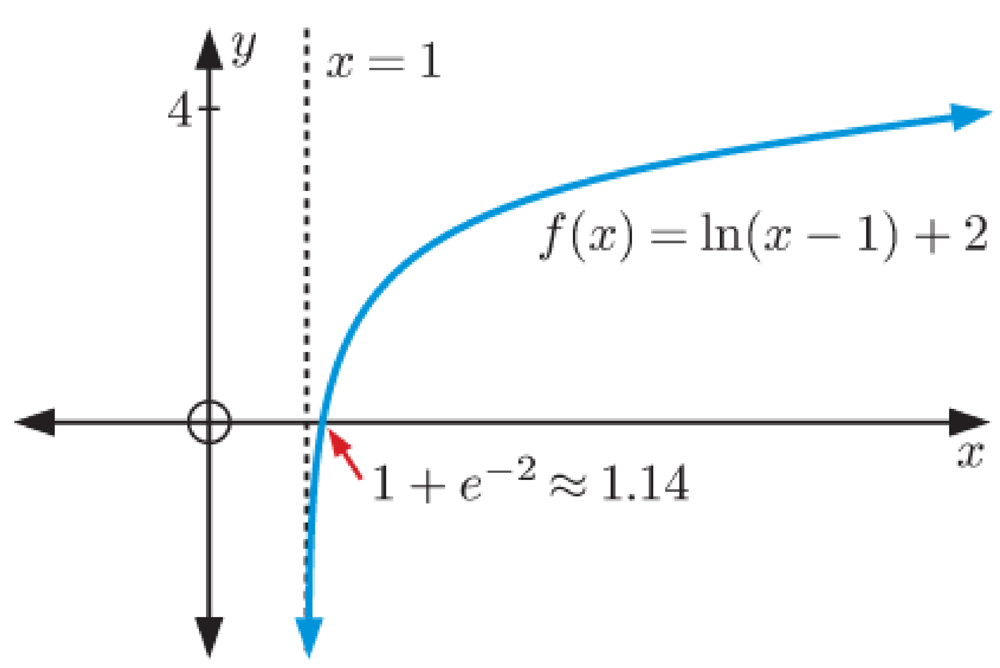
v $f^{-1}(x) = e^{x+4}$

b **i** A translation through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

ii Domain is $\{x \mid x > 1\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii vertical asymptote is $x = 1$, x-intercept $1 + e^{-2}$, no y-intercept

iv



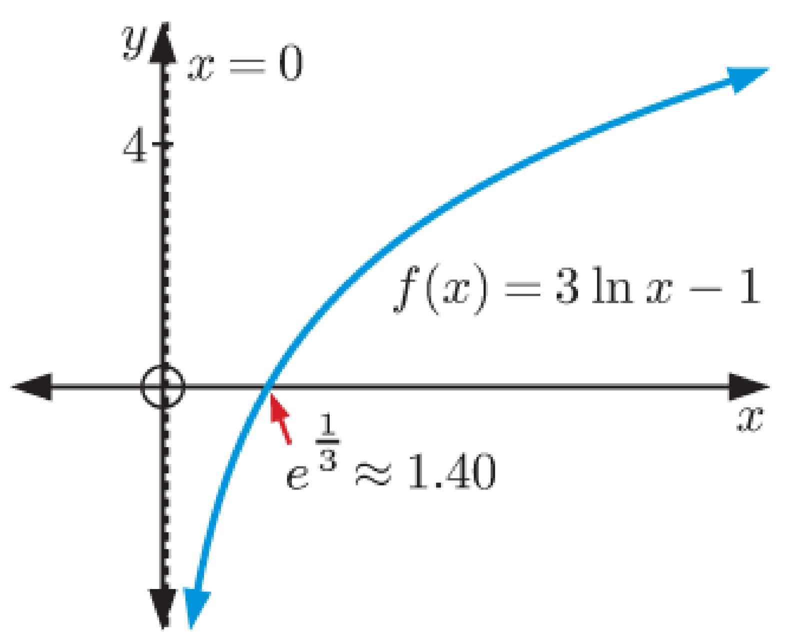
v $f^{-1}(x) = e^{x-2} + 1$

c **i** A vertical stretch with scale factor 3, then a translation through $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

ii Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

iii vertical asymptote is $x = 0$, x-intercept $e^{\frac{1}{3}}$, no y-intercept

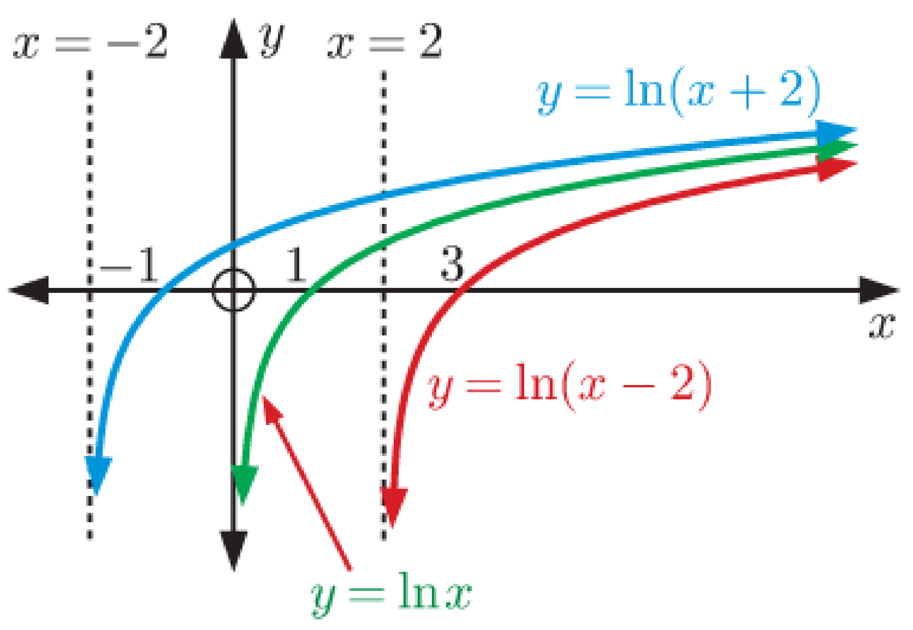
iv



v $f^{-1}(x) = e^{\frac{x+1}{3}}$

3 **a** A is $y = \ln x$ as its x-intercept is 1. B is $y = \ln(x-2)$.

