33 $f(x) = \frac{x+2}{x-1} = \frac{x-1+3}{x-1} = 1 + \frac{3}{x-1}$

a The domain is $\{x \mid x \neq 1\}$.

The range is $\{y \mid y \neq 1\}$.

b The vertical asymptote is $x = 1$.

The horizontal asymptote is $y = 1$.

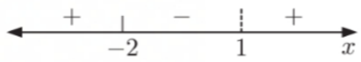
c $f(0) = \frac{2}{-1} = -2$, so the y-intercept is -2.

$f(x) = 0$ when $x + 2 = 0$

$\therefore \quad x = -2$

$\therefore$ the x-intercept is -2.

d



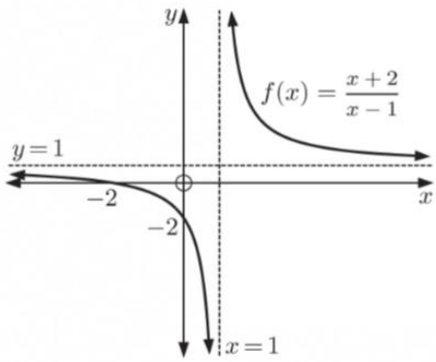
e As $x \to 1^-$, $f(x) \to -\infty$

As $x \to 1^+$, $f(x) \to \infty$

As $x \to -\infty$, $f(x) \to 1^-$

As $x \to \infty$, $f(x) \to 1^+$

f



34 $f(x) = 2 + \frac{4}{x+1}$

a $f(0) = 2 + \frac{4}{1} = 6$, so the y-intercept is 6.

$f(x) = 0$ when $2 + \frac{4}{x+1} = 0$

$\therefore \quad 2(x+1) + 4 = 0$

$\therefore \quad 2(x+1) = -4$

$\therefore \quad x + 1 = -2$

$\therefore \quad x = -3$

$\therefore$ the x-intercept is -3.

b $f(-2) = 2 + \frac{4}{-2+1}$

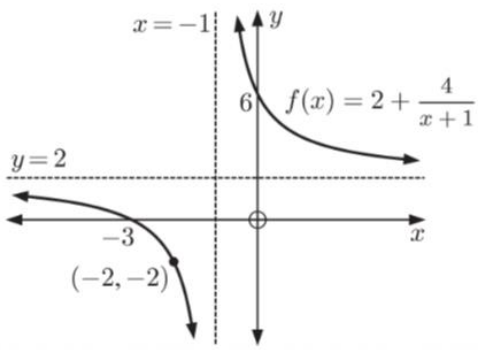
$= 2 + \frac{4}{-1}$

$= -2$

c **i** The horizontal asymptote is $y = 2$.

ii The vertical asymptote is $x = -1$.

d



35 **a** **i** The horizontal asymptote is $y = 0$.

The function is undefined when $x^2 + 4x - 21 = 0$

$$\therefore \quad (x+7)(x-3) = 0$$

$$\therefore \quad x = -7 \text{ or } 3$$

$\therefore$ the vertical asymptotes are $x = -7$ and $x = 3$.

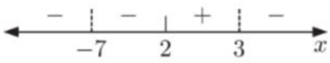
ii When $x = 0$, $y = -\frac{2}{21}$, so the y-intercept is $-\frac{2}{21}$.

When $y = 0$, $2 - x = 0$

$$\therefore \quad x = 2$$

$\therefore$ the x-intercept is 2.

iii $y = \dfrac{2-x}{x^2+4x-21}$ has sign diagram



iv As $x \to -7^-$, $y \to \infty$

As $x \to -7^+$, $y \to -\infty$

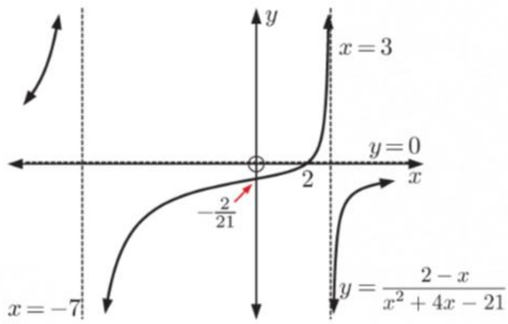
As $x \to 3^-$, $y \to \infty$

As $x \to 3^+$, $y \to -\infty$

As $x \to -\infty$, $y \to 0^+$

As $x \to \infty$, $y \to 0^-$

v



b **i** The horizontal asymptote is $y = 0$.

