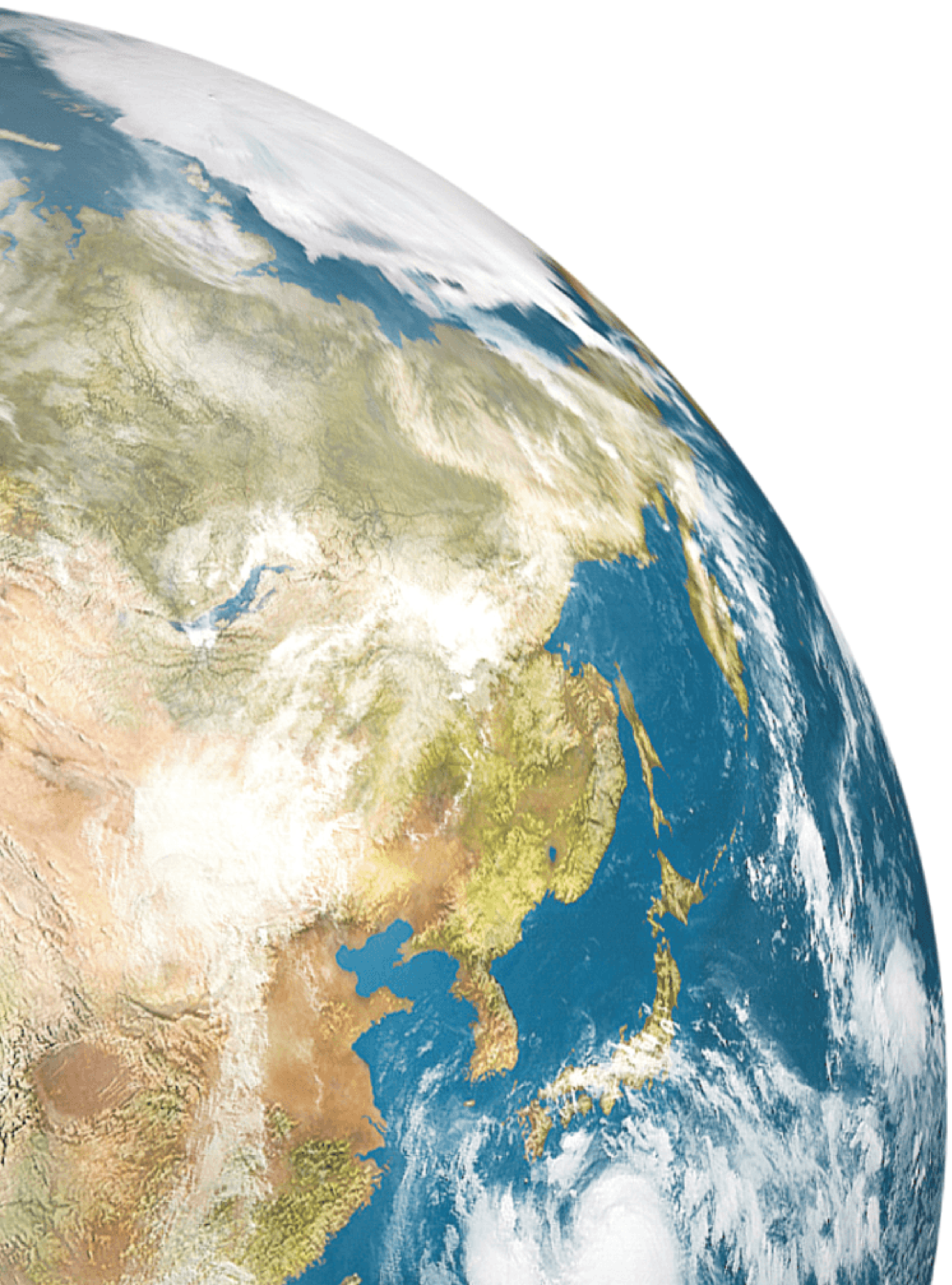
Chapter 1

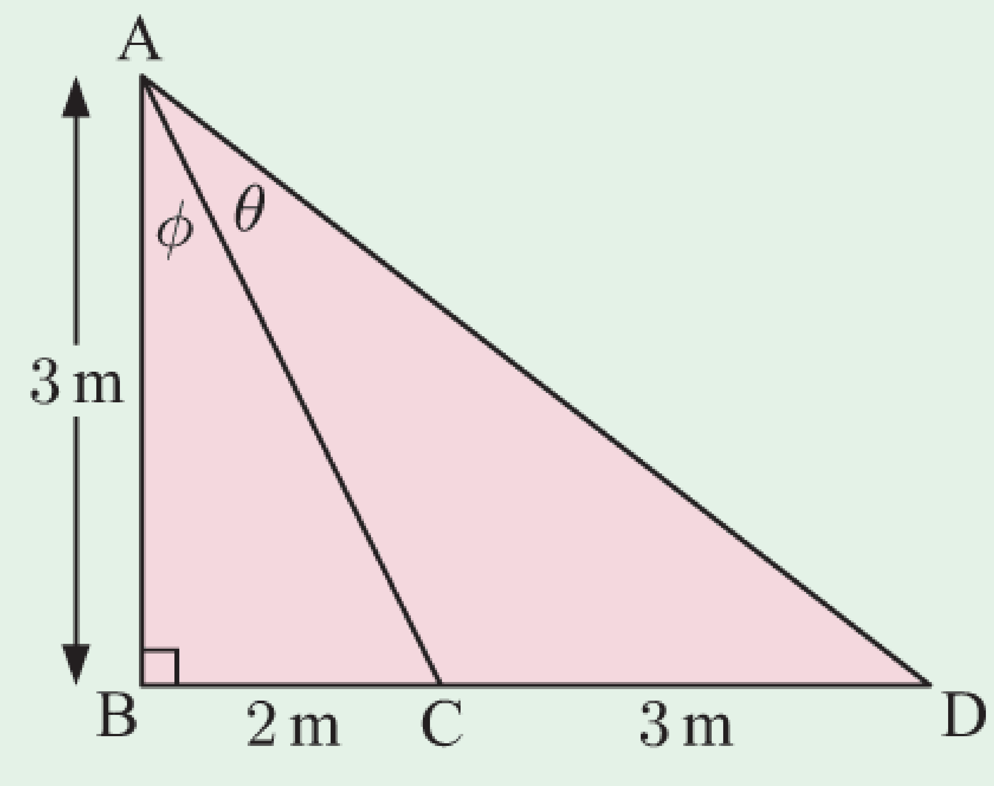
Further trigonometry

Contents:



OPENING PROBLEM

Consider the triangle alongside.



Things to think about:

a Can you find $\tan\theta$ using a combination of Pythagoras' theorem, the cosine rule, and your knowledge of the unit circle?

b Write down the ratios $\tan\phi$ and $\tan(\theta+\phi)$. Is there an easier way to calculate $\tan\theta$ exactly, which involves these ratios?

A RECIPROCAL TRIGONOMETRIC FUNCTIONS

We define the functions cosec x, secant x, and cotangent x as the **reciprocal** functions of $\sin x$, $\cos x$, and $\tan x$ respectively:

$$\operatorname{cosec} x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

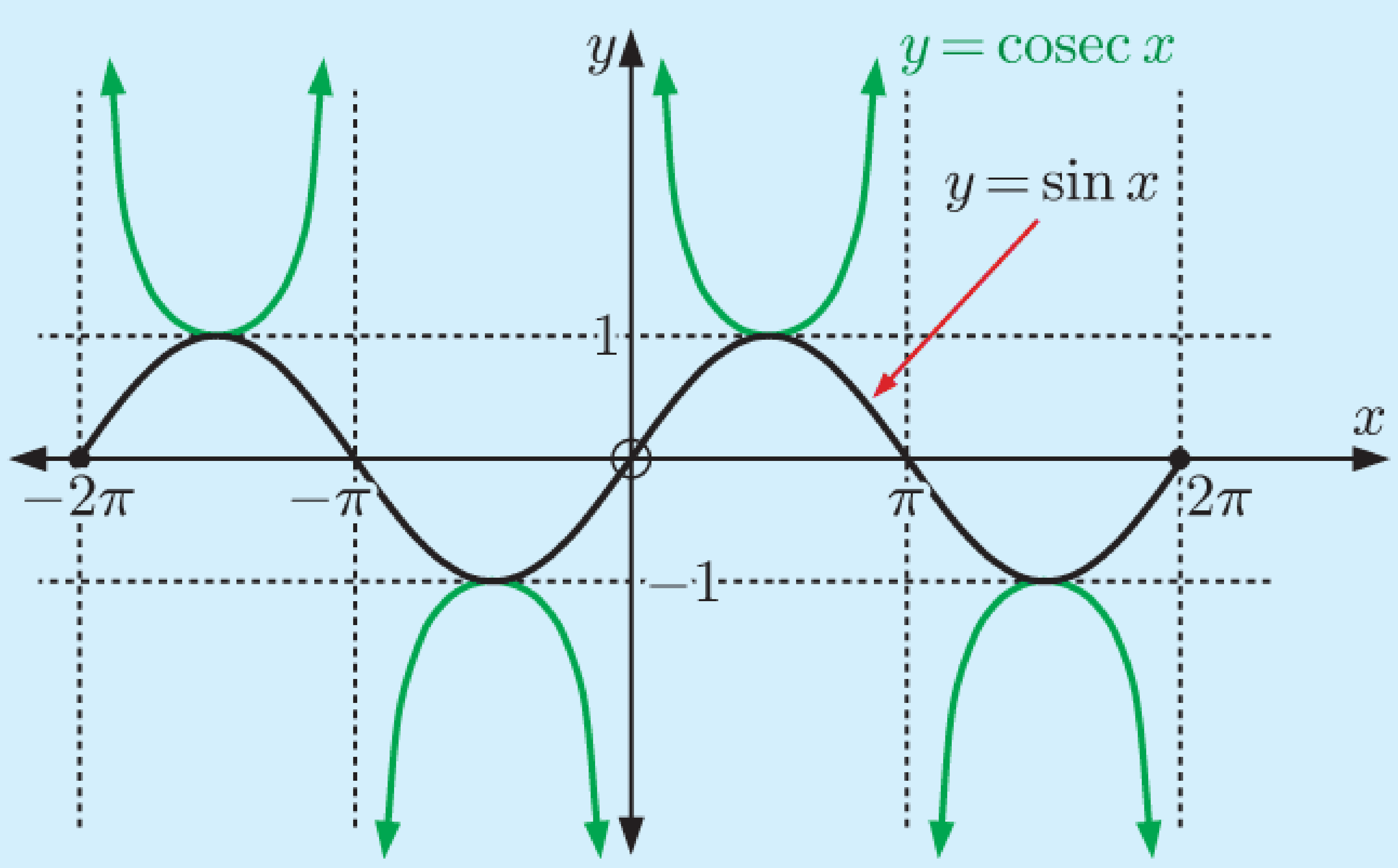
For example, $\operatorname{cosec}\frac{\pi}{6} = \dfrac{1}{\sin\frac{\pi}{6}} = \dfrac{1}{\frac{1}{2}} = 2$.

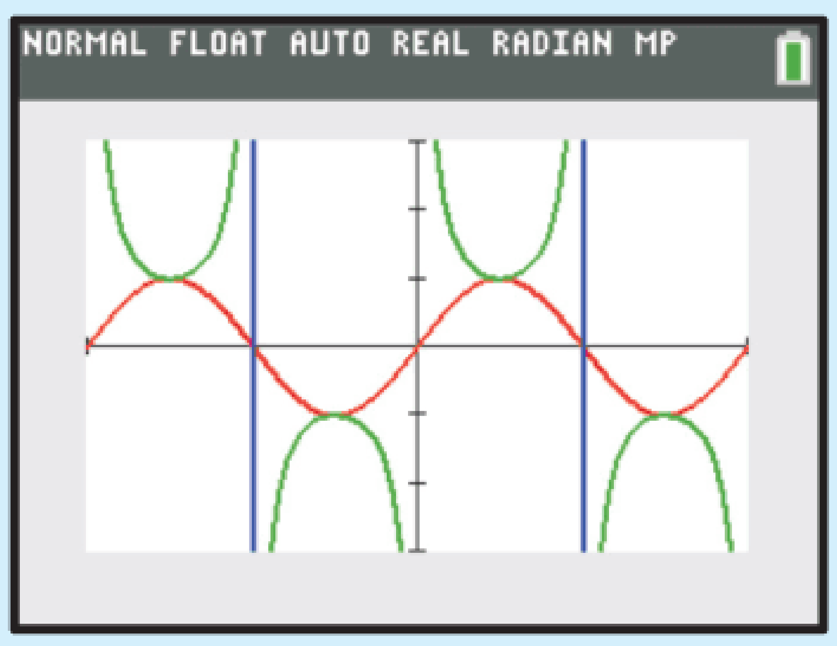
Example 1

Self Tutor

Using the graph of $y=\sin x$, sketch the graph of $y=\dfrac{1}{\sin x}=\operatorname{cosec} x$ for $-2\pi \leqslant x \leqslant 2\pi$.
Check your answer using technology.

- The zeros of $y=\sin x$ become vertical asymptotes of $y=\dfrac{1}{\sin x}$.
- The local maxima of $y=\sin x$ become local minima of $y=\dfrac{1}{\sin x}$.
- The local minima of $y=\sin x$ become local maxima of $y=\dfrac{1}{\sin x}$.
- When $\sin x=1$, $\dfrac{1}{\sin x}=1$ and when $\sin x=-1$, $\dfrac{1}{\sin x}=-1$.





EXERCISE 1A

1 Without using a calculator, find:

a $\operatorname{cosec} \frac{\pi}{3}$ **b** $\cot \frac{2\pi}{3}$ **c** $\sec \frac{5\pi}{6}$ **d** $\cot \pi$

2 Without using a calculator, find $\operatorname{cosec} x$, $\sec x$, and $\cot x$ given:

a $\sin x = \frac{3}{5}$, $0 \leqslant x \leqslant \frac{\pi}{2}$ **b** $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$

3 Find the other *five* trigonometric ratios if:

a $\cos x = \frac{3}{4}$ and $\frac{3\pi}{2} < x < 2\pi$ **b** $\sin x = -\frac{2}{3}$ and $\pi < x < \frac{3\pi}{2}$

c $\sec x = 2\frac{1}{2}$ and $0 < x < \frac{\pi}{2}$ **d** $\operatorname{cosec} x = 2$ and $\frac{\pi}{2} < x < \pi$

e $\tan \beta = \frac{1}{2}$ and $\pi < \beta < \frac{3\pi}{2}$ **f** $\cot \theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$.

4 Using the graph of $y = \cos x$, sketch the graph of $y = \frac{1}{\cos x} = \sec x$ for $-2\pi \leqslant x \leqslant 2\pi$. Check your answer using technology.

5 Using the graph of $y = \tan x$, sketch the graph of $y = \frac{1}{\tan x} = \cot x$ for $-2\pi \leqslant x \leqslant 2\pi$.

6 Use technology to sketch $y = \sec x$ and $y = \operatorname{cosec}\left(x + \frac{\pi}{2}\right)$ on the same set of axes for $-2\pi \leqslant x \leqslant 2\pi$. Explain your answer.

7 Solve exactly for $0 \leqslant x \leqslant 2\pi$:

a $\sec x = 2$ **b** $\operatorname{cosec} x = -\sqrt{2}$

c $\sqrt{3}\sec 2x = -2$ **d** $\operatorname{cosec}\left(x + \frac{\pi}{6}\right) + \sqrt{2} = 0$

e $\cot x + 1 = 0$ **f** $\cot\left(2x - \frac{\pi}{4}\right) - \sqrt{3} = 0$

8 Solve exactly for $0 \leqslant x \leqslant 2\pi$:

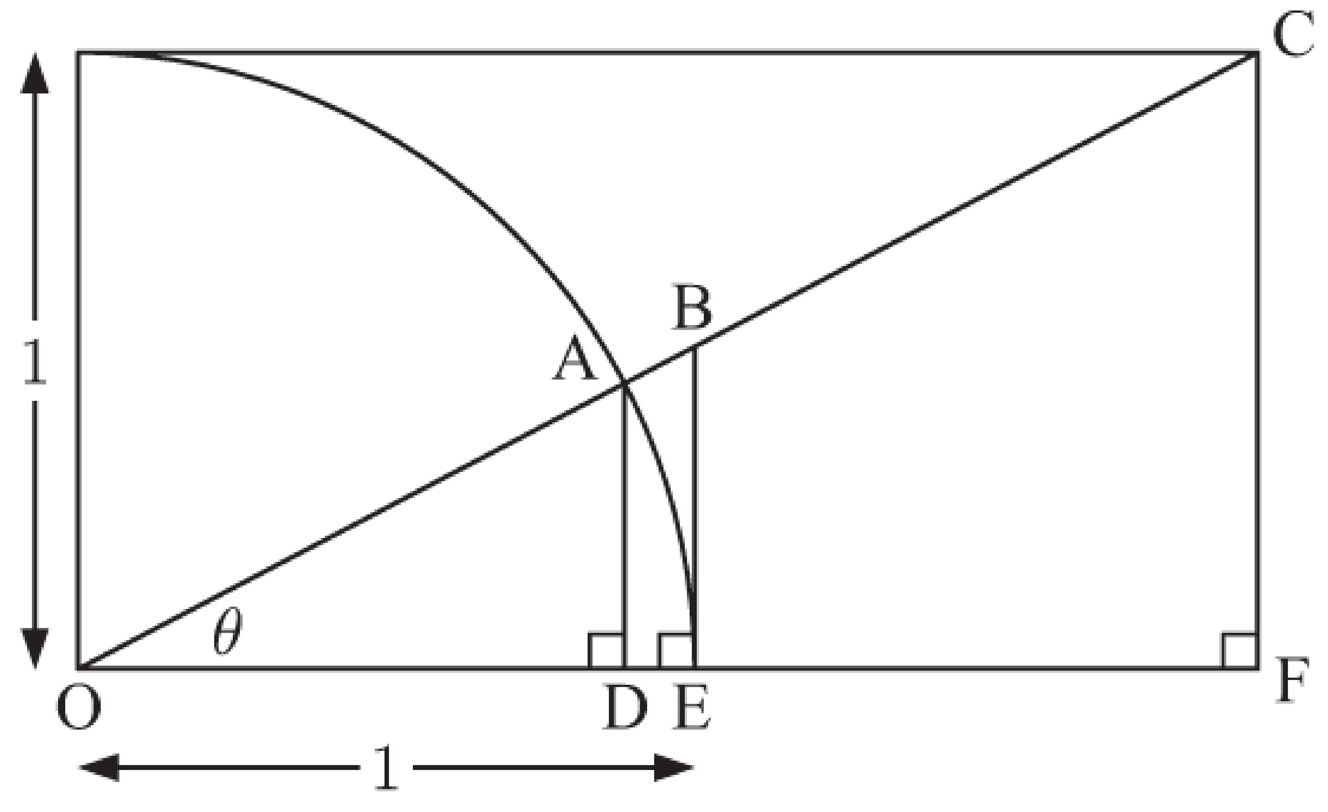
a $2\sin x + \operatorname{cosec} x = 3$ **b** $2\cos x = \sec x$

c $\tan^2 x - 2 - 3\cot^2 x = 0$ **d** $4\sin x = \sqrt{3}\operatorname{cosec} x + 2 - 2\sqrt{3}$