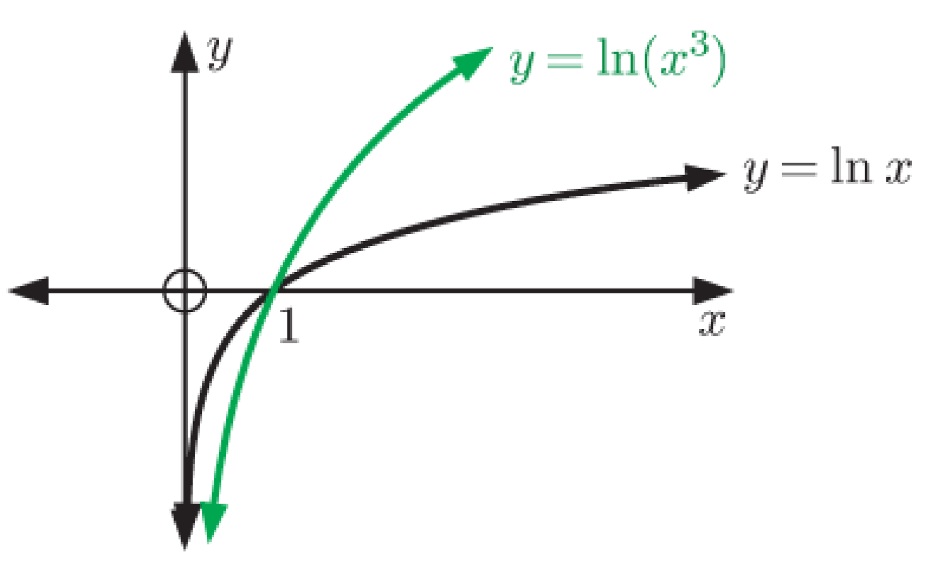
b



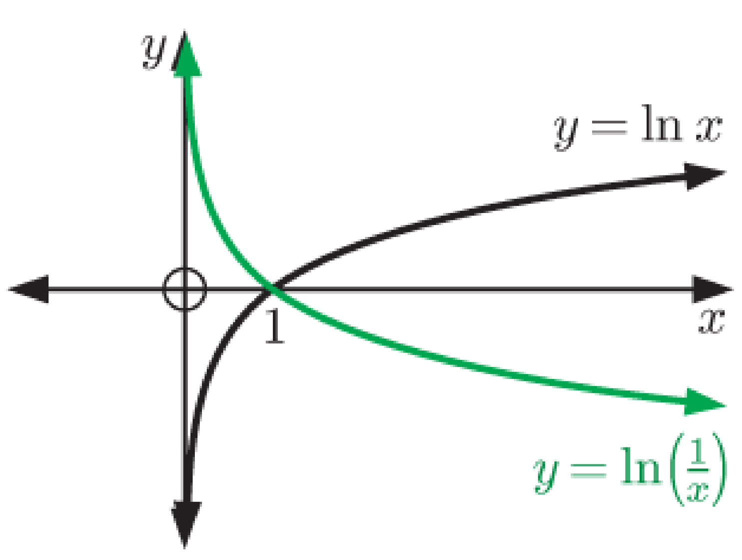
c $y = \ln x$ has vertical asymptote $x = 0$
$y = \ln(x-2)$ has vertical asymptote $x = 2$
$y = \ln(x+2)$ has vertical asymptote $x = -2$

4 $y = \ln(x^2) = 2\ln x$, so she is correct.
This is because the y-values are twice as large for $y = \ln(x^2)$ as they are for $y = \ln x$.

