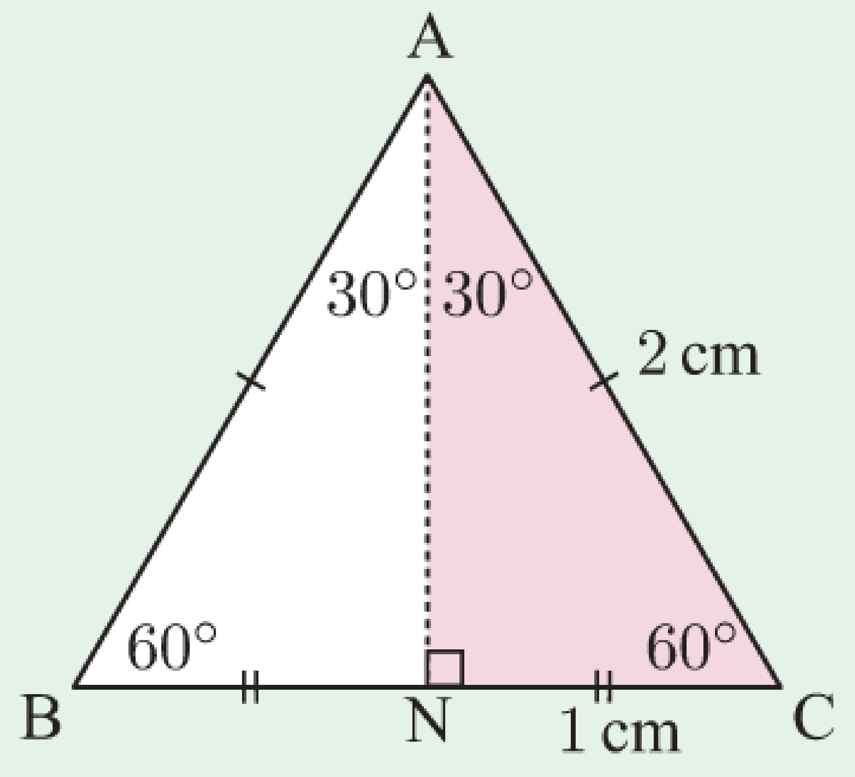
Chapter 8

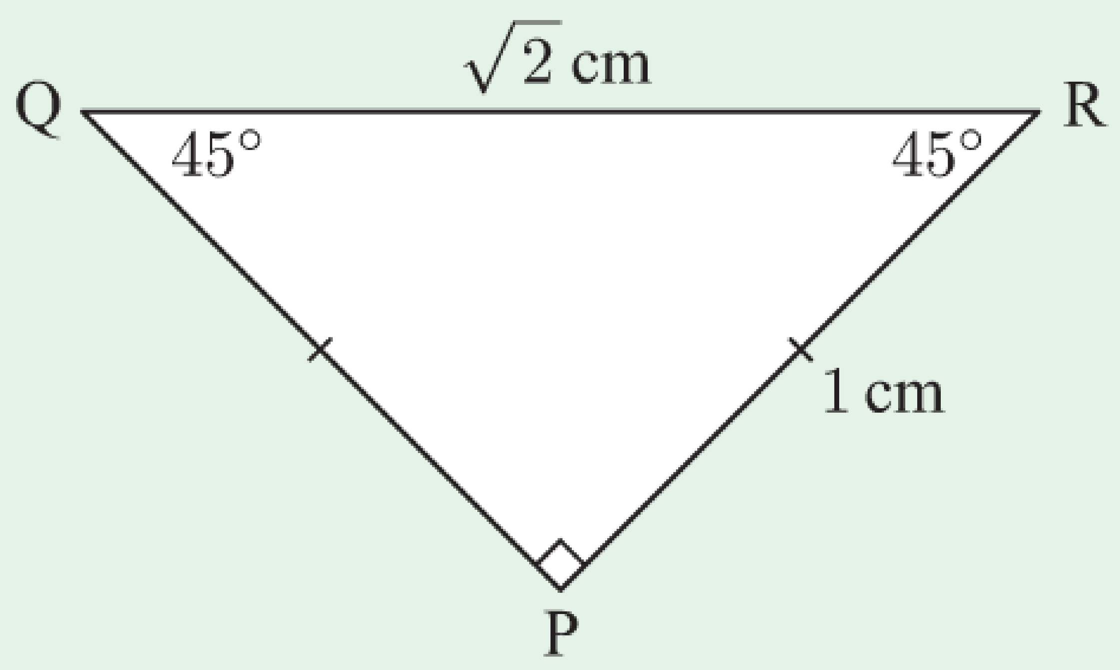
The unit circle and radian measure

Contents:

OPENING PROBLEM

Consider the triangles below:





Things to think about:

a Triangle ABC is an equilateral triangle with sides 2 cm long. Altitude [AN] bisects side [BC] and the vertical angle BAC.
Can you use this figure to find:
i $\sin 30°$ **ii** $\cos 60°$ **iii** $\cos 30°$ **iv** $\sin 60°$?

b Triangle PQR is a right angled isosceles triangle with hypotenuse $\sqrt{2}$ cm long.
Can you use this figure to find:
i $\cos 45°$ **ii** $\sin 45°$ **iii** $\tan 45°$?

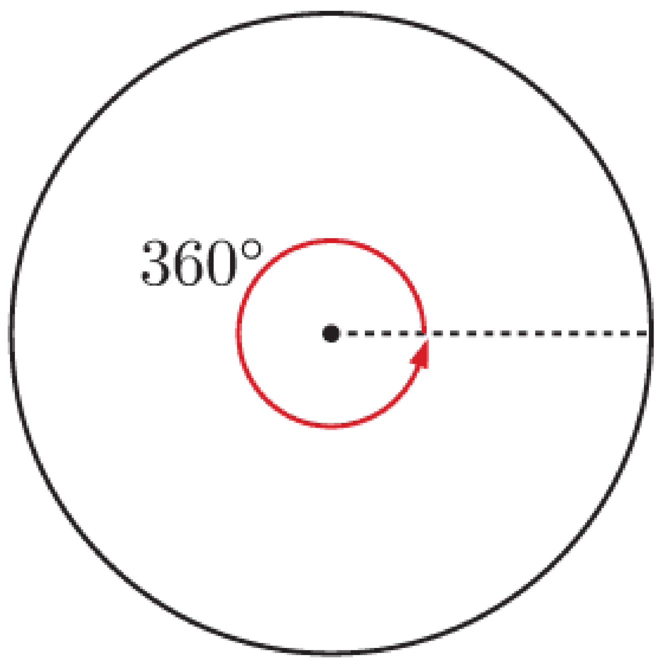
In this Chapter we build on our knowledge of angles and trigonometry. We consider:

- **radian** measure as an alternative to degrees
- the **unit circle** which helps us give meaning to the trigonometric ratios for *any* angle.

A RADIAN MEASURE

DEGREE MEASUREMENT OF ANGLES

We have seen previously that one full revolution makes an angle of 360°, and the angle on a straight line is 180°. Hence, one **degree**, 1°, can be defined as $\frac{1}{360}$th of one full revolution. This measure of angle is commonly used by surveyors and architects.



For greater accuracy we define one **minute**, $1'$, as $\frac{1}{60}$th of one degree and one **second**, $1''$, as $\frac{1}{60}$th of one minute. Obviously a minute and a second are very small angles.

Most graphics calculators can convert fractions of angles measured in degrees into minutes and seconds. This is also useful for converting fractions of hours into minutes and seconds for time measurement, as one minute is $\frac{1}{60}$th of one hour, and one second is $\frac{1}{60}$th of one minute.

GRAPHICS CALCULATOR INSTRUCTIONS

OPENING PROBLEM

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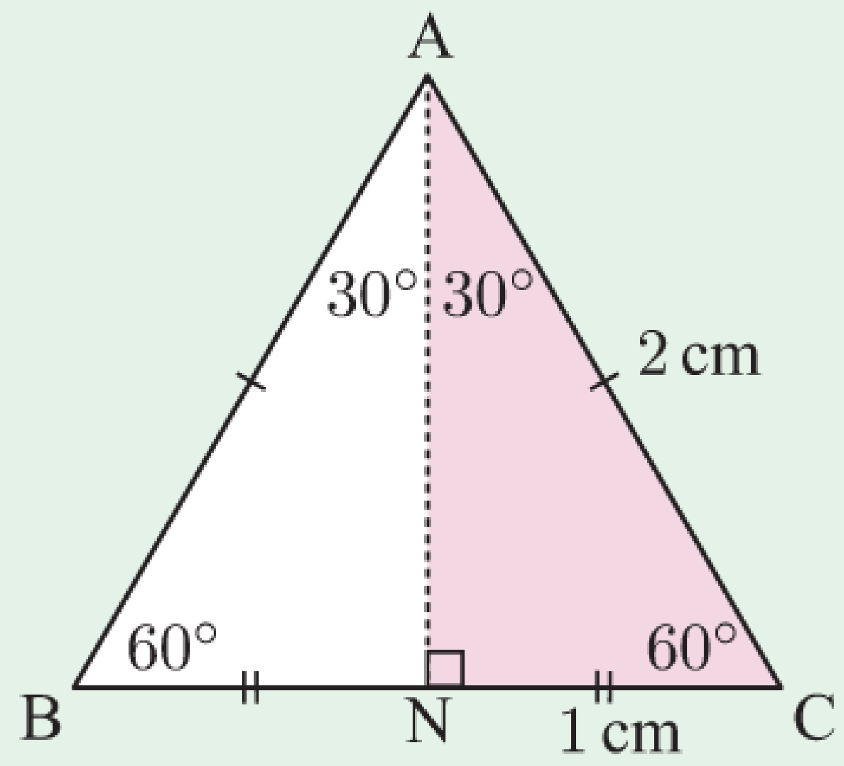

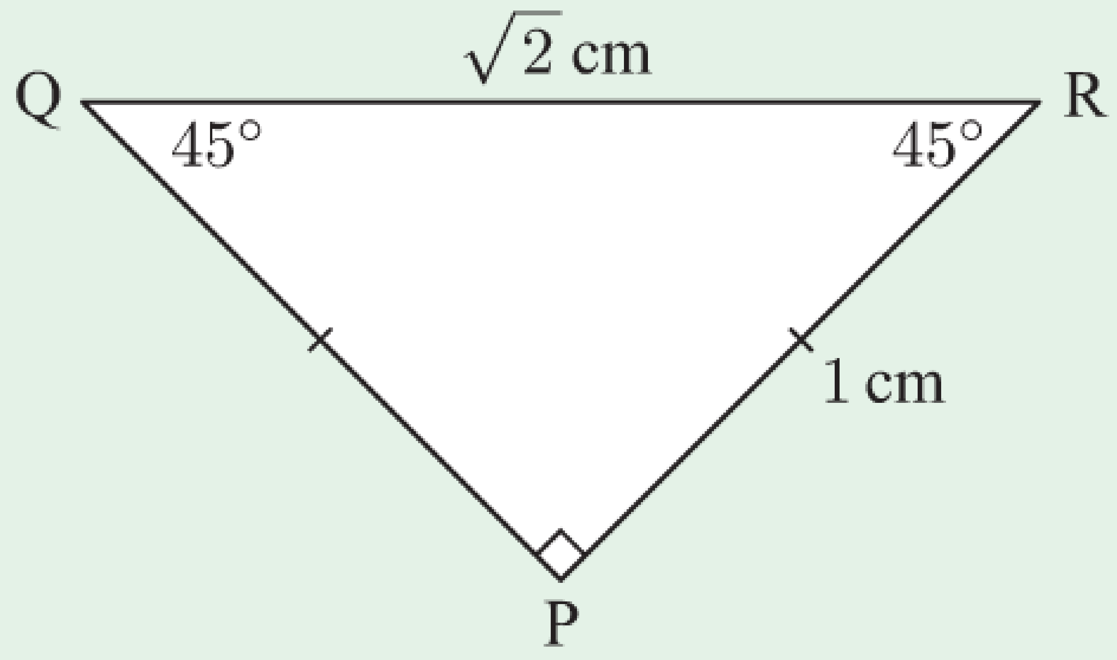

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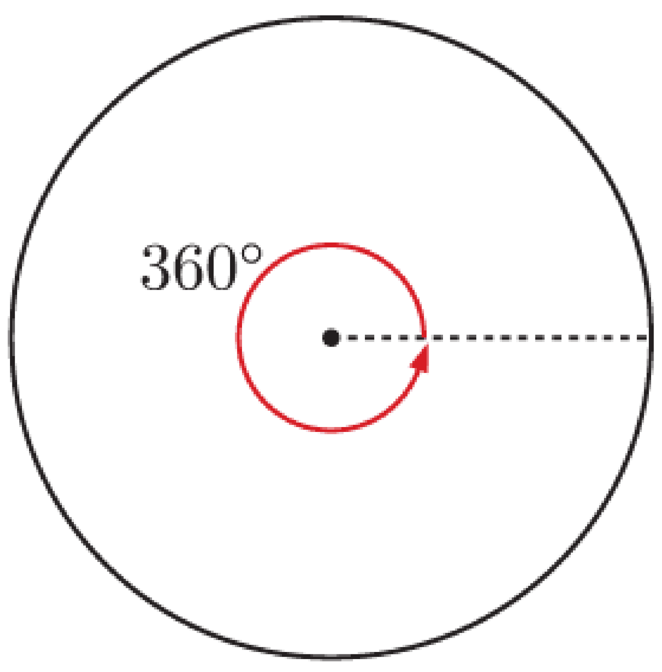

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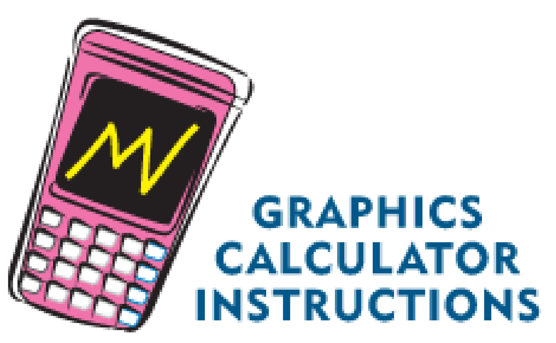


EXERCISE 8A

1 Convert to radians, in terms of π:

a 90° **b** 60° **c** 30° **d** 18° **e** 9°
f 135° **g** 225° **h** 270° **i** 360° **j** 720°
k 315° **l** 540° **m** 36° **n** 80° **o** 230°

2 Convert to radians, correct to 3 significant figures:

a 36.7° **b** 137.2° **c** 317.9° **d** 219.6° **e** 396.7°

Example 2 **Self Tutor**

Convert to degrees:

a $\frac{5\pi}{6}$ **b** 0.638 radians.

a $\frac{5\pi}{6}$

$= \left(\frac{5\pi}{6} \times \frac{180}{\pi}\right)^\circ$

$= 150^\circ$

b 0.638 radians

$= \left(0.638 \times \frac{180}{\pi}\right)^\circ$

$\approx 36.6^\circ$

3 Convert to degrees:

a $\frac{\pi}{5}$ **b** $\frac{3\pi}{5}$ **c** $\frac{3\pi}{4}$ **d** $\frac{\pi}{18}$ **e** $\frac{\pi}{9}$
f $\frac{7\pi}{9}$ **g** $\frac{\pi}{10}$ **h** $\frac{3\pi}{20}$ **i** $\frac{7\pi}{6}$ **j** $\frac{\pi}{8}$

4 Convert to degrees, correct to 2 decimal places:

a 2 **b** 1.53 **c** 0.867 **d** 3.179 **e** 5.267

5 Copy and complete, giving answers in terms of π:

a

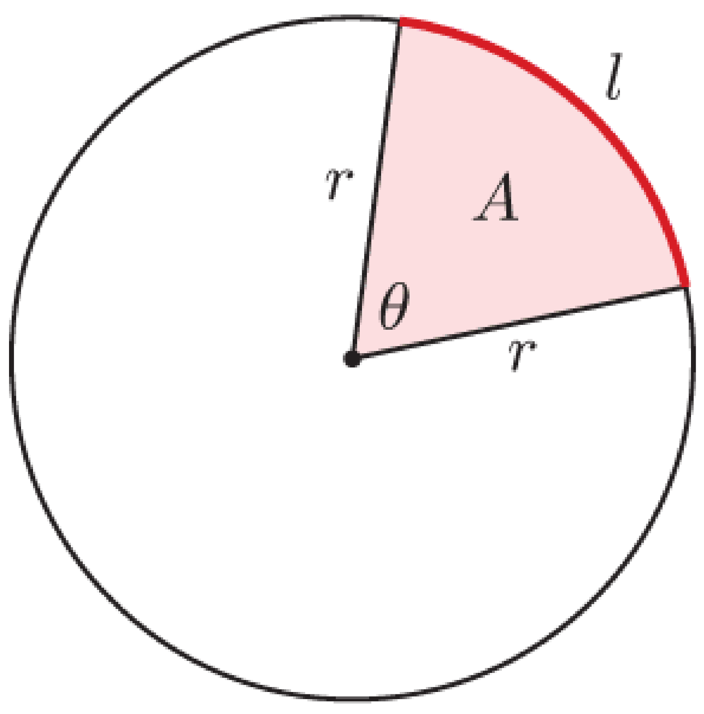
Degrees	0	45	90	135	180	225	270	315	360
Radians									

b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians													

B ARC LENGTH AND SECTOR AREA

You should have previously seen formulae for the length of an arc and the area of a sector, for an angle given in degrees.