9 Consider the diagram alongside.
Identify a line segment which has length:

a $\cos\theta$ **b** $\sin\theta$ **c** $\tan\theta$

d $\sec\theta$ **e** $\operatorname{cosec}\theta$ **f** $\cot\theta$



HISTORICAL NOTE ASTRONOMY AND TRIGONOMETRY

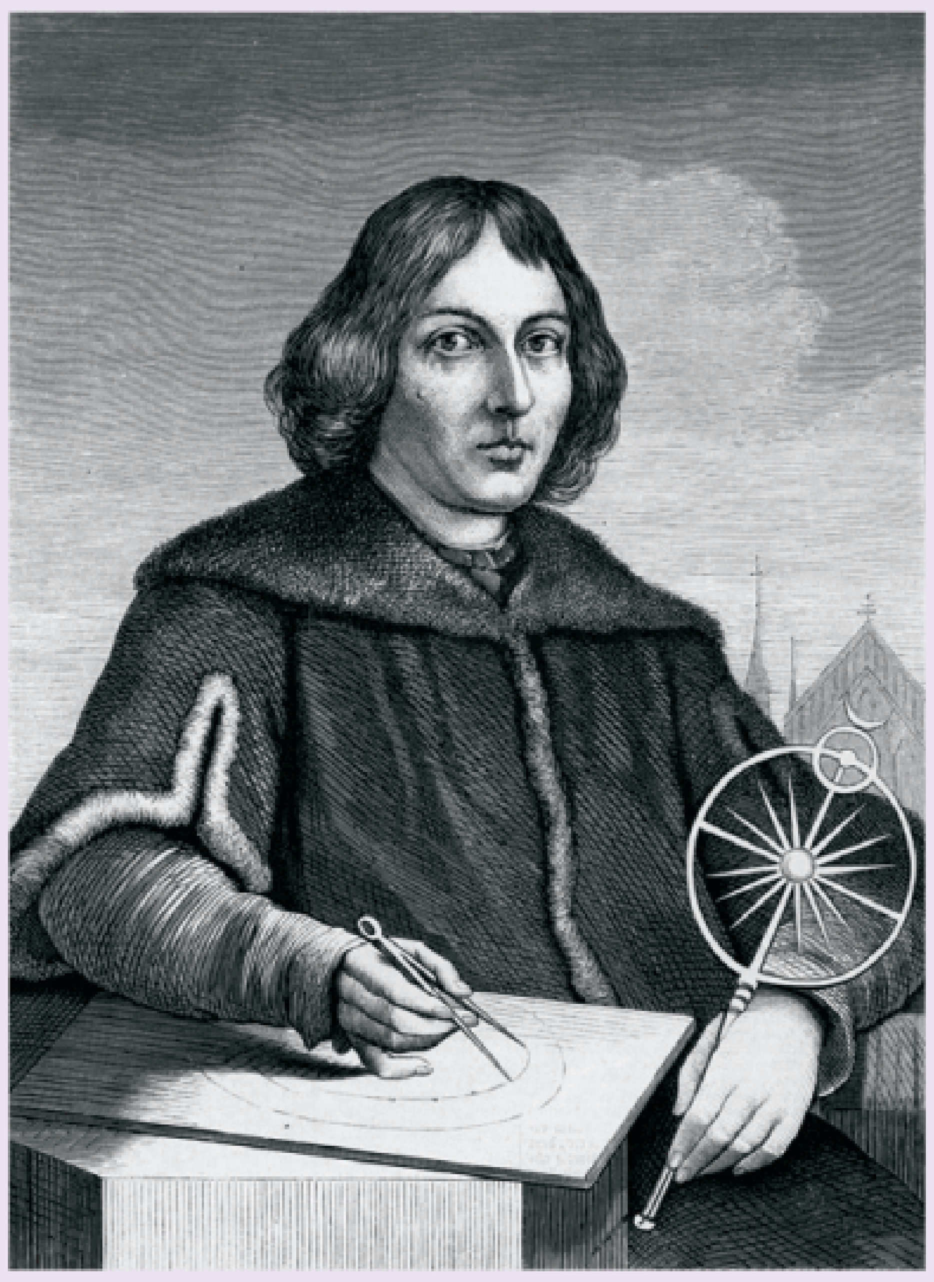
The Greek astronomer **Hipparchus** (140 BC) is credited with being the founder of trigonometry. To aid his astronomical calculations, he produced a table of numbers in which the lengths of chords of a circle were related to the length of the radius.

Ptolemy, another great Greek astronomer of the time, extended this table in his major published work *Almagest*, which was used by astronomers for the next 1000 years. In fact, much of Hipparchus' work is known through the writings of Ptolemy. These writings found their way to Hindu and Arab scholars.

Aryabhata, a Hindu mathematician in the 5th and 6th Century AD, constructed a table of the lengths of half-chords of a circle with radius one unit. This was the first table of **sine** values.

In the late 16th century, **Georg Joachim de Porris**, also known as **Rheticus**, produced comprehensive and remarkably accurate tables of all six trigonometric ratios. These involved a tremendous number of tedious calculations, all without the aid of calculators or computers.

Rheticus was the only student of **Nicolaus Copernicus**, and helped his tutor publish his work *De revolutionibus orbium coelestium* (On the Revolutions of the Heavenly Spheres).

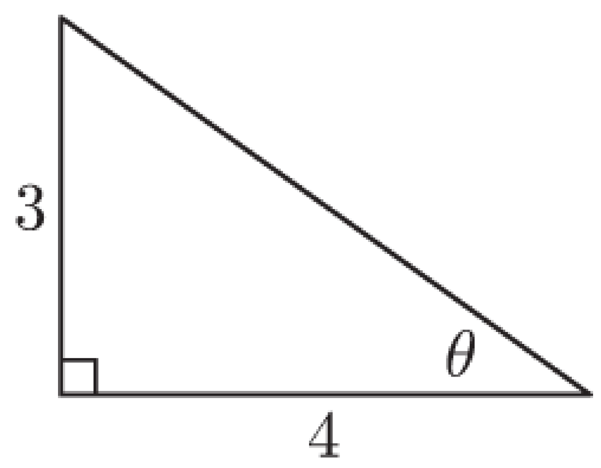


Nicolaus Copernicus

B INVERSE TRIGONOMETRIC FUNCTIONS

In many problems we need to know what angle results in a particular trigonometric ratio. We have previously seen this for right angled triangle problems and when using the cosine and sine rules.

For example:



$\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{3}{4}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{3}{4}\right)$

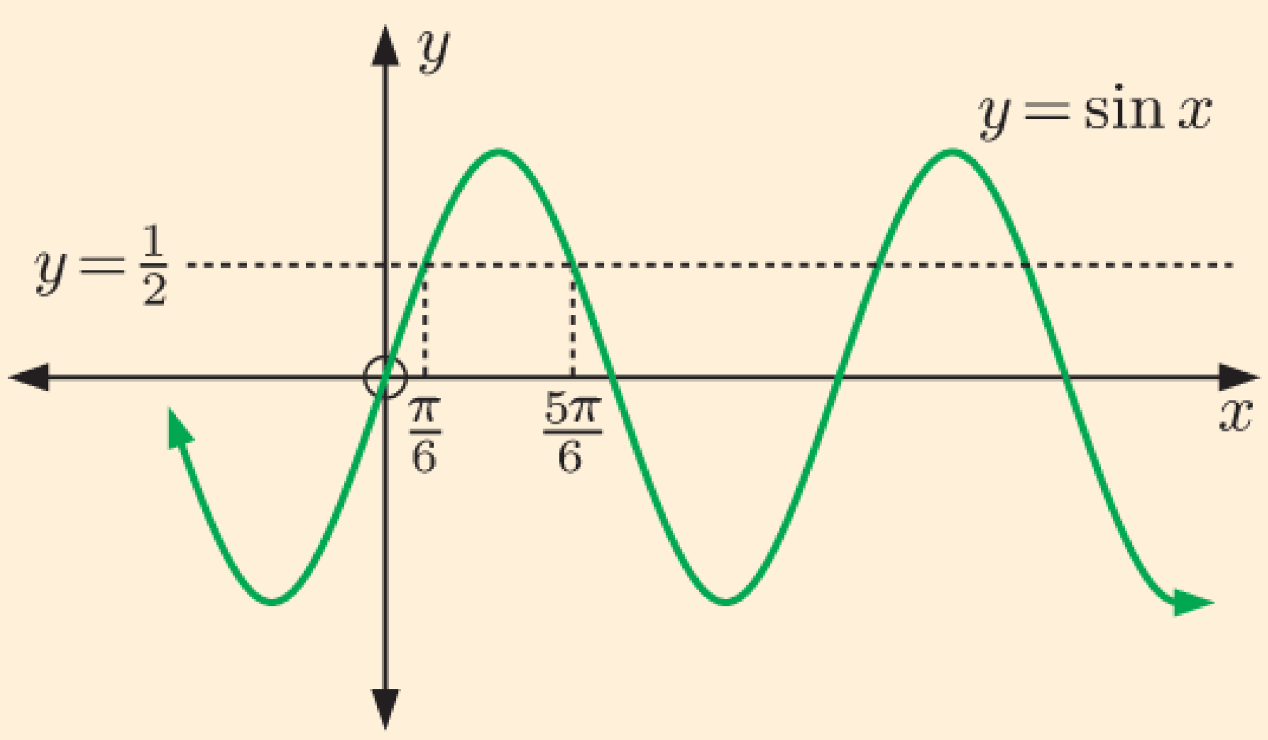
$\therefore \quad \theta \approx 36.9°$

Rather than writing the inverse trigonometric functions as $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$, which can be confused with reciprocals, mathematicians more formally refer to these functions as the **inverse trigonometric functions arcsine**, **arccosine**, and **arctangent**.

INVESTIGATION 1 INVERSE TRIGONOMETRIC FUNCTIONS

The function $y = \sin x$ fails the horizontal line test on its natural domain. For example, when $y = \frac{1}{2}$, there is more than one corresponding value of x, since $\sin x = \frac{1}{2}$ has infinitely many solutions, including $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

Therefore, if we want to find inverse functions for $\sin x$, $\cos x$, and $\tan x$, we will need to apply suitable domain restrictions.



What to do:

1. Draw the graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$ on the domain $-3\pi \leqslant x \leqslant 3\pi$.

2 Copy and complete the following table, indicating whether each function is one-to-one on the given restricted domain.

Restricted domain	$\sin x$	$\cos x$	$\tan x$
$0 \leqslant x \leqslant 2\pi$			
$-\pi \leqslant x \leqslant \pi$			
$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$			
$0 \leqslant x \leqslant \pi$			
$-\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$			
$\pi \leqslant x \leqslant 2\pi$			

3 Discuss which domain restriction you think would be most suitable for the inverse function of:

a $\sin x$ **b** $\cos x$ **c** $\tan x$.

4 The function $f(x) = \sin x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ has inverse $f^{-1}(x) = \arcsin x$.

Sketch the graph of $y = f(x)$, and hence sketch the graph of $f^{-1}(x) = \arcsin x$ on the same set of axes.

5 The function $f(x) = \cos x$, $0 \leqslant x \leqslant \pi$ has inverse $f^{-1}(x) = \arccos x$.

Sketch the graph of $y = f(x)$, and hence sketch the graph of $f^{-1}(x) = \arccos x$ on the same set of axes.

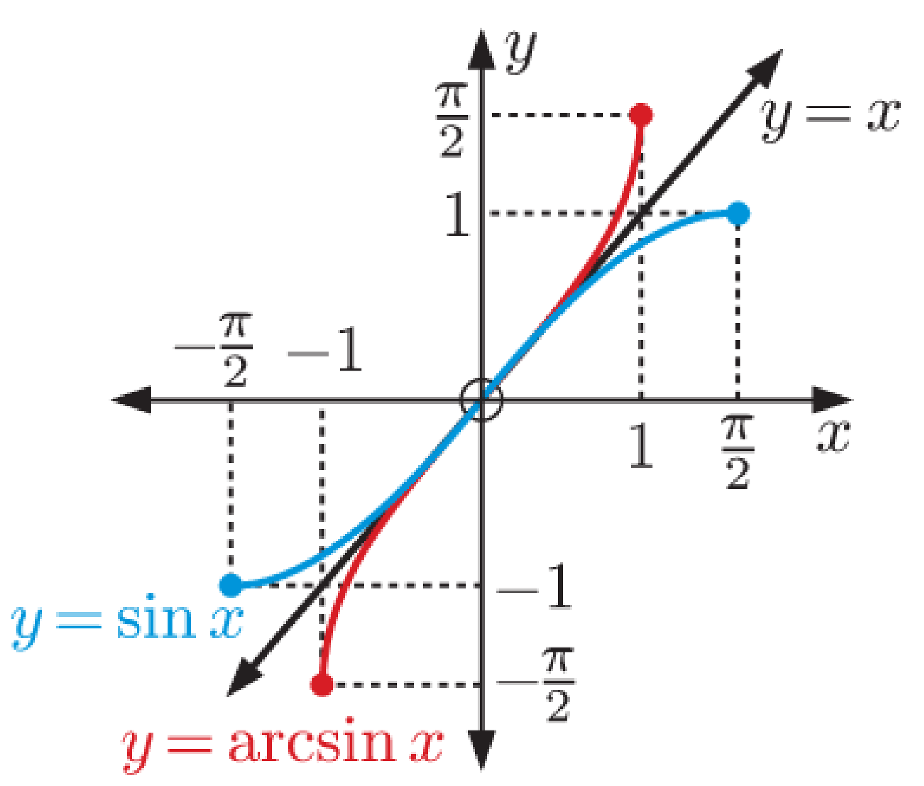
6 The function $f(x) = \tan x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ has inverse $f^{-1}(x) = \arctan x$.

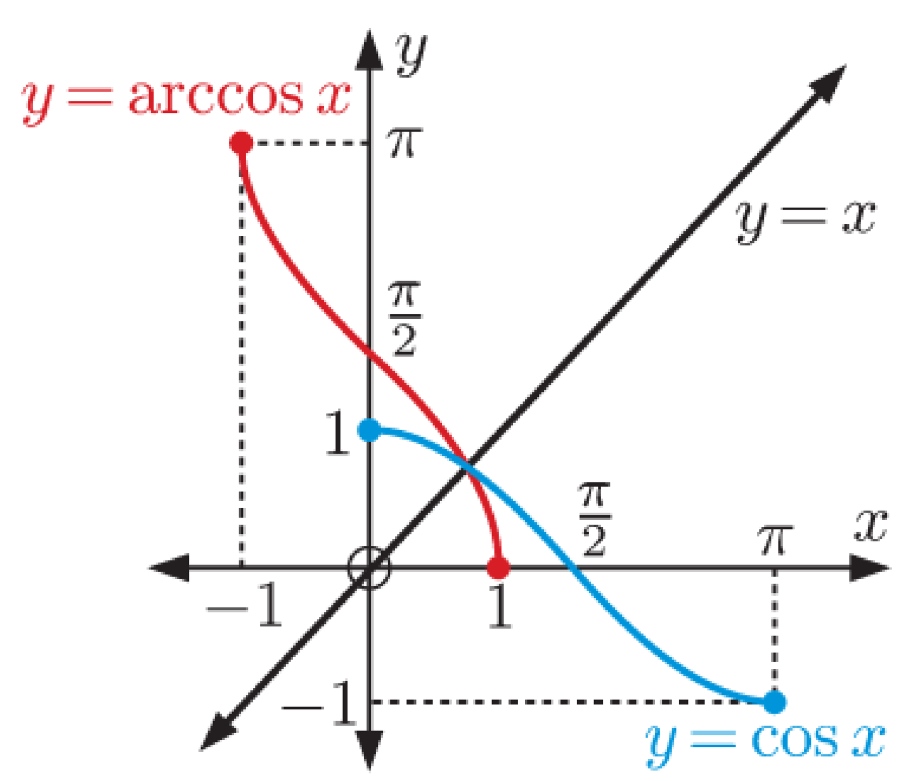
Sketch the graph of $y = f(x)$, and hence sketch the graph of $f^{-1}(x) = \arctan x$ on the same set of axes.

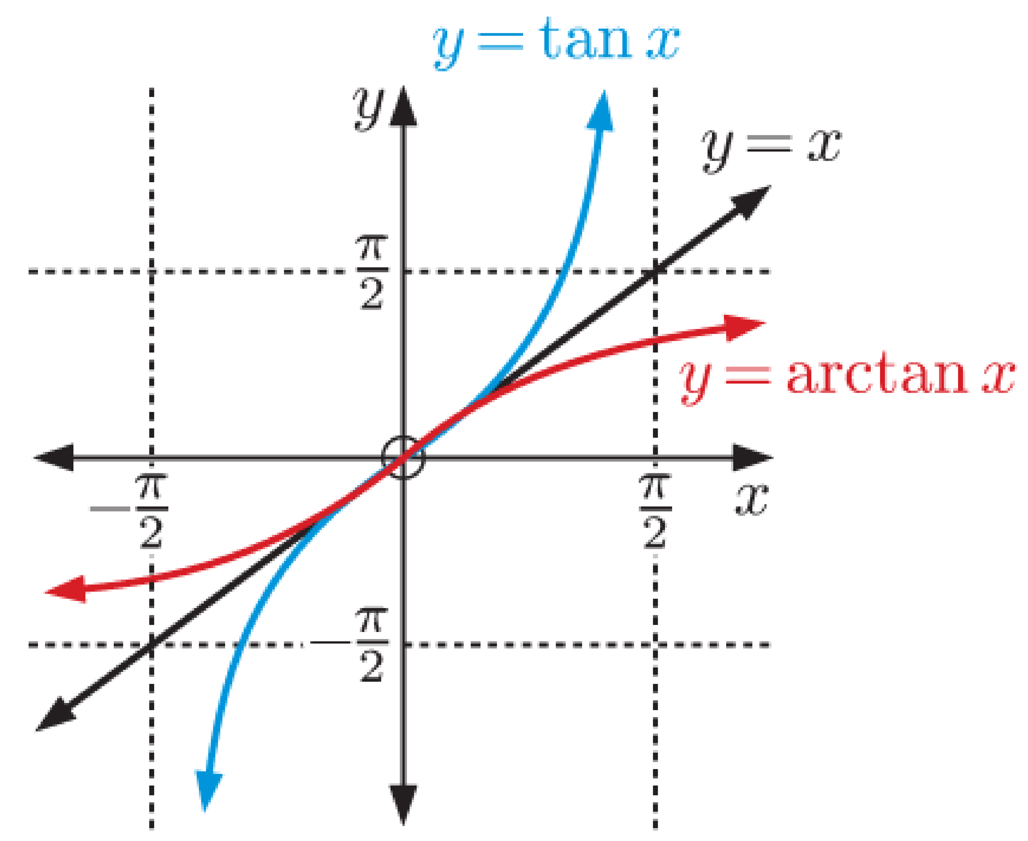
The **inverse trigonometric functions** are defined as:

Function	*Definition*	*Range*
$y = \arcsin x$	$x = \sin y$, $-1 \leqslant x \leqslant 1$	$-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$
$y = \arccos x$	$x = \cos y$, $-1 \leqslant x \leqslant 1$	$0 \leqslant y \leqslant \pi$
$y = \arctan x$	$x = \tan y$, $x \in \mathbb{R}$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

The graphs of these functions are illustrated below, along with the corresponding graphs of $\sin x$, $\cos x$, and $\tan x$ on their restricted domains.







EXERCISE 1B

1 Find, giving your answer in radians:

a $\arccos 1$ **b** $\arcsin(-1)$ **c** $\arctan 1$ **d** $\arctan(-1)$

e $\arcsin \frac{1}{2}$ **f** $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ **g** $\arctan \sqrt{3}$ **h** $\arccos\left(-\frac{1}{\sqrt{2}}\right)$

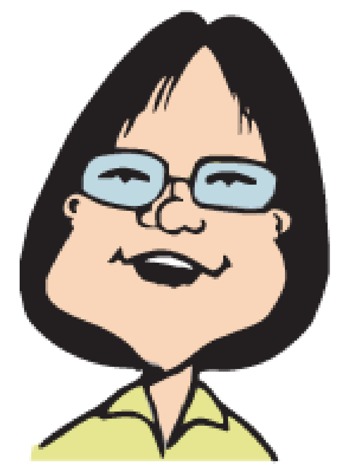
i $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ **j** $\sin^{-1}(-0.767)$ **k** $\cos^{-1} 0.327$ **l** $\tan^{-1}(-50)$

2 Find the invariant point for the inverse transformation from:

a $y = \sin x$ to $y = \arcsin x$

b $y = \tan x$ to $y = \arctan x$

c $y = \cos x$ to $y = \arccos x$.



3 **a** State the equations of the asymptotes of $y = \arctan x$.

b Do the functions $y = \arcsin x$ and $y = \arccos x$ have vertical asymptotes? Explain your answer.