The function is undefined when

$$2x^2 + 9x + 9 = 0$$

$$\therefore \quad 2x^2 + 6x + 3x + 9 = 0$$

$$\therefore \quad 2x(x+3) + 3(x+3) = 0$$

$$\therefore \quad (x+3)(2x+3) = 0$$

$$\therefore \quad x = -3 \text{ or } -\tfrac{3}{2}$$

$\therefore$ the vertical asymptotes are $x = -3$ and $x = -\frac{3}{2}$.

ii When $x = 0$, $y = -\frac{2}{9}$, so the y-intercept is $-\frac{2}{9}$.

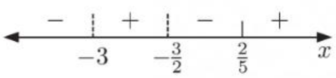
When $y = 0$, $5x - 2 = 0$

$$\therefore \quad 5x = 2$$

$$\therefore \quad x = \tfrac{2}{5}$$

$\therefore$ the x-intercept is $\frac{2}{5}$.

iii $y = \dfrac{5x-2}{2x^2+9x+9}$ has sign diagram



iv As $x \to -3^-$, $y \to -\infty$

As $x \to -3^+$, $y \to \infty$

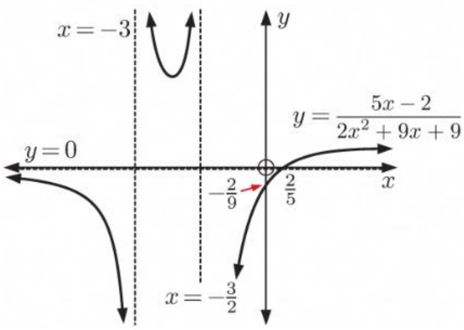
As $x \to -\frac{3}{2}^-$, $y \to \infty$

As $x \to -\frac{3}{2}^+$, $y \to -\infty$

As $x \to -\infty$, $y \to 0^-$

As $x \to \infty$, $y \to 0^+$

v



36 $f(x) = \dfrac{x+2}{x^2+bx+3}$

a $f(0) = \frac{2}{3}$, so the y-intercept is $\frac{2}{3}$.

$f(x) = 0$ when $x + 2 = 0$

$\therefore\ x = -2$

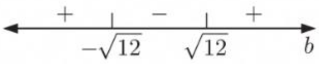
$\therefore$ the x-intercept is -2.

b $x^2 + bx + 3 = 0$ .... $(*)$ has discriminant $\Delta = b^2 - 4(1)(3)$

$= b^2 - 12$

$= (b - \sqrt{12})(b + \sqrt{12})$

Δ has sign diagram



i $f(x)$ has no vertical asymptotes when $(*)$ has no real solutions.

$\therefore\ \Delta < 0$

$\therefore\ -\sqrt{12} < b < \sqrt{12}$

ii $f(x)$ has one vertical asymptote when $(*)$ has 1 real solution.

$\therefore\ \Delta = 0$

$\therefore\ b = \pm\sqrt{12}$

iii $f(x)$ has two vertical asymptotes when $(*)$ has 2 real solutions.

$\therefore\ \Delta > 0$

$\therefore\ b < -\sqrt{12}$ or $b > \sqrt{12}$

37 **a** $y = \dfrac{x^2 - 2x - 8}{x - 3}$

i The vertical asymptote is $x = 3$.

ii When $x = 0$, $y = \frac{-8}{-3} = \frac{8}{3}$, so the y-intercept is $\frac{8}{3}$.

