Harmonic form

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When we want to calculate for example $\sin\frac{\pi}{12}$, we will write it as $\sin(\frac{\pi}{3}-\frac{\pi}{4})$ and apply the second formula. Now we want to learn to use the formulae in the opposite direction.

Sidenote

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Recall that $\sin^{-1}(x)$ denotes the inverse of sine. Similarly $\cos^{-1}(x)$ and $\tan^{-1}(x)$ are inverses of cosine and tangent respectively. Another way to write these is $\arcsin(x)$, $\arccos(x)$ and $\arctan(x)$.

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So $\arctan(x)$ is simply the inverse of tangent (i.e. it is the same thing as $\tan^{-1}(x)$).

Example 1

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I will use the following argument: the range of $\cos x$ is $[-1, 1]$, so the range of $3\cos x$ is $[-3, 3]$, similarly the range of $4\sin x$ is $[-4, 4]$, so the range of $3\cos x + 4\sin x$ is $[-7, 7]$.

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The range of $f(x) = 3\cos x + 4\sin x$ is **not** $[-7, 7]$.

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The range of $f(x) = 3\cos x + 4\sin x$ is **not** $[-7, 7]$. The reason the above argument is wrong is that $\cos x$ and $\sin x$ are maximal/minimal for different values of x (there is no x for which $\cos x = 1$ and $\sin x = 1$ simultaneously).

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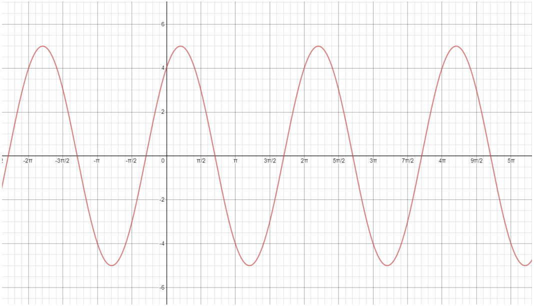

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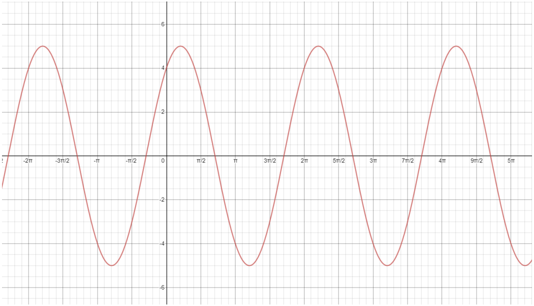

We can actually see what the range is from the graph, but that won't always be possible. What's more important is that the graph is a trigonometric function. So we should be able to write $f(x)$ as a single trigonometric function.

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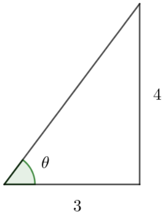

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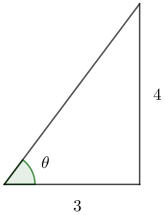

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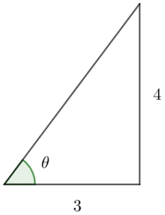

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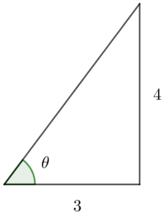

The hypotenuse is then 5, so we have $\cos\theta = \dfrac{3}{5}$ and $\sin\theta = \dfrac{4}{5}$. Also $\theta = \arctan\left(\dfrac{4}{3}\right)$. We can actually calculate that $\theta \approx 0.927$, but I'll stick with $\theta = \arctan\left(\dfrac{4}{3}\right)$.

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where $\theta = \arctan\left(\frac{4}{3}\right)$.
Now we use the compound angle formula to get:

$$5(\cos\theta\cos x + \sin\theta\sin x) = 5\cos(x - \theta)$$

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θ corresponds to a horizontal shift, so it doesn't influence the range. The amplitude is 5, so the range of f is $[-5, 5]$.

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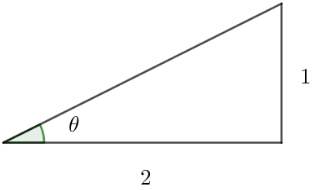

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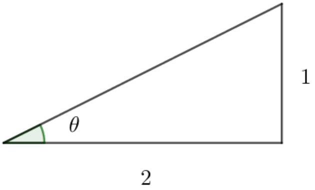

The hypotenuse is $\sqrt{5}$ and $\theta = \arctan\left(\frac{1}{2}\right)$.

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We can now write:

$$\begin{aligned} 2\sin x - \cos x =& \sqrt{5}\left(\frac{2}{\sqrt{5}}\sin x - \frac{1}{\sqrt{5}}\cos x\right) = \\ =& \sqrt{5}\left(\cos\theta\sin x - \sin\theta\cos x\right) \\ =& \sqrt{5}\sin(x-\theta) \end{aligned}$$

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where $\theta = \arctan\left(\frac{1}{2}\right)$.

So $f(x) = \sqrt{5}\sin(x-\theta)$, which means that the range of $f(x)$ is $[-\sqrt{5}, \sqrt{5}]$.

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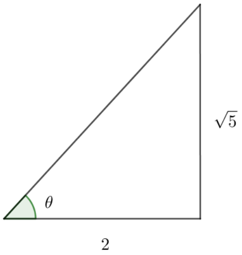

The hypotenuse is 3 and $\theta = \arctan\left(\frac{\sqrt{5}}{2}\right)$.

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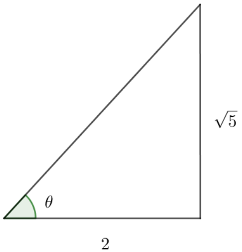

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We get:

$$
\begin{aligned}
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=& 3\left(\cos\theta\sin x + \sin\theta\cos x\right) \\
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where $\theta = \arctan\left(\frac{\sqrt{5}}{2}\right)$.

So $f(x) = 3\sin(x+\theta)$, which means that the range of $f(x)$ is $[-3, 3]$.

Make sure you study this presentation carefully. If there are any questions, we will discuss them on Wednesday.