(d) $\left(\frac{a^{m+n}}{a^n}\right)^m \times \left(\frac{a^{n-m}}{a^n}\right)^{m-n}$ (e) $\frac{p^{-2}-q^{-2}}{p^{-1}-q^{-1}}$ (f) $\frac{1}{1+a^{\frac{1}{2}}} - \frac{1}{1-a^{\frac{1}{2}}}$

(g) $\frac{2^{n+4}-2(2^n)}{2(2^{n+3})}$ (h) $\sqrt{a\sqrt{a\sqrt{a}}}$

8. Simplify the following

(a) $\frac{\sqrt{x}\times\sqrt[3]{x^2}}{\sqrt[4]{x}}$ (b) $\frac{b^{n+1}\times 8a^{2n-1}}{(2b)^2(ab)^{-n+1}}$ (c) $\frac{2^n-6^n}{1-3^n}$

(d) $\frac{7^{m+1}-7^m}{7^n-7^{n+2}}$ (e) $\frac{5^{2n+1}+25^n}{5^{2n}+5^{1+n}}$ (f) $\left(x-2x^{\frac{1}{2}}+1\right)^{\frac{1}{2}} \times \frac{x+1}{\sqrt{x}-1}$

7.1.2 INDICIAL EQUATIONS

Solving equations of the form $x^{\frac{1}{2}} = 3$, where the **variable is the base**, requires that we square both sides of the equation so that $\left(x^{\frac{1}{2}}\right)^2 = 3^2 \Rightarrow x = 9$. However, when the **variable is the power** and not the base we need to take a different approach.

> Indicial (exponential) equations take on the general form $b^x = a$, where the unknown (variable), x, is the power

Consider the case where we wish to solve for x given that $2^x = 8$.

In this case we need to think of a value of x so that when 2 is raised to the power of x the answer is 8. Using trial and error, it is not too difficult to arrive at $x = 3$ $(2^3 = 2\times 2\times 2 = 8)$.

Next consider the equation $3^{x+1} = 27$.
Again, we need to find a number such that when 3 is raised to that number, the answer is 27. Here we have that $27 = 3^3$. Therefore we can rewrite the equation as $3^{x+1} = 3^3$.
As the base on both sides of the equality is the same we can then equate the powers, that is,

$$3^{x+1} = 27 \Leftrightarrow 3^{x+1} = 3^3$$
$$\Leftrightarrow x+1 = 3$$
$$\Leftrightarrow x = 2$$

Such an approach can be used for a variety of equations. We summarise this process for simple exponential equations:

	Solve for x : $b^x = N$	Example: Solve $5^x = 625$
Step 1:	Express the number N in the form b^{number}	$625 = 5^4$
Step 2:	Write the equation $b^x = b^{\text{number}}$	$\therefore 5^x = 5^4$
Step 3:	Equate exponents, x = number	$\Leftrightarrow x = 4$

EXAMPLE 7.4

Solve the following

(a) $3^x = 81$ (b) $2 \times 5^u = 250$ (c) $2^x = \frac{1}{32}$

Solution

(a) $3^x = 81 \Leftrightarrow 3^x = 3^4$

$\Leftrightarrow x = 4$

(b) $2 \times 5^u = 250 \Leftrightarrow 5^u = 125$

$\Leftrightarrow 5^u = 5^3$

$\Leftrightarrow u = 3$

(c) $2^x = \frac{1}{32} \Leftrightarrow 2^x = \frac{1}{2^5}$

$\Leftrightarrow 2^x = 2^{-5}$

$\Leftrightarrow x = -5$

EXAMPLE 7.5

Find

(a) $\left\{x \mid \left(\frac{1}{2}\right)^x = 16\right\}$ (b) $\{x \mid 3^{x+1} = 3\sqrt{3}\}$ (c) $\{x \mid 4^{x-1} = 64\}$

Solution

(a) $\left(\frac{1}{2}\right)^x = 16 \Leftrightarrow (2^{-1})^x = 16$

$\Leftrightarrow 2^{-x} = 2^4$

$\Leftrightarrow -x = 4$

$\Leftrightarrow x = -4$

i.e., solution set is $\{-4\}$.

(b) $3^{x+1} = 3\sqrt{3} \Leftrightarrow 3^{x+1} = 3 \times 3^{1/2}$

$\Leftrightarrow 3^{x+1} = 3^{3/2}$

$\Leftrightarrow x + 1 = \frac{3}{2}$

$\Leftrightarrow x = \frac{1}{2}$

i.e., solution set is $\{0.5\}$.

(c) $4^{x-1} = 64 \Leftrightarrow (2^2)^{x-1} = 2^6$

$\Leftrightarrow 2^{2x-2} = 2^6$

$\Leftrightarrow 2x - 2 = 6$

$\Leftrightarrow 2x = 8$

$\Leftrightarrow x = 4$

i.e., solution set is $\{4\}$.