4 Simplify:

a $\arcsin\left(\sin \frac{\pi}{3}\right)$ **b** $\arccos\left(\cos\left(-\frac{\pi}{6}\right)\right)$

c $\tan(\arctan(0.3))$ **d** $\cos\left(\arccos\left(-\frac{1}{2}\right)\right)$

e $\arctan(\tan \pi)$ **f** $\arcsin\left(\sin \frac{4\pi}{3}\right)$

Example 2 Self Tutor

Find, where possible, the exact solutions of:

a $\arctan x = \frac{\pi}{3}$ **b** $\arccos(x-1) = \frac{2\pi}{3}$ **c** $\arcsin x = \frac{2\pi}{3}$

a The range of $y = \arctan x$ is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. $\frac{\pi}{3}$ is within the range.

$\therefore \quad x = \tan \frac{\pi}{3} = \sqrt{3}$

b The range of $y = \arccos(x-1)$ is $0 \leqslant y \leqslant \pi$. $\frac{2\pi}{3}$ is within the range.

$\therefore \quad x - 1 = \cos \frac{2\pi}{3}$

$\therefore \quad x - 1 = -\frac{1}{2}$

$\therefore \quad x = \frac{1}{2}$

c The range of $y = \arcsin x$ is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, and $\frac{2\pi}{3}$ is outside this range.

$\therefore$ no solution exists, even though we can find $\sin \frac{2\pi}{3}$.

5 Find, where possible, the exact solutions of:

a $\arctan x = \frac{\pi}{4}$ **b** $\arcsin x = -\frac{\pi}{3}$ **c** $\arccos x = \frac{3\pi}{4}$

d $\arcsin(x+1) = \frac{\pi}{6}$ **e** $\arccos x = -\frac{\pi}{4}$ **f** $\arctan(x - \sqrt{3}) = -\frac{\pi}{3}$

ACTIVITY 1 — arctan x

Carl Friedrich Gauss used his **Gaussian hypergeometric series** to analyse the continued fraction:

$$\cfrac{x}{1+\cfrac{(1x)^2}{3+\cfrac{(2x)^2}{5+\cfrac{(3x)^2}{7+\cfrac{(4x)^2}{9+\dots}}}}}$$

What to do:

1. Evaluate the fraction with $x = 1$ for as many levels as necessary for the answer to be accurate to 5 decimal places. You may wish to use a spreadsheet.
2. Compare your result with $\arctan 1$.
3. Compare the continued fraction and $\arctan x$ for another value of x of your choosing.

C ALGEBRA WITH TRIGONOMETRIC FUNCTIONS

For any given angle θ, $\sin\theta$ and $\cos\theta$ are real numbers. $\tan\theta$ is also real whenever it is defined. The algebra of trigonometry is therefore identical to the algebra of real numbers.

An expression like $2\sin\theta + 3\sin\theta$ compares with $2x + 3x$ when we wish to do simplification, and so $2\sin\theta + 3\sin\theta = 5\sin\theta$.

Example 3 — Self Tutor

Simplify:

a $3\cos\theta + 4\cos\theta$ **b** $\tan\alpha - 3\tan\alpha$

a $3\cos\theta + 4\cos\theta = 7\cos\theta$
{compare with $3x + 4x = 7x$}

b $\tan\alpha - 3\tan\alpha = -2\tan\alpha$
{compare with $x - 3x = -2x$}

ANGLE RELATIONSHIPS

The **negative angle formulae** are established by reflection in the x-axis:

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

The **supplementary angle formulae** are established by reflection in the y-axis:

$$\sin(\pi - \theta) = \sin\theta \qquad \cos(\pi - \theta) = -\cos\theta$$

The **complementary angle formulae** are established by reflection in the line $y = x$:

$$\sin\left(\tfrac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\tfrac{\pi}{2} - \theta\right) = \sin\theta$$

ANGLE RELATIONSHIPS

The tangent definition $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ enables us to calculate the tangent ratio in each case.

Example 4 — Self Tutor

Simplify:

a $\sin(-\theta) + 2\sin\theta$

b $\dfrac{\cos(\frac{\pi}{2} - \theta)}{\cos(\pi - \theta)}$

a $\sin(-\theta) + 2\sin\theta$
$= -\sin\theta + 2\sin\theta$
$= \sin\theta$

b $\dfrac{\cos(\frac{\pi}{2} - \theta)}{\cos(\pi - \theta)}$
$= \dfrac{\sin\theta}{-\cos\theta}$
$= -\tan\theta$

EXERCISE 1C.1

1 Simplify:

a $\sin\theta + \sin\theta$ **b** $2\cos\theta + \cos\theta$ **c** $3\sin\theta - \sin\theta$
d $3\sin\theta - 2\sin\theta$ **e** $\tan\theta - 3\tan\theta$ **f** $2\cos^2\theta - 5\cos^2\theta$

2 Simplify:

a $3\tan x - \dfrac{\sin x}{\cos x}$ **b** $\dfrac{\sin^2 x}{\cos^2 x}$ **c** $\tan x \cos x$
d $\dfrac{\sin x}{\tan x}$ **e** $3\sin x + 2\cos x \tan x$ **f** $\dfrac{2\tan x}{\sin x}$

3 Simplify:

a $\tan x \cot x$ **b** $\sin x \operatorname{cosec} x$ **c** $\operatorname{cosec} x \cot x$
d $\sin x \cot x$ **e** $\dfrac{\cot x}{\operatorname{cosec} x}$

4 Simplify:

a $3\cos\theta - \cos(-\theta)$ **b** $\tan(-\theta)$ **c** $\sin(-\theta) + \cos\left(\frac{\pi}{2} - \theta\right)$
d $\tan(\pi - \theta)$ **e** $\tan\left(\frac{\pi}{2} - \theta\right)$ **f** $\sin\left(\frac{\pi}{2} - \theta\right) - \cos(\pi - \theta)$
g $\dfrac{\sin(-\theta)}{\cos(\pi - \theta)}$ **h** $\dfrac{\cos(\frac{\pi}{2} - \theta)}{\cos(-\theta)}$ **i** $\dfrac{\sin(\pi - \theta) - \sin(-\theta)}{\cos(-\theta)}$

THE PYTHAGOREAN IDENTITY

The **Pythagorean identity** is established by applying Pythagoras' theorem on the unit circle:

$$\boldsymbol{\sin^2\theta + \cos^2\theta = 1}$$

We commonly use rearrangements of these formulae such as:

$$\sin^2\theta = 1 - \cos^2\theta \qquad \cos^2\theta = 1 - \sin^2\theta$$

Using these definitions we can also derive the identities:

$$\boldsymbol{\tan^2\theta + 1 = \sec^2\theta} \quad \text{and} \quad \boldsymbol{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$$

Proof (for the first case):

Using $\sin^2\theta + \cos^2\theta = 1$,

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \quad \{\text{dividing each term by } \cos^2\theta\}$$

$$\therefore \tan^2\theta + 1 = \sec^2\theta$$

Example 5 Self Tutor

Simplify:

a $2 - 2\sin^2\theta$

b $\cos^2\theta\sin\theta + \sin^3\theta$

a $2 - 2\sin^2\theta$
$= 2(1 - \sin^2\theta)$
$= 2\cos^2\theta$

b $\cos^2\theta\sin\theta + \sin^3\theta$
$= \sin\theta(\cos^2\theta + \sin^2\theta)$
$= \sin\theta$

EXERCISE 1C.2

1 Simplify:

a $3\sin^2\theta + 3\cos^2\theta$

b $-2\sin^2\theta - 2\cos^2\theta$

c $-\cos^2\theta - \sin^2\theta$

d $3 - 3\sin^2\theta$

e $4 - 4\cos^2\theta$

f $\cos^3\theta + \cos\theta\sin^2\theta$

g $\cos^2\theta - 1$

h $\sin^2\theta - 1$

i $2\cos^2\theta - 2$

j $\dfrac{1 - \sin^2\theta}{\cos^2\theta}$

k $\dfrac{1 - \cos^2\theta}{\sin\theta}$

l $\dfrac{\cos^2\theta - 1}{-\sin\theta}$

2 Prove that $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

Example 6 Self Tutor

Expand and simplify: $(\cos\theta - \sin\theta)^2$

$(\cos\theta - \sin\theta)^2$
$= \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta \quad \{\text{using } (a-b)^2 = a^2 - 2ab + b^2\}$
$= \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta$
$= 1 - 2\cos\theta\sin\theta$

3 Expand and simplify, if possible:

a $(1 + \sin\theta)^2$

b $(\sin\alpha - 2)^2$

c $(\tan\alpha - 1)^2$

d $(\sin\alpha + \cos\alpha)^2$

e $(\sin\beta - \cos\beta)^2$

f $-(2 - \cos\alpha)^2$

4 Simplify:

a $1 - \sec^2\beta$

b $\dfrac{\tan^2\theta(\cot^2\theta + 1)}{\tan^2\theta + 1}$

c $\cos^2\alpha(\sec^2\alpha - 1)$

d $(\sin x + \tan x)(\sin x - \tan x)$

e $(2\sin\theta + 3\cos\theta)^2 + (3\sin\theta - 2\cos\theta)^2$

f $(1 + \operatorname{cosec}\theta)(\sin\theta - \sin^2\theta)$

g $\sec A - \sin A\tan A - \cos A$

FACTORISING TRIGONOMETRIC EXPRESSIONS

Example 7 — Self Tutor

Factorise:

a $\cos^2\alpha - \sin^2\alpha$

b $\tan^2\theta - 3\tan\theta + 2$

a $\cos^2\alpha - \sin^2\alpha$

$= (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha)$ $\{a^2 - b^2 = (a+b)(a-b)\}$

b $\tan^2\theta - 3\tan\theta + 2$

$= (\tan\theta - 2)(\tan\theta - 1)$ $\{x^2 - 3x + 2 = (x-2)(x-1)\}$

EXERCISE 1C.3

1 Factorise:

a $1 - \sin^2\theta$

b $\sin^2\alpha - \cos^2\alpha$

c $\tan^2\alpha - 1$

d $2\sin^2\beta - \sin\beta$

e $2\cos\phi + 3\cos^2\phi$

f $3\sin^2\theta - 6\sin\theta$

g $\tan^2\theta + 5\tan\theta + 6$

h $2\cos^2\theta + 7\cos\theta + 3$

i $6\cos^2\alpha - \cos\alpha - 1$

j $3\tan^2\alpha - 2\tan\alpha$

k $\sec^2\beta - \operatorname{cosec}^2\beta$

l $2\cot^2 x - 3\cot x + 1$

m $2\sin^2 x + 7\sin x\cos x + 3\cos^2 x$

Example 8 — Self Tutor

Simplify:

a $\dfrac{2 - 2\cos^2\theta}{1 + \cos\theta}$

b $\dfrac{\cos\theta - \sin\theta}{\cos^2\theta - \sin^2\theta}$

a $\dfrac{2 - 2\cos^2\theta}{1 + \cos\theta}$

$= \dfrac{2(1 - \cos^2\theta)}{1 + \cos\theta}$

$= \dfrac{2(1 + \cos\theta)(1 - \cos\theta)}{(1 + \cos\theta)}$

$= 2(1 - \cos\theta)$

b $\dfrac{\cos\theta - \sin\theta}{\cos^2\theta - \sin^2\theta}$

$= \dfrac{(\cos\theta - \sin\theta)}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)}$

$= \dfrac{1}{\cos\theta + \sin\theta}$

2 Simplify:

a $\dfrac{1 - \sin^2\alpha}{1 - \sin\alpha}$

b $\dfrac{\tan^2\beta - 1}{\tan\beta + 1}$

c $\dfrac{\cos^2\phi - \sin^2\phi}{\cos\phi + \sin\phi}$

d $\dfrac{\cos^2\phi - \sin^2\phi}{\cos\phi - \sin\phi}$

e $\dfrac{\sin\alpha + \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$

f $\dfrac{3 - 3\sin^2\theta}{6\cos\theta}$

g $1 - \dfrac{\cos^2\theta}{1 + \sin\theta}$

h $\dfrac{1 + \cot\theta}{\operatorname{cosec}\theta} - \dfrac{\sec\theta}{\tan\theta + \cot\theta}$

i $\dfrac{\tan^2\theta}{\sec\theta - 1}$

3 Show that:

a $(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2 = 2$

b $(\sin\theta + 4\cos\theta)^2 + (4\sin\theta - \cos\theta)^2 = 17$

c $(1 - \cos\theta)\left(1 + \dfrac{1}{\cos\theta}\right) = \tan\theta\sin\theta$

d $\left(1 + \dfrac{1}{\sin\theta}\right)(\sin\theta - \sin^2\theta) = \cos^2\theta$

e $\sec A - \cos A = \tan A \sin A$

f $\dfrac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$

g $\dfrac{\cos\alpha}{1-\tan\alpha} + \dfrac{\sin\alpha}{1-\cot\alpha} = \sin\alpha + \cos\alpha$

h $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = 2\,\text{cosec}\,\theta$

i $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{\sin\theta}{1+\cos\theta} = 2\cot\theta$

j $\dfrac{1}{1-\sin\theta} + \dfrac{1}{1+\sin\theta} = 2\sec^2\theta$

GRAPHING PACKAGE

Check these simplifications by graphing both sides of the equations on the same set of axes.

4 Solve for $0 \leqslant x \leqslant 2\pi$:

a $2\cos^2 x = \sin x + 1$

b $\sin^2 x = 2 - \cos x$

c $2\cos^2 x = 3\sin x$

d $2\tan^2 x + 3\sec^2 x = 7$

5 For $-\pi \leqslant x \leqslant \pi$, find the exact solutions of $3\sec 2x = \cot 2x + 3\tan 2x$.

D DOUBLE ANGLE IDENTITIES

INVESTIGATION 2 DOUBLE ANGLE IDENTITIES

What to do:

1 Copy and complete this table using your calculator. Include extra lines for angles of your choice.

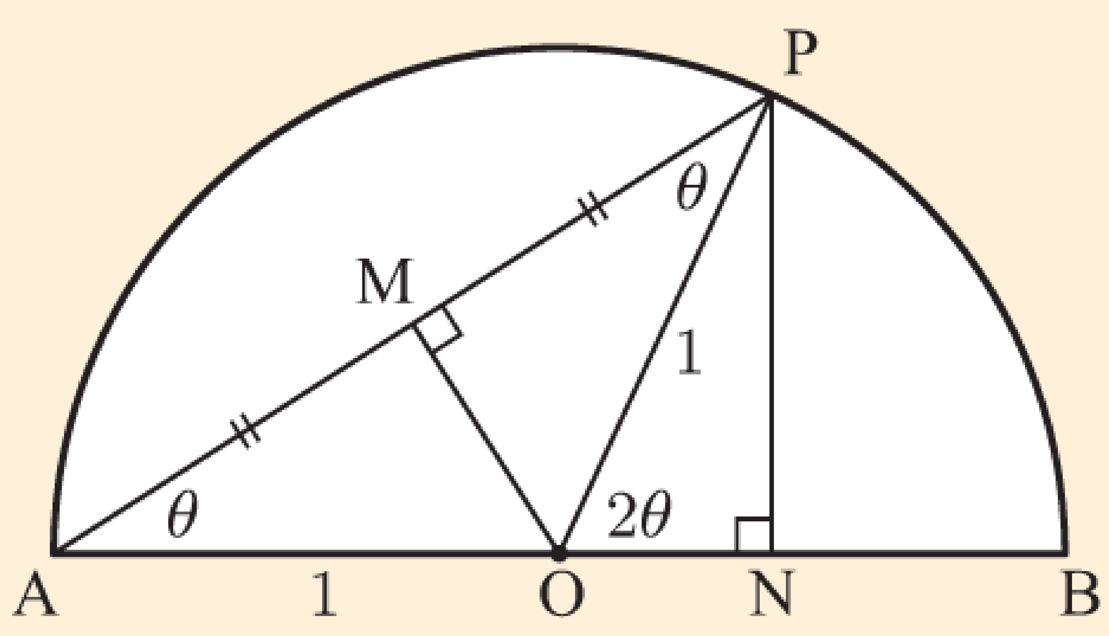
θ	$\sin 2\theta$	$2\sin\theta$	$2\sin\theta\cos\theta$	$\cos 2\theta$	$2\cos\theta$	$\cos^2\theta - \sin^2\theta$
0.631						
57.81°						
−3.697						
⋮						

2 Write down any discoveries from your table of values.

3 In the diagram alongside, the semi-circle has radius 1 unit, and $\widehat{PAB} = \theta$.

$\widehat{APO} = \theta$ {$\triangle$AOP is isosceles}

$\widehat{PON} = 2\theta$ {exterior angle of a triangle}



a Find in terms of θ, the lengths of:

i [OM]

ii [AM]

iii [ON]

iv [PN]

b Use $\triangle$ANP and the lengths in **a** to show that:

i $\cos\theta = \dfrac{\sin 2\theta}{2\sin\theta}$

ii $\cos\theta = \dfrac{1+\cos 2\theta}{2\cos\theta}$

c Hence deduce that:

i $\sin 2\theta = 2\sin\theta\cos\theta$

ii $\cos 2\theta = 2\cos^2\theta - 1$

d For what values of θ have we proven the identities in **c**?

The formulae $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 2\cos^2\theta - 1$ found in the **Investigation** are in fact true for all angles θ.

Using $\cos^2\theta = 1 - \sin^2\theta$, we can also show that $\cos 2\theta = \cos^2\theta - \sin^2\theta$
and $\cos 2\theta = 1 - 2\sin^2\theta$.

Using the definition of tangent, we find

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{\frac{2\sin\theta\cos\theta}{\cos^2\theta}}{\frac{\cos^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}} = \frac{2\tan\theta}{1-\tan^2\theta}$$

So, the **double angle identities** are:

$$\mathbf{\sin 2\theta = 2\sin\theta\cos\theta}$$
$$\mathbf{\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1}$$
$$\mathbf{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}}$$

Example 9 | Self Tutor

Given that $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = -\frac{4}{5}$ find:

a $\sin 2\alpha$ **b** $\cos 2\alpha$ **c** $\tan 2\alpha$

a $\sin 2\alpha$
$= 2\sin\alpha\cos\alpha$
$= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)$
$= -\frac{24}{25}$

b $\cos 2\alpha$
$= \cos^2\alpha - \sin^2\alpha$
$= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2$
$= \frac{7}{25}$

c $\tan 2\alpha$
$= \frac{\sin 2\alpha}{\cos 2\alpha}$
$= \frac{-\frac{24}{25}}{\frac{7}{25}}$ {using **a**, **b**}
$= -\frac{24}{7}$

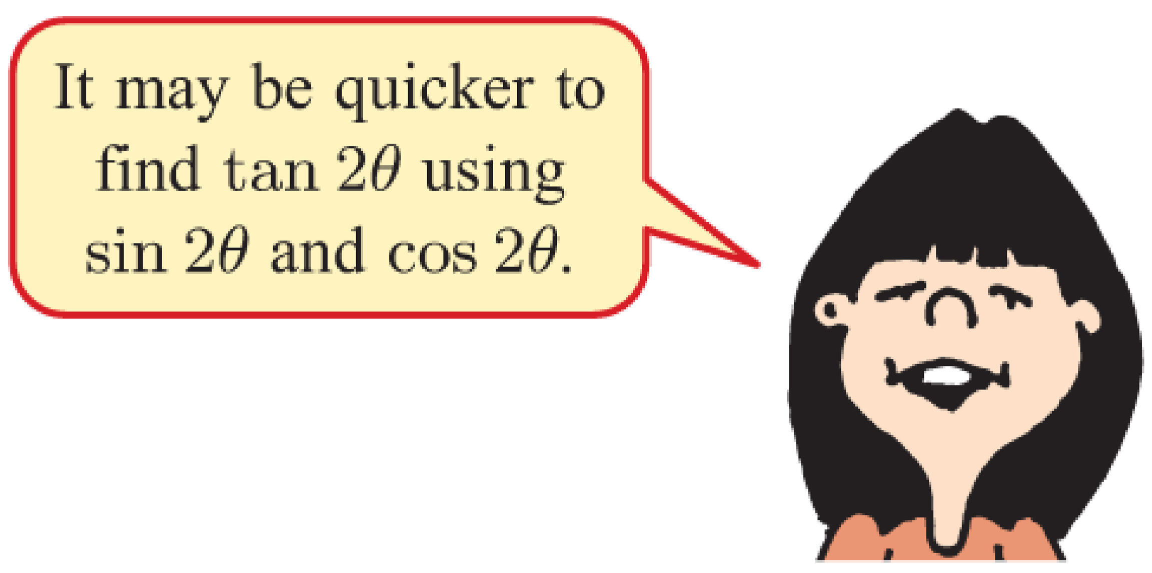
EXERCISE 1D

1 For $\theta = 30°$, verify that:

a $\sin 2\theta = 2\sin\theta\cos\theta$ **b** $\cos 2\theta = \cos^2\theta - \sin^2\theta$ **c** $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

2 If $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$ find the exact value of:

a $\sin 2\theta$ **b** $\cos 2\theta$ **c** $\tan 2\theta$



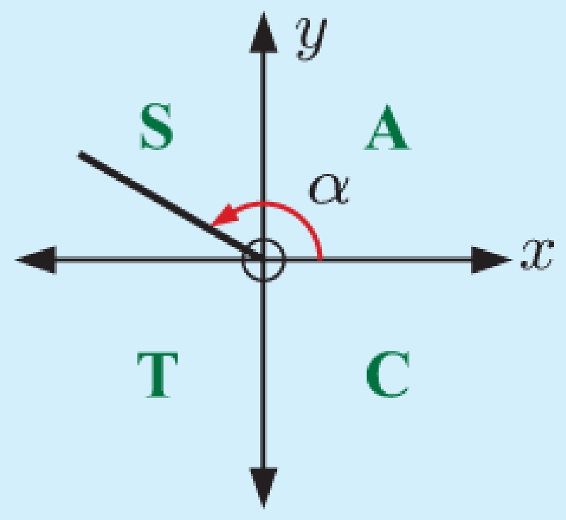
3 **a** If $\cos A = \frac{1}{3}$, find $\cos 2A$.

b If $\sin\phi = -\frac{2}{3}$, find $\cos 2\phi$.