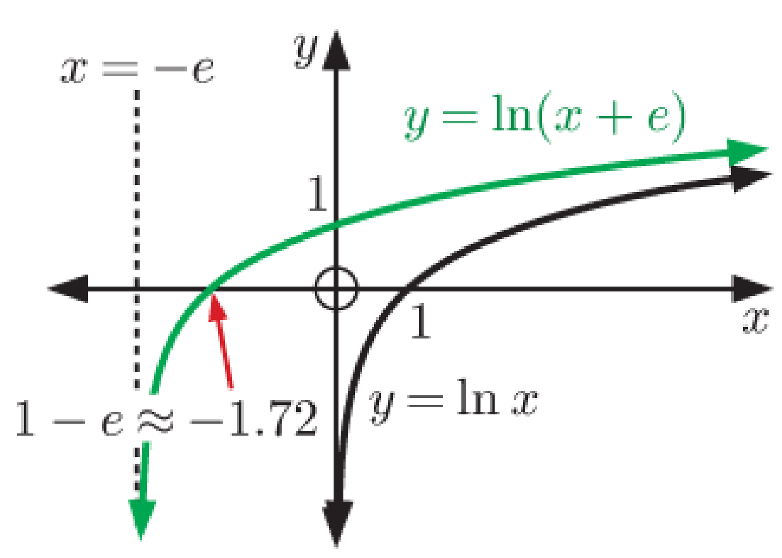
5 **a**



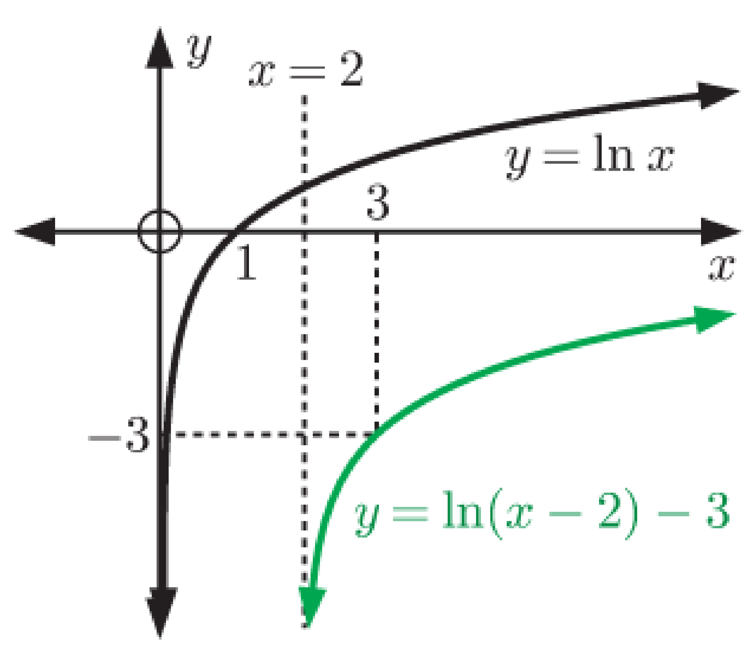
b



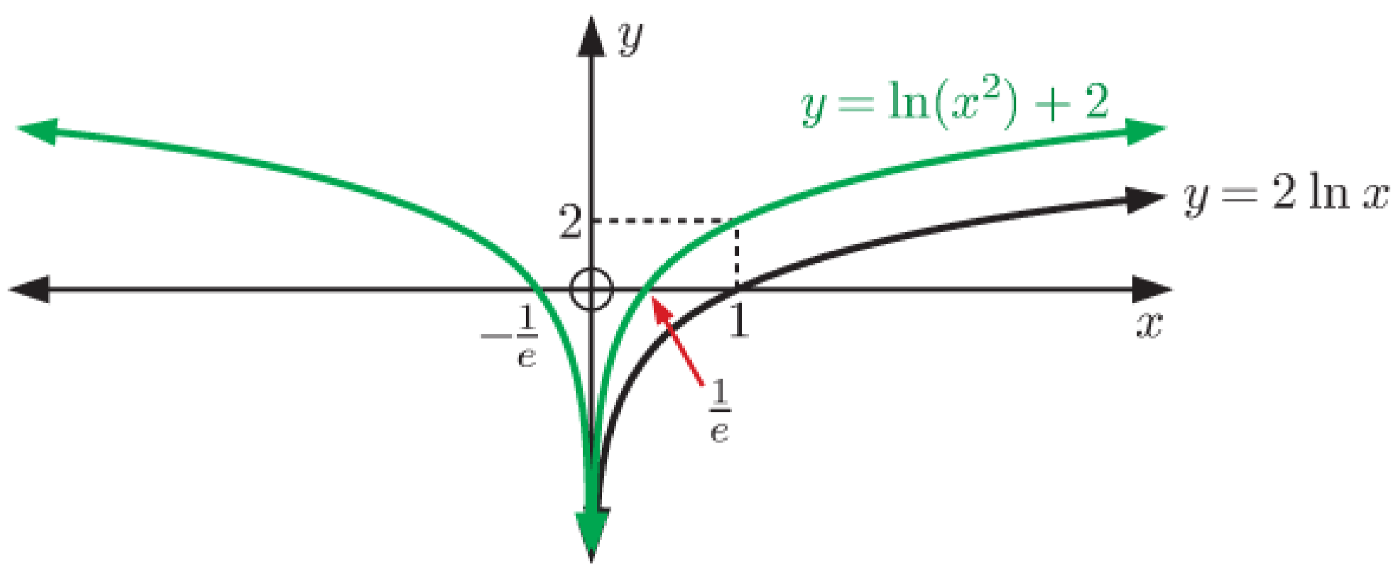
c



d



e



6 A vertical stretch with scale factor $\log_5 2$.

7 **a** **A** is $y = \log_2(x+2)$ since it is increasing.
B is $y = 3 - \log_2(3x+1)$ since it is decreasing.

b **A**: x-intercept -1, y-intercept 1, vertical asymptote $x = -2$
B: x-intercept $\frac{7}{3}$, y-intercept 3, vertical asymptote $x = -\frac{1}{3}$

c $(\frac{2}{3}, 3 - \log_2 3)$

8 **a** $f^{-1}(x) = \log_3 x$
Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

b $f^{-1}(x) = \log_2 x - 1$
Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

c $f^{-1}(x) = \frac{1}{2}\ln x$
Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$

d $f^{-1}(x) = \log_5(x+3)$
Domain is $\{x \mid x > -3\}$, Range is $\{y \mid y \in \mathbb{R}\}$

9 **a** $b^2 x$ **b** $2\ln b + x$ **c** $x = \dfrac{2\ln b}{b^2 - 1}$

10 **a** $\dfrac{x}{e}$, Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \in \mathbb{R}\}$

b $x - 1$, Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \in \mathbb{R}\}$

11 **a** Domain is $\{x \mid x < \frac{1}{3}\}$, Range is $\{y \mid y \in \mathbb{R}\}$

b **i** $x = -\frac{17}{3}$ **ii** $x = \frac{1}{3} - \frac{1}{3\sqrt{2}}$

c $f^{-1}(x) = \dfrac{1 - 2^x}{3}$
Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y < \frac{1}{3}\}$

12 $f^{-1}(x) = \frac{1}{2}\ln x$

a $(f^{-1} \circ g)(x) = \frac{1}{2}\ln(2x-1)$

b $(g \circ f)^{-1}(x) = \frac{1}{2}\ln\left(\dfrac{x+1}{2}\right)$

13 **a** $k = 3$ **b** x-intercepts $1 \pm 2\sqrt{2}$, y-intercept $\log_3 8$

c $g^{-1}(x) = 1 - \sqrt{9 - 3^x}$
Domain is $\{x \mid x \leqslant 2\}$, Range is $\{y \mid -2 < y \leqslant 1\}$

REVIEW SET 3A

1 **a** $\frac{1}{2}$ **b** $-\frac{1}{3}$ **c** $a + b + 1$

2 **a** 3 **b** 8 **c** -2 **d** $\frac{1}{2}$ **e** 0
f $\frac{1}{4}$ **g** -1 **h** $\frac{1}{2}$, $k > 0$, $k \neq 1$