For a sector with radius r and angle θ given in *degrees*,

arc length $\boldsymbol{l = \frac{\theta}{360} \times 2\pi r}$

area $\boldsymbol{A = \frac{\theta}{360} \times \pi r^2}$

However, if the angle θ is measured in radians, the formulae become much simpler.

- θ measures how many times longer the arc length is than the radius.

$$\therefore \ \theta = \frac{l}{r}$$

$$\therefore \ l = \theta r$$

- There are 2π radians in a circle so

$$\text{area of sector} = \frac{\theta}{2\pi} \times \text{area of circle}$$

$$\therefore \ A = \frac{\theta}{2\pi} \times \pi r^2$$

$$\therefore \ A = \tfrac{1}{2}\theta r^2$$

For a sector with radius r and angle θ given in *radians*:

- arc length $\boldsymbol{l = \theta r}$
- area $\boldsymbol{A = \frac{1}{2}\theta r^2}$

Example 3 **Self Tutor**

A sector has radius 12 cm and angle 3 radians. Find its:

a arc length **b** area

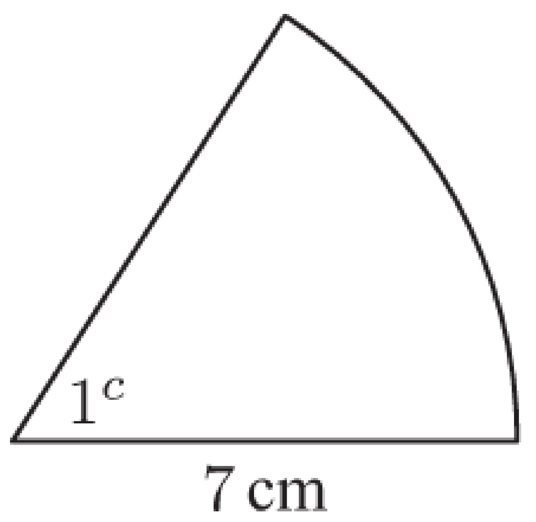
a arc length $= \theta r$
$= 3 \times 12$
$= 36$ cm

b area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 3 \times 12^2$
$= 216$ cm^2

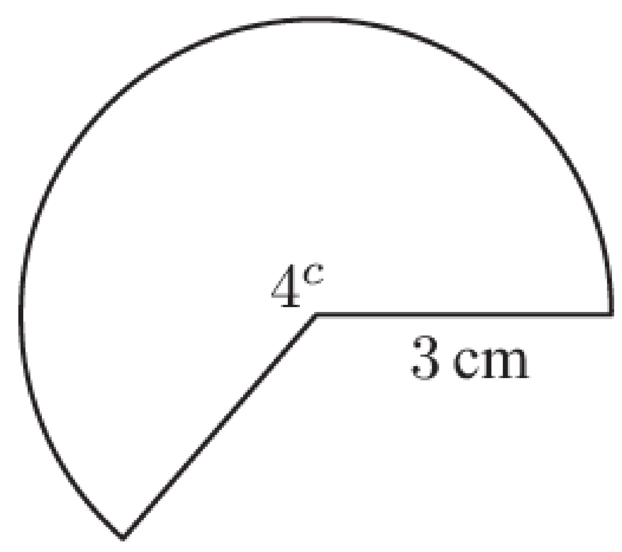
EXERCISE 8B

1 Find the arc length of each sector:

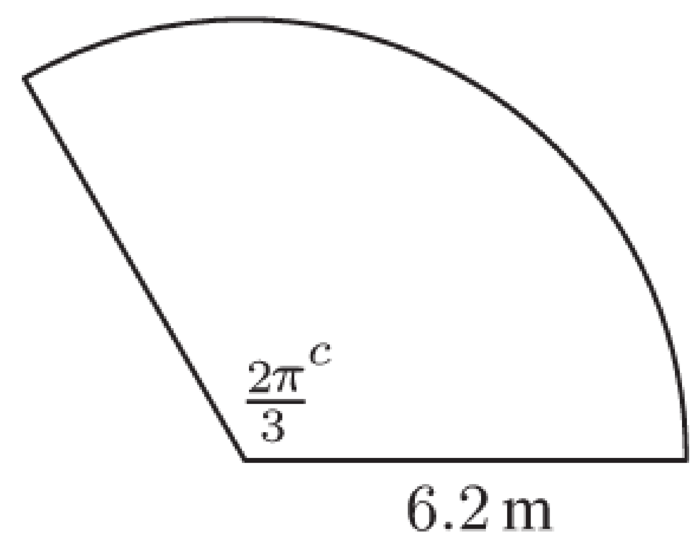
a



b

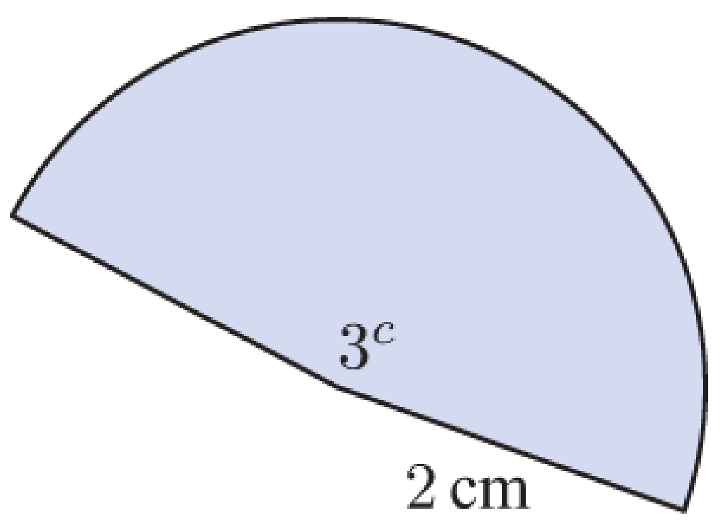


c



2 Find the area of each sector:

a



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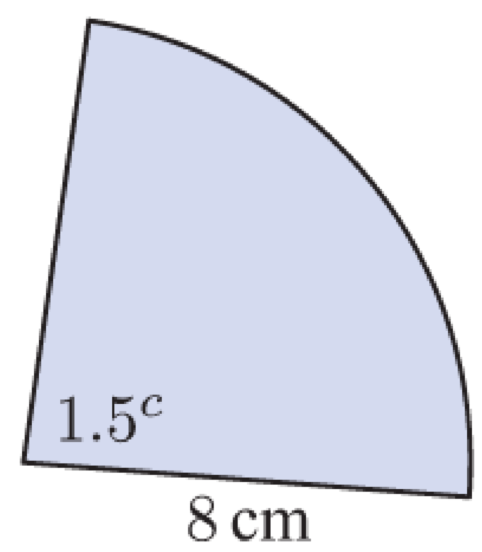

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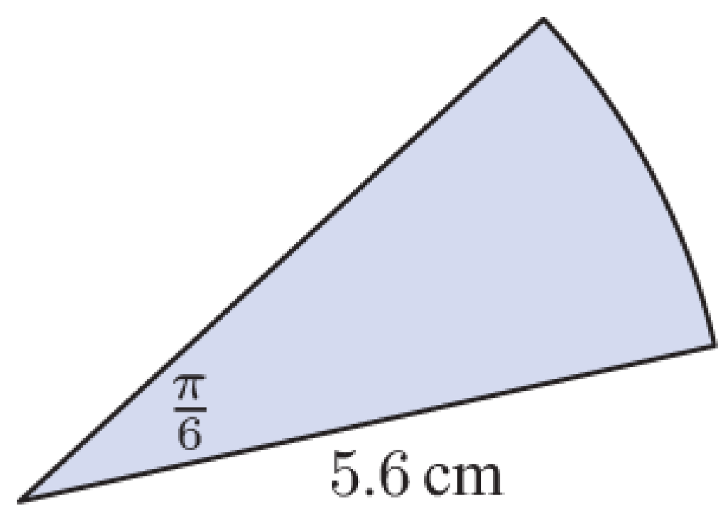


3 Find the arc length and area of a sector of a circle with:

a radius 9 cm and angle $\frac{7\pi}{4}$ **b** radius 4.93 cm and angle 4.67 radians.

Example 4 **Self Tutor**

A sector has radius 8.2 cm and arc length 12.3 cm. Find its:

a angle **b** area.

a $l = \theta r$ $\{\theta$ in radians$\}$

$$\therefore \ \theta = \frac{l}{r} = \frac{12.3}{8.2} = 1.5 \text{ radians}$$

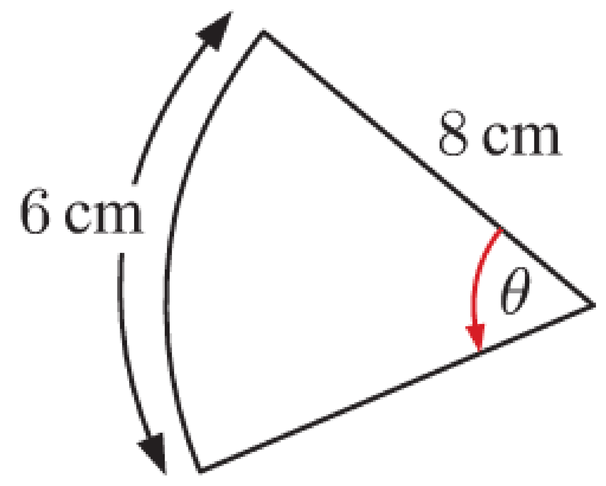
b area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 1.5 \times 8.2^2$
$= 50.43$ cm^2

4 Find, in radians, the angle of a sector of:

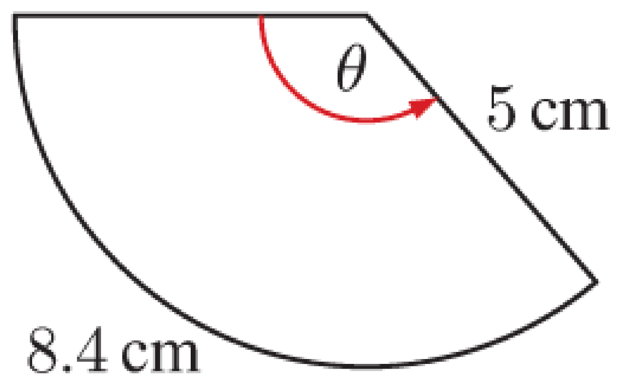
- **a** radius 4.3 m and arc length 2.95 m
- **b** radius 10 cm and area 30 cm^2.

5 Find θ (in radians) for each of the following, and hence find the area of each figure:

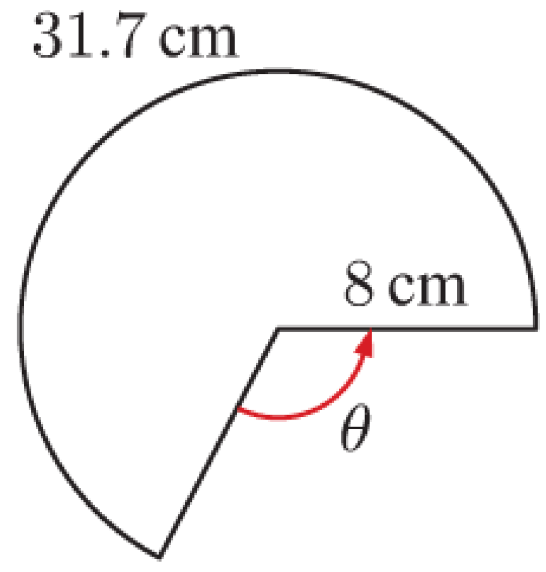
a



b



c



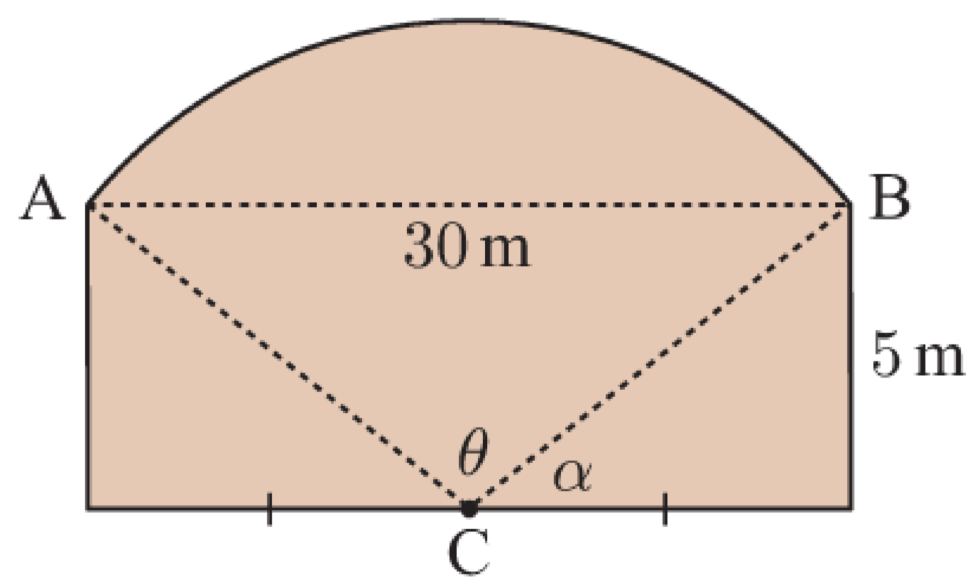
6 A sector has an angle of 1.88 radians and an arc length of 5.92 m. Find its:

- **a** radius
- **b** area.