When $y = 0$, $x^2 - 2x - 8 = 0$

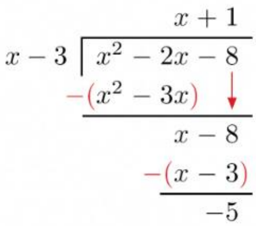
$\therefore\ (x+2)(x-4) = 0$

$\therefore\ x = -2$ or 4

$\therefore$ the x-intercepts are -2 and 4.

iii $y = \dfrac{x^2 - 2x - 8}{x - 3}$

$= x + 1 - \dfrac{5}{x - 3}$



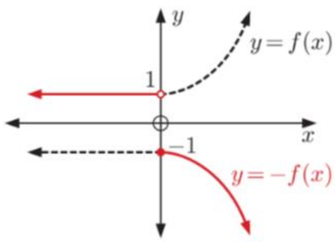
$\therefore$ the oblique asymptote is $y = x + 1$.

70 **a** $f(x)$ —horizontal stretch scale factor $\frac{1}{2}$→ $f(2x)$ —translation $\begin{pmatrix}1\\3\end{pmatrix}$→ $f(2(x-1)) + 3$

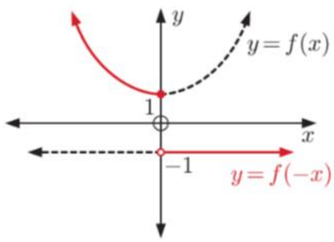
b $f(x)$ —horizontal stretch scale factor 4→ $f(\frac{1}{4}x)$ —vertical stretch scale factor 2→ $2f(\frac{1}{4}x)$ —reflection in x-axis→ $-2f(\frac{1}{4}x)$ —translation $\begin{pmatrix}0\\5\end{pmatrix}$→ $5 - 2f(\frac{1}{4}x)$

c $f(x)$ —translation $\begin{pmatrix}2\\0\end{pmatrix}$→ $f(x-2)$ —horizontal stretch scale factor 3→ $f(\frac{1}{3}x - 2)$ —vertical stretch scale factor 6→ $6f(\frac{1}{3}x - 2)$ —translation $\begin{pmatrix}0\\4\end{pmatrix}$→ $6f(\frac{1}{3}x - 2) + 4$

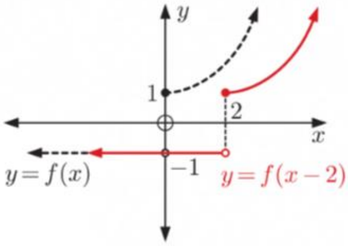
71 **a** The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x-axis.



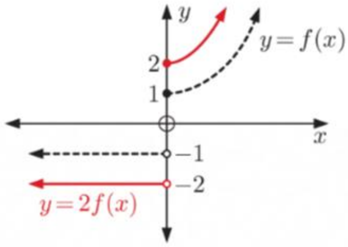
b The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y-axis.



c The graph of $y = f(x-2)$ is found by translating $y = f(x)$ 2 units to the right.



d The graph of $y = 2f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 2.

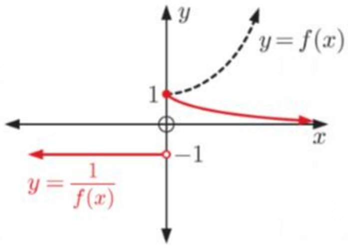


e $f(x) = -1$ for $x < 0$, so

$\frac{1}{f(x)} = -1$ for $x < 0$.

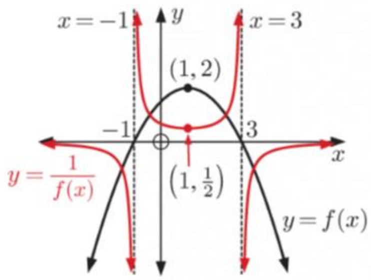
$f(x) \geqslant 1$ for $x \geqslant 0$, so

$0 < \frac{1}{f(x)} \leqslant 1$ for $x \geqslant 0$.