Example 10 — Self Tutor

If $\sin\alpha = \frac{5}{13}$ where $\frac{\pi}{2} < \alpha < \pi$, find the exact value of: **a** $\sin 2\alpha$ **b** $\tan 2\alpha$.

α is in quadrant 2, so $\cos\alpha$ is negative.



Now $\cos^2\alpha + \sin^2\alpha = 1$

$\therefore \cos^2\alpha + \frac{25}{169} = 1$

$\therefore \cos^2\alpha = \frac{144}{169}$

$\therefore \cos\alpha = \pm\frac{12}{13}$

$\therefore \cos\alpha = -\frac{12}{13}$ {as $\frac{\pi}{2} < \alpha < \pi$}

a $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\therefore \sin 2\alpha = 2\left(\frac{5}{13}\right)\left(-\frac{12}{13}\right)$

$= -\frac{120}{169}$

b $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = -\frac{5}{12}$

$\therefore \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$= \frac{2\left(-\frac{5}{12}\right)}{1-\left(-\frac{5}{12}\right)^2}$

$= \frac{-\frac{5}{6}}{1-\frac{25}{144}}$

$= -\frac{120}{119}$

4 If $\sin\alpha = -\frac{2}{3}$ where $\pi < \alpha < \frac{3\pi}{2}$, find the exact value of:

a $\cos\alpha$ **b** $\sin 2\alpha$ **c** $\tan 2\alpha$

5 If $\cos\beta = \frac{2}{5}$ where $270° < \beta < 360°$, find the exact value of:

a $\sin\beta$ **b** $\sin 2\beta$ **c** $\tan 2\beta$

Example 11 — Self Tutor

If α is acute and $\cos 2\alpha = \frac{3}{4}$ find the exact value of: **a** $\cos\alpha$ **b** $\sin\alpha$.

a $\cos 2\alpha = 2\cos^2\alpha - 1$

$\therefore \frac{3}{4} = 2\cos^2\alpha - 1$

$\therefore \cos^2\alpha = \frac{7}{8}$

$\therefore \cos\alpha = \pm\frac{\sqrt{7}}{2\sqrt{2}}$

$\therefore \cos\alpha = \frac{\sqrt{7}}{2\sqrt{2}}$

{as α is acute, $\cos\alpha > 0$}

b $\sin\alpha = \sqrt{1-\cos^2\alpha}$

{as α is acute, $\sin\alpha > 0$}

$\therefore \sin\alpha = \sqrt{1-\frac{7}{8}}$

$\therefore \sin\alpha = \sqrt{\frac{1}{8}}$

$\therefore \sin\alpha = \frac{1}{2\sqrt{2}}$

6 If α is acute and $\cos 2\alpha = -\frac{7}{9}$, find without a calculator:

a $\cos\alpha$ **b** $\sin\alpha$

7 If θ is obtuse and $\cos 2\theta = -\frac{1}{3}$, find the exact value of:

a $\cos\theta$ **b** $\sin\theta$

8 Find the exact value of $\tan A$ if:

a $\tan 2A = \frac{21}{20}$ and A is obtuse **b** $\tan 2A = -\frac{12}{5}$ and A is acute.

9 Find the exact value of $\tan\frac{\pi}{8}$.

Example 12 **Self Tutor**

Use an appropriate double angle identity to simplify:

a $3\sin\theta\cos\theta$ **b** $4\cos^2 2B - 2$

a $3\sin\theta\cos\theta$
$= \frac{3}{2}(2\sin\theta\cos\theta)$
$= \frac{3}{2}\sin 2\theta$

b $4\cos^2 2B - 2$
$= 2(2\cos^2 2B - 1)$
$= 2\cos 2(2B)$
$= 2\cos 4B$

10 Use an appropriate double angle identity to simplify:

a $2\sin\alpha\cos\alpha$ **b** $4\cos\alpha\sin\alpha$ **c** $\sin\alpha\cos\alpha$
d $2\cos^2\beta - 1$ **e** $1 - 2\cos^2\phi$ **f** $1 - 2\sin^2 N$
g $2\sin^2 M - 1$ **h** $\cos^2\alpha - \sin^2\alpha$ **i** $\sin^2\alpha - \cos^2\alpha$
j $2\sin 2A\cos 2A$ **k** $2\cos 3\alpha\sin 3\alpha$ **l** $2\cos^2 4\theta - 1$
m $1 - 2\cos^2 3\beta$ **n** $1 - 2\sin^2 5\alpha$ **o** $2\sin^2 3D - 1$
p $\cos^2 2A - \sin^2 2A$ **q** $\cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)$ **r** $2\sin^2 3P - 2\cos^2 3P$

11 Find the exact value of $\left[\cos\frac{\pi}{12} + \sin\frac{\pi}{12}\right]^2$.

12 Show that:

a $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$ **b** $\cos^4\theta - \sin^4\theta = \cos 2\theta$

Check these identities by graphing both sides of the equations on the same set of axes.

GRAPHING PACKAGE

13 Solve exactly for x where $0 \leqslant x \leqslant 2\pi$:

a $\sin 2x + \sin x = 0$ **b** $\sin 2x - 2\cos x = 0$ **c** $\sin 2x + 3\sin x = 0$

14 **a** Use a double angle identity to show that:

i $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ **ii** $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

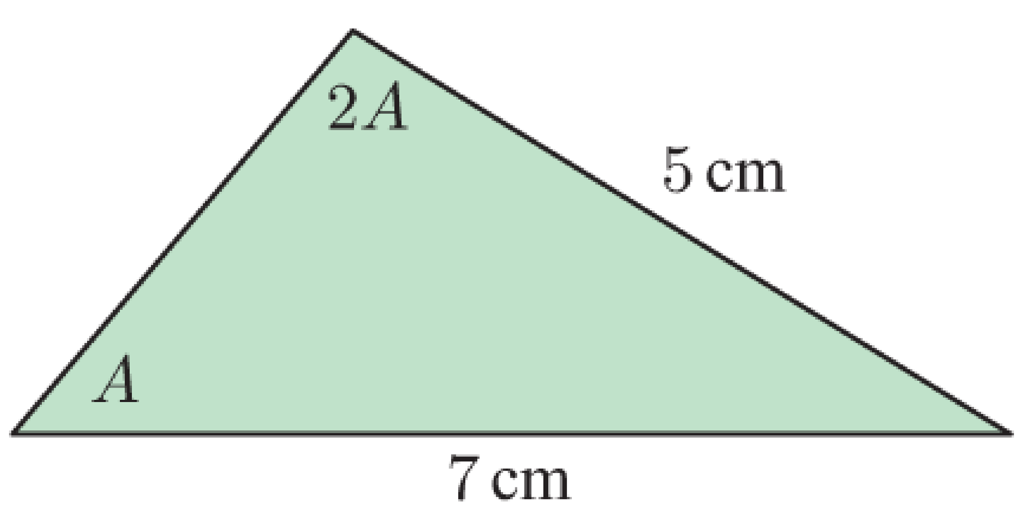
b Hence show that:

i $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2} - \frac{1}{2}\cos\theta$ **ii** $\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos\theta$

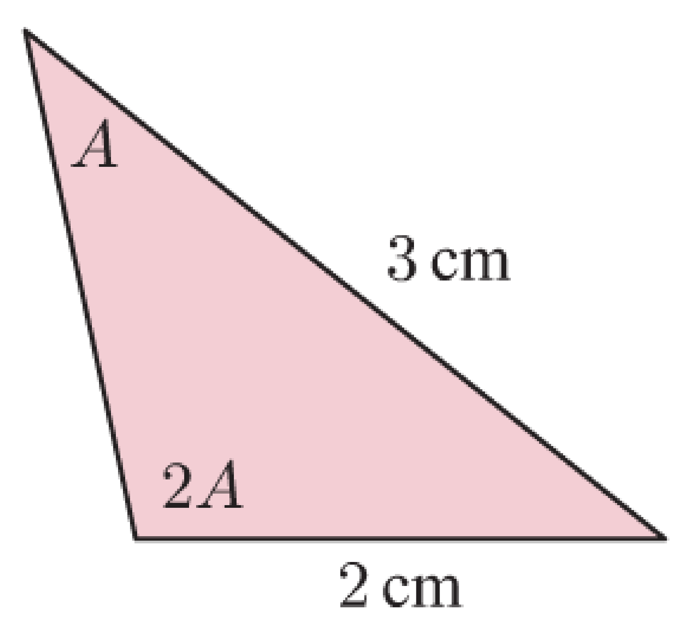
15 Solve $\sin\theta\cos\theta = \frac{1}{4}$ for $-\pi \leqslant \theta \leqslant \pi$.

16 Find the exact value of $\cos A$ in the diagram:

a



b



17 Prove that:

a $\dfrac{\sin 2\theta}{1 - \cos 2\theta} = \cot\theta$ **b** $\dfrac{\sin\theta + \sin 2\theta}{1 + \cos\theta + \cos 2\theta} = \tan\theta$ **c** $\dfrac{\sin 2\theta}{1 + \cos 2\theta} = \tan\theta$

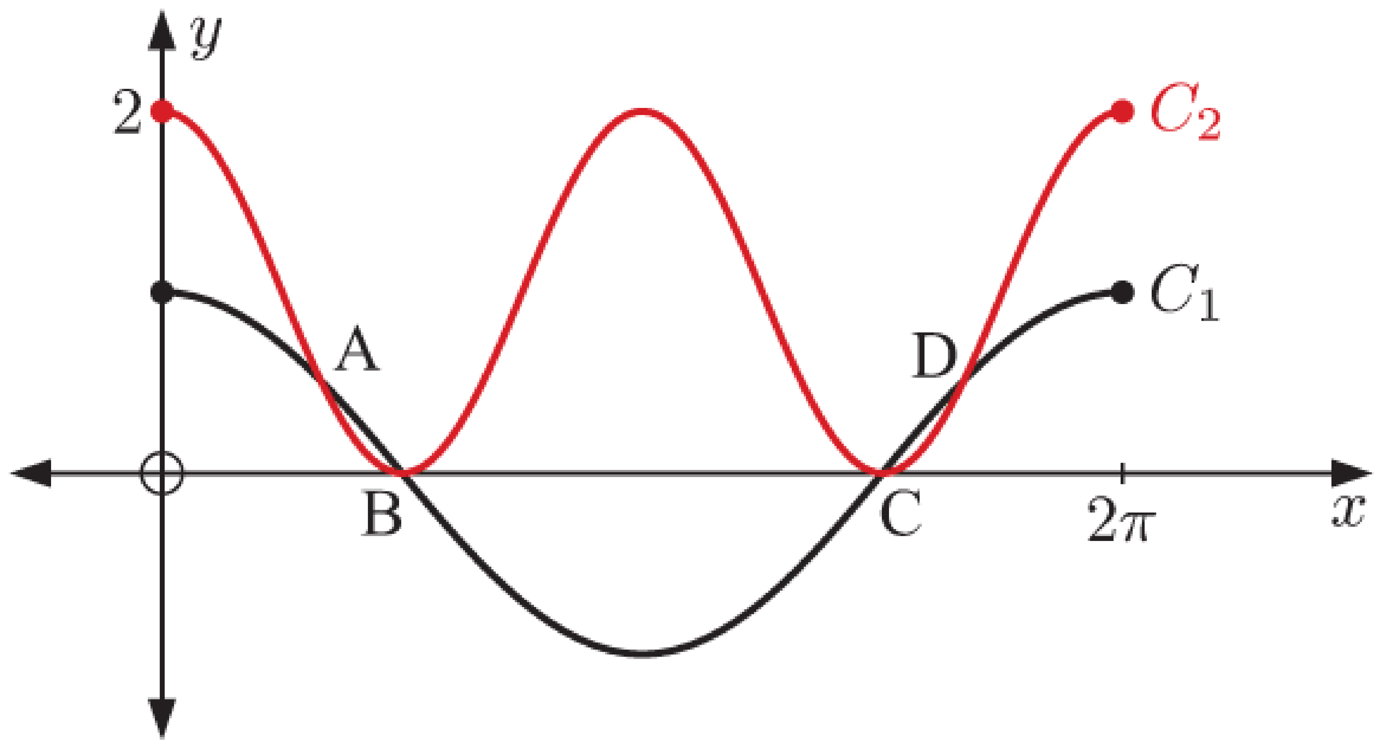
d $\operatorname{cosec} 2\theta = \tan\theta + \cot 2\theta$ **e** $\dfrac{\sin 2\theta}{\sin\theta} - \dfrac{\cos 2\theta}{\cos\theta} = \sec\theta$

18 Solve for $0 \leqslant x \leqslant 2\pi$, giving exact answers:

- **a** $\cos 2x - \cos x = 0$
- **b** $\cos 2x + 3\cos x = 1$
- **c** $\cos 2x + \sin x = 0$
- **d** $\sin 4x = \sin 2x$
- **e** $2\cos 2x + 9\sin x = 7$
- **f** $\sin x + \cos x = \sqrt{2}$
- **g** $2\cos^2 x = 3\sin x$
- **h** $\sin 2x + \cos x - 2\sin x - 1 = 0$

19 The curves $y = \cos x$ and $y = \cos 2x + 1$ are graphed alongside for $0 \leqslant x \leqslant 2\pi$.

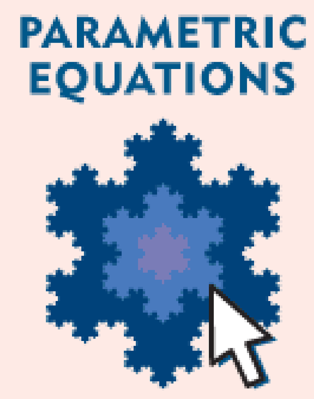
- **a** Identify each curve.
- **b** Find the exact coordinates of A, B, C, and D.



ACTIVITY 2 — PARAMETRIC EQUATIONS

In a **parametric equation**, the variables x and y are each expressed in terms of a third variable, called the **parameter**.

Click on the icon to obtain an Activity about parametric equations.



E COMPOUND ANGLE IDENTITIES

INVESTIGATION 3 — ANGLE SUM AND DIFFERENCE IDENTITIES

What to do:

1 Copy and complete for angles A and B in radians or degrees. Include some angles of your own choosing.

A	B	$\cos A$	$\cos B$	$\cos(A-B)$	$\cos A - \cos B$	$\cos A\cos B + \sin A\sin B$
$47°$	$24°$					
$138°$	$49°$					
3^c	2^c					
⋮	⋮					

2 Write down any discoveries from your table of values.

3 Copy and complete for four pairs of angles A and B of your own choosing:

A	B	$\sin A$	$\sin B$	$\sin(A+B)$	$\sin A + \sin B$	$\sin A\cos B + \cos A\sin B$
⋮	⋮					

4 Write down any discoveries from your table of values.

If A and B are **any** two angles then:

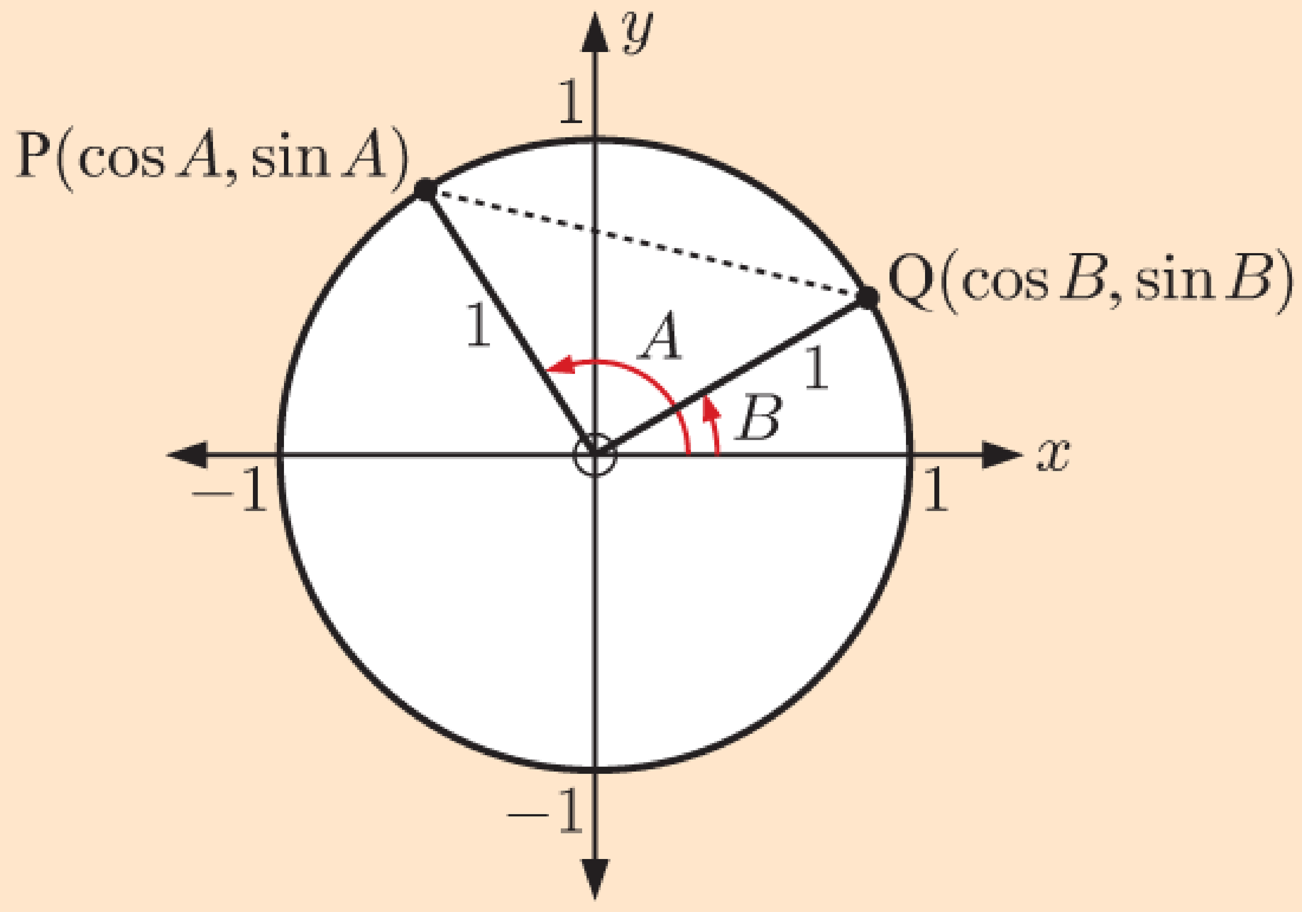
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

These are known as the **compound angle identities**. There are many ways of establishing them, but many are unsatisfactory as the arguments limit the angles A and B to being acute.