7 A sector has an angle of 1.19 radians and an area of 20.8 cm^2. Find its:

- **a** radius
- **b** perimeter.

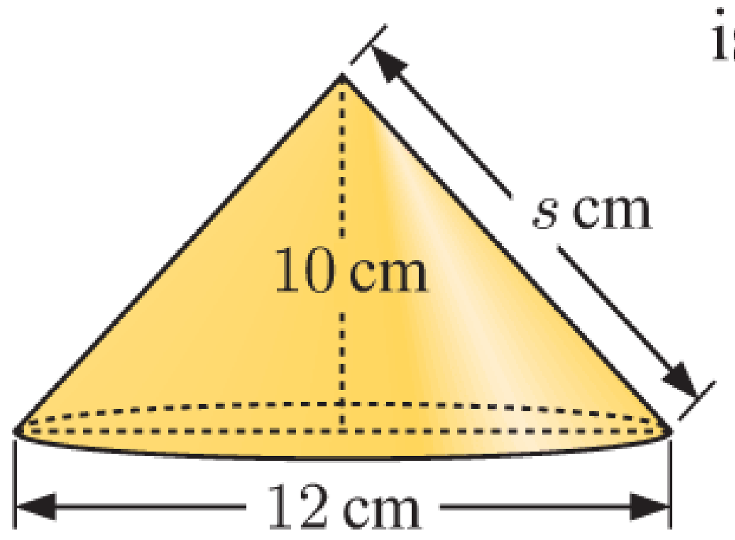
8



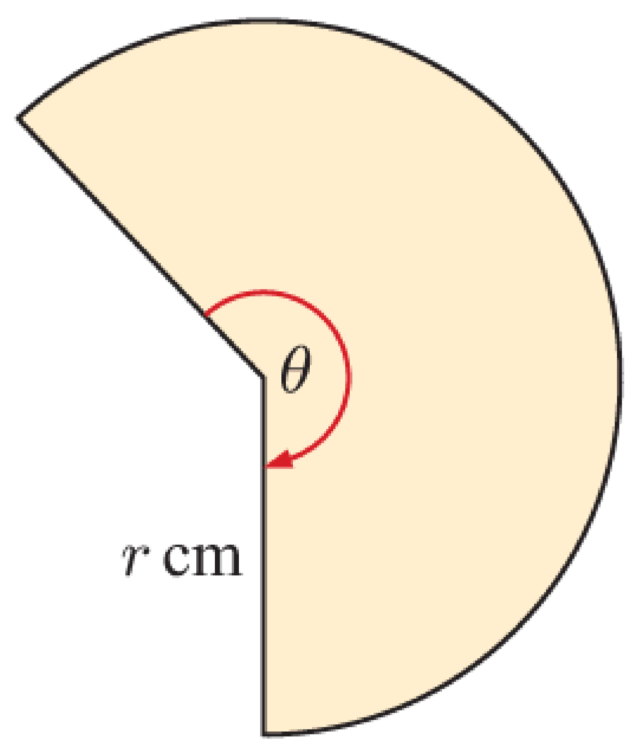
The end wall of a building has the shape illustrated, where the centre of arc AB is at C. Find:

- **a** α in radians to 4 significant figures
- **b** θ in radians to 4 significant figures
- **c** the area of the wall.

9 The cone



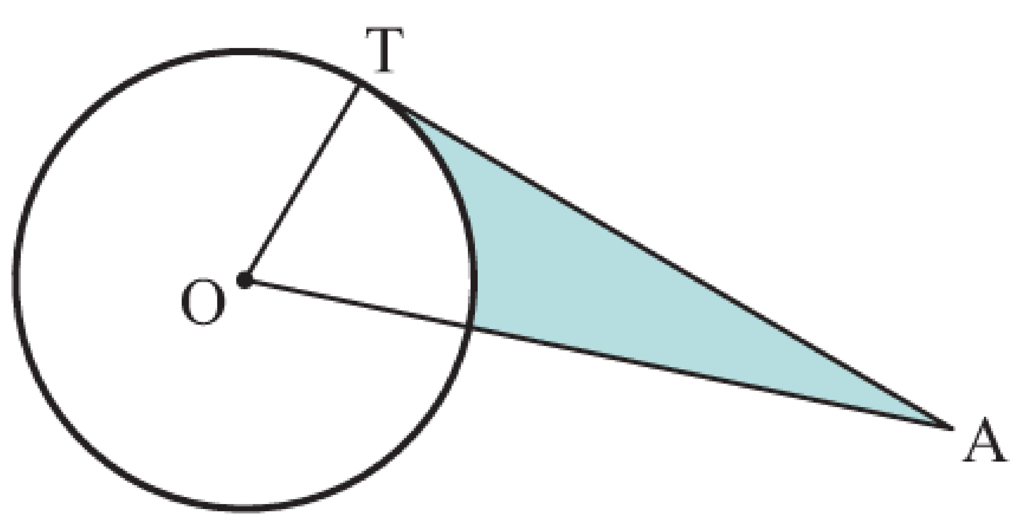
is made from this sector:



Find, correct to 3 significant figures:

- **a** the slant length s cm
- **b** the value of r
- **c** the arc length of the sector
- **d** the sector angle θ in radians.

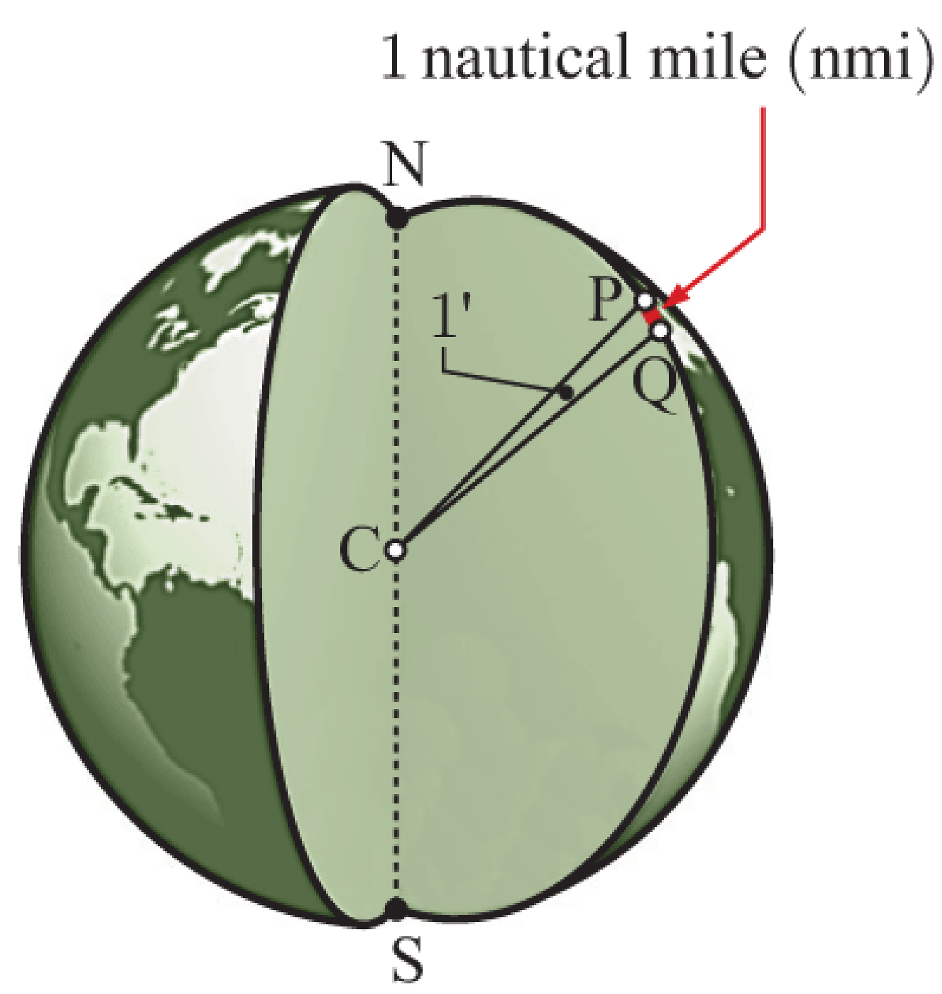
10

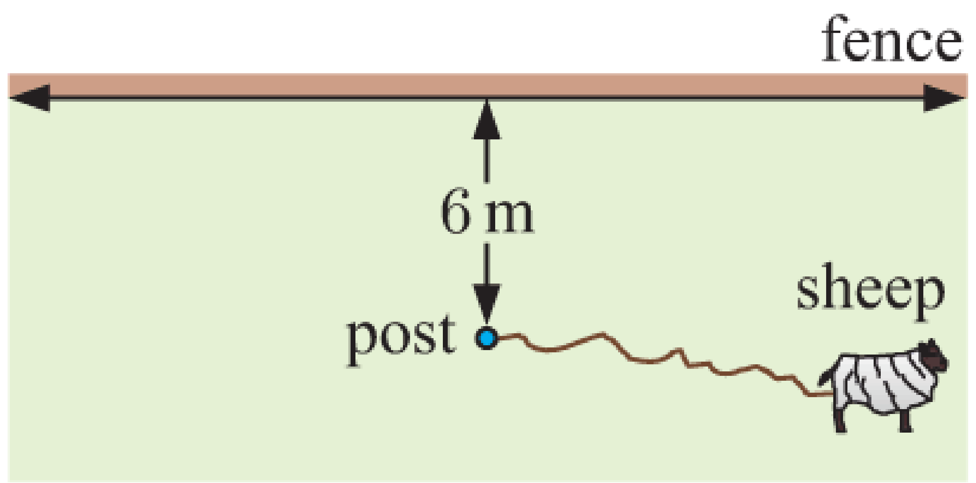


[AT] is a tangent to the given circle. OA = 13 cm and the circle has radius 5 cm. Find the perimeter of the shaded region.

11 A **nautical mile** (nmi) is the distance on the Earth's surface that subtends an angle of 1 minute (or $\frac{1}{60}$th of a degree) of the Great Circle arc measured from the centre of the Earth. A **knot** is a speed of 1 nautical mile per hour.

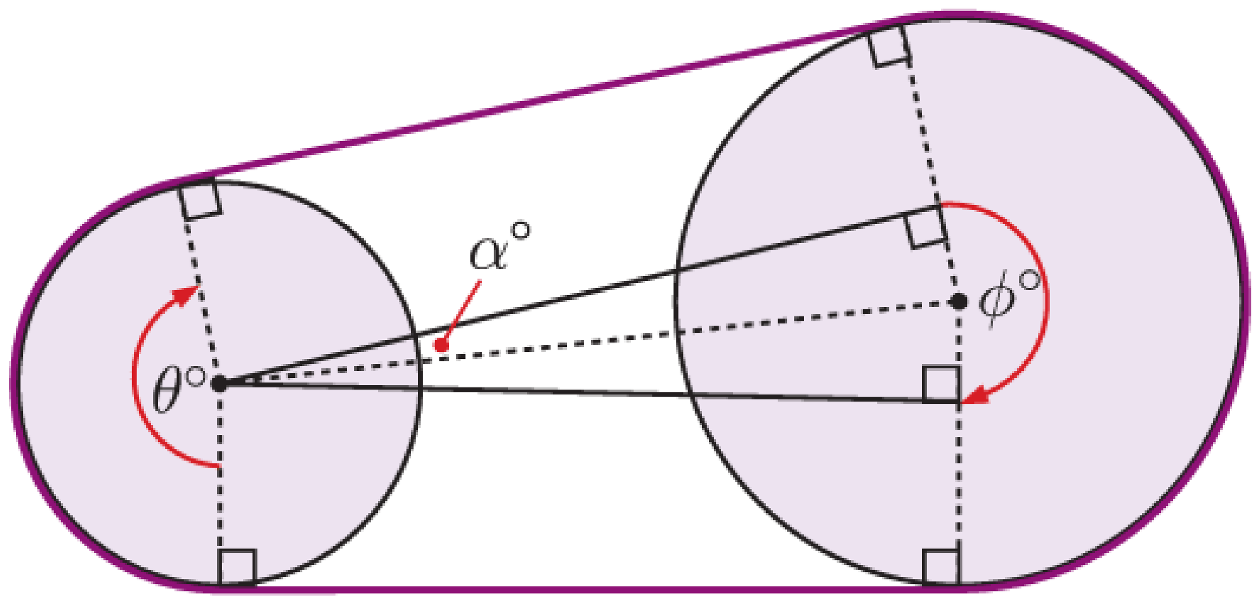
- **a** Given that the radius of the Earth is 6370 km, show that 1 nmi is approximately 1.853 km.
- **b** Calculate how long it would take a plane to fly 2130 km from Perth to Adelaide if the plane can fly at 480 knots.



12 

A sheep is tethered to a post which is 6 m from a long fence. The length of the rope is 9 m. Find the area which the sheep can feed on.

13 A belt fits tightly around two pulleys with radii 4 cm and 6 cm respectively. The distance between their centres is 20 cm.

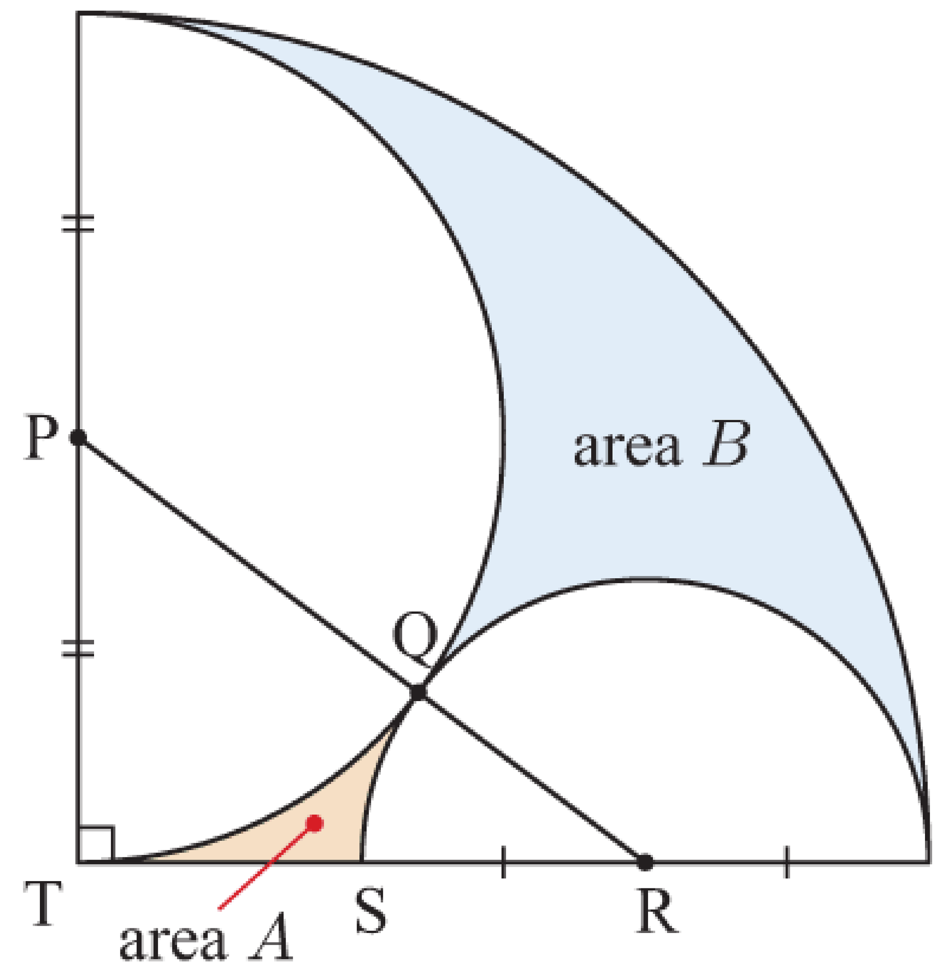


Find, correct to 4 significant figures:

a α **b** θ **c** ϕ

d the length of the belt.