72 **a** $y = f(x)$ has x-intercepts -1 and 3, so $y = \frac{1}{f(x)}$ has vertical asymptotes $x = -1$ and $x = 3$.

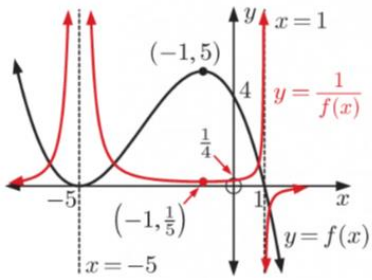
$y = f(x)$ has a local maximum at $(1, 2)$, so $y = \frac{1}{f(x)}$ has local minimum at $(1, \frac{1}{2})$.



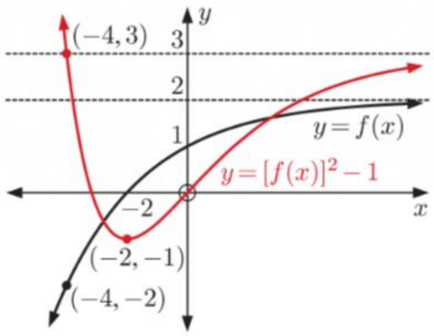
b $y = f(x)$ has x-intercepts -5 and 1, so $y = \frac{1}{f(x)}$ has vertical asymptotes $x = -5$ and $x = 1$.

$y = f(x)$ has local maximum $(-1, 5)$, so $y = \frac{1}{f(x)}$ has local minimum $(-1, \frac{1}{5})$.

$y = f(x)$ has y-intercept 4, so $y = \frac{1}{f(x)}$ has y-intercept $\frac{1}{4}$.



73



74 $f(x) = \dfrac{6-2x}{x+3}$

a $f(0) = \dfrac{6}{3} = 2$, so the y-intercept is 2.

$f(x) = 0$ when $6 - 2x = 0$

$\therefore\ 2x = 6$

$\therefore\ x = 3$

$\therefore$ the x-intercept is 3.

The vertical asymptote is $x = -3$.

$$f(x) = \frac{6-2x}{x+3} = \frac{6-2(x+3)+6}{x+3} = \frac{12}{x+3} - 2$$

$\therefore$ the horizontal asymptote is $y = -2$.

b The y-intercept of $y = [f(x)]^2$ is $2^2 = 4$.

The x-intercept of $y = [f(x)]^2$ is 3.

The vertical asymptote of $y = [f(x)]^2$ is $x = -3$.

The horizontal asymptote of $y = [f(x)]^2$ is $y = (-2)^2 = 4$.

c Invariant points occur where $f(x) = 0$ or $f(x) = 1$

$\therefore\ \dfrac{6-2x}{x+3} = 1$

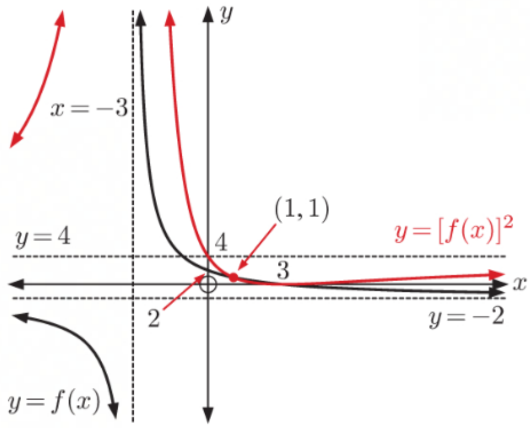
$\therefore\ 6 - 2x = x + 3$

$\therefore\ -3x = -3$

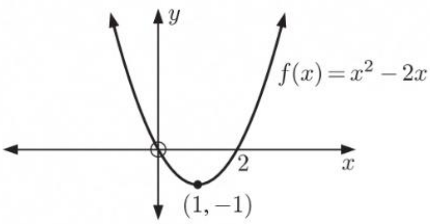
$\therefore\ x = 1$

$\therefore$ the invariant points are $(1, 1)$ and $(3, 0)$.