Proof:

Consider $P(\cos A, \sin A)$ and $Q(\cos B, \sin B)$ as any two points on the unit circle, as shown.

Angle POQ is $A - B$.



Using the distance formula:

$$PQ = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$
$$\therefore\ (PQ)^2 = \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B$$
$$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B) - 2(\cos A \cos B + \sin A \sin B)$$
$$= 2 - 2(\cos A \cos B + \sin A \sin B) \quad .... \ (1)$$

But, by the Cosine Rule in $\triangle POQ$,

$$(PQ)^2 = 1^2 + 1^2 - 2(1)(1)\cos(A - B)$$
$$= 2 - 2\cos(A - B) \quad .... \ (2)$$
$$\therefore\ \cos(A - B) = \cos A \cos B + \sin A \sin B \quad \{\text{comparing (1) and (2)}\}$$

From this formula the other formulae can be established:

- $\cos(A + B) = \cos(A - (-B))$
  $= \cos A \cos(-B) + \sin A \sin(-B)$
  $= \cos A \cos B + \sin A(-\sin B)$
  $\{\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta\}$
  $= \cos A \cos B - \sin A \sin B$

- $\sin(A - B)$
  $= \cos\left(\frac{\pi}{2} - (A - B)\right)$
  $= \cos\left(\left(\frac{\pi}{2} - A\right) + B\right)$
  $= \cos\left(\frac{\pi}{2} - A\right)\cos B - \sin\left(\frac{\pi}{2} - A\right)\sin B$
  $= \sin A \cos B - \cos A \sin B$

- $\sin(A + B)$
  $= \sin(A - (-B))$
  $= \sin A \cos(-B) - \cos A \sin(-B)$
  $= \sin A \cos B - \cos A(-\sin B)$
  $= \sin A \cos B + \cos A \sin B$

- $$\begin{aligned}&\tan(A+B)\\ &=\frac{\sin(A+B)}{\cos(A+B)}\\ &=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}\\ &=\frac{\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}}\\ &=\frac{\tan A+\tan B}{1-\tan A\tan B}\end{aligned}$$

- $$\begin{aligned}&\tan(A-B)\\ &=\tan(A+(-B))\\ &=\frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}\\ &=\frac{\tan A-\tan B}{1+\tan A\tan B}\end{aligned}$$
  $\{\tan(-B)=-\tan B\}$

Example 13 — Self Tutor

Expand and simplify $\sin(270° + \alpha)$.

$$\begin{aligned}&\sin(270°+\alpha)\\ &=\sin 270°\cos\alpha+\cos 270°\sin\alpha \qquad \{\text{compound angle identity}\}\\ &=-1\times\cos\alpha\ +\ 0\times\sin\alpha\\ &=-\cos\alpha\end{aligned}$$

EXERCISE 1E

1 Use your calculator to verify all six of the compound angle identities for $A = 57°$ and $B = 21°$.

2 Expand and simplify:

a $\sin(90° + \theta)$ **b** $\cos(90° + \theta)$ **c** $\sin(180° - \theta)$

d $\cos(\pi + \alpha)$ **e** $\sin(2\pi - A)$ **f** $\cos\left(\frac{3\pi}{2} - \theta\right)$

g $\tan\left(\frac{\pi}{4} + \theta\right)$ **h** $\tan\left(\theta - \frac{3\pi}{4}\right)$ **i** $\tan(\pi + \theta)$

3 Expand, then simplify and write your answer in the form $A\sin\theta + B\cos\theta$:

a $\sin\left(\theta + \frac{\pi}{3}\right)$ **b** $\cos\left(\frac{2\pi}{3} - \theta\right)$ **c** $\cos\left(\theta + \frac{\pi}{4}\right)$ **d** $\sin\left(\frac{\pi}{6} - \theta\right)$

Example 14 — Self Tutor

Simplify $\cos 3\theta\cos\theta - \sin 3\theta\sin\theta$.

$$\begin{aligned}&\cos 3\theta\cos\theta-\sin 3\theta\sin\theta\\ &=\cos(3\theta+\theta) \qquad \{\text{compound angle identity in reverse}\}\\ &=\cos 4\theta\end{aligned}$$

4 Simplify using the compound angle identities in reverse:

a $\cos 2\theta\cos\theta + \sin 2\theta\sin\theta$ **b** $\sin 2A\cos A + \cos 2A\sin A$

c $\cos A\sin B - \sin A\cos B$ **d** $\sin\alpha\sin\beta + \cos\alpha\cos\beta$

e $\sin\phi\sin\theta - \cos\phi\cos\theta$ **f** $2\sin\alpha\cos\beta - 2\cos\alpha\sin\beta$

g $\dfrac{\tan 2\theta - \tan\theta}{1 + \tan 2\theta\tan\theta}$ **h** $\dfrac{\tan 2A + \tan A}{1 - \tan 2A\tan A}$

5 Use the compound angle identities to prove the double angle identities:

a $\sin 2\theta = 2\sin\theta\cos\theta$ **b** $\cos 2\theta = \cos^2\theta - \sin^2\theta$ **c** $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

Example 15 — Self Tutor

Without using your calculator, show that $\sin 75° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

$$\begin{aligned}\sin 75° &= \sin(45° + 30°)\\ &= \sin 45°\cos 30° + \cos 45°\sin 30°\\ &= \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{\sqrt{3}}{2}\right) + \left(\tfrac{1}{\sqrt{2}}\right)\left(\tfrac{1}{2}\right)\\ &= \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

6 Without using your calculator, show that:

a $\cos 75° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ **b** $\sin 105° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$ **c** $\cos\frac{13\pi}{12} = \dfrac{-\sqrt{6}-\sqrt{2}}{4}$

7 Find the exact value of:

a $\tan\frac{5\pi}{12}$ **b** $\tan 105°$

8 If $\tan A = \frac{2}{3}$ and $\tan B = -\frac{1}{5}$, find the exact value of $\tan(A+B)$.

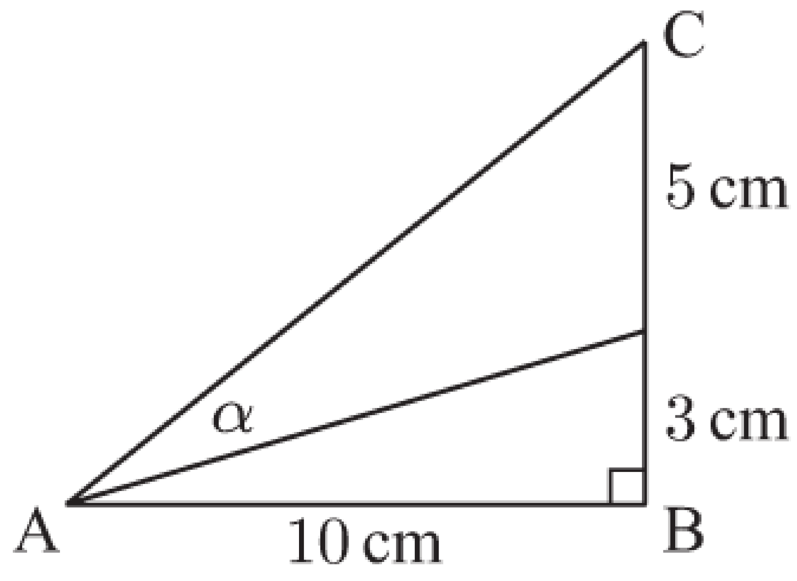
9 If $\tan A = \frac{3}{4}$, find $\tan\left(A + \frac{\pi}{4}\right)$.

10 If $\sin A = -\frac{1}{3}$, $\pi \leqslant A \leqslant \frac{3\pi}{2}$ and $\cos B = \frac{1}{\sqrt{5}}$, $0 \leqslant B \leqslant \frac{\pi}{2}$, find $\tan(A+B)$.

11 Simplify, giving your answer exactly: $\dfrac{\tan 80° - \tan 20°}{1 + \tan 80°\tan 20°}$

12 If $\tan(A+B) = \frac{3}{5}$ and $\tan B = \frac{2}{3}$, find the exact value of $\tan A$.

13 Find the exact value of $\tan\alpha$:



14 Find exactly the tangent of the acute angle between two lines with gradients $\frac{1}{2}$ and $\frac{2}{3}$.

15 Simplify:

a $\cos(\alpha+\beta)\cos(\alpha-\beta) - \sin(\alpha+\beta)\sin(\alpha-\beta)$

b $\sin(\theta-2\phi)\cos(\theta+\phi) - \cos(\theta-2\phi)\sin(\theta+\phi)$

c $\cos\alpha\cos(\beta-\alpha) - \sin\alpha\sin(\beta-\alpha)$

d $\tan\left(A+\frac{\pi}{4}\right)\tan\left(A-\frac{\pi}{4}\right)$ **e** $\dfrac{\tan(A+B)+\tan(A-B)}{1-\tan(A+B)\tan(A-B)}$

16 If $\sin A = \frac{2}{3}$, $\frac{\pi}{2} \leqslant A \leqslant \pi$ and $\cos B = -\frac{4}{5}$, $\pi \leqslant B \leqslant \frac{3\pi}{2}$, find:

a $\tan(A+B)$ **b** $\tan 2A$.

17 Find $\tan A$ if $\tan(A-B)\tan(A+B) = 1$.

18 Express $\tan(A+B+C)$ in terms of $\tan A$, $\tan B$, and $\tan C$.
Hence show that if A, B, and C are the angles of a triangle, then
$\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

19 Show that:

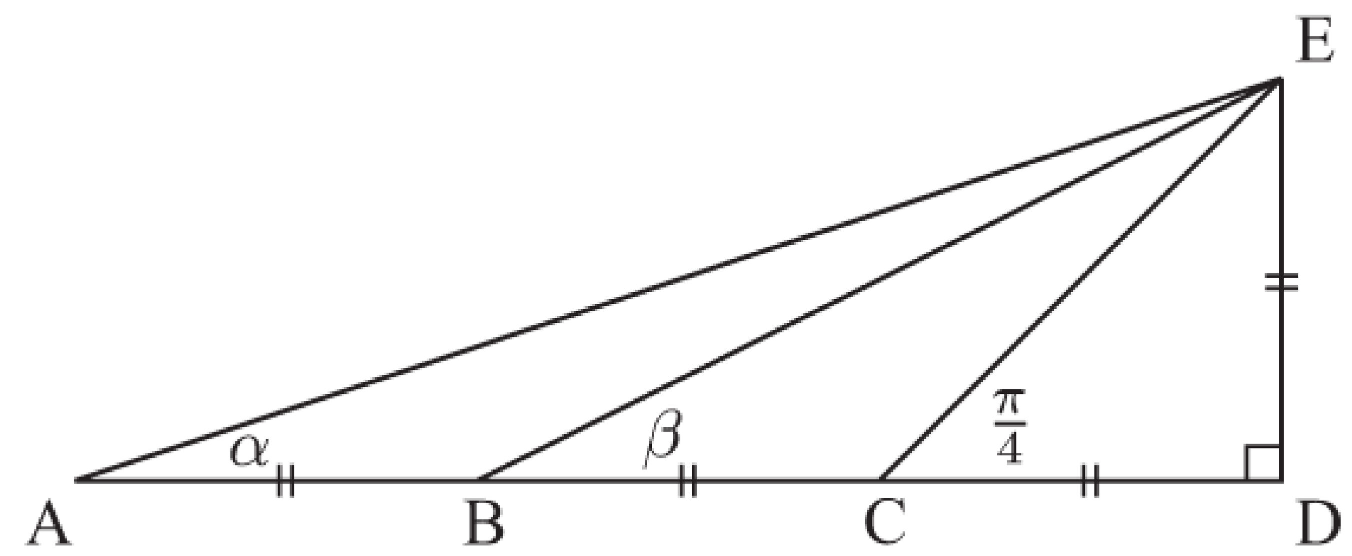
a $\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right) = \cos\theta - \sin\theta$

b $2\cos\left(\theta - \frac{\pi}{3}\right) = \cos\theta + \sqrt{3}\sin\theta$

c $\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$

d $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$.

20 Prove that, in the given figure, $\alpha + \beta = \frac{\pi}{4}$.



21 **a** Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ by replacing 3θ by $(2\theta + \theta)$.

b Hence solve the equation $8\cos^3\theta - 6\cos\theta + 1 = 0$ for $-\pi \leqslant \theta \leqslant \pi$.

22 **a** Write $\sin 3\theta$ in the form $a\sin^3\theta + b\sin\theta$ where $a, b \in \mathbb{Z}$.

b Hence solve the equation $\sin 3\theta = \sin\theta$ for $0 \leqslant \theta \leqslant 3\pi$.

Example 16 **Self Tutor**

Suppose $\sin x - \sqrt{3}\cos x = k\cos(x+b)$ for $k > 0$ and $0 < b < 2\pi$. Find k and b.

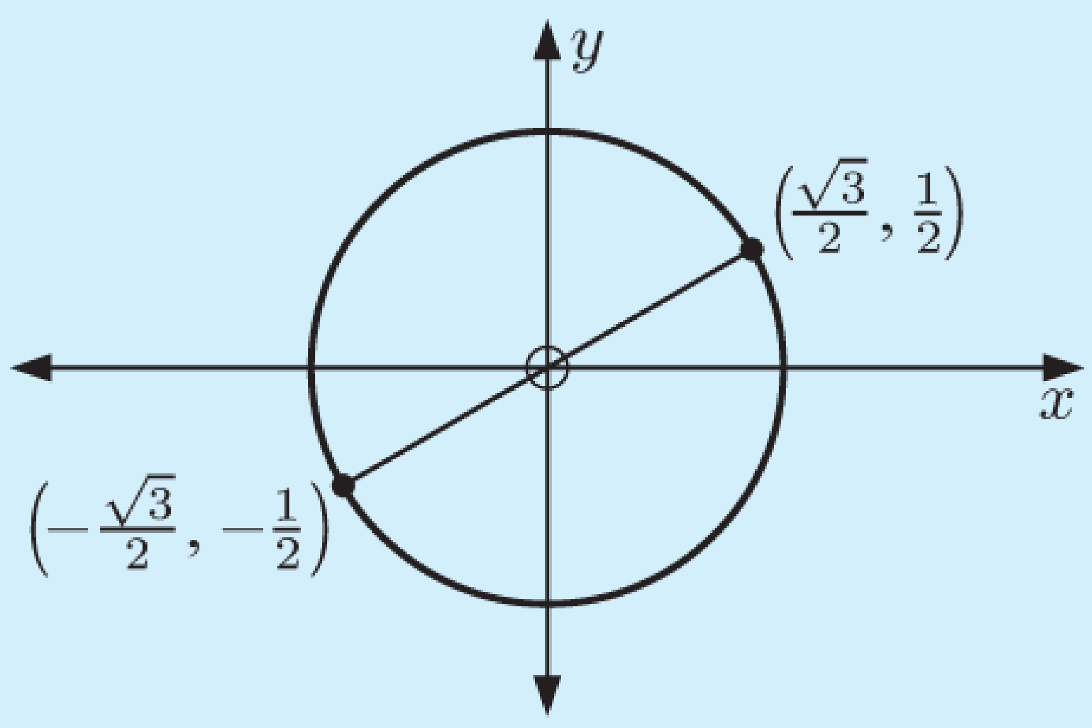
$$\begin{aligned}\sin x - \sqrt{3}\cos x &= k\cos(x+b)\\ &= k[\cos x\cos b - \sin x\sin b]\\ &= k\cos x\cos b - k\sin x\sin b\end{aligned}$$

Equating coefficients of $\cos x$ and $\sin x$, $k\cos b = -\sqrt{3}$ .... (1) and $-k\sin b = 1$ .... (2)

$\dfrac{-k\sin b}{k\cos b} = -\dfrac{1}{\sqrt{3}}$ {dividing (2) by (1)}

$\therefore \tan b = \frac{1}{\sqrt{3}}$

$\therefore b = \frac{\pi}{6}$ or $\frac{7\pi}{6}$



Substituting $b = \frac{\pi}{6}$ into (1) gives $k \times \frac{\sqrt{3}}{2} = -\sqrt{3}$
$\therefore k = -2$

We reject this solution as $k > 0$.

Substituting $b = \frac{7\pi}{6}$ into (1) gives $k \times -\frac{\sqrt{3}}{2} = -\sqrt{3}$
$\therefore k = 2$

So, $k = 2$ and $b = \frac{7\pi}{6}$.

23 Suppose $\sqrt{3}\sin x + \cos x = k\sin(x+a)$ for $k > 0$ and $0 < a < 2\pi$. Find k and a.

24 **a** Write $2\cos x + 2\sin x$ in the form $k\cos(x+a)$ for $k > 0$, $0 < a < 2\pi$.

b Hence solve the equation $2\cos x + 2\sin x = \sqrt{2}$ for $0 \leqslant x \leqslant 2\pi$.

25 **a** Find a sequence of transformations which map the graph of $y = \sin x$ onto the graph of $y = \cos x + 3\sin x$.

b Find the greatest and least values of $(\cos x + 3\sin x)^2 + 2$.

26 Use the basic definition of periodicity to show algebraically that the period of $f(x) = \sin(nx)$ is $\frac{2\pi}{n}$, for all $n > 0$.

27 **a** Write $2\cos x - 5\sin x$ in the form $k\cos(x+b)$ for $k > 0$, $0 < b < 2\pi$.

b Hence solve $2\cos x - 5\sin x = -2$ for $0 \leqslant x \leqslant \pi$.

c Given that $t = \tan\frac{x}{2}$, prove that $\sin x = \dfrac{2t}{1+t^2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$.

d Hence solve $2\cos x - 5\sin x = -2$ for $0 \leqslant x \leqslant \pi$.

28 Use $\tan(\theta - \phi) = \dfrac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}$ to show that:

a $\arctan 5 - \arctan\frac{2}{3} = \frac{\pi}{4}$

b $\arctan\frac{1}{5} + \arctan\frac{2}{3} = \frac{\pi}{4}$

29 Find the exact value of:

a $\arctan\frac{4}{3} - 2\arctan\frac{1}{2}$

b $4\arctan\frac{1}{5} - \arctan\frac{1}{239}$

30 **a** Show that $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.

b Hence show that $\sin A\cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$.

c Hence write the following as sums:

i $\sin 3\theta\cos\theta$ **ii** $\sin 6\alpha\cos\alpha$ **iii** $2\sin 5\beta\cos\beta$

iv $4\cos\theta\sin 4\theta$ **v** $6\cos 4\alpha\sin 3\alpha$ **vi** $\frac{1}{3}\cos 5A\sin 3A$

31 **a** Show that $\cos(A+B) + \cos(A-B) = 2\cos A\cos B$.

b Hence show that $\cos A\cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$.

c Hence write the following as a *sum* of cosines:

i $\cos 4\theta\cos\theta$ **ii** $\cos 7\alpha\cos\alpha$ **iii** $2\cos 3\beta\cos\beta$

iv $6\cos x\cos 7x$ **v** $3\cos P\cos 4P$ **vi** $\frac{1}{4}\cos 4x\cos 2x$

32 **a** Show that $\cos(A-B) - \cos(A+B) = 2\sin A\sin B$.

b Hence show that $\sin A\sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$.

c Hence write the following as a *difference* of cosines:

i $\sin 3\theta\sin\theta$ **ii** $\sin 6\alpha\sin\alpha$ **iii** $2\sin 5\beta\sin\beta$

iv $4\sin\theta\sin 4\theta$ **v** $10\sin 2A\sin 8A$ **vi** $\frac{1}{5}\sin 3M\sin 7M$

33 The **products to sums formulae** are:

$$\sin A\cos B = \tfrac{1}{2}\sin(A+B) + \tfrac{1}{2}\sin(A-B) \quad .... (1)$$

$$\cos A\cos B = \tfrac{1}{2}\cos(A+B) + \tfrac{1}{2}\cos(A-B) \quad .... (2)$$

$$\sin A\sin B = \tfrac{1}{2}\cos(A-B) - \tfrac{1}{2}\cos(A+B) \quad .... (3)$$

a What formulae result if we replace B by A in each of these formulae?

b Suppose $A+B=S$ and $A-B=D$.

i Show that $A=\dfrac{S+D}{2}$ and $B=\dfrac{S-D}{2}$.

ii Using the substitutions $A+B=S$ and $A-B=D$, show that equation (1) becomes $\sin S+\sin D=2\sin\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$ .... (4)

iii By replacing D by $(-D)$ in (4), show that $\sin S-\sin D=2\cos\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$.

c What formula results when the substitution $A=\dfrac{S+D}{2}$ and $B=\dfrac{S-D}{2}$ is made into (2)?

d What formula results when the substitution $A=\dfrac{S+D}{2}$ and $B=\dfrac{S-D}{2}$ is made into (3)?