14 Two semi-circles touch each other within a quarter circle as shown. P, Q, and R are collinear. The radius of the quarter circle is 12 cm.

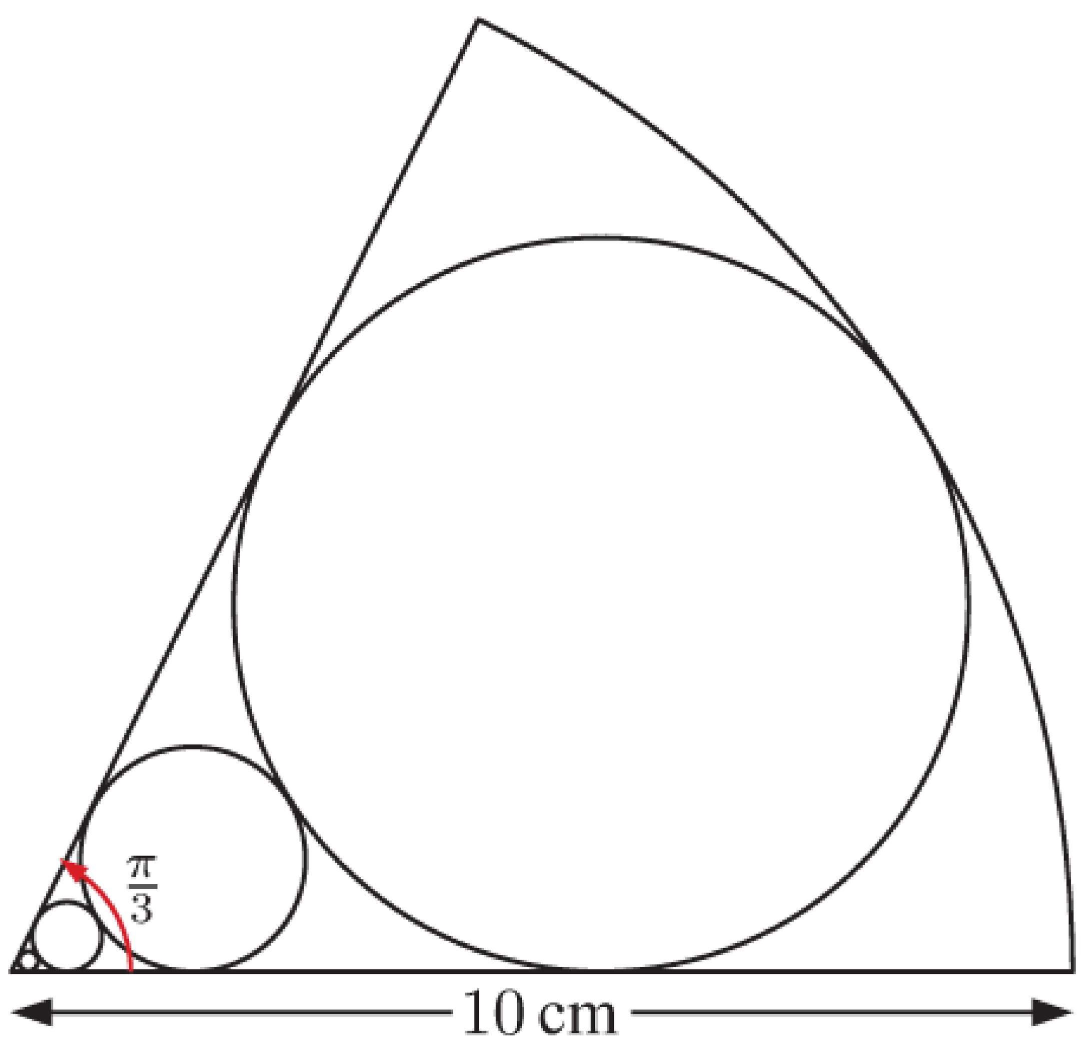


a Find the radius of the smaller semi-circle.

b Calculate the area of:

i A **ii** B.

15 An infinite number of circles are drawn in a sector of a circle as shown.



a Show that the largest circle has radius $\frac{10}{3}$ cm.

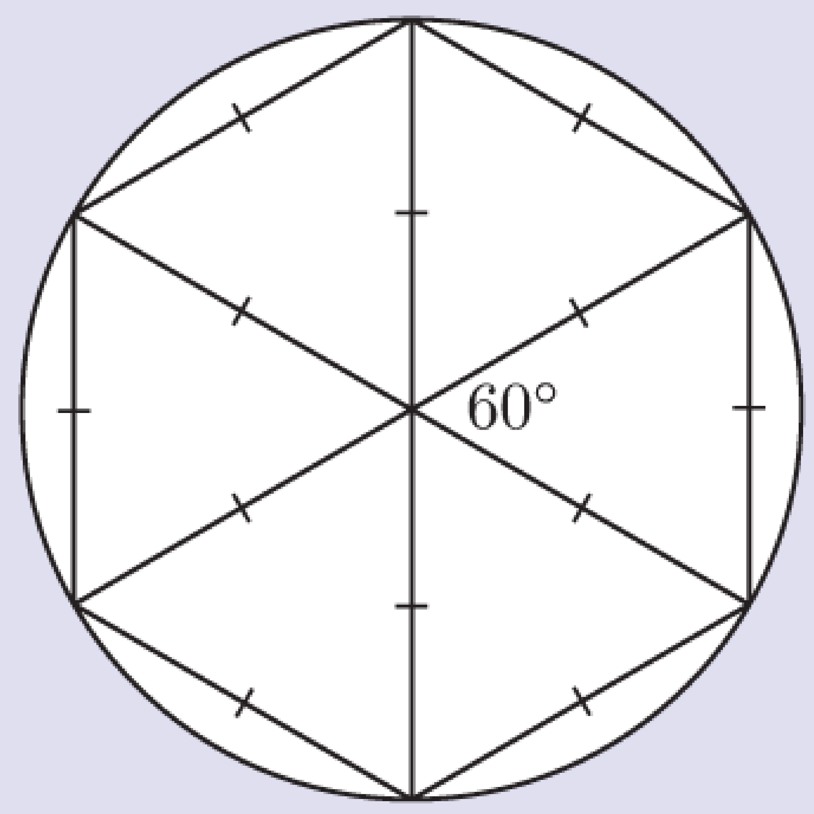
b Find the total area of this infinite series of circles.

c What fraction of the sector is occupied by the circles?

THEORY OF KNOWLEDGE

There are several theories for why one complete turn was divided into 360 degrees:

- 360 is approximately the number of days in a year.
- The Babylonians used a counting system in base 60. If they drew 6 equilateral triangles within a circle as shown, and divided each angle into 60 subdivisions, then there were 360 subdivisions in one turn. The division of an hour into 60 minutes, and a minute into 60 seconds, is from this base 60 counting system.
- 360 has 24 divisors, including every integer from 1 to 10 except 7.

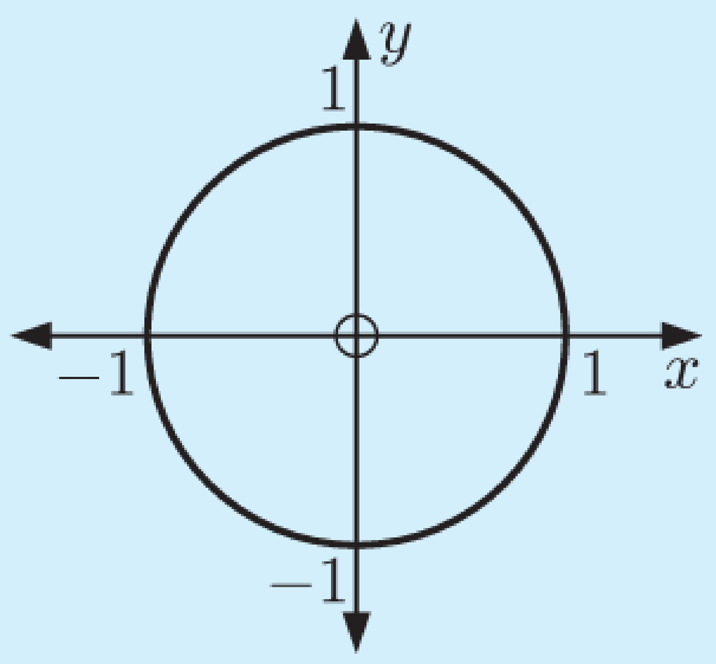


By contrast, we have seen how radians are convenient in simplifying formulae which relate angles with distances and areas.

1 Which angle measure do you think is more:
   a *practical*   b *natural*   c *mathematical*?
2 What other measures of angle are there, and for what purpose were they defined?
3 Which temperature scale, Celsius, Kelvin, or Fahrenheit, do you think is more:
   a *practical*   b *natural*?
4 What other measures have we defined as a way of convenience?
5 What things are done differently around the world, but would be useful to globally standardise? For example, why are there different power voltages in different countries? Why have they not been standardised?
6 What things do we measure in a particular way simply for reasons of history rather than practical purpose?

C THE UNIT CIRCLE

The **unit circle** is the circle with centre $(0, 0)$ and radius 1 unit.

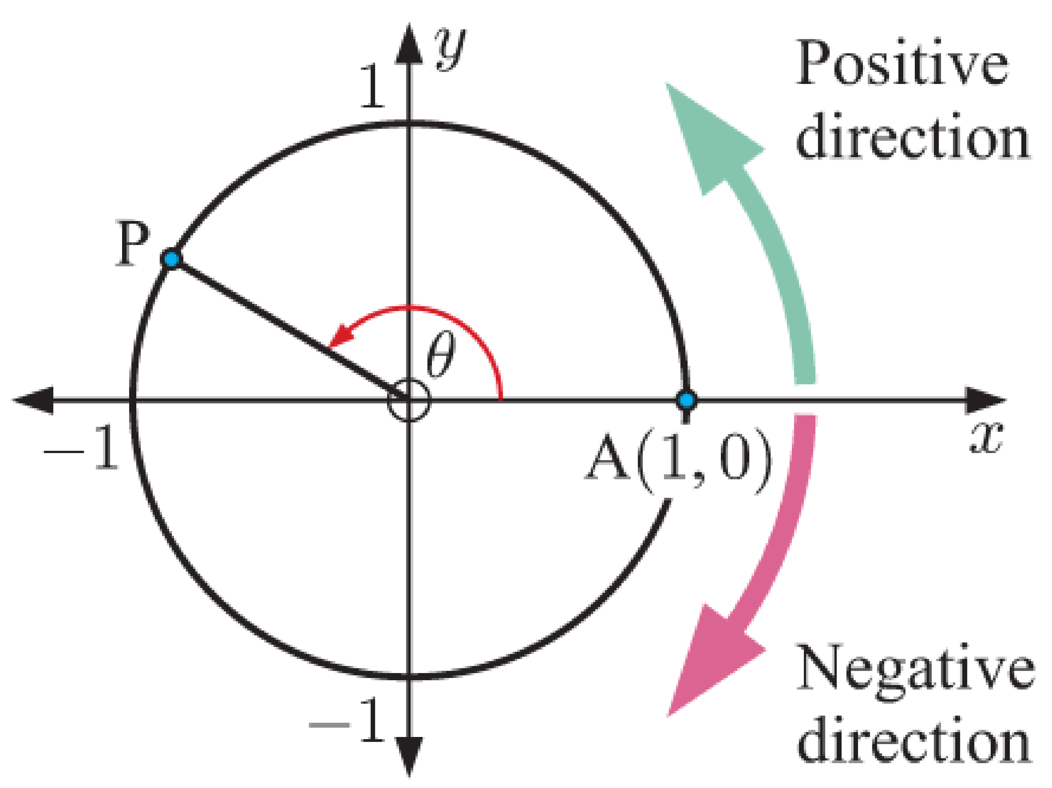


Applying the distance formula to a general point (x, y) on the circle, we find the equation of the unit circle is $x^2 + y^2 = 1$.

ANGLE MEASUREMENT

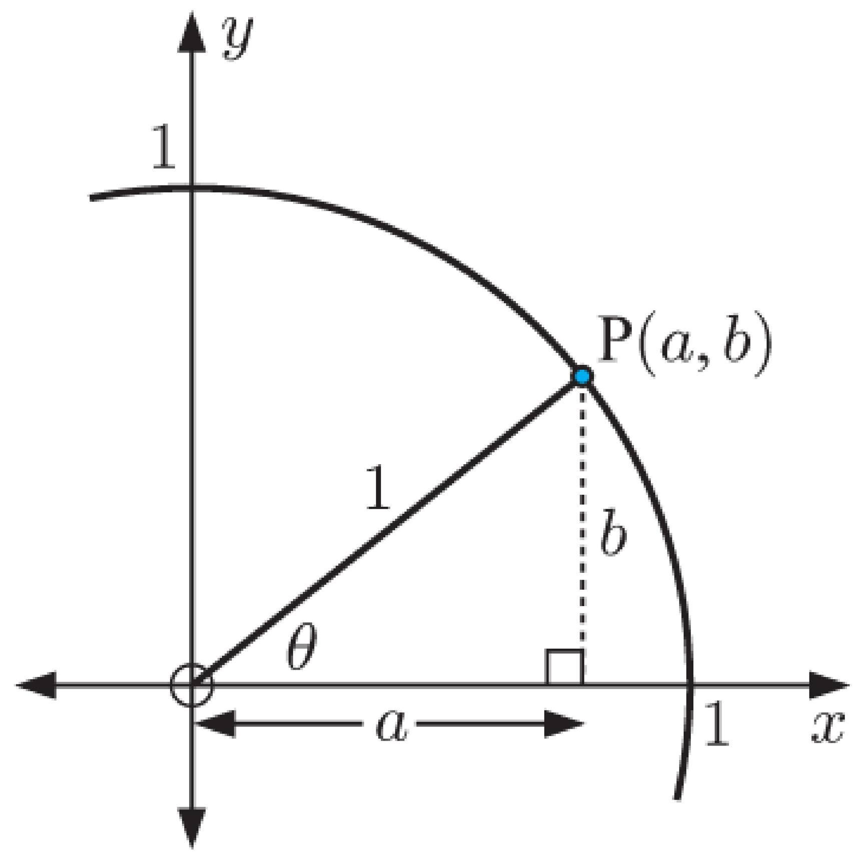
Suppose P lies anywhere on the unit circle, and A is $(1, 0)$. Let θ be the angle measured anticlockwise from [OA] on the positive x-axis.

θ is **positive** for anticlockwise rotations and **negative** for clockwise rotations.



DEFINITION OF SINE AND COSINE

Consider a point P(a, b) which lies on the unit circle in the first quadrant. [OP] makes an angle θ with the x-axis as shown.