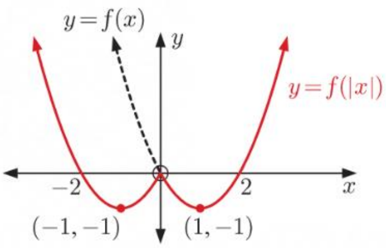
d



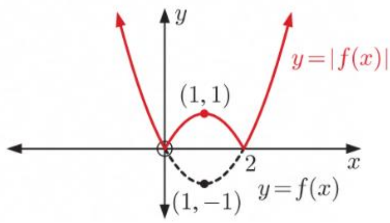
75 **a**



b **i**



ii



77 **a** If $|3 - 2x| = 5$, then $3 - 2x = \pm 5$

$\therefore\ 3 - 2x = 5$ or $3 - 2x = -5$

$\therefore\ -2x = 2$ or $-2x = -8$

$\therefore\ x = -1$ or $x = 4$

So, $x = -1$ or 4.

b If $\left|\dfrac{2x+5}{3-x}\right| = 2$, then $\dfrac{2x+5}{3-x} = \pm 2$

$\therefore\ \dfrac{2x+5}{3-x} = 2$ or $\dfrac{2x+5}{3-x} = -2$

$\therefore\ 2x + 5 = 6 - 2x$ or $2x + 5 = -6 + 2x$

$\therefore\ 4x = 1$ $\qquad 5 = -6$ ✘

$\therefore\ x = \frac{1}{4}$

So, $x = \frac{1}{4}$.

82 $f(x) = (5x-2)\left(\frac{a}{x}+3\right)$ is an odd function

$$\therefore\ f(-x) = -f(x)$$

$$\therefore\ (5(-x)-2)\left(\frac{a}{-x}+3\right) = -(5x-2)\left(\frac{a}{x}+3\right)$$

$$\therefore\ -(5x+2)\left(3-\frac{a}{x}\right) = -(5x-2)\left(\frac{a}{x}+3\right)$$

$$\therefore\ (5x+2)\left(3-\frac{a}{x}\right) = (5x-2)\left(\frac{a}{x}+3\right)$$

$$\therefore\ (5x+2)(3x-a) = (5x-2)(a+3x), \quad x \neq 0$$

$$\therefore\ 15x^2 - 5ax + 6x - 2a = 5ax + 15x^2 - 2a - 6x$$

$$\therefore\ (12-10a)x = 0$$

$$\therefore\ 12 - 10a = 0 \quad \{x \neq 0\}$$

$$\therefore\ 10a = 12$$

$$\therefore\ a = \tfrac{6}{5}$$

83 **a** A function is even if $f(-x) = f(x)$ for all x in the domain of the function.

$\therefore$ for any value in the range of f, there exists at least two corresponding values of x in the domain (except in the trivial case $f(x) = k$ with domain $x = 0$)

$\therefore$ the function fails the horizontal line test, and does not have an inverse.

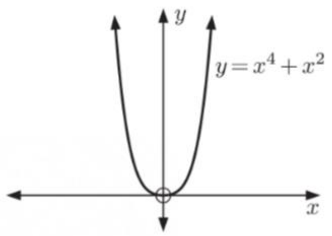
b If $f(x) = x^4 + x^2$

then $f(-x) = (-x)^4 + (-x)^2$

$= x^4 + x^2$

$= f(x)$

$\therefore$ $f(x)$ is even.



From the graph, we observe that the function is strictly increasing for $x \geqslant 0$.

$\therefore$ we can choose the domain restriction $x \geqslant 0$ for the function to have an inverse.