Sum/difference - to - product formulae

When we solve an equation like:

$$x^2 - 4x - 12 = 0 \tag{1}$$

what we want to do is to turn it (if possible) into:

$$(x-6)(x+2) = 0 \tag{2}$$

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We turned equation (1) where we add terms to an equation (2) where we multiply terms. This is often useful when solving equations.

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can we somehow turn this sum into a product?

The answer is of course yes and we will see how to do this on the next few slides.

Recall that we already have the following four formulae:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

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Now let $A=\alpha+\beta$ and $B=\alpha-\beta$. This gives $\alpha=\dfrac{A+B}{2}$ and $\beta=\dfrac{A-B}{2}$.

If we add the first two equations we get:

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$$

Now let $A = \alpha + \beta$ and $B = \alpha - \beta$. This gives $\alpha = \dfrac{A+B}{2}$ and $\beta = \dfrac{A-B}{2}$. So we get the following formula:

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

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Now let $A=\alpha+\beta$ and $B=\alpha-\beta$. This gives $\alpha=\dfrac{A+B}{2}$ and $\beta=\dfrac{A-B}{2}$. So we get the following formula:

$$\sin A+\sin B=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Similarly if we subtracted the second equation from the first one we would get:

$$\sin A-\sin B=2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

We can also add the third and fourth equations to get:

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta$$

and again using the substitutions $A = \alpha + \beta$ and $B = \alpha - \beta$ we get:

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$$\cos A+\cos B=2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

Subtracting the fourth equation from the third one gives:

$$\cos A-\cos B=-2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

In the end we get the following four formulae:

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
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The IB does not require you to know these, but they are very helpful. I would highly recommend to learn them by heart (and since they're not required by the IB, they're not in the formula booklet).

Exercise 1

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$$\sin 105^\circ - \sin 15^\circ = 2\sin 45^\circ \cos 60^\circ = 2 \times \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{2}}{2}$$

Exercise 2

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$$\cos 105^\circ - \sin 75^\circ = \cos 105^\circ - \cos 15^\circ = -2\sin 60^\circ \sin 45^\circ = -\frac{\sqrt{6}}{2}$$

Exercise 3

Express $\sin\theta + \sin 3\theta$ as a product of two trigonometric functions. Hence solve $\sin\theta + \sin 2\theta + \sin 3\theta = 0$ for $0 \leq \theta \leq \pi$.

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We have:

$$\sin\theta + \sin 3\theta = 2\sin\left(\frac{\theta + 3\theta}{2}\right)\cos\left(\frac{\theta - 3\theta}{2}\right) =$$
$$= 2\sin 2\theta\cos(-\theta) = 2\sin 2\theta\cos\theta$$

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So we get:

$$\sin\theta + \sin 2\theta + \sin 3\theta = 0$$
$$2\sin 2\theta\cos\theta + \sin 2\theta = 0$$
$$\sin 2\theta(2\cos\theta + 1) = 0$$

Which gives $\sin 2\theta = 0$ or $2\cos\theta + 1 = 0$

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The first equation gives us $\theta \in \{0, \frac{\pi}{2}, \pi\}$, the second gives us $\theta = \frac{2\pi}{3}$, so in the end we have four solutions: $0, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$.

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Hence we get:

$$\cos 5\theta - \cos\theta = \sin 2\theta \quad \Rightarrow \quad -2\sin 3\theta\sin 2\theta = \sin 2\theta$$

This gives:

$$\sin 2\theta + 2\sin 3\theta \sin 2\theta = 0$$

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From the first equation we get $\theta \in \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$. The second equation gives $\theta \in \{\frac{7\pi}{18}, \frac{11\pi}{18}, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{31\pi}{18}, \frac{35\pi}{18}\}$

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So in the end we have 11 solutions:
$\theta \in \{0, \frac{7\pi}{18}, \frac{\pi}{2}, \frac{11\pi}{18}, \pi, \frac{19\pi}{18}, \frac{23\pi}{18}, \frac{3\pi}{2}, \frac{31\pi}{18}, \frac{35\pi}{18}, 2\pi\}$

In case of any questions you can email me at T.J.Lechowski@gmail.com.