34 The **factor formulae** are:

$$\sin S+\sin D=2\sin\left(\tfrac{S+D}{2}\right)\cos\left(\tfrac{S-D}{2}\right) \qquad \cos S+\cos D=2\cos\left(\tfrac{S+D}{2}\right)\cos\left(\tfrac{S-D}{2}\right)$$

$$\sin S-\sin D=2\cos\left(\tfrac{S+D}{2}\right)\sin\left(\tfrac{S-D}{2}\right) \qquad \cos S-\cos D=-2\sin\left(\tfrac{S+D}{2}\right)\sin\left(\tfrac{S-D}{2}\right)$$

Use these formulae to convert the following to products:

a $\sin 5x+\sin x$ **b** $\cos 8A+\cos 2A$ **c** $\cos 3\alpha-\cos\alpha$

d $\sin 5\theta-\sin 3\theta$ **e** $\cos 7\alpha-\cos\alpha$ **f** $\sin 3\alpha+\sin 7\alpha$

g $\cos 2B-\cos 4B$ **h** $\sin(x+h)-\sin x$ **i** $\cos(x+h)-\cos x$

35 In triangle ABC it is known that $\sin A=\cos B+\cos C$. Show that the triangle is right angled.

HISTORICAL NOTE

In the late sixteenth century, there were no calculators. Instead, values of trigonometric functions for different angles were calculated by hand and recorded in tables. The values could then readily be used for trigonometric applications, and also in other surprising ways. For example, the compound angle identities could be used to quickly multiply two numbers together.

Using the compound angle identities, we can show that

$$\cos A\times\cos B=\frac{\cos(A+B)+\cos(A-B)}{2} \qquad \text{.... } (*)$$

Now, suppose you need to find 0.1362×0.4573. By consulting trigonometric tables, it could be found that $\cos 82.172^\circ\approx 0.1362$ and $\cos 62.787^\circ\approx 0.4573$.

So, 0.1362×0.4573

$\approx \cos 82.172^\circ\times\cos 62.787^\circ$

$\approx \dfrac{\cos(82.172^\circ+62.787^\circ)+\cos(82.172^\circ-62.787^\circ)}{2}$ {using $(*)$}

$\approx \dfrac{\cos(144.959^\circ)+\cos(19.385^\circ)}{2}$

$\approx \dfrac{-0.8187+0.9433}{2}$ {using trigonometric tables}

≈ 0.0623

This method may seem complicated, but in the late sixteenth century it was much faster than performing long multiplication!

ACTIVITY 3

A **trigonometric series** is the sum of a sequence of trigonometric expressions which follow a rule.

In this Investigation we will explore patterns formed by a trigonometric series.

What to do:

1 Consider the function $f(x) = \sin x + \dfrac{\sin 3x}{3}$.

- **a** Show that $f(x) = \frac{2}{3}\sin x(2\cos^2 x + 1)$.

  Hint: Write $\sin 3x$ as $\sin(2x + x)$.
- **b** Hence find the x-intercepts of $y = f(x)$ on $-4\pi \leqslant x \leqslant 4\pi$.
- **c** Use the graphing package to sketch $y = f(x)$ on $-4\pi \leqslant x \leqslant 4\pi$. Discuss the similarities and differences between this graph and the graph of $y = \sin x$.

2
- **a** Write the function $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5}$ in terms of $\sin x$ and $\cos x$.
- **b** Hence find the x-intercepts of this function on $-4\pi \leqslant x \leqslant 4\pi$.
- **c** Use the graphing package to sketch the graph of the function on $-4\pi \leqslant x \leqslant 4\pi$. Compare your graph with the graphs in **1**.

3 Use the graphing package to graph on $-4\pi \leqslant x \leqslant 4\pi$:

- **a** $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7}$
- **b** $f(x) = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7} + \dfrac{\sin 9x}{9} + \dfrac{\sin 11x}{11}$

4 Predict the graph of $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{\sin(2k-1)}{2k-1}$.

REVIEW SET 1A

1 Without using a calculator, find:

a $\operatorname{cosec} \frac{\pi}{4}$ **b** $\cot \frac{5\pi}{6}$ **c** $\sec \frac{5\pi}{3}$

2 Find, giving your answer in radians:

a $\arccos \frac{1}{\sqrt{2}}$ **b** $\arctan \frac{1}{\sqrt{3}}$ **c** $\arcsin\left(-\frac{1}{2}\right)$

3 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sec x = \sqrt{2}$ **b** $\sqrt{3}\cos x \operatorname{cosec} x + 1 = 0$

4 Simplify:

a $3\cos(-\theta) - 2\cos\theta$ **b** $\cos\left(\frac{3\pi}{2} - \theta\right)$ **c** $\sin\left(\theta + \frac{\pi}{2}\right)$

d $\dfrac{1 - \cos^2\theta}{1 + \cos\theta}$ **e** $\dfrac{\sin\alpha - \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$ **f** $\dfrac{4\sin^2\alpha - 4}{8\cos\alpha}$

5 If $\sin\alpha = -\frac{3}{4}$, $\pi \leqslant \alpha \leqslant \frac{3\pi}{2}$, find the exact value of:

a $\cos\alpha$ **b** $\sin 2\alpha$ **c** $\cos 2\alpha$ **d** $\tan 2\alpha$ **e** $\cos \frac{\alpha}{2}$ **f** $\sin \frac{\alpha}{2}$

6 Show that:

a $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) = \sin\theta - \cos\theta$

b $\sin\theta\cos(\phi-\theta) + \cos\theta\sin(\phi-\theta) = \sin\phi$

c $\dfrac{\sin 2\alpha - \sin\alpha}{\cos 2\alpha - \cos\alpha + 1} = \tan\alpha$

d $\operatorname{cosec} 2x + \cot 2x = \cot x$

7 Find the exact value of:

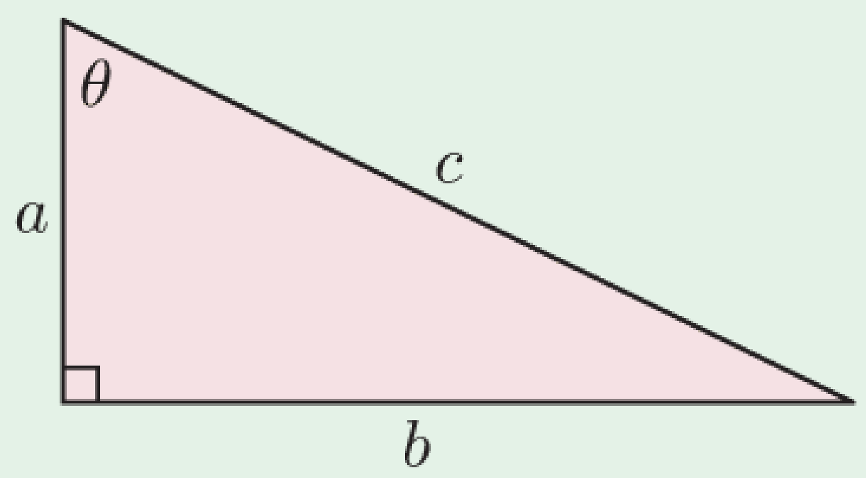
a $\cos 165°$

b $\tan\frac{\pi}{12}$

8 From ground level, a shooter is aiming at targets on a vertical brick wall. At the current angle of elevation of his rifle, he will hit a target 20 m above ground level. If he doubles the angle of elevation of the rifle, he will hit a target 45 m above ground level. How far is the shooter from the wall?

9 Solve for $0 \leqslant x \leqslant 2\pi$: $\sec^2 x = \tan x + 1$

10

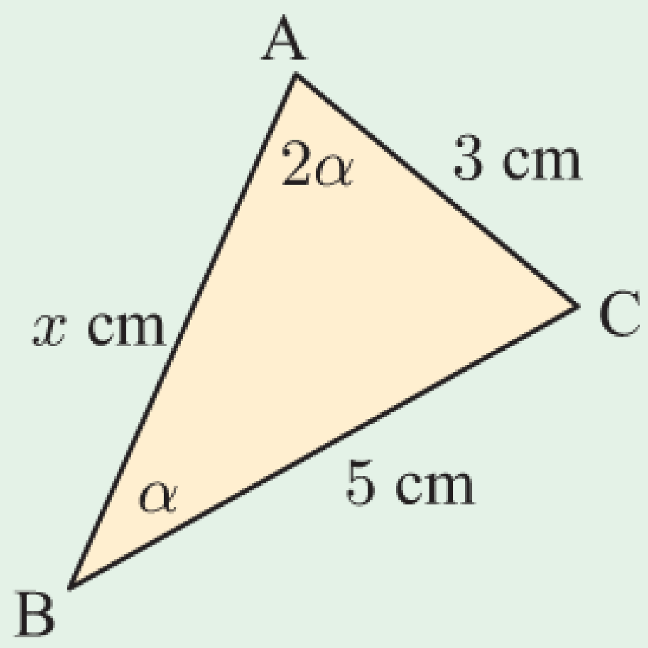


For the diagram alongside, prove that:

a $\sin 2\theta = \dfrac{2ab}{c^2}$

b $\cos 2\theta = \dfrac{a^2 - b^2}{c^2}$

11 Consider triangle ABC shown.



a Show that $\cos\alpha = \frac{5}{6}$.

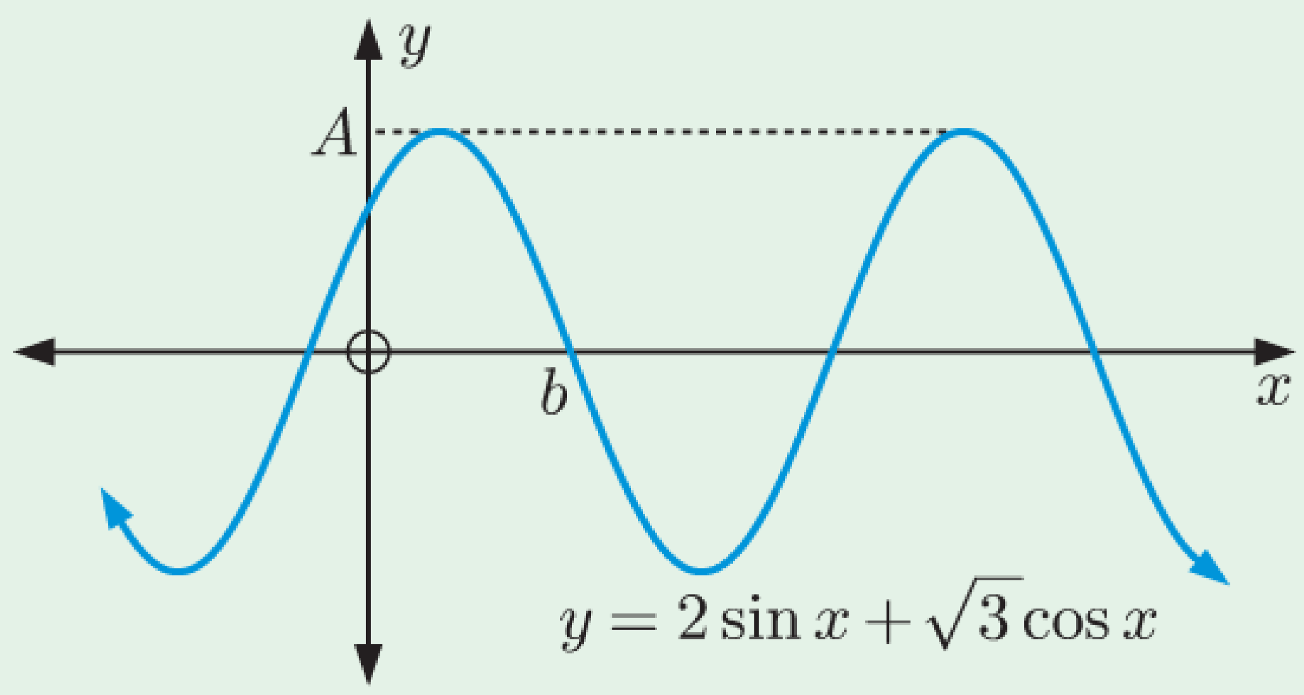
b Show that x is a solution of $3x^2 - 25x + 48 = 0$.

c Find x by solving the equation in **b**.

12 If α and β are the other angles of a right angled triangle, show that $\sin 2\alpha = \sin 2\beta$.

13 Write $3\sin x - 5\cos x$ in the form $k\cos(x + a)$ where $k > 0$ and $0 < a < 2\pi$.

14 The graph of $y = 2\sin x + \sqrt{3}\cos x$ is shown alongside.



a Write $2\sin x + \sqrt{3}\cos x$ in the form $k\sin(x + a)$ where $k > 0$ and $0 < a < 2\pi$.

b Hence find:

i the exact value of A

ii b correct to 2 decimal places.

15 Find $\arctan\frac{1}{7} + 2\arctan\frac{1}{3}$.

REVIEW SET 1B

1 Find the other *five* trigonometric ratios if $\cos x = -\frac{1}{3}$ and $\pi < x < \frac{3\pi}{2}$.

2 **a** Sketch the graphs of $y = \sec x$ and $y = \operatorname{cosec} x$ on the same set of axes for $-2\pi \leqslant x \leqslant 2\pi$.

b State a transformation which maps $y = \sec x$ onto $y = \operatorname{cosec} x$ for all $x \in \mathbb{R}$.

3 Solve $\cot x = \sqrt{3}$ for $-\pi \leqslant x \leqslant \pi$.

4 Solve exactly:

a $\arcsin x = \frac{\pi}{3}$

b $\arctan(x-2) = \frac{\pi}{6}$

5 Simplify:

a $\operatorname{cosec} x \tan x$

b $\dfrac{\tan x}{\sec x}$

c $\sec x - \tan x \sin x$

6 Simplify:

a $\cos^3\theta + \sin^2\theta\cos\theta$

b $\dfrac{\cos^2\theta - 1}{\sin\theta}$

c $5 - 5\sin^2\theta$

d $\dfrac{\sin^2\theta - 1}{\cos\theta}$

e $\dfrac{\tan\theta + \cot\theta}{\sec\theta}$

f $\cos^2\theta(\tan\theta + 1)^2 - 1$

7 If $\sin A = \frac{5}{13}$ and $\cos A = \frac{12}{13}$, find:

a $\sin 2A$

b $\cos 2A$

c $\tan 2A$

8 Show that:

a $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = 2\sec\theta$

b $\left(1 + \dfrac{1}{\cos\theta}\right)\left(\cos\theta - \cos^2\theta\right) = \sin^2\theta$

9 Show that $\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$ using a suitable double angle identity.

10 Solve:

a $\sqrt{3}\cos x + \sin 2x = 0$ for $-\pi \leqslant x \leqslant \pi$

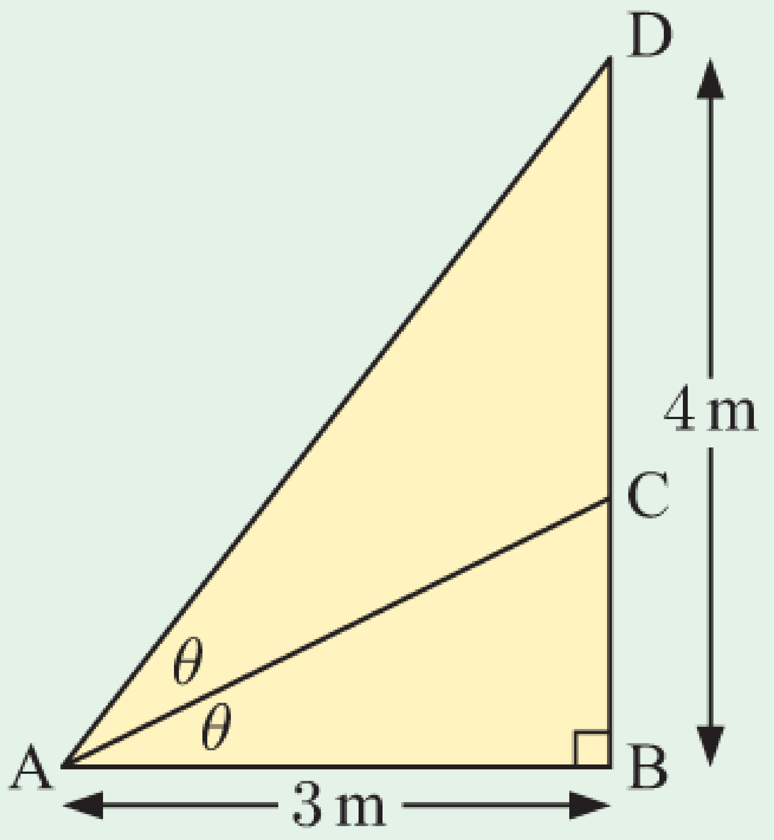
b $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \sqrt{3}$ for $0 < \theta < \frac{\pi}{2}$.

11 Given $\sin\theta = \frac{3}{4}$ and $\frac{\pi}{2} < \theta < \pi$, find $\sin\left(\theta + \frac{\pi}{6}\right)$.

12 Consider the figure in the **Opening Problem** on page **18**. Find $\tan\theta$ using the ratios $\tan\phi$ and $\tan(\theta+\phi)$.

13 Write $3\sin x + 4\cos x$ in the form $k\sin(x+a)$, where $k > 0$ and $0 < a < 2\pi$.

14 Find exactly the length of [BC].



15 **a** Show that $\dfrac{1}{1+\sqrt{2}\sin x} + \dfrac{1}{1-\sqrt{2}\sin x} = 2\sec 2x$.

b Hence explain why $\dfrac{1}{1+\sqrt{2}\sin x} + \dfrac{1}{1-\sqrt{2}\sin x} = 1$ has no solutions.