Using right angled triangle trigonometry:

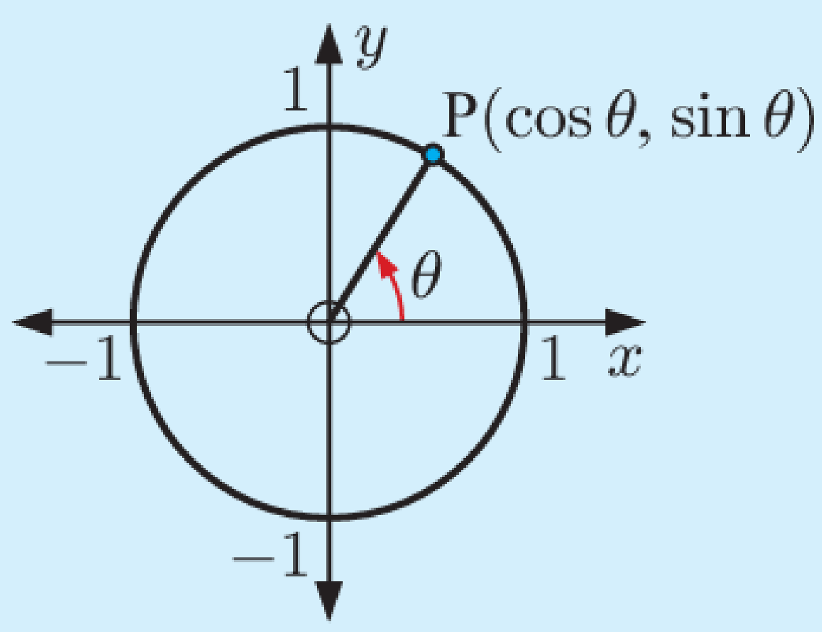
$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{a}{1} = a$$

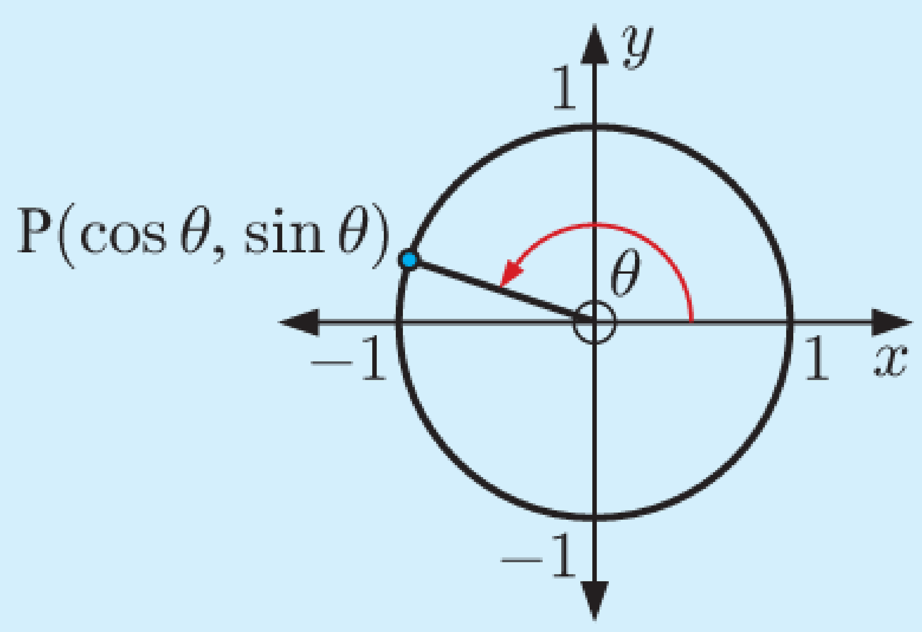
$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{b}{1} = b$$

More generally, we define:

If P is any point on the unit circle such that [OP] makes an angle θ measured anticlockwise from the positive x-axis:

- $\boldsymbol{\cos\theta}$ is the x-coordinate of P
- $\boldsymbol{\sin\theta}$ is the y-coordinate of P





For all points on the unit circle, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$, and $x^2 + y^2 = 1$. We therefore conclude:

For any angle θ:

- $\boldsymbol{-1 \leqslant \cos\theta \leqslant 1}$ and $\boldsymbol{-1 \leqslant \sin\theta \leqslant 1}$
- $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$

DEFINITION OF TANGENT

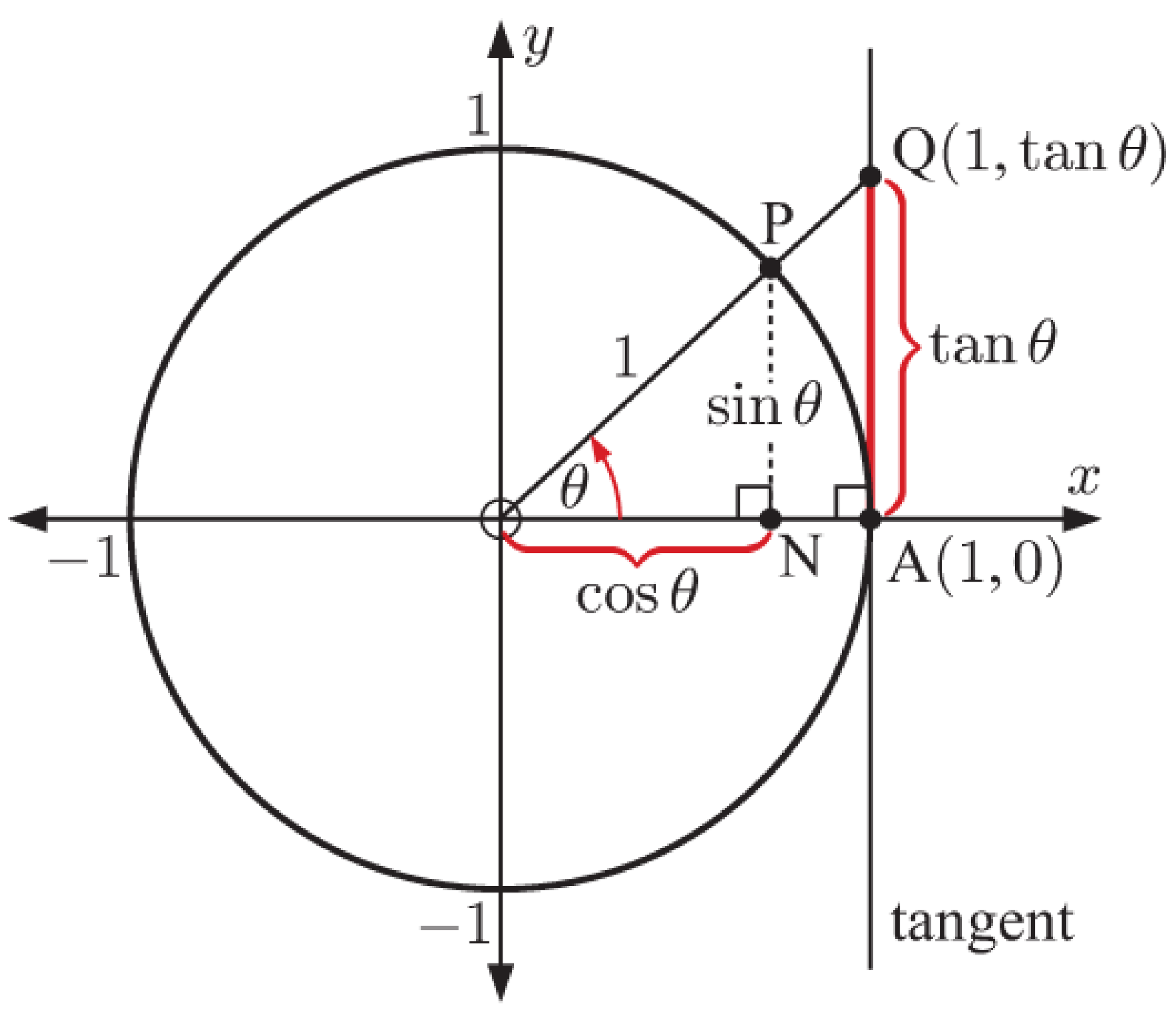
Suppose we extend [OP] to meet the tangent from A$(1, 0)$.

We let the intersection between these lines be point Q.

Note that as P moves, so does Q.

The position of Q relative to A is defined as the **tangent function**.

Notice that triangles ONP and OAQ are equiangular and therefore similar.



Consequently $\dfrac{\text{AQ}}{\text{OA}} = \dfrac{\text{NP}}{\text{ON}}$ and hence $\dfrac{\text{AQ}}{1} = \dfrac{\sin\theta}{\cos\theta}$.

Under the definition that $\text{AQ} = \tan\theta$, $\boldsymbol{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$.

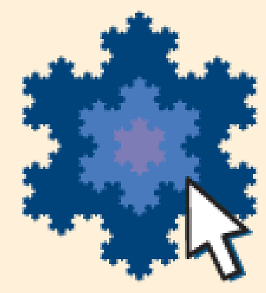
Since [OP] has gradient $\dfrac{\sin\theta}{\cos\theta}$, we can also say that $\tan\theta$ is the **gradient** of [OP].

INVESTIGATION THE TRIGONOMETRIC RATIOS

In this Investigation we explore the signs of the trigonometric ratios in each quadrant of the unit circle.

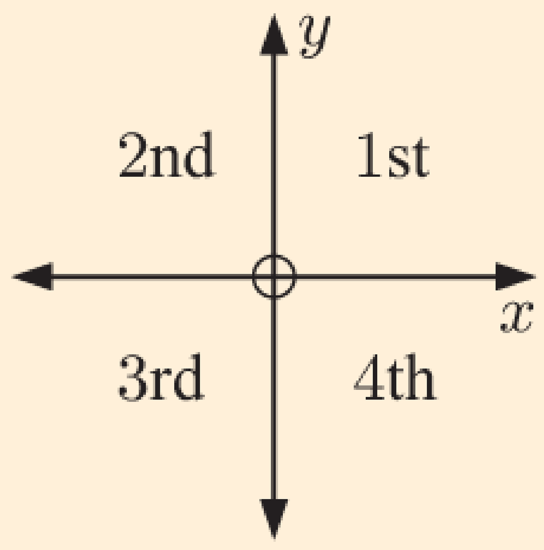
What to do:

THE UNIT CIRCLE



1 Click on the icon to run the Unit Circle software.
Drag the point P slowly around the circle.
Note the *sign* of each trigonometric ratio in each quadrant.

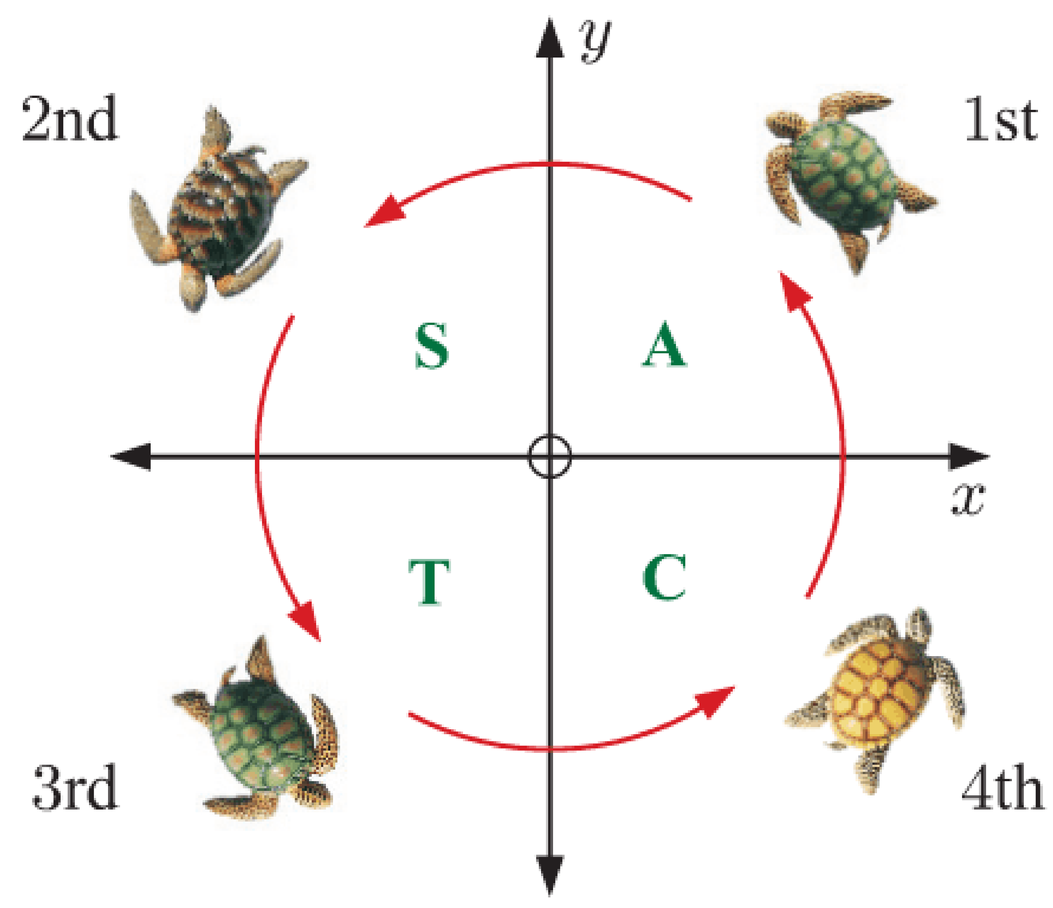
Quadrant	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	positive		
2			
3			
4			



2 Hence write down the trigonometric ratios which are *positive* for each quadrant.

From the **Investigation** you should have discovered that:

- $\sin\theta$, $\cos\theta$, and $\tan\theta$ are all positive in quadrant 1
- only $\sin\theta$ is positive in quadrant 2
- only $\tan\theta$ is positive in quadrant 3
- only $\cos\theta$ is positive in quadrant 4.



We can use a letter to show which trigonometric ratios are positive in each quadrant. The A stands for *all* of the ratios.

You might like to remember them using

All Silly Turtles Crawl.

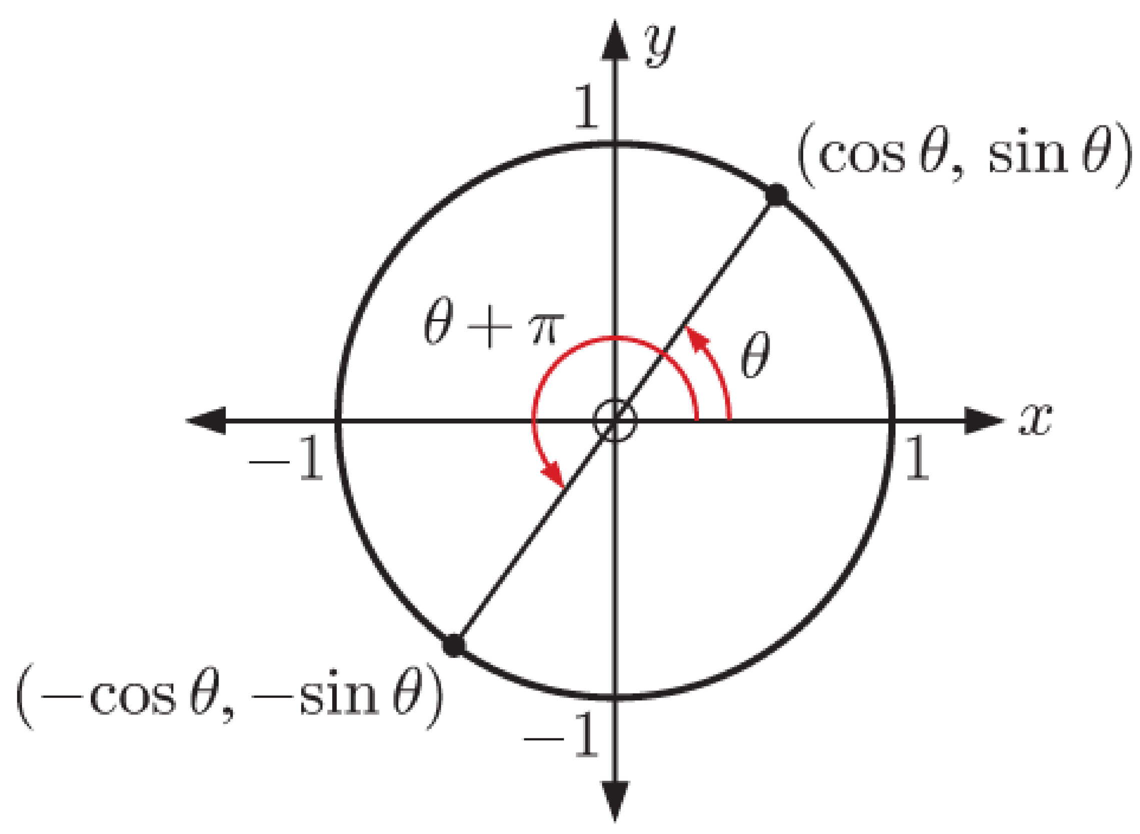
PERIODICITY OF TRIGONOMETRIC RATIOS

Since there are 2π radians in a full revolution, if we add any integer multiple of 2π to θ (in radians) then the position of P on the unit circle is unchanged.

For θ in radians and $k \in \mathbb{Z}$,
$\mathbf{\cos(\theta + 2k\pi) = \cos\theta}$ and $\mathbf{\sin(\theta + 2k\pi) = \sin\theta}$.

We notice that for any point $(\cos\theta, \sin\theta)$ on the unit circle, the point directly opposite is $(-\cos\theta, -\sin\theta)$.

$$\therefore \quad \cos(\theta+\pi) = -\cos\theta$$
$$\sin(\theta+\pi) = -\sin\theta$$
$$\text{and} \quad \tan(\theta+\pi) = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$



For θ in radians and $k \in \mathbb{Z}$, $\mathbf{\tan(\theta + k\pi) = \tan\theta}$.

CALCULATOR USE

When using your calculator to find trigonometric ratios for angles, you must make sure your calculator is correctly set to either **degree** or **radian** mode. Click on the icon for instructions.

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 8C

1 With the aid of a unit circle, complete the following table:

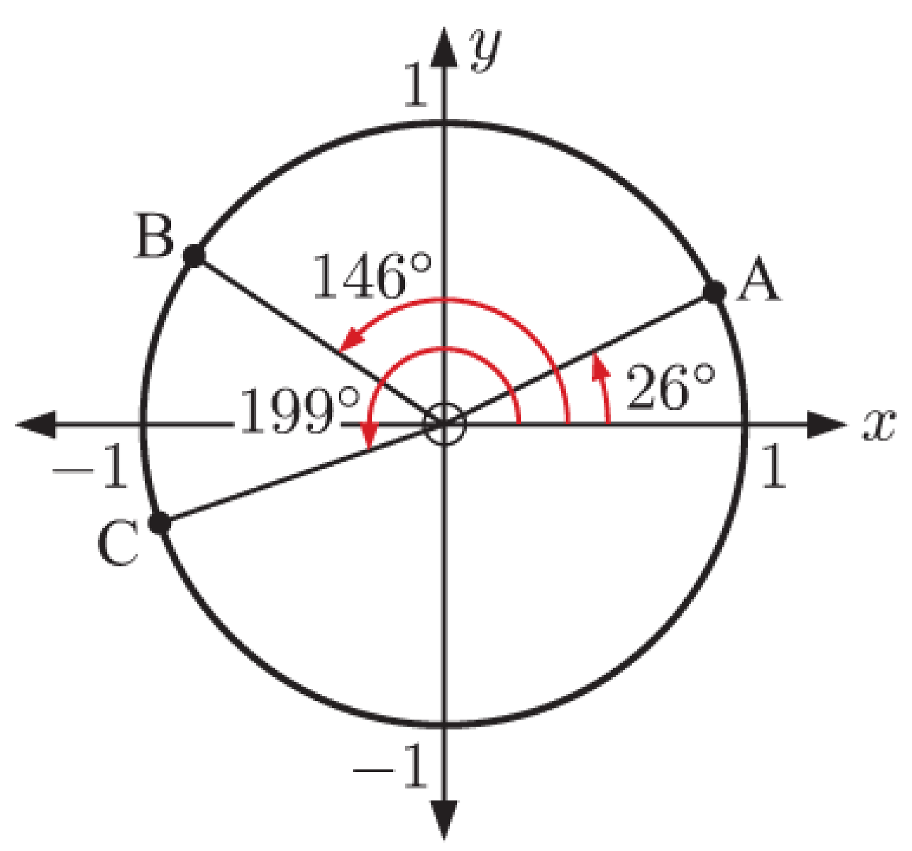
θ (degrees)	$0°$	$90°$	$180°$	$270°$	$360°$	$450°$
θ (radians)						
sine						
cosine						
tangent						

2 For each unit circle illustrated:

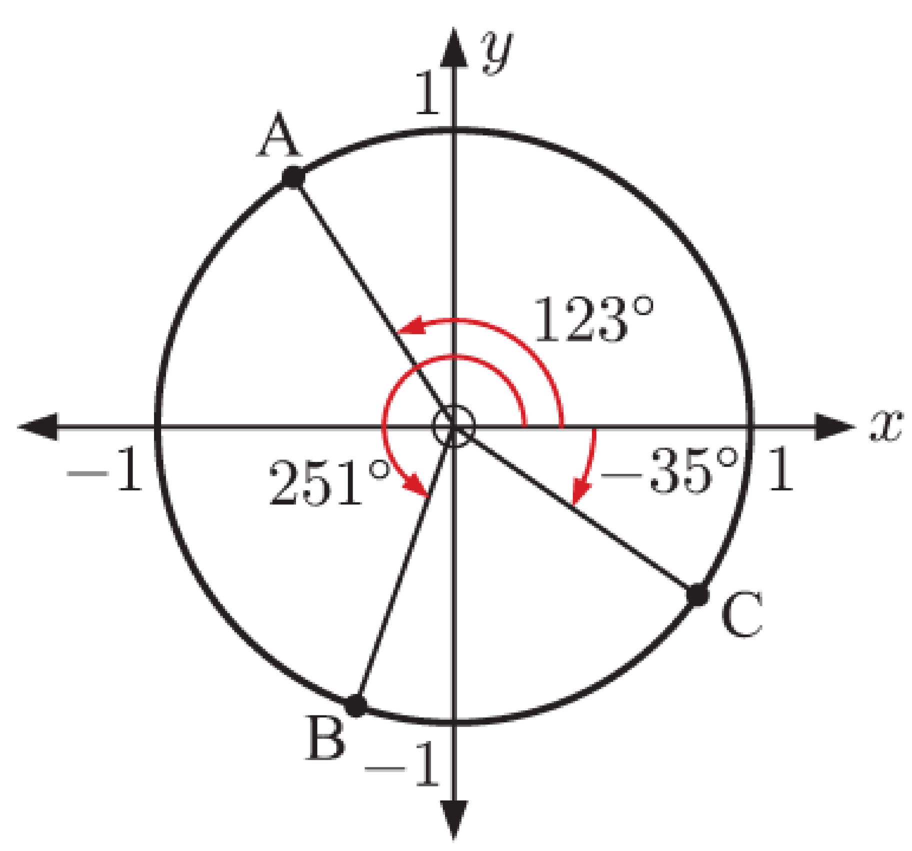
i State the exact coordinates of points A, B, and C in terms of sine and cosine.

ii Use your calculator to give the coordinates of A, B, and C correct to 3 significant figures.

a



b



3 **a** Use your calculator to evaluate:

i $\frac{1}{\sqrt{2}}$ **ii** $\frac{\sqrt{3}}{2}$

b Copy and complete the following table. Use your calculator to evaluate the trigonometric ratios, then **a** to write them exactly.

θ (degrees)	$30°$	$45°$	$60°$	$135°$	$150°$	$240°$	$315°$
θ (radians)							
sine							
cosine							
tangent							

4 Copy and complete:

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	positive	positive	
2					
3					
4					

5 In which quadrants are the following true?

a $\cos\theta$ is positive.
b $\cos\theta$ is negative.
c $\cos\theta$ and $\sin\theta$ are both negative.
d $\cos\theta$ is negative and $\sin\theta$ is positive.

6 Explain why:

a $\cos 400° = \cos 40°$
b $\sin\frac{5\pi}{7} = \sin\frac{19\pi}{7}$
c $\tan\frac{13\pi}{8} = \tan\left(-\frac{11\pi}{8}\right)$

7 Which two of these have the same value?

A $\tan 15°$ **B** $\tan 50°$ **C** $\tan 200°$ **D** $\tan 230°$ **E** $\tan 300°$

8 Which two of these have the same value?

A $\sin 220°$ **B** $\sin\frac{2\pi}{9}$ **C** $\sin\left(-\frac{2\pi}{9}\right)$ **D** $\sin 120°$ **E** $\sin 40°$

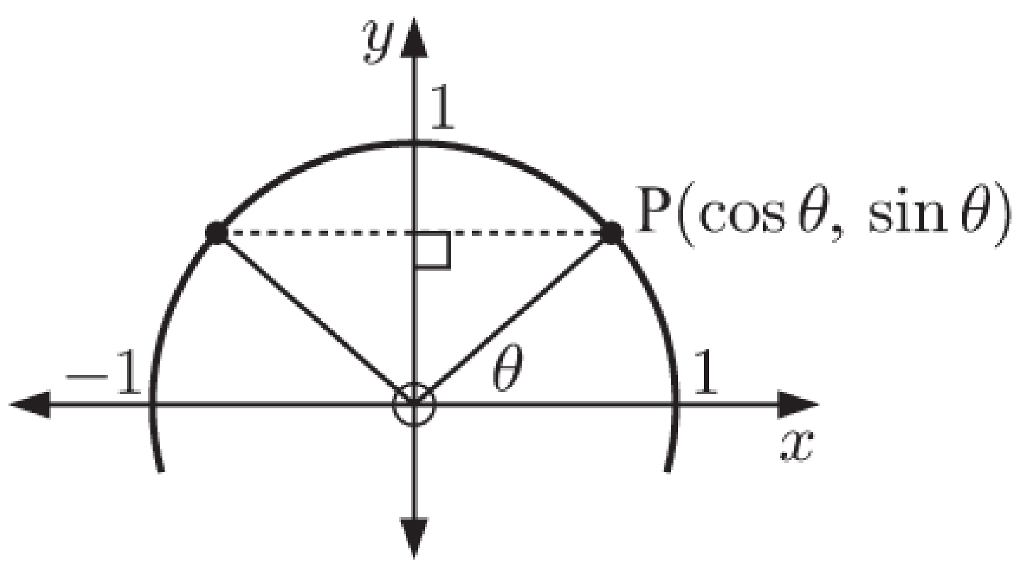
9 **a** Use your calculator to evaluate:

i $\sin 100°$ **ii** $\sin 80°$ **iii** $\sin 120°$ **iv** $\sin 60°$
v $\sin 150°$ **vi** $\sin 30°$ **vii** $\sin 45°$ **viii** $\sin 135°$

b Use the results from **a** to copy and complete: $\sin(180° - \theta) = ...$

c Write the rule you have just found in terms of radians.

d Justify your answer using the diagram alongside:



e Find the obtuse angle with the same sine as:

i $45°$ **ii** $51°$ **iii** $\frac{\pi}{3}$ **iv** $\frac{\pi}{6}$

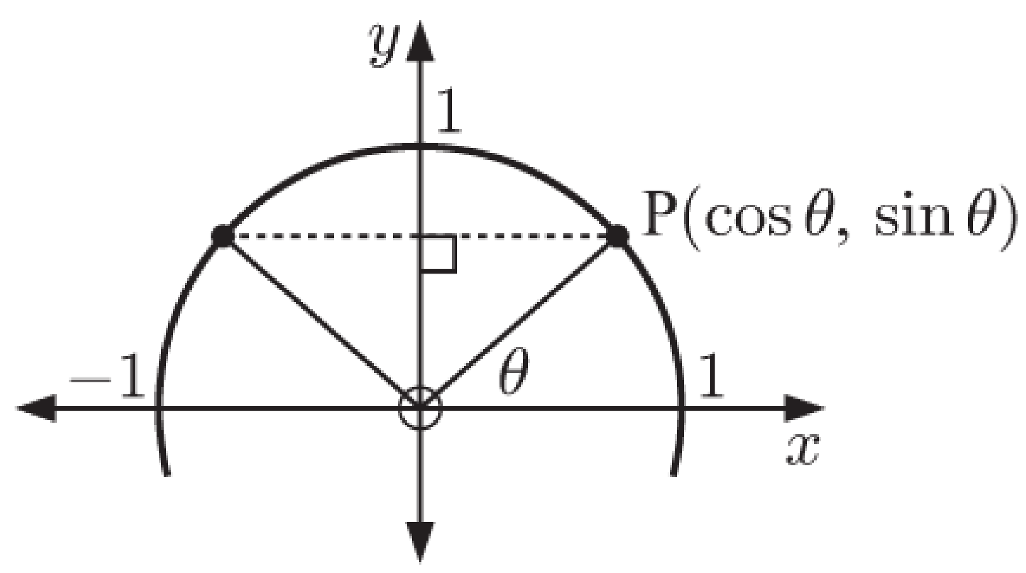
10 **a** Use your calculator to evaluate:

i $\cos 70°$ **ii** $\cos 110°$ **iii** $\cos 60°$ **iv** $\cos 120°$
v $\cos 25°$ **vi** $\cos 155°$ **vii** $\cos 80°$ **viii** $\cos 100°$

b Use the results from **a** to copy and complete: $\cos(180° - \theta) = ...$

c Write the rule you have just found in terms of radians.

d Justify your answer using the diagram alongside:



e Find the obtuse angle which has the negative cosine of:

i $40°$ **ii** $19°$ **iii** $\frac{\pi}{5}$ **iv** $\frac{2\pi}{5}$

11 Use the definition $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and your results from **9** and **10** to write $\tan(\pi - \theta)$ in terms of $\tan\theta$.

12 Without using your calculator, find:

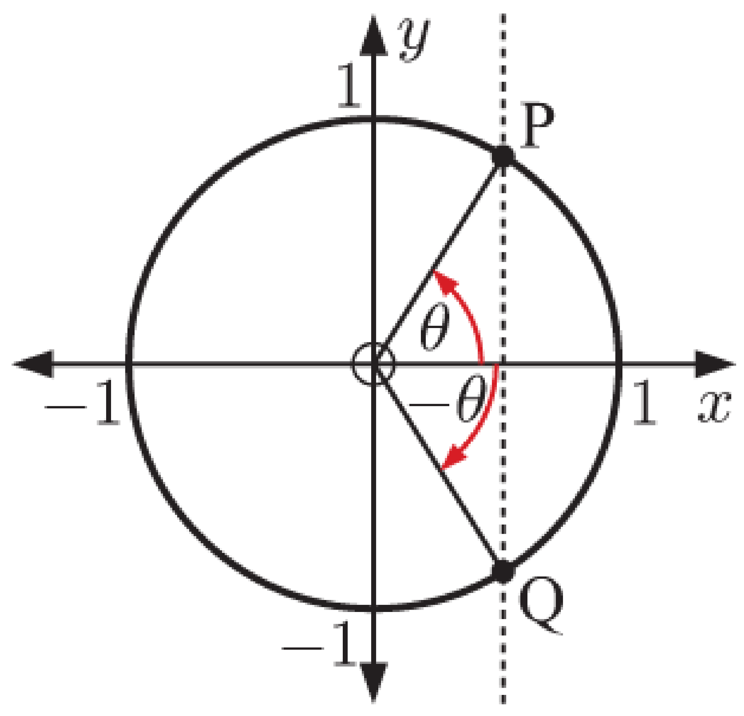
a $\sin 137°$ if $\sin 43° \approx 0.6820$
b $\sin 59°$ if $\sin 121° \approx 0.8572$
c $\cos 143°$ if $\cos 37° \approx 0.7986$
d $\cos 24°$ if $\cos 156° \approx -0.9135$
e $\sin 115°$ if $\sin 65° \approx 0.9063$
f $\cos 132°$ if $\cos 48° \approx 0.6691$

13 **a** Copy and complete:

θ (radians)	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75				
1.772				
3.414				
6.25				
-1.17				

b What trigonometric formulae can be deduced from your results in **a**?

c The coordinates of P in the figure are $(\cos\theta, \sin\theta)$. By finding the coordinates of Q in terms of θ in *two different* ways, prove your formulae in **b**.