3 **a** ≈ 1.431 **b** ≈ -0.237 **c** ≈ 2.602 **d** ≈ 3.689

4 $x = 2 - 3^{2y}$

5 **a** $\ln 144$ **b** $\ln\left(\frac{3}{2}\right)$ **c** $\ln\left(\dfrac{25}{e}\right)$ **d** $\ln 3$

6 **a** $\log 144$ **b** $\log_2\left(\frac{16}{9}\right)$ **c** $\log_4 80$

7 **a** $2A + 2B$ **b** $A + 3B$ **c** $3A + \frac{1}{2}B$
d $\frac{1}{2}(A + B)$ **e** $4B - 2A$ **f** $3A - 2B$

8 **a** $\log M = \log a + n \log b$ **b** $\log T = \log 5 - \frac{1}{2}\log l$
c $\log G = 2\log a + \log b - \log c$

9 **a** $x \approx 5.19$ **b** $x \approx 4.29$ **c** $x \approx -0.839$

10 **a** $x = \ln 3$ **b** $x = \ln 3$ or $\ln 4$

11 **a** $P = TQ^{1.5}$ **b** $M = \dfrac{e^{1.2}}{\sqrt{N}}$

12 **a** $x = 0$ or $\ln\left(\frac{2}{3}\right)$ **b** $x = e^2$

13 **a** $x = \frac{1}{8}$ **b** $x \approx 82.7$ **c** $x \approx 0.0316$

14 **a** **i** $x = \dfrac{\log 50}{\log 2}$ **ii** $x \approx 5.64$
b **i** $x = \dfrac{\log 4}{\log 7}$ **ii** $x \approx 0.71$
c **i** $x = \dfrac{-2}{\log(0.6)}$ **ii** $x \approx 9.02$

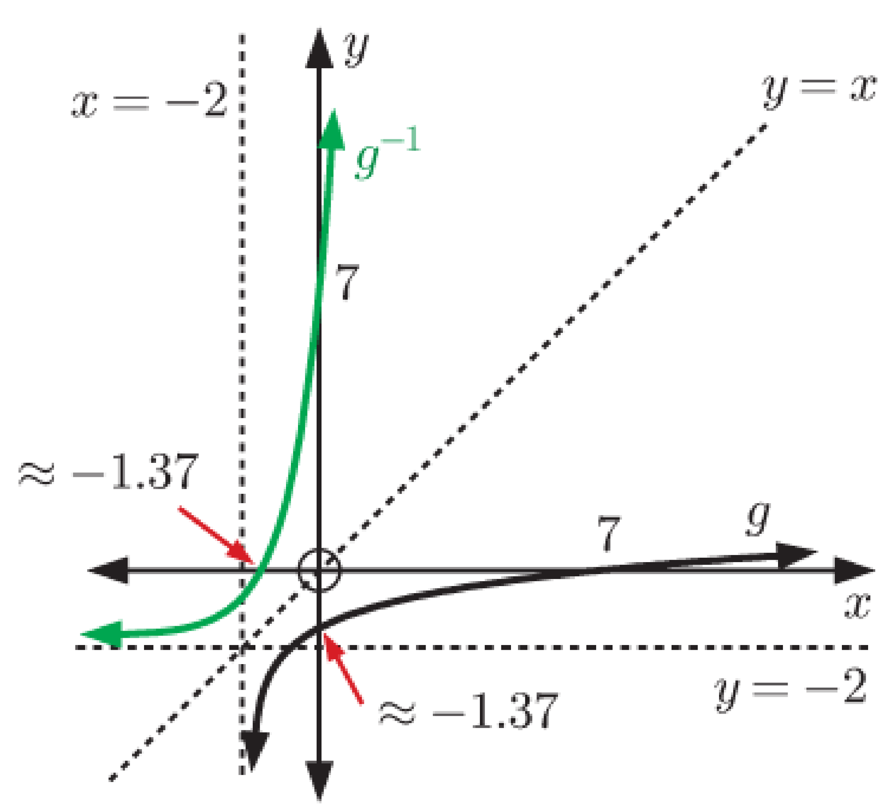
15 $\log_a\left(\dfrac{1}{b}\right) = -x$ **16** $4\log_3 5$

17 **Hint:** $2^{4x} - 5 \times 2^{3x} = 0$

18 **a** $x = e^5$ **b** $x = e^{-\frac{2}{3}}$ **c** $x = \ln 400$
d $x = \dfrac{\ln 11 - 1}{2}$ **e** $x = 2\ln 30$

19 **b** $\frac{1}{3}\ln\left(\frac{7}{2}\right)$ **20** $x = 2,\ y = \frac{1}{8}$ or $x = 64,\ y = 4$