16 Prove that if A, B, and C are the angles of a triangle, then
$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$.

EXERCISE 1A

1 **a** $\frac{2}{\sqrt{3}}$ **b** $-\frac{1}{\sqrt{3}}$ **c** $-\frac{2}{\sqrt{3}}$ **d** undefined

2 **a** $\operatorname{cosec} x = \frac{5}{3}$, $\sec x = \frac{5}{4}$, $\cot x = \frac{4}{3}$

b $\operatorname{cosec} x = -\frac{3}{\sqrt{5}}$, $\sec x = \frac{3}{2}$, $\cot x = -\frac{2}{\sqrt{5}}$

3 **a** $\sin x = -\frac{\sqrt{7}}{4}$, $\tan x = -\frac{\sqrt{7}}{3}$, $\operatorname{cosec} x = -\frac{4}{\sqrt{7}}$, $\sec x = \frac{4}{3}$, $\cot x = -\frac{3}{\sqrt{7}}$

b $\cos x = -\frac{\sqrt{5}}{3}$, $\tan x = \frac{2}{\sqrt{5}}$, $\operatorname{cosec} x = -\frac{3}{2}$, $\sec x = -\frac{3}{\sqrt{5}}$, $\cot x = \frac{\sqrt{5}}{2}$

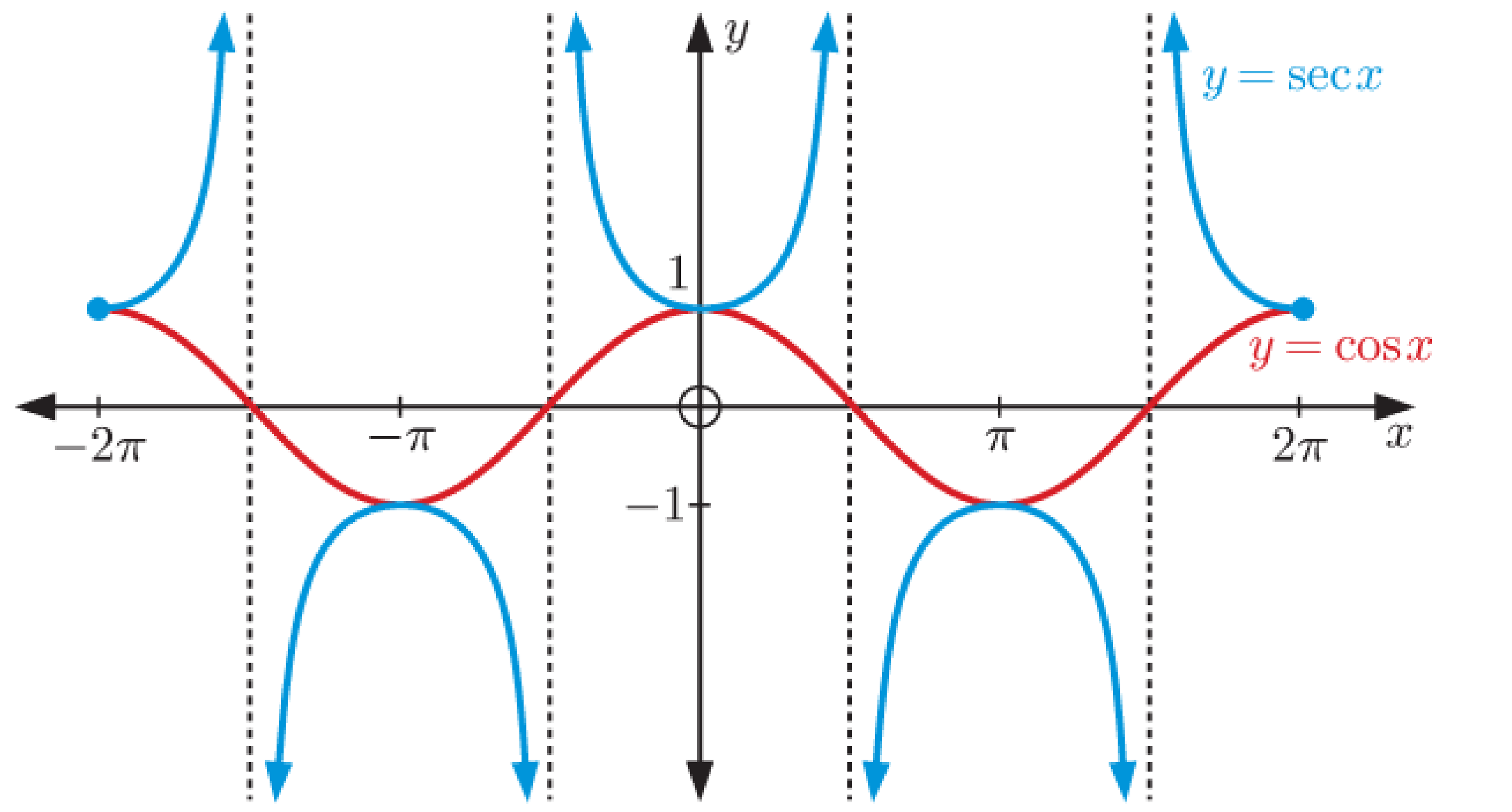
c $\sin x = \frac{\sqrt{21}}{5}$, $\cos x = \frac{2}{5}$, $\tan x = \frac{\sqrt{21}}{2}$, $\operatorname{cosec} x = \frac{5}{\sqrt{21}}$, $\cot x = \frac{2}{\sqrt{21}}$

d $\sin x = \frac{1}{2}$, $\cos x = -\frac{\sqrt{3}}{2}$, $\tan x = -\frac{1}{\sqrt{3}}$, $\sec x = -\frac{2}{\sqrt{3}}$, $\cot x = -\sqrt{3}$

e $\sin\beta = -\frac{1}{\sqrt{5}}$, $\cos\beta = -\frac{2}{\sqrt{5}}$, $\operatorname{cosec}\beta = -\sqrt{5}$, $\sec\beta = -\frac{\sqrt{5}}{2}$, $\cot\beta = 2$

f $\sin\theta = -\frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\operatorname{cosec}\theta = -\frac{5}{3}$, $\sec\theta = -\frac{5}{4}$

4



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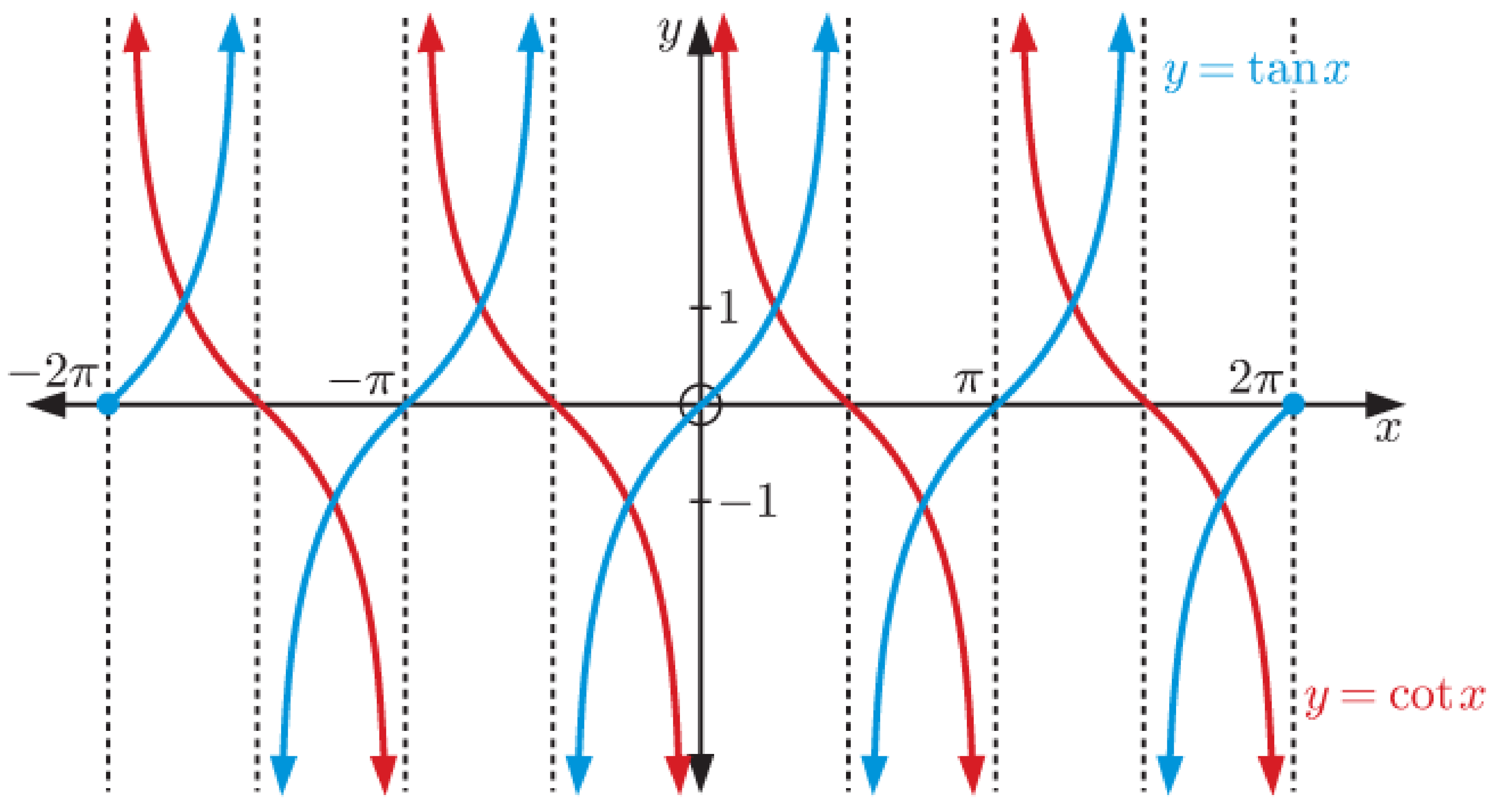

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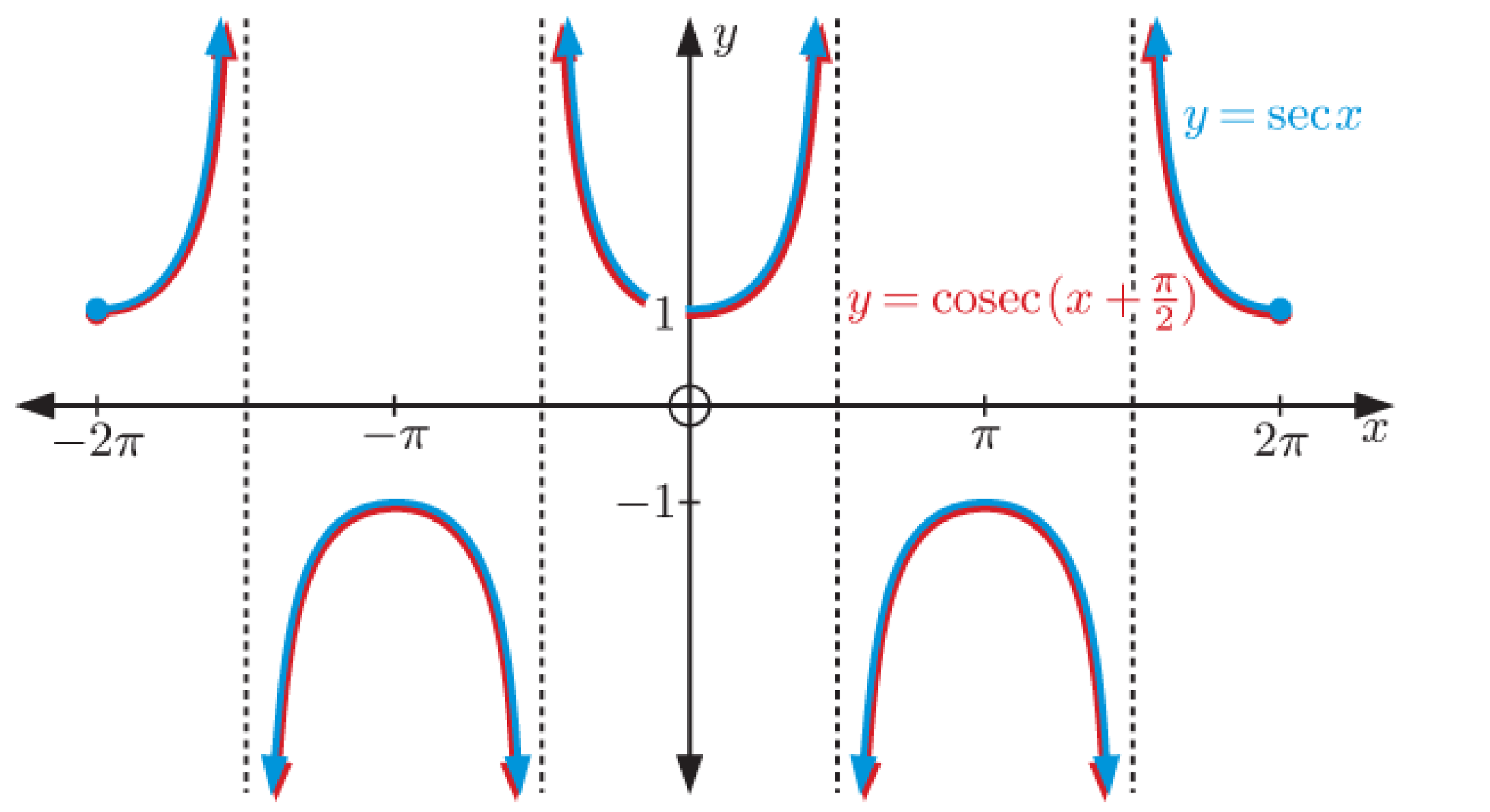


$\sin(x + \frac{\pi}{2}) = \cos x \quad \therefore \quad \frac{1}{\sin(x+\frac{\pi}{2})} = \frac{1}{\cos x}$

$\therefore \quad \operatorname{cosec}(x + \frac{\pi}{2}) = \sec x$

7 **a** $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ **b** $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

c $x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}$, or $\frac{19\pi}{12}$ **d** $x = \frac{13\pi}{12}$ or $\frac{19\pi}{12}$

e $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ **f** $x = \frac{5\pi}{24}, \frac{17\pi}{24}, \frac{29\pi}{24}$, or $\frac{41\pi}{24}$

8 **a** $x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$ **b** $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$

c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$ **d** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$

9 **a** [OD] **b** [AD] **c** [BE] **d** [OB] **e** [OC] **f** [OF]

EXERCISE 1B

1 **a** 0 **b** $-\frac{\pi}{2}$ **c** $\frac{\pi}{4}$ **d** $-\frac{\pi}{4}$ **e** $\frac{\pi}{6}$

f $\frac{5\pi}{6}$ **g** $\frac{\pi}{3}$ **h** $\frac{3\pi}{4}$ **i** $-\frac{\pi}{6}$ **j** ≈ -0.874

k ≈ 1.24 **l** ≈ -1.55

2 **a** $(0, 0)$ **b** $(0, 0)$ **c** $(0.739, 0.739)$

3 **a** horizontal asymptotes $y = -\frac{\pi}{2}$, $y = \frac{\pi}{2}$

b No, $y = \sin x$ and $y = \cos x$ do not have horizontal asymptotes.

4 **a** $\arcsin(\sin\frac{\pi}{3}) = \frac{\pi}{3}$ **b** $\arccos(\cos(-\frac{\pi}{6})) = \frac{\pi}{6}$

c $\tan(\arctan(0.3)) = 0.3$ **d** $\cos(\arccos(-\frac{1}{2})) = -\frac{1}{2}$

e $\arctan(\tan\pi) = 0$ **f** $\arcsin(\sin\frac{4\pi}{3}) = -\frac{\pi}{3}$

5 **a** $x = 1$ **b** $x = -\frac{\sqrt{3}}{2}$ **c** $x = -\frac{1}{\sqrt{2}}$

d $x = -\frac{1}{2}$ **e** no solutions **f** $x = 0$

EXERCISE 1C.1

1 **a** $2\sin\theta$ **b** $3\cos\theta$ **c** $2\sin\theta$ **d** $\sin\theta$

e $-2\tan\theta$ **f** $-3\cos^2\theta$

2 **a** $2\tan x$ **b** $\tan^2 x$ **c** $\sin x$ **d** $\cos x$

e $5\sin x$ **f** $2\sec x$

3 **a** 1 **b** 1 **c** $\frac{\cos x}{\sin^2 x}$ **d** $\cos x$ **e** $\cos x$

4 **a** $2\cos\theta$ **b** $-\tan\theta$ **c** 0 **d** $-\tan\theta$ **e** $\cot\theta$

f $2\cos\theta$ **g** $\tan\theta$ **h** $\tan\theta$ **i** $2\tan\theta$

EXERCISE 1C.2

1 **a** 3 **b** -2 **c** -1 **d** $3\cos^2\theta$

e $4\sin^2\theta$ **f** $\cos\theta$ **g** $-\sin^2\theta$ **h** $-\cos^2\theta$

i $-2\sin^2\theta$ **j** 1 **k** $\sin\theta$ **l** $\sin\theta$

3 **a** $1 + 2\sin\theta + \sin^2\theta$ **b** $\sin^2\alpha - 4\sin\alpha + 4$

c $\sec^2\alpha - 2\tan\alpha$ **d** $1 + 2\sin\alpha\cos\alpha$

e $1 - 2\sin\beta\cos\beta$ **f** $-4 + 4\cos\alpha - \cos^2\alpha$

4 **a** $-\tan^2\beta$ **b** 1 **c** $\sin^2\alpha$

d $\sin^2 x - \tan^2 x$ **e** 13 **f** $\cos^2\theta$ **g** 0

EXERCISE 1C.3

1 **a** $(1 + \sin\theta)(1 - \sin\theta)$ **b** $(\sin\alpha + \cos\alpha)(\sin\alpha - \cos\alpha)$

c $(\tan\alpha + 1)(\tan\alpha - 1)$ **d** $\sin\beta(2\sin\beta - 1)$

e $\cos\phi(2 + 3\cos\phi)$ **f** $3\sin\theta(\sin\theta - 2)$

g $(\tan\theta + 2)(\tan\theta + 3)$ **h** $(2\cos\theta + 1)(\cos\theta + 3)$

i $(3\cos\alpha + 1)(2\cos\alpha - 1)$ **j** $\tan\alpha(3\tan\alpha - 2)$

k $(\sec\beta + \operatorname{cosec}\beta)(\sec\beta - \operatorname{cosec}\beta)$

l $(2\cot x - 1)(\cot x - 1)$

m $(2\sin x + \cos x)(\sin x + 3\cos x)$

2 **a** $1+\sin\alpha$ **b** $\tan\beta-1$ **c** $\cos\phi-\sin\phi$
d $\cos\phi+\sin\phi$ **e** $\dfrac{1}{\sin\alpha-\cos\alpha}$ **f** $\dfrac{\cos\theta}{2}$
g $\sin\theta$ **h** $\cos\theta$ **i** $\sec\theta+1$

4 **a** $x=\frac{\pi}{6}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$ **b** no real solutions
c $x=\frac{\pi}{6}$ or $\frac{5\pi}{6}$ **d** $x\approx 0.730,\ 2.41,\ 3.87,$ or 5.55

5 $x=-\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}$, or $\frac{5\pi}{12}$

EXERCISE 1D

2 **a** $\frac{24}{25}$ **b** $-\frac{7}{25}$ **c** $-\frac{24}{7}$ **3** **a** $-\frac{7}{9}$ **b** $\frac{1}{9}$

4 **a** $\cos\alpha=-\frac{\sqrt5}{3}$ **b** $\sin 2\alpha=\frac{4\sqrt5}{9}$ **c** $\tan 2\alpha=4\sqrt5$

5 **a** $\sin\beta=-\frac{\sqrt{21}}{5}$ **b** $\sin 2\beta=-\frac{4\sqrt{21}}{25}$
c $\tan 2\beta=\frac{4\sqrt{21}}{17}$

6 **a** $\frac13$ **b** $\frac{2\sqrt2}{3}$ **7** **a** $-\frac{1}{\sqrt3}$ **b** $\sqrt{\frac23}$

8 **a** $\tan A=-\frac{7}{3}$ **b** $\tan A=\frac32$ **9** $\tan\frac{\pi}{8}=\sqrt2-1$

10 **a** $\sin 2\alpha$ **b** $2\sin 2\alpha$ **c** $\frac12\sin 2\alpha$ **d** $\cos 2\beta$
e $-\cos 2\phi$ **f** $\cos 2N$ **g** $-\cos 2M$ **h** $\cos 2\alpha$
i $-\cos 2\alpha$ **j** $\sin 4A$ **k** $\sin 6\alpha$ **l** $\cos 8\theta$
m $-\cos 6\beta$ **n** $\cos 10\alpha$ **o** $-\cos 6D$ **p** $\cos 4A$
q $\cos\alpha$ **r** $-2\cos 6P$

11 $\frac32$

13 **a** $x=0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$, or 2π **b** $x=\frac{\pi}{2}$ or $\frac{3\pi}{2}$
c $x=0, \pi$, or 2π

15 $\theta=-\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}$, or $\frac{5\pi}{12}$