EXERCISES 7.1.2

1. Solve the following equations.

(a) $\{x|\ 4^x = 16\}$ (b) $\left\{x|\ 7^x = \frac{1}{49}\right\}$ (c) $\{x|\ 8^x = 4\}$

(d) $\{x|\ 3^x = 243\}$ (e) $\{x|\ 3^{x-2} = 81\}$ (f) $\left\{x|\ 4^x = \frac{1}{32}\right\}$

(g) $\{x|\ 3^{2x-4} = 1\}$ (h) $\{x|\ 4^{2x+1} = 128\}$ (i) $\{x|\ 27^x = 3\}$

2. Solve the following equations.

(a) $\{x|\ 7^{x+6} = 1\}$ (b) $\left\{x|\ 8^x = \frac{1}{4}\right\}$ (c) $\{x|\ 10^x = 0.001\}$

(d) $\{x|\ 9^x = 27\}$ (e) $\{x|\ 2^{4x-1} = 1\}$ (f) $\{x|\ 25^x = \sqrt{5}\}$

(g) $\left\{x|\ 16^x = \frac{1}{\sqrt{2}}\right\}$ (h) $\{x|\ 4^{-x} = 32\sqrt{2}\}$ (i) $\{x|\ 9^{-2x} = 243\}$

7.1.3 EQUATIONS OF THE FORM $b^{f(x)} = b^{g(x)}$.

This is an extension of the previous section, in that now we will consider exponential equations of the form $b^{f(x)} = N$ where N can be expressed as a number having base b so that $N = b^{g(x)}$.

Consider the equation $2^{x^2-1} = 8$. Our first step is to express 8 as 2^3 so that we can then write

$$2^{x^2-1} = 8 \Leftrightarrow 2^{x^2-1} = 2^3$$

Then, equating powers we have: $\Leftrightarrow x^2 - 1 = 3$

So that, $\Leftrightarrow x^2 - 4 = 0$

$$\Leftrightarrow (x-2)(x+2) = 0$$

$$\therefore x = 2 \text{ or } x = -2$$

Checking these values by substituting back into the origial equation shows them to be correct.

i.e., when $x = 2$, L.H.S $= 2^{2^2-1} = 2^{4-1} = 2^3 = 8 =$ R.H.S

when $x = -2$, L.H.S $= 2^{(-2)^2-1} = 2^{4-1} = 2^3 = 8 =$ R.H.S

However, had the equation been, $2^{x^2-1} = 2^{5-x}$, then the solution would have been

$$2^{x^2-1} = 2^{5-x} \Leftrightarrow x^2 - 1 = 5 - x \quad \text{[equating powers]}$$

$$\Leftrightarrow x^2 + x - 6 = 0$$

$$\Leftrightarrow (x-2)(x+3) = 0$$

$$\therefore x = 2 \text{ or } x = -3$$

Again, we can check that these solutions satisfy the original equation.
The thing to note here is that the solution process has not altered. Rather than having one of the powers represented by a constant, we now have both powers containing the variable.

EXAMPLE 7.6

Find $\{x|\ 3^{x^2-5x+2} = 9^{x+1}\}$.

Solution

We need to first express the equation in the form $b^{f(x)} = b^{g(x)}$ where, in this case, $b = 3$:

$$3^{x^2-5x+2} = 9^{x+1} \Leftrightarrow 3^{x^2-5x+2} = (3^2)^{x+1}$$
$$\Leftrightarrow 3^{x^2-5x+2} = 3^{2x+2}$$
$$\Leftrightarrow x^2-5x+2 = 2x+2$$
$$\Leftrightarrow x^2-7x = 0$$
$$\Leftrightarrow x(x-7) = 0$$
$$\Leftrightarrow x = 0 \text{ or } x = 7$$

Again, checking our solutions we have, $x=0$: L.H.S $= 3^{0-0+2} = 9 = 9^{0+1} =$ R.H.S

$x=7$: L.H.S $= 3^{7^2-5\times7+2} = 3^{16} = 9^{7+1} =$ R.H.S

Therefore, the solution set is $\{0, 7\}$

We now have a more general statement for solving exponential equations:

$b^{f(x)} = b^{g(x)} \Leftrightarrow f(x) = g(x)$, where $b > 0$ and $b \neq 1$.

It is important to realise that this will only be true if **the base is the same on both sides** of the equality sign.

EXERCISES 7.1.3

1. Solve the following for the unknown

(a) $9^{2x-1} = 3^{2x+5}$ (b) $4^{x+1} = 8^{2x-4}$ (c) $25^{2x+3} = 125^{x+1}$

(d) $2^{4x+1} = 4^{x+2}$ (e) $16^{2x-1} = 8^{2x+1}$ (f) $\sqrt{3}\times 27^{x+1} = 9^{2x+1}$

(g) $(\sqrt{3})^{x-1} = 9^{-x+2}$ (h) $8^x = \dfrac{1}{16^{x+1}}$ (i) $4^{x+2}\times 8^{x-1} = 2$

2. Solve for the unknown

(a) $8^{x+1} = \dfrac{1}{2^x}$ (b) $8^{x+1} = 2^{x^2-1}$ (c) $3^{x-1} = 3^{x^2-1}$

(d) $4^{x^2-7x+12} = 1$ (e) $6^{\sqrt{n^2-3n}} = 36$ (f) $(5^x)^2 = 5^{x^2}$

3. Solve the following

(a) $(x^2-x-1)^{x^2} = x^2-x-1$

(b) $(x-2)^{x^2-x-12} = 1$

(c) $(3x-4)^{2x^2} = (3x-4)^{5x-2}$

(d) $|x|^{x^2-2x} = 1$

(e) $(x^2+x-57)^{3x^2+3} = (x^2+x-57)^{10x}$