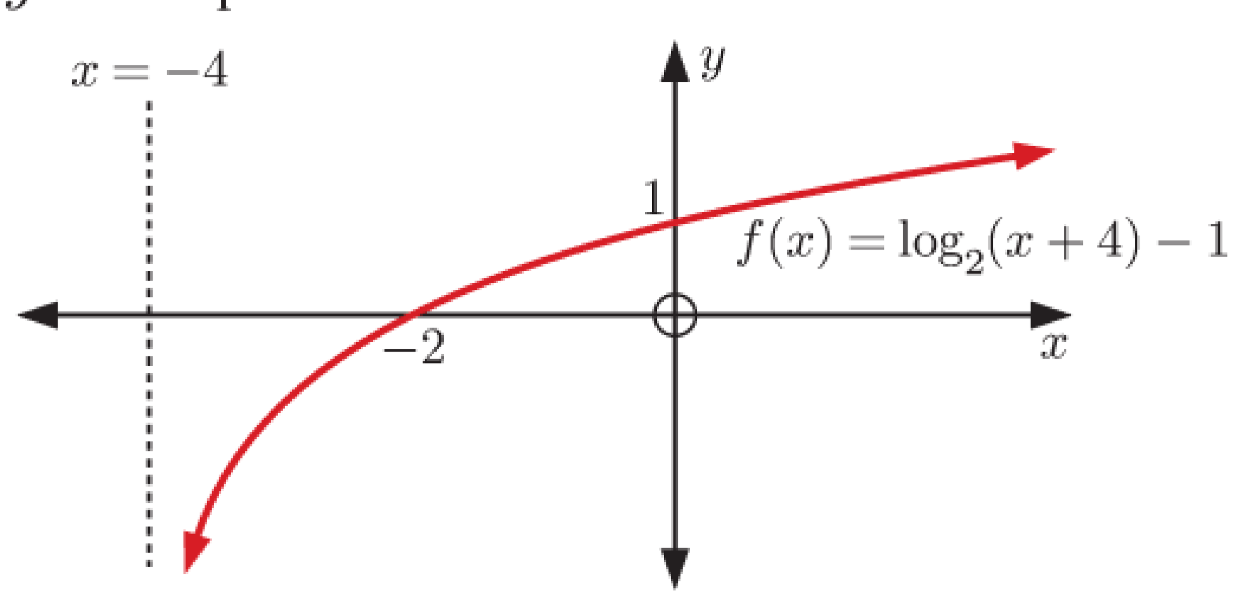
21 **a** A translation through $\begin{pmatrix} -2 \\ -2 \end{pmatrix}$.
b Domain is $\{x \mid x > -2\}$, Range is $\{y \mid y \in \mathbb{R}\}$
c vertical asymptote is $x = -2$, x-intercept is 7, y-intercept is ≈ -1.37
d $g^{-1}(x) = 3^{x+2} - 2$
e



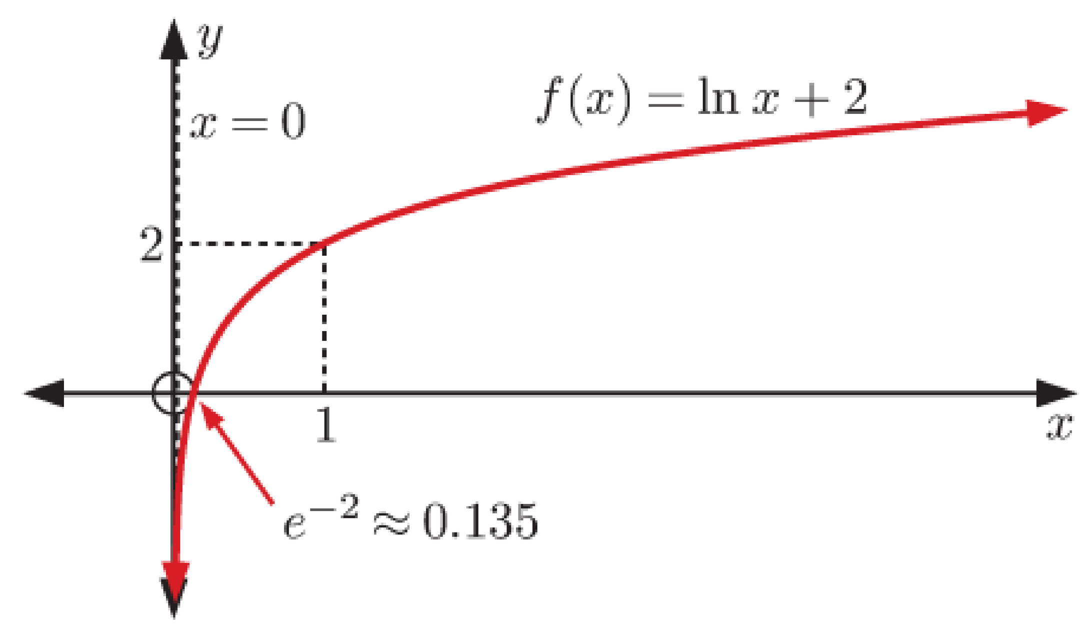
22 **a** ≈ 13.9 weeks **b** ≈ 41.6 weeks **c** ≈ 138 weeks

23 **a** ≈ 4.96 years or ≈ 4 years $11\frac{1}{2}$ months **b** $\approx 74.9\%$

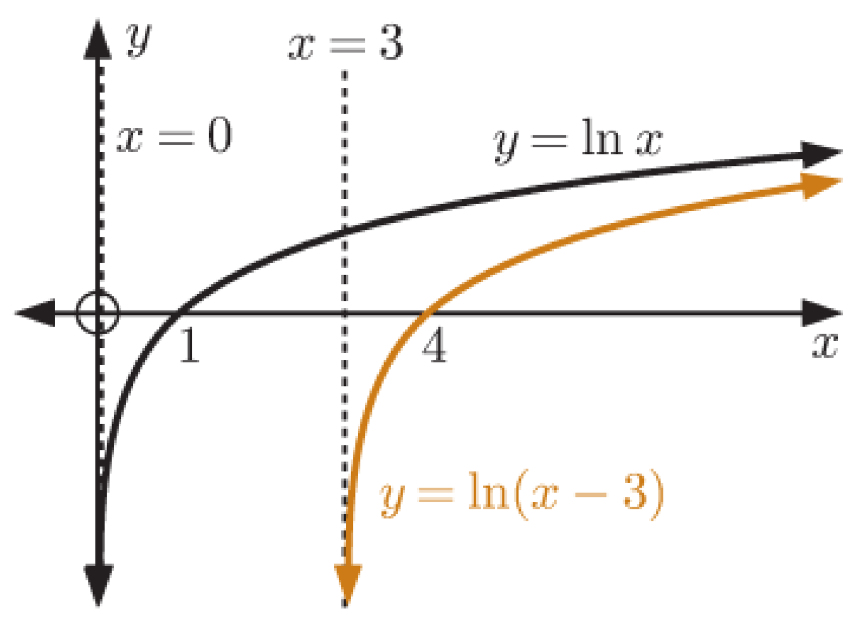
24 **a** **i** Domain is $\{x \mid x > -4\}$, Range is $\{y \mid y \in \mathbb{R}\}$
ii vertical asymptote is $x = -4$, x-intercept -2, y-intercept 1
iii



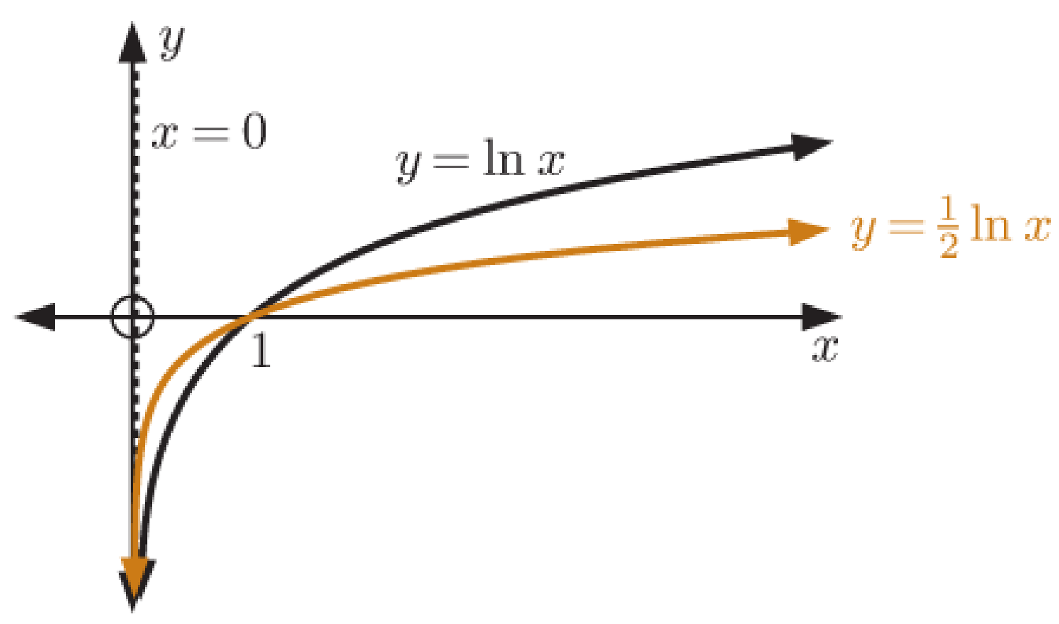
b **i** Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y \in \mathbb{R}\}$
ii vertical asymptote is $x = 0$, x-intercept e^{-2}, no y-intercept
iii



25 **a**



b



26 **a** 9 **b** $\ln 5$

REVIEW SET 3B

1 **a** $\frac{3}{2}$ **b** $\frac{2}{3}$ **c** $a + b$

2 **a** 7 **b** -3 **c** $-\frac{1}{2}$

3 **a** $\approx 10^{1.5051}$ **b** $\approx 10^{-2.8861}$ **c** $\approx 10^{-4.0475}$

4 **a** $\frac{3}{2}$ **b** -3 **c** $2x$ **d** $1 - x$

5 **a** $\frac{2}{3}$ **b** $\frac{6}{5}$ **c** 8

6 **a** x^4 **b** 5 **c** $\frac{1}{2}$ **d** $3x$ **e** $-x$ **f** $\log x$

7 **a** $\approx e^{2.9957}$ **b** $\approx e^{8.0064}$ **c** $\approx e^{-2.5903}$

8 **a** **i** $x = \dfrac{\log 7}{\log 5}$ **ii** $x \approx 1.21$
b **i** $x = -\dfrac{1}{\log 2}$ **ii** $x \approx -3.32$

9 **a** $\ln 3$ **b** $\ln 4$ **c** $\ln 125$

10 **a** $x = \dfrac{\ln 70}{2}$ **b** $x = \dfrac{\log\left(\frac{11}{3}\right)}{\log 1.3}$ **c** $x = \dfrac{10\log\left(\frac{16}{5}\right)}{3\log 2}$

11 $x = 1$

13 **a** $\log P = \log 3 + x\log b$ **b** $\log m = 3\log n - 2\log p$

14 **Hint:** Use the change of base rule.

15 **a** $x = 2^{\frac{18}{13}}$ **b** $x = 10^{\frac{3}{11}}$

16 **a** $T = \dfrac{x^2}{y}$ **b** $K = n\sqrt{t}$

17 **a** $5\ln 2$ **b** $3\ln 5$ **c** $6\ln 3$

18

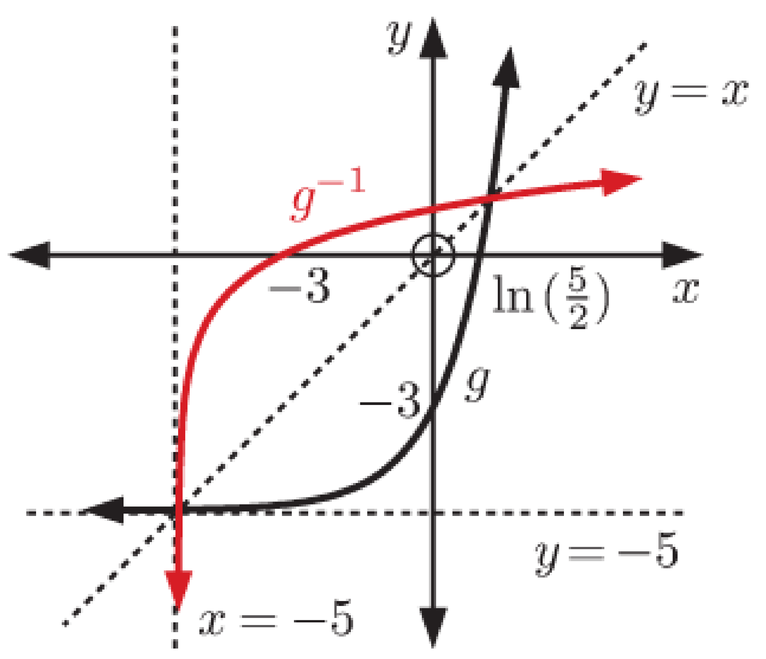
	$y = \log_2 x$	$y = \ln(x+5)$
Domain	$x > 0$	$x > -5$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

19 **a** $(2^x + 4)(2^x - 5)$ **b** $x = \dfrac{\log 5}{\log 2}$

c **i** $x = \dfrac{1}{p}$ **ii** $x = \dfrac{1}{3p+1}$

20 **a** $g^{-1}(x) = \ln\left(\dfrac{x+5}{2}\right)$

b



c Domain of g is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > -5\}$
Domain of g^{-1} is $\{x \mid x > -5\}$, Range is $\{y \mid y \in \mathbb{R}\}$

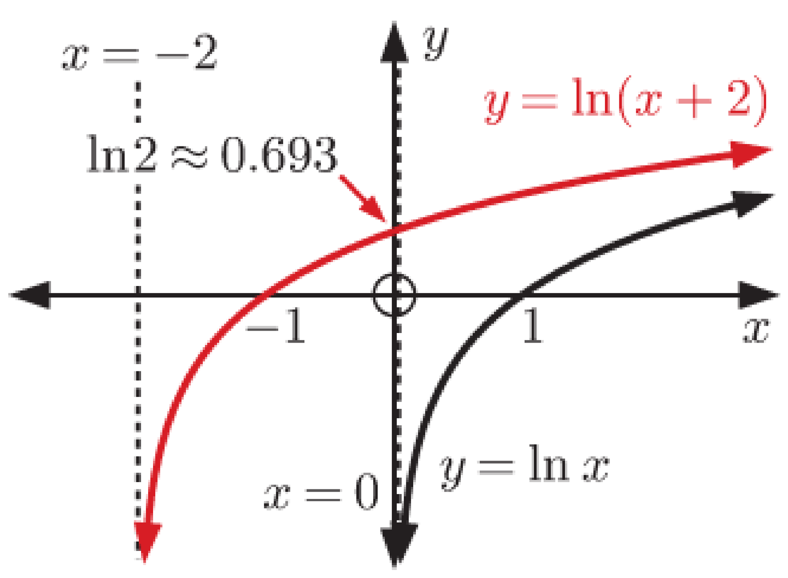
d g has horizontal asymptote $y = -5$,
x-intercept is $\ln\left(\frac{5}{2}\right) \approx 0.916$, y-intercept is -3
g^{-1} has vertical asymptote $x = -5$,
x-intercept is -3, y-intercept is ≈ 0.916

21 **Hint:** Set $T = 40$, and solve for t.

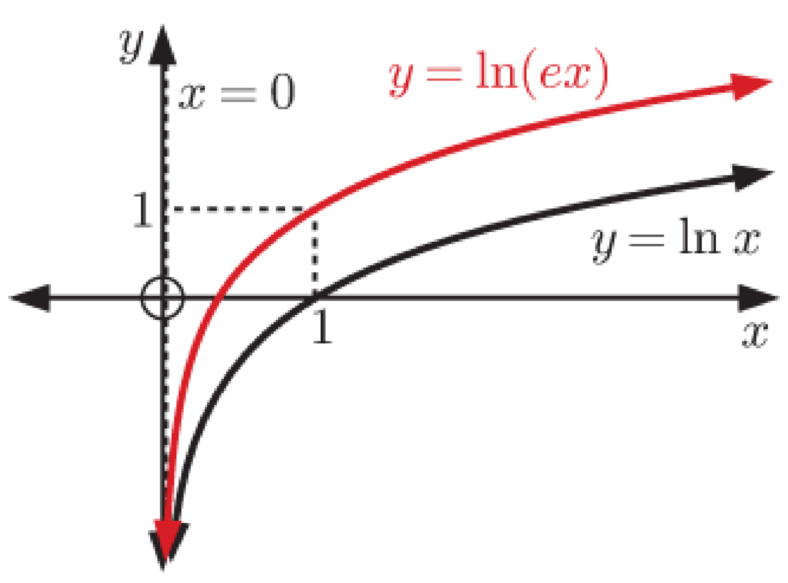
22 **a** 2500 g **b** ≈ 3288 years **c** $\approx 42.3\%$

23 **a** $x = \dfrac{2\log 9}{\log 5}$ **b** $x = \ln 30$ **c** $x = \dfrac{1-\ln 2}{3}$

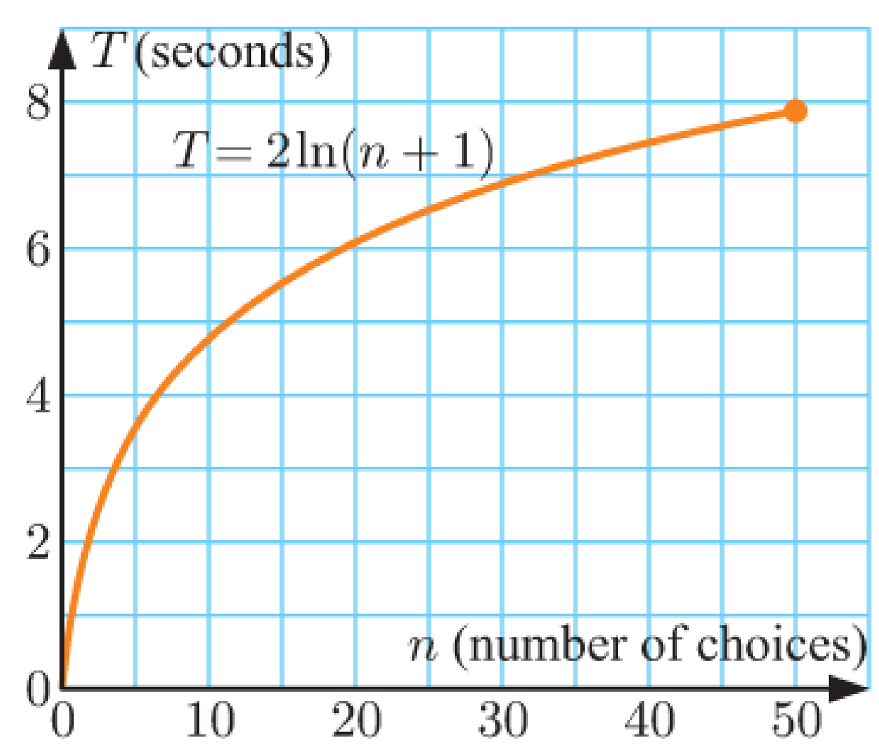
24 **a**



b



25 **a**



b **i** ≈ 3.58 seconds **ii** ≈ 5.55 seconds

c ≈ 1.34 seconds longer

26 **a** **i** $f^{-1}(x) = 3^{x+2}$
Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y > 0\}$