16 **a** $\cos A=\frac{7}{10}$ **b** $\cos A=\frac34$

18 **a** $x=0, \frac{2\pi}{3}, \frac{4\pi}{3}$, or 2π **b** $x=\frac{\pi}{3}$ or $\frac{5\pi}{3}$
c $x=\frac{\pi}{2}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$
d $x=0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$, or 2π **e** $x=\frac{\pi}{2}$
f $x=\frac{\pi}{4}$ **g** $x=\frac{\pi}{6}$ or $\frac{5\pi}{6}$
h $x=0, \frac{7\pi}{6}, \frac{11\pi}{6}$, or 2π

19 **a** C_1 is $y=\cos x$, C_2 is $y=\cos 2x+1$
b A$(\frac{\pi}{3}, \frac12)$, B$(\frac{\pi}{2}, 0)$, C$(\frac{3\pi}{2}, 0)$, D$(\frac{5\pi}{3}, \frac12)$

EXERCISE 1E

2 **a** $\cos\theta$ **b** $-\sin\theta$ **c** $\sin\theta$
d $-\cos\alpha$ **e** $-\sin A$ **f** $-\sin\theta$
g $\dfrac{1+\tan\theta}{1-\tan\theta}$ **h** $\dfrac{1+\tan\theta}{1-\tan\theta}$ **i** $\tan\theta$

3 **a** $\frac12\sin\theta+\frac{\sqrt3}{2}\cos\theta$ **b** $\frac{\sqrt3}{2}\sin\theta-\frac12\cos\theta$
c $-\frac{1}{\sqrt2}\sin\theta+\frac{1}{\sqrt2}\cos\theta$ **d** $-\frac{\sqrt3}{2}\sin\theta+\frac12\cos\theta$

4 **a** $\cos\theta$ **b** $\sin 3A$ **c** $\sin(B-A)$
d $\cos(\alpha-\beta)$ **e** $-\cos(\phi+\theta)$ **f** $2\sin(\alpha-\beta)$
g $\tan\theta$ **h** $\tan 3A$

7 **a** $2+\sqrt3$ **b** $-2-\sqrt3$ **8** $\frac{7}{17}$ **9** 7

10 $\dfrac{9+5\sqrt2}{2}$ **11** $\sqrt3$ **12** $\tan A=-\frac{1}{21}$

13 $\tan\alpha=\frac{25}{62}$ **14** $\frac18$

15 **a** $\cos 2\alpha$ **b** $-\sin 3\phi$ **c** $\cos\beta$ **d** -1 **e** $\tan 2A$

16 **a** $\dfrac{54-25\sqrt5}{22}$ **b** $-4\sqrt5$ **17** $\tan A=\pm1$

18 $\tan(A+B+C)$
$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan A\tan C-\tan B\tan C}$$

21 **b** $\theta=-\frac{8\pi}{9}, -\frac{4\pi}{9}, -\frac{2\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}$, or $\frac{8\pi}{9}$

22 **a** $\sin 3\theta=-4\sin^3\theta+3\sin\theta$
b $\theta=0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi, \frac{9\pi}{4}, \frac{11\pi}{4}$, or 3π

23 $k=2$, $a=\frac{\pi}{6}$

24 **a** $2\cos x+2\sin x=2\sqrt2\cos\left(x+\frac{7\pi}{4}\right)$
b $x=\frac{7\pi}{12}$ or $\frac{23\pi}{12}$

25 **a** A vertical stretch with scale factor $\sqrt{10}$, then a translation of ≈ 0.322 units left.
b greatest value $=12$, least value $=2$

27 **a** $2\cos x-5\sin x\approx\sqrt{29}\cos(x+1.19)$
b $x\approx 0.761$ or π
d $x\approx 0.761$ (the solution $x=\pi$ has been lost)

29 **a** 0 **b** $\frac{\pi}{4}$

30 **c** **i** $\frac12\sin 4\theta+\frac12\sin 2\theta$ **ii** $\frac12\sin 7\alpha+\frac12\sin 5\alpha$
iii $\sin 6\beta+\sin 4\beta$ **iv** $2\sin 5\theta+2\sin 3\theta$
v $3\sin 7\alpha-3\sin\alpha$ **vi** $\frac16\sin 8A-\frac16\sin 2A$

31 **c** **i** $\frac12\cos 5\theta+\frac12\cos 3\theta$ **ii** $\frac12\cos 8\alpha+\frac12\cos 6\alpha$
iii $\cos 4\beta+\cos 2\beta$ **iv** $3\cos 8x+3\cos 6x$
v $\frac32\cos 5P+\frac32\cos 3P$ **vi** $\frac18\cos 6x+\frac18\cos 2x$

32 **c** **i** $\frac12\cos 2\theta-\frac12\cos 4\theta$ **ii** $\frac12\cos 5\alpha-\frac12\cos 7\alpha$
iii $\cos 4\beta-\cos 6\beta$ **iv** $2\cos 3\theta-2\cos 5\theta$
v $5\cos 6A-5\cos 10A$ **vi** $\frac{1}{10}\cos 4M-\frac{1}{10}\cos 10M$

33 **a** $\sin A\cos A=\frac12\sin 2A$, $\cos^2 A=\frac12\cos 2A+\frac12$,
$\sin^2 A=\frac12-\frac12\cos 2A$
c $\cos S+\cos D=2\cos\left(\frac{S+D}{2}\right)\cos\left(\frac{S-D}{2}\right)$
d $\cos D-\cos S=2\sin\left(\frac{S+D}{2}\right)\sin\left(\frac{S-D}{2}\right)$

34 **a** $2\sin 3x\cos 2x$ **b** $2\cos 5A\cos 3A$ **c** $-2\sin 2\alpha\sin\alpha$
d $2\cos 4\theta\sin\theta$ **e** $-2\sin 4\alpha\sin 3\alpha$ **f** $2\sin 5\alpha\cos 2\alpha$
g $2\sin 3B\sin B$ **h** $2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$
i $-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}$

REVIEW SET 1A

1 **a** $\sqrt2$ **b** $-\sqrt3$ **c** 2

2 **a** $\frac{\pi}{4}$ **b** $\frac{\pi}{6}$ **c** $-\frac{\pi}{6}$

3 **a** $x=\frac{\pi}{4}$ or $\frac{7\pi}{4}$ **b** $x=\frac{2\pi}{3}$ or $\frac{5\pi}{3}$

4 **a** $\cos\theta$ **b** $-\sin\theta$ **c** $\cos\theta$
d $1-\cos\theta$ **e** $\dfrac{1}{\sin\alpha+\cos\alpha}$ **f** $\dfrac{-\cos\alpha}{2}$

5 **a** $-\frac{\sqrt7}{4}$ **b** $\frac{3\sqrt7}{8}$ **c** $-\frac18$
d $-3\sqrt7$ **e** $-\sqrt{\frac{4-\sqrt7}{8}}$ **f** $\sqrt{\frac{4+\sqrt7}{8}}$

7 **a** $\frac{-\sqrt2-\sqrt6}{4}$ **b** $2-\sqrt3$ **8** 60 m

9 $x=0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$, or 2π **11** **c** $x=\frac{16}{3}$ or 3

13 $3\sin x - 5\cos x \approx \sqrt{34}\cos(x+3.68)$

14 **a** $2\sin x + \sqrt{3}\cos x \approx \sqrt{7}\sin(x+0.714)$

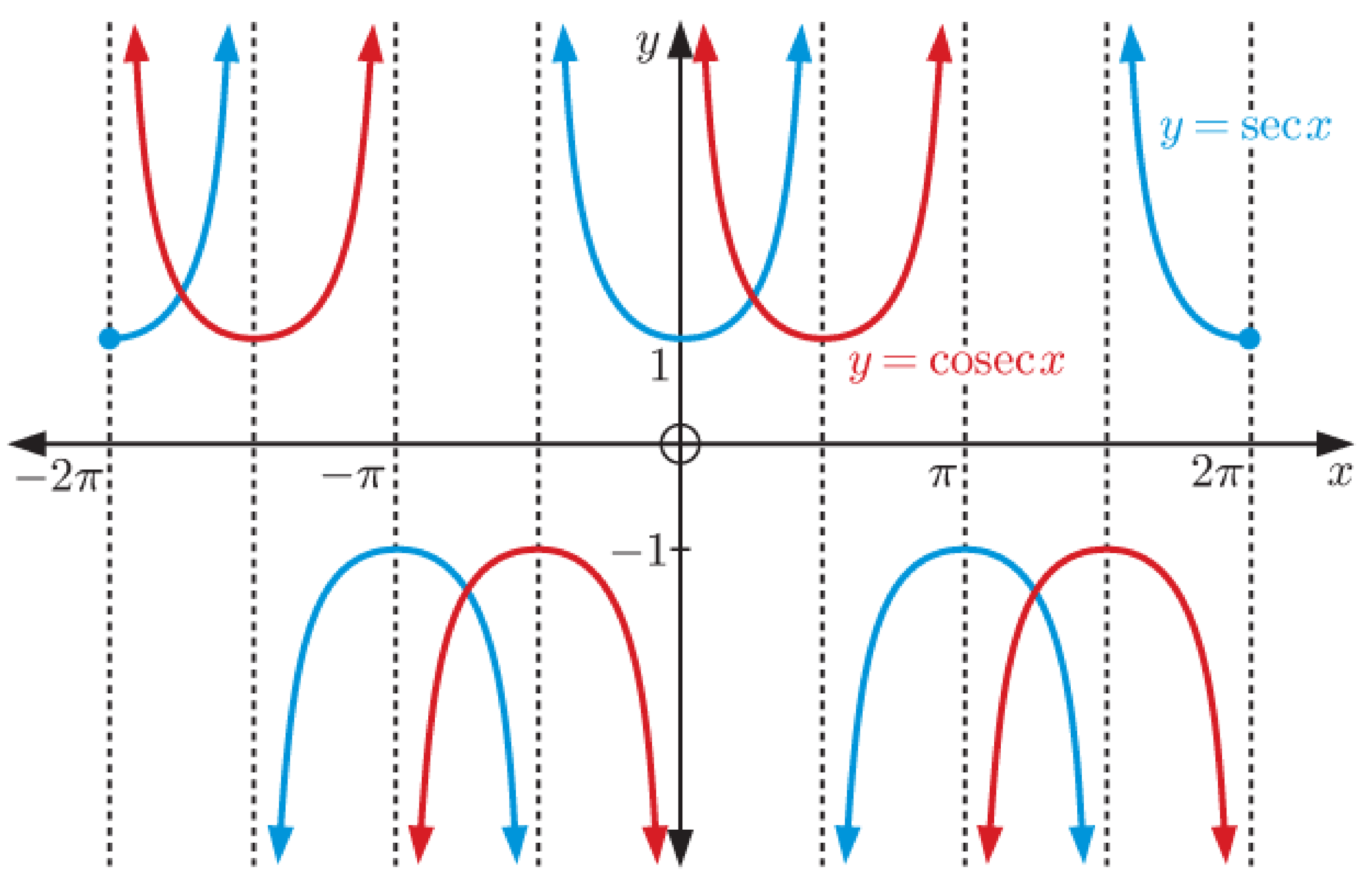
b **i** $A=\sqrt{7}$ **ii** $b \approx 2.43$

15 $\frac{\pi}{4}$

REVIEW SET 1B

1 $\sin x = -\frac{2\sqrt{2}}{3}$, $\tan x = 2\sqrt{2}$, $\operatorname{cosec} x = -\frac{3}{2\sqrt{2}}$, $\sec x = -3$, $\cot x = \frac{1}{2\sqrt{2}}$

2 **a**



b translation $\frac{\pi}{2}$ units right

3 $x = -\frac{5\pi}{6}$ or $\frac{\pi}{6}$ **4** **a** $x = \frac{\sqrt{3}}{2}$ **b** $x = 2 + \frac{1}{\sqrt{3}}$

5 **a** $\sec x$ **b** $\sin x$ **c** $\cos x$

6 **a** $\cos\theta$ **b** $-\sin\theta$ **c** $5\cos^2\theta$ **d** $-\cos\theta$ **e** $\operatorname{cosec}\theta$ **f** $\sin 2\theta$

7 **a** $\frac{120}{169}$ **b** $\frac{119}{169}$ **c** $\frac{120}{119}$

10 **a** $x = -\frac{2\pi}{3}, -\frac{\pi}{2}, -\frac{\pi}{3}$, or $\frac{\pi}{2}$ **b** $\theta = \frac{\pi}{3}$

11 $\sin\left(\theta + \frac{\pi}{6}\right) = \frac{3\sqrt{3}-\sqrt{7}}{8}$ **12** $\tan\theta = \frac{9}{19}$

13 $3\sin x + 4\cos x \approx 5\sin(x+0.927)$ **14** 1.5 m

15 **b** $y = 2\sec 2x$ has range $\{y \mid y \leqslant -2 \text{ or } y \geqslant 2\}$

$\therefore \dfrac{1}{1+\sqrt{2}\sin x} + \dfrac{1}{1-\sqrt{2}\sin x} = 1$ has no solutions.

EXERCISE 2A

1 **a** $2^{\frac{1}{5}}$ **b** $2^{-\frac{1}{5}}$ **c** $2^{\frac{3}{2}}$ **d** $2^{\frac{5}{2}}$ **e** $2^{-\frac{1}{3}}$ **f** $2^{\frac{4}{3}}$ **g** $2^{\frac{3}{2}}$ **h** $2^{\frac{3}{2}}$ **i** $2^{-\frac{4}{3}}$ **j** $2^{-\frac{3}{2}}$

2 **a** $3^{\frac{1}{3}}$ **b** $3^{-\frac{1}{3}}$ **c** $3^{\frac{1}{4}}$ **d** $3^{\frac{3}{2}}$ **e** $3^{-\frac{5}{2}}$

3 **a** $7^{\frac{1}{3}}$ **b** $3^{\frac{3}{4}}$ **c** $2^{\frac{4}{5}}$ **d** $2^{\frac{5}{3}}$ **e** $7^{\frac{2}{7}}$ **f** $7^{-\frac{1}{3}}$ **g** $3^{-\frac{3}{4}}$ **h** $2^{-\frac{4}{5}}$ **i** $2^{-\frac{5}{3}}$ **j** $7^{-\frac{2}{7}}$

4 **a** $x^{\frac{1}{2}}$ **b** $x^{\frac{3}{2}}$ **c** $x^{-\frac{1}{2}}$ **d** $x^{\frac{5}{2}}$ **e** $x^{-\frac{3}{2}}$

5 **a** ≈ 2.28 **b** ≈ 0.435 **c** ≈ 1.68 **d** ≈ 1.93 **e** ≈ 0.523

6 **a** $\sqrt[3]{5}$ **b** $\frac{1}{\sqrt{3}}$ **c** $9\sqrt{3}$ **d** $m\sqrt{m}$ **e** $x^3\sqrt{x}$

7 **a** 8 **b** 32 **c** 8 **d** 125 **e** 4 **f** $\frac{1}{2}$ **g** $\frac{1}{27}$ **h** $\frac{1}{16}$ **i** $\frac{1}{81}$ **j** $\frac{1}{25}$

EXERCISE 2B

1 **a** 1 **b** x **c** $x^{\frac{1}{2}}$ or $\sqrt{x}$

2 **a** $x^5+2x^4+x^2$ **b** $2^{2x}+2^x$ **c** $x+1$ **d** $7^{2x}+2(7^x)$ **e** $2(3^x)-1$ **f** x^2+2x+3 **g** $1+5(2^{-x})$ **h** 5^x+1 **i** $x^{\frac{3}{2}}+x^{\frac{1}{2}}+1$ **j** $3^{2x}+5(3^x)+1$ **k** $2x^{\frac{3}{2}}-x^{\frac{1}{2}}+5$ **l** $2^{3x}-3(2^{2x})-1$

3 **a** $2^{2x}+2^{x+1}-3$ **b** $3^{2x}+7(3^x)+10$ **c** $5^{2x}-6(5^x)+8$ **d** $2^{2x}+6(2^x)+9$ **e** $3^{2x}-2(3^x)+1$ **f** $4^{2x}+14(4^x)+49$ **g** $x-4$ **h** 4^x-9 **i** $x-\frac{1}{x}$ **j** $x^2+4+\frac{4}{x^2}$ **k** $7^{2x}-2+7^{-2x}$ **l** $25-10(2^{-x})+2^{-2x}$

4 **a** $5^x(5^x+1)$ **b** $10(3^n)$ **c** $7^n(1+7^{2n})$ **d** $5(5^n-1)$ **e** $6(6^{n+1}-1)$ **f** $16(4^n-1)$ **g** $2^n(2^n-8)$ **h** $\frac{5}{2}(2^n)$ **i** $\frac{9}{2}(2^{2n})$

5 **a** $(3^x+2)(3^x-2)$ **b** $(2^x+5)(2^x-5)$ **c** $(4+3^x)(4-3^x)$ **d** $(5+2^x)(5-2^x)$ **e** $(3^x+2^x)(3^x-2^x)$ **f** $(2^x+3)^2$ **g** $(3^x+5)^2$ **h** $(2^x-7)^2$ **i** $(5^x-2)^2$

6 **a** $(2^x+1)(2^x-2)$ **b** $(3^x+3)(3^x-2)$ **c** $(2^x-3)(2^x-4)$ **d** $(2^x+3)(2^x+6)$ **e** $(2^x+4)(2^x-5)$ **f** $(3^x+2)(3^x+7)$ **g** $(3^x+5)(3^x-1)$ **h** $(5^x+2)(5^x-1)$ **i** $(7^x-4)(7^x-3)$

7 **a** 2^n **b** 10^a **c** 3^b **d** $\frac{1}{5^n}$ **e** 5^x **f** $(\frac{3}{4})^a$ **g** $(\frac{8}{3})^k$ **h** 5 **i** 5^n

8 **a** 3^m+1 **b** $1+6^n$ **c** 4^n+2^n **d** 4^x-1 **e** 6^n **f** 5^n **g** 4 **h** 2^n-1 **i** $\frac{1}{2}$

9 **a** $n\,2^{n+1}$ **b** -3^{n-1}

EXERCISE 2C

1 **a** $x=5$ **b** $x=2$ **c** $x=4$ **d** $x=0$ **e** $x=-1$ **f** $x=\frac{1}{2}$ **g** $x=-3$ **h** $x=2$ **i** $x=-3$ **j** $x=-4$ **k** $x=2$ **l** $x=\frac{3}{4}$

2 **a** $x=\frac{5}{3}$ **b** $x=-\frac{3}{2}$ **c** $x=-\frac{3}{2}$ **d** $x=-\frac{1}{2}$ **e** $x=-\frac{2}{3}$ **f** $x=-\frac{5}{4}$ **g** $x=\frac{3}{2}$ **h** $x=\frac{5}{2}$ **i** $x=\frac{1}{8}$ **j** $x=\frac{9}{2}$ **k** $x=-4$ **l** $x=-\frac{7}{2}$ **m** $x=0$ **n** $x=\frac{7}{2}$ **o** $x=-\frac{2}{3}$ **p** $x=-6$

3 **a** $x=\frac{1}{7}$ **b** no solution **c** $x=\frac{5}{2}$ **d** $x=\frac{1}{3}$ **e** $x=-\frac{1}{4}$ **f** $x=-1$ or 3

4 **a** $x=3$ **b** $x=2$ **c** $x=-1$ **d** $x=2$ **e** $x=-2$ **f** $x=-2$

5 **a** $x=1$ or 2 **b** $x=1$ **c** $x=1$ or 2 **d** $x=1$ **e** $x=2$ **f** $x=0$ **g** $x=1$ **h** $x=1$ or -1 **i** $x=2$ **j** $x=-2$ or 1 **k** $x=2$ **l** $x=\frac{1}{2}$

6 $x=\frac{15}{7}$, $y=\frac{10}{7}$

EXERCISE 2D

1 **a** **i** ≈ 1.4 **ii** ≈ 1.7 **iii** ≈ 2.8 **iv** ≈ 0.4

b **i** $x \approx 1.6$ **ii** $x \approx -0.7$

c $y=2^x$ has a horizontal asymptote of $y=0$.

2 **a** **C** **b** **B** **c** **E** **d** **A** **e** **D**