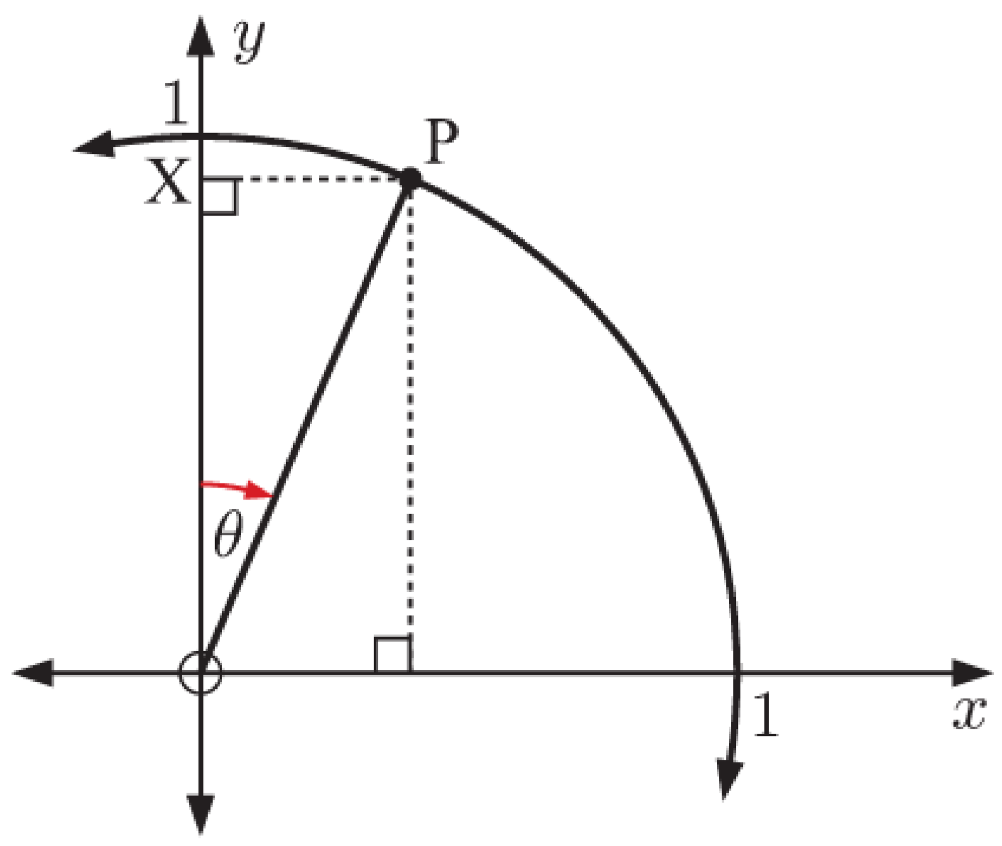
d Hence explain why

$$\cos(2\pi - \theta) = \cos\theta$$

and $$\sin(2\pi - \theta) = -\sin\theta.$$

e Write $\tan(2\pi - \theta)$ in terms of $\tan\theta$.

14 **a** Explain why P has coordinates $(\cos(\frac{\pi}{2} - \theta), \sin(\frac{\pi}{2} - \theta))$.



b Show that:

i $XP = \sin\theta$ **ii** $OX = \cos\theta$

c Hence, copy and complete:

i $\cos(\frac{\pi}{2} - \theta) = ...$ **ii** $\sin(\frac{\pi}{2} - \theta) = ...$

d Check your answer to **c** by calculating:

i $\cos\frac{\pi}{5}$ and $\sin\frac{3\pi}{10}$ **ii** $\sin\frac{\pi}{8}$ and $\cos\frac{3\pi}{8}$.

e Write $\tan(\frac{\pi}{2} - \theta)$ in terms of $\tan\theta$.

DISCUSSION IDENTITIES

In the previous Exercise you should have proven the following trigonometric **identities**:

	Degree form	*Radian form*
Supplementary angles	$\cos(180° - \theta) = -\cos\theta$ $\sin(180° - \theta) = \sin\theta$ $\tan(180° - \theta) = -\tan\theta$	$\cos(\pi - \theta) = -\cos\theta$ $\sin(\pi - \theta) = \sin\theta$ $\tan(\pi - \theta) = -\tan\theta$
Negative angles	$\cos(-\theta) = \cos\theta$ $\sin(-\theta) = -\sin\theta$ $\tan(-\theta) = -\tan\theta$	$\cos(-\theta) = \cos\theta$ $\sin(-\theta) = -\sin\theta$ $\tan(-\theta) = -\tan\theta$
Complementary angles	$\cos(90° - \theta) = \sin\theta$ $\sin(90° - \theta) = \cos\theta$ $\tan(90° - \theta) = \frac{1}{\tan\theta}$	$\cos(\frac{\pi}{2} - \theta) = \sin\theta$ $\sin(\frac{\pi}{2} - \theta) = \cos\theta$ $\tan(\frac{\pi}{2} - \theta) = \frac{1}{\tan\theta}$

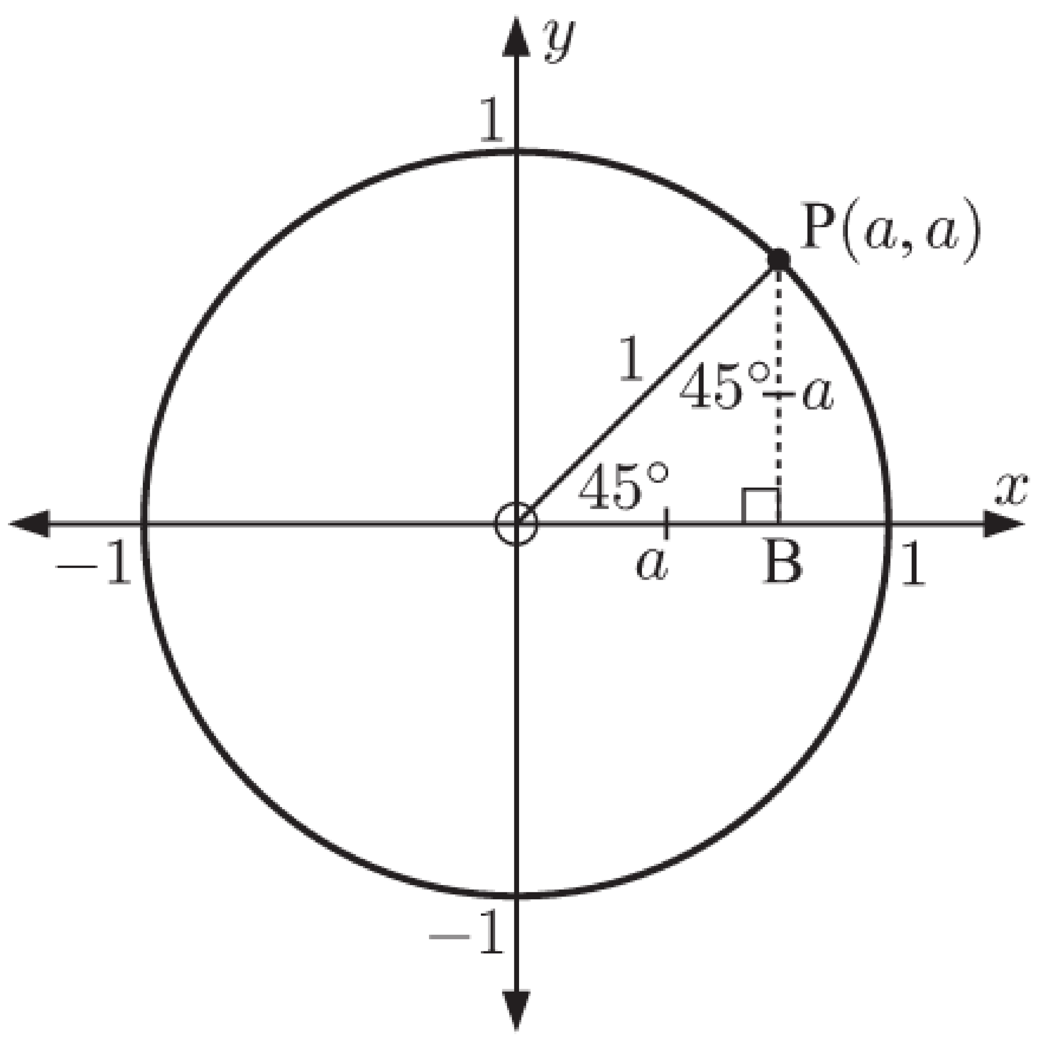
- What do we mean by the word "identity"?
- Why are identities important?
- What other identities do we use?

D MULTIPLES OF $\frac{\pi}{6}$ AND $\frac{\pi}{4}$

Angles which are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ occur frequently in geometry, so it is important for us to write their trigonometric ratios exactly.

MULTIPLES OF $\frac{\pi}{4}$ OR 45°

Consider $\theta = 45°$.



Angle OPB also measures 45°, so triangle OBP is isosceles.

$\therefore$ we let $\text{OB} = \text{BP} = a$

Now $a^2 + a^2 = 1^2$ {Pythagoras}

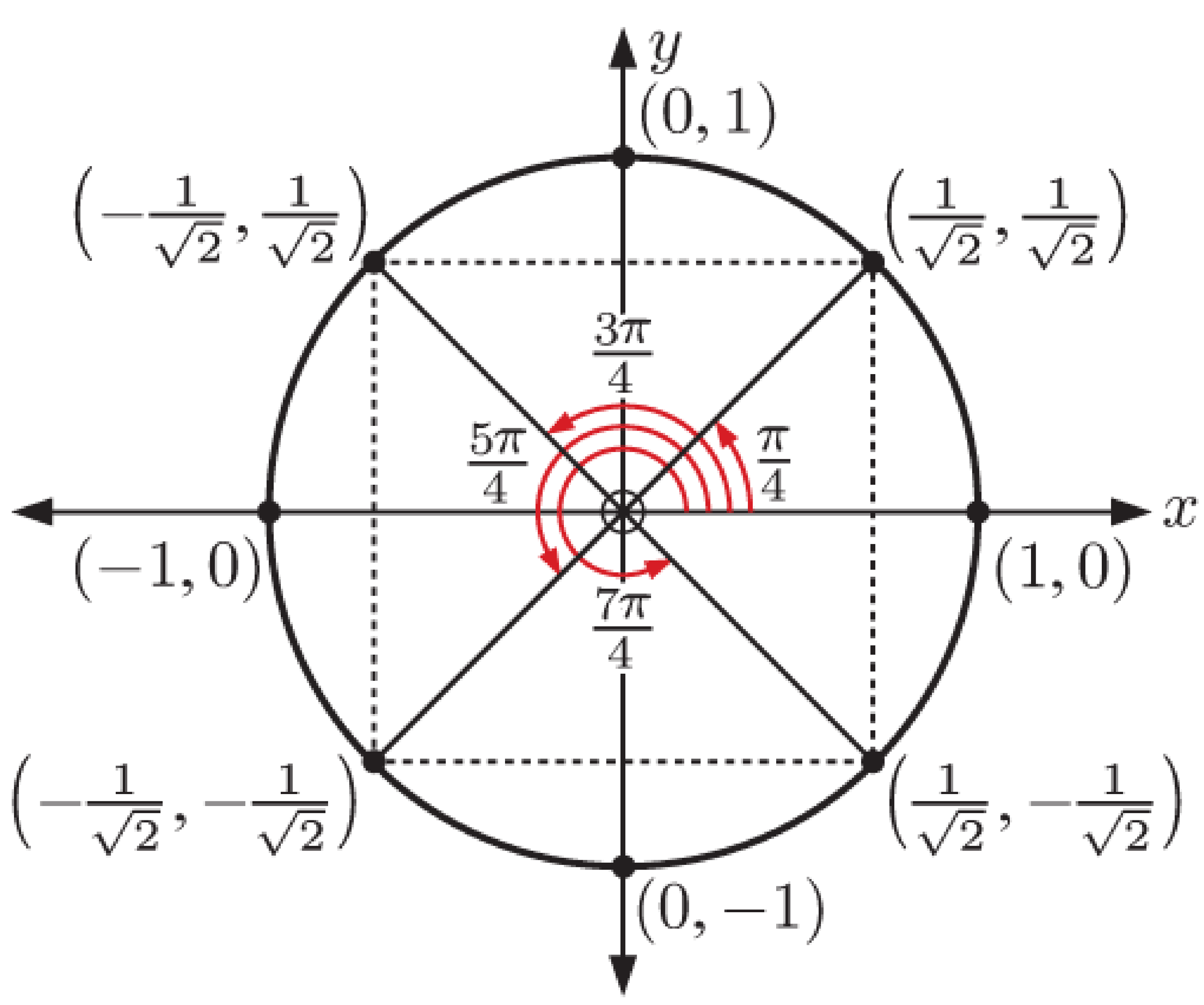
$\therefore a^2 = \frac{1}{2}$

$\therefore a = \frac{1}{\sqrt{2}}$ {since $a > 0$}

$\therefore$ P is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ where $\frac{1}{\sqrt{2}} \approx 0.707$.

So, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

We can now find the coordinates of all points on the unit circle corresponding to multiples of $\frac{\pi}{4}$ by symmetry.