ii $(f \circ g)(x) = \log_3(3 - \sqrt{x}) - 2$
Domain is $\{x \mid 0 \leqslant x < 9\}$, Range is $\{y \mid y \leqslant -1\}$

iii $(g \circ f)(x) = 3 - \sqrt{\log_3 x - 2}$
Domain is $\{x \mid x \geqslant 9\}$, Range is $\{y \mid y \leqslant 3\}$

b **i** $x = 4$ **ii** $x = 3^{11} = 177\,147$

c $(g \circ f)^{-1}(x) = 3^{x^2 - 6x + 11}$, $x \leqslant 3$
Domain is $\{x \mid x \leqslant 3\}$, Range is $\{y \mid y \geqslant 9\}$

EXERCISE 4A

1

z	$\mathcal{Re}(z)$	$\mathcal{Im}(z)$	z^*
$3+2i$	3	2	$3-2i$
$5-i$	5	-1	$5+i$
3	3	0	3
0	0	0	0
$-3+4i$	-3	4	$-3-4i$
$-7-2i$	-7	-2	$-7+2i$
$-11i$	0	-11	$11i$
$i\sqrt{3}$	0	$\sqrt{3}$	$-i\sqrt{3}$

2 **a** $3i$ **b** $8i$ **c** $\frac{1}{2}i$ **d** $i\sqrt{5}$ **e** $i\sqrt{8}$

3 **a** $x = \pm 5$ **b** $x = \pm 5i$ **c** $x = \pm\sqrt{5}$
d $x = \pm i\sqrt{5}$ **e** $x = \pm\frac{3}{2}$ **f** $x = \pm\frac{3}{2}i$

4 **a** $x = 5 \pm 2i$ **b** $x = -3 \pm 4i$ **c** $x = -7 \pm i$
d $x = \frac{3}{2} \pm \frac{1}{2}i$ **e** $x = \sqrt{3} \pm i$ **f** $x = \frac{1}{4} \pm i\frac{\sqrt{7}}{4}$

EXERCISE 4B

1 **a** $(x+3)(x-3)$ **b** $(x+3i)(x-3i)$
c $(x+\sqrt{7})(x-\sqrt{7})$ **d** $(x+i\sqrt{7})(x-i\sqrt{7})$
e $(2x+1)(2x-1)$ **f** $(2x+i)(2x-i)$
g $(\sqrt{2}x+3)(\sqrt{2}x-3)$ **h** $(\sqrt{2}x+3i)(\sqrt{2}x-3i)$

2 $(a+bi)(a-bi) = a^2 - b^2i^2 = a^2 + b^2$ {since $i^2 = -1$}

3 **a** $x = \pm 4i$ **b** $x = \pm 6i$ **c** $x = \pm i\sqrt{5}$
d $x = \pm\frac{1}{3}i$ **e** $x = \pm\frac{5}{2}i$ **f** $x = \pm i\frac{\sqrt{2}}{\sqrt{3}}$

4 **a** $x(x+1)(x-1)$ **b** $x(x+i)(x-i)$
c $(x+1)(x-1)(x+i)(x-i)$
d $(x+2)(x-2)(x+2i)(x-2i)$

5 **a** $x = 0$ or ± 2 **b** $x = 0$ or $\pm 2i$
c $x = 0$ or $\pm\sqrt{3}$ **d** $x = 0$ or $\pm i\sqrt{3}$
e $x = \pm 1$ or $\pm i$ **f** $x = \pm 3$ or $\pm 3i$

6 **a** $x = \pm i\sqrt{3}$ or ± 1 **b** $x = \pm\sqrt{3}$ or $\pm i\sqrt{2}$
c $x = \pm 3i$ or ± 2 **d** $x = \pm i\sqrt{7}$ or $\pm i\sqrt{2}$
e $x = \pm 1$ **f** $x = \pm i$

EXERCISE 4C

1 **a** $7-i$ **b** $10-4i$ **c** $-1+2i$ **d** $3-3i$
e $4-7i$ **f** $12+i$ **g** $3+4i$ **h** $21-20i$

2 **a** $-3+7i$ **b** $2i$ **c** $-2+2i$ **d** $-1+i$
e $-5-12i$ **f** $-5+i$ **g** $-6-4i$ **h** $-1-5i$

3 $(a + \cancel{bi}) + (a - \cancel{bi}) = 2a$ which is real

4 **a** $i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$,
$i^6 = -1$, $i^7 = -i$, $i^8 = 1$, $i^9 = i$, $i^{-1} = -i$,
$i^{-2} = -1$, $i^{-3} = i$, $i^{-4} = 1$, $i^{-5} = -i$
b $i^{4n+3} = -i$

5 $(1+i)^4 = -4$, $(1+i)^{101} = -2^{50}(1+i)$

6 **a** -6 **b** No, $\sqrt{-4} \times \sqrt{-9} = -6 \neq \sqrt{36}$.

7 **a** $-\frac{1}{10} - \frac{7}{10}i$ **b** $-\frac{1}{5} + \frac{2}{5}i$ **c** $\frac{7}{5} + \frac{1}{5}i$ **d** $\frac{3}{25} + \frac{4}{25}i$

8 **a** $-\frac{2}{5} + \frac{1}{5}i$ **b** $-\frac{1}{13} + \frac{8}{13}i$ **c** $-\frac{2}{5} + \frac{3}{5}i$
d $\dfrac{2}{x^2+1}$

9 **a** -2 **b** -4 **c** 3 **d** 0

10 $z = 65 - 72i$ **11** $z = i\sqrt{2}$