Sum and difference of angles

The goal is to learn the identities for the sine and cosine of a sum or a difference of two angles.

Identities

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
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Easy consequences of the above identities:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

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If you don't think that the above are easy consequences, let me know and I will explain in class.

These:

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are called double angle identities. They're important. You will need to use those to solve the exercises posted.

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are called double angle identities. They're important. You will need to use those to solve the exercises posted.
They're easy consequences of the first four identities. For instance if we set $\alpha = \beta$ into the first identity, we get:

$$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \sin\alpha\cos\alpha$$

Which gives:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

Identities

We will prove:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

The other 3 follow easily by substituting certain angles.

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Proof

Let's put $\alpha + \beta$ on the unit circle.

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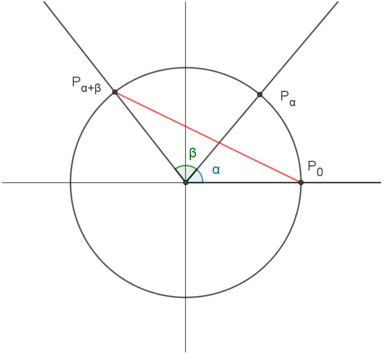

We then have from the definition of cosine that $\cos(\alpha + \beta)$ is the x-coordinate of point $P_{\alpha+\beta}$. We will calculate the length of $P_0 P_{\alpha+\beta}$.

Proof

Using Pythagorean Theorem:

$$|P_0 P_{\alpha+\beta}|^2 = (1 - \cos(\alpha + \beta))^2 + \sin^2(\alpha + \beta)$$

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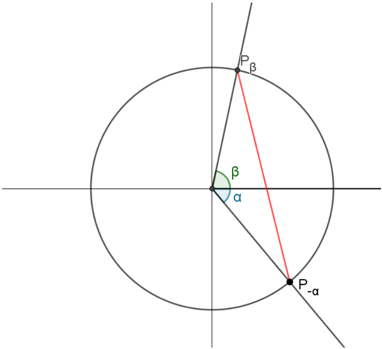
We get:

$$|P_0P_{\alpha+\beta}|^2 = 1-2\cos(\alpha+\beta)+\cos^2(\alpha+\beta)+\sin^2(\alpha+\beta) = 2-2\cos(\alpha+\beta)$$

Proof

Now if we rotate our triangle we get:



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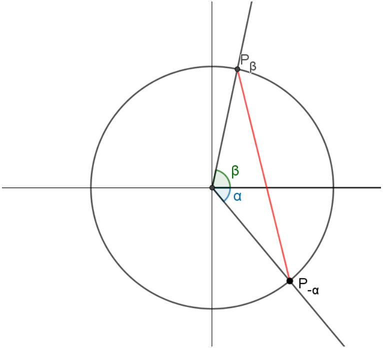

Of course the triangle did not change, so the length of the red segment is the same:

$$|P_0 P_{\alpha+\beta}| = |P_{-\alpha} P_{\beta}|$$

Proof

We will calculate $|P_{-\alpha} P_{\beta}|^2$.

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$$|P_{-\alpha}P_{\beta}|^2 = (\cos\beta - \cos(-\alpha))^2 + (\sin(-\alpha) - \sin\beta)^2$$

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$$|P_{-\alpha}P_\beta|^2 = (\cos\beta - \cos(-\alpha))^2 + (\sin(-\alpha) - \sin\beta)^2$$

We get:

$$\begin{aligned}|P_{-\alpha}P_\beta|^2 =& \cos^2\beta - 2\cos\beta\cos(-\alpha) + \cos^2(-\alpha) + \\ &+ \sin^2(-\alpha) - 2\sin(-\alpha)\sin\beta + \sin^2\beta = \\ =& 2 - 2\cos\beta\cos\alpha + 2\sin\alpha\sin\beta\end{aligned}$$

Proof

So finally we got:

$$2 - 2\cos(\alpha + \beta) = 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$$

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So:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Summary

Please try and understand this proof. If you think some details are missing, let me know.

Applications

Let's calculate $\sin 105^\circ$.

Applications

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$$\begin{aligned}\sin(105^\circ) = \sin(45^\circ + 60^\circ) = \\ = \sin 45^\circ \cos 60^\circ + \sin 60^\circ \cos 45^\circ = \\ = \frac{\sqrt{2}}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \\ = \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

Applications

How about $\cos\dfrac{\pi}{12}$?

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$$
\begin{aligned}
\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) &= \\
&= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \\
&= \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \\
&= \frac{\sqrt{2}+\sqrt{6}}{4}
\end{aligned}
$$

Applications

How about $\cos\frac{\pi}{12}$?

$$\begin{aligned}\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) &= \\ &= \cos\frac{\pi}{3}\cos\frac{\pi}{4}+\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \\ &= \frac{1}{2}\times\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} = \\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

We could've predicted the result, since:

$$\sin 105^\circ = \sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{2}+\frac{\pi}{12}\right) = \cos\frac{\pi}{12}$$

In case of any questions, you can email me at T.J.Lechowski@gmail.com.