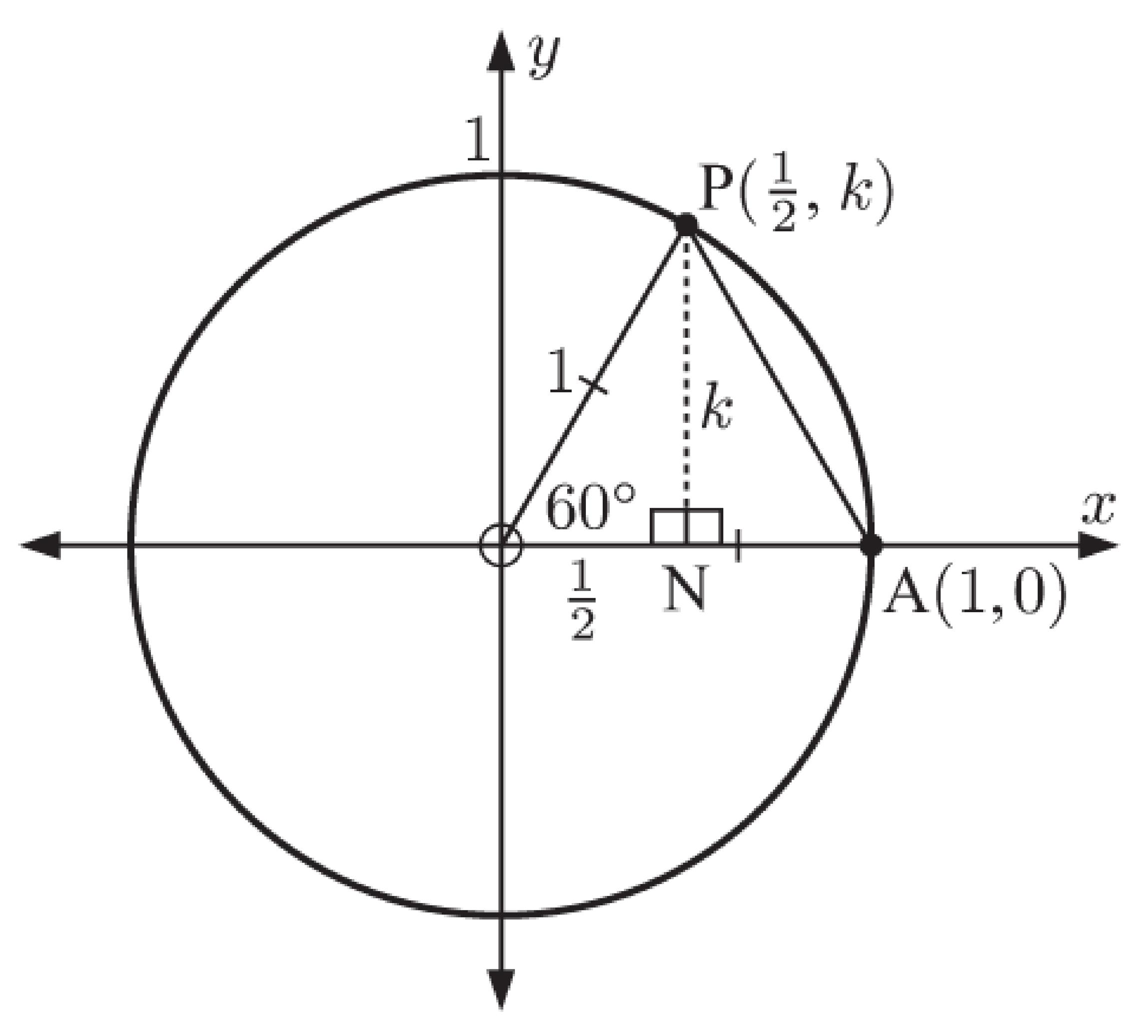


MULTIPLES OF $\frac{\pi}{6}$ OR 30°

Consider $\theta = 60°$.

Since OA = OP, triangle OAP is isosceles.

Now $\text{A}\widehat{\text{O}}\text{P} = 60°$, so the remaining angles are therefore also 60°. Triangle AOP is therefore equilateral.



The altitude [PN] bisects base [OA], so $\text{ON} = \frac{1}{2}$.

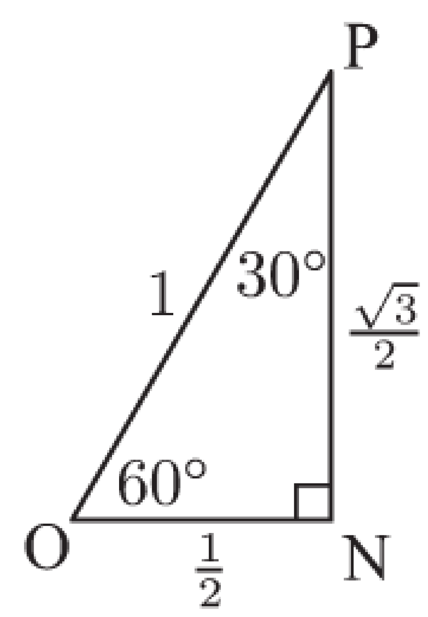
If P is $(\frac{1}{2}, k)$, then $(\frac{1}{2})^2 + k^2 = 1$

$\therefore k^2 = \frac{3}{4}$

$\therefore k = \frac{\sqrt{3}}{2}$ {since $k > 0$}

$\therefore$ P is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ where $\frac{\sqrt{3}}{2} \approx 0.866$.

So, $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$



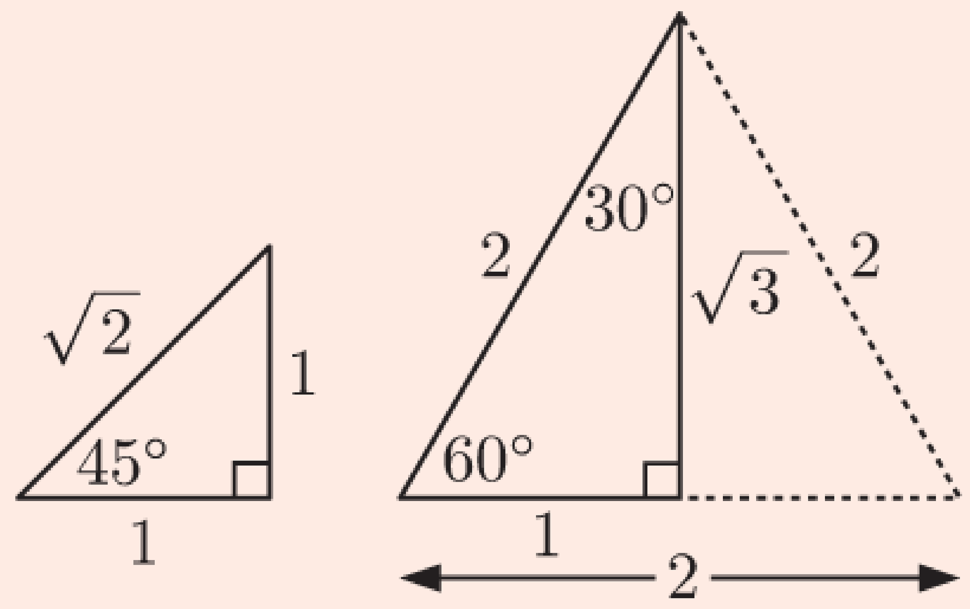
Now $\text{N}\widehat{\text{P}}\text{O} = \frac{\pi}{6} = 30°$. Hence $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$

DISCUSSION

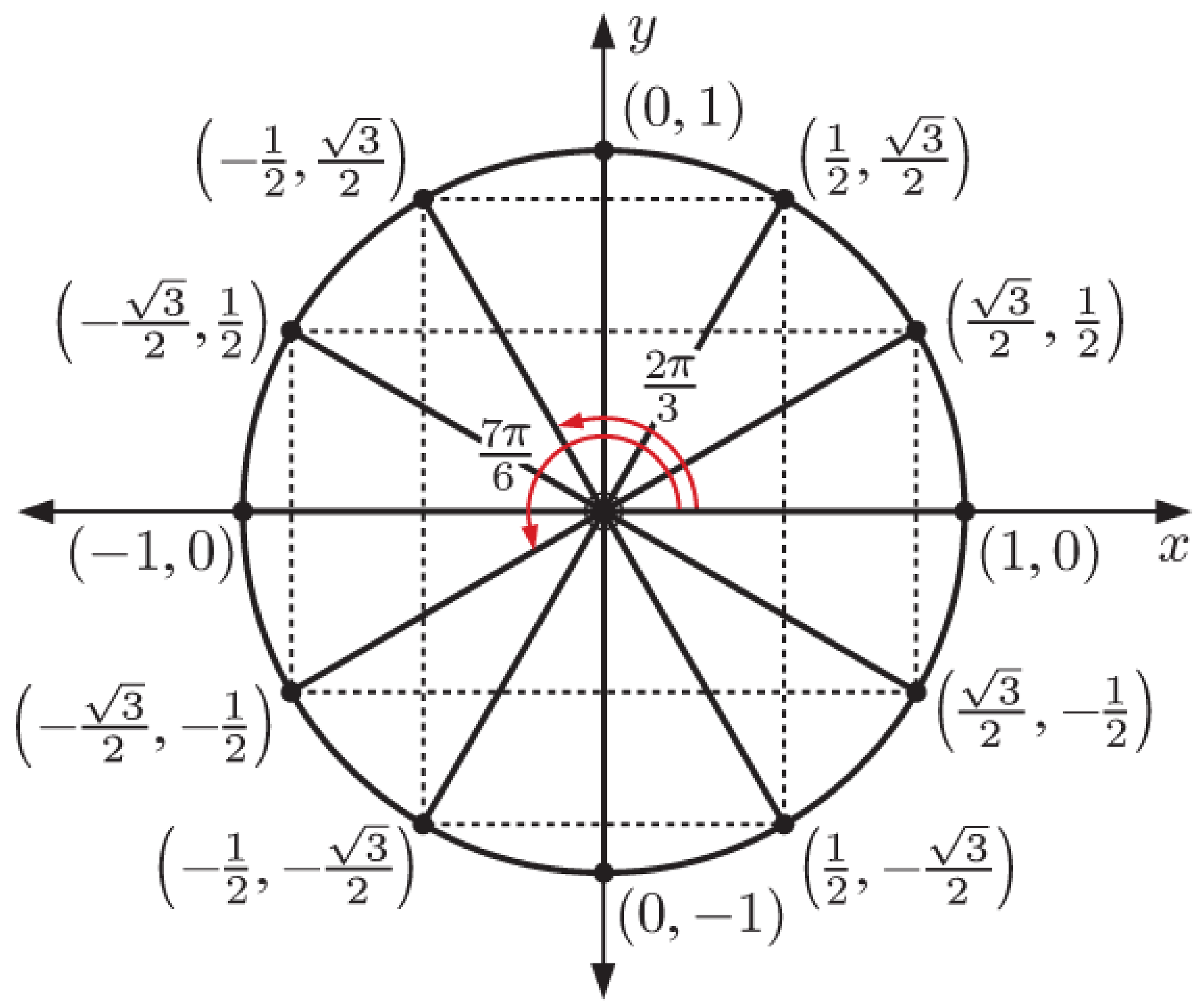
You should remember the values of cosine and sine for angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$.

However, if you forget, you can use these diagrams to quickly generate the results.

Discuss how you can do this.



We can now find the coordinates of all points on the unit circle corresponding to multiples of $\frac{\pi}{6}$ by symmetry.



SUMMARY

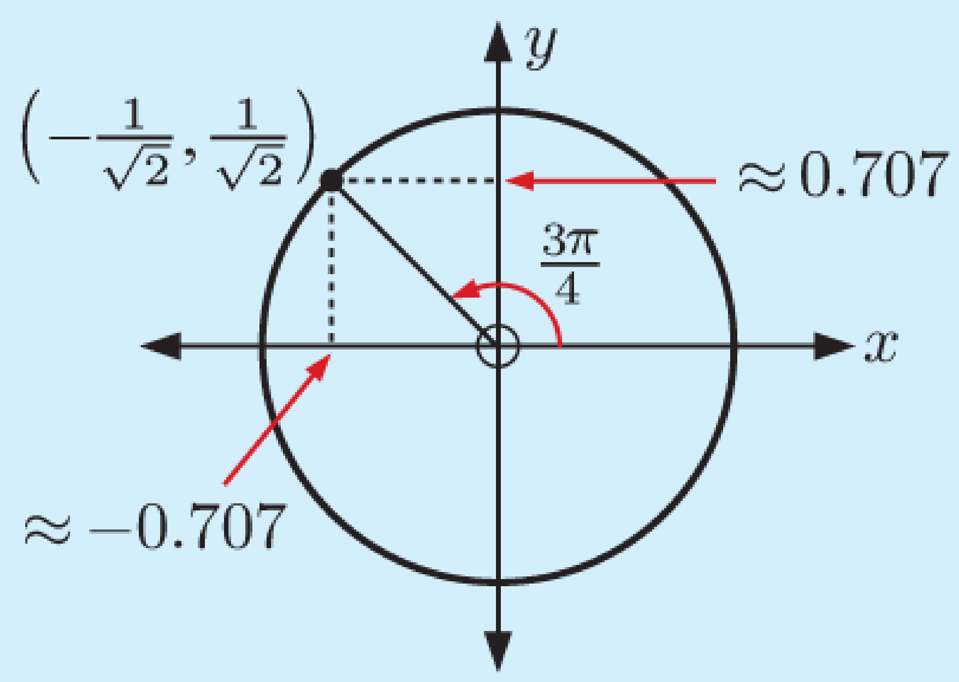
- For **multiples of** $\frac{\pi}{2}$, the coordinates of the points on the unit circle involve 0 and ± 1.
- For *other* **multiples of** $\frac{\pi}{4}$, the coordinates involve $\pm\frac{1}{\sqrt{2}}$.
- For *other* **multiples of** $\frac{\pi}{6}$, the coordinates involve $\pm\frac{1}{2}$ and $\pm\frac{\sqrt{3}}{2}$.

Example 5 Self Tutor

Find the exact values of $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ for:

a $\alpha = \frac{3\pi}{4}$ **b** $\alpha = \frac{4\pi}{3}$

a

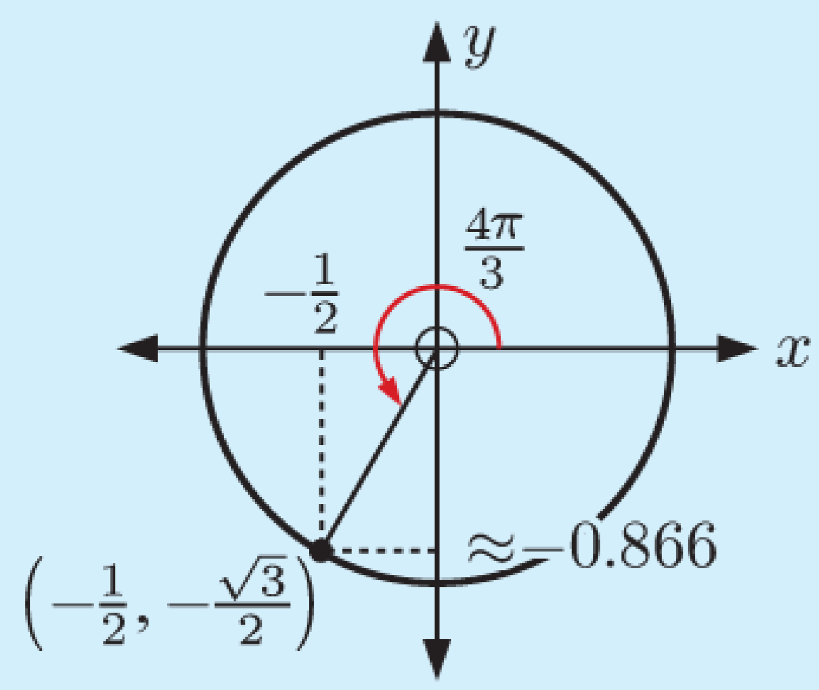


$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$

$\tan\frac{3\pi}{4} = -1$

b



$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

$\cos\frac{4\pi}{3} = -\frac{1}{2}$

$\tan\frac{4\pi}{3} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$

EXERCISE 8D

1 Use a unit circle diagram to find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ for θ equal to:

a $\frac{\pi}{4}$ **b** $\frac{3\pi}{4}$ **c** $\frac{7\pi}{4}$ **d** π **e** $-\frac{3\pi}{4}$

2 Use a unit circle diagram to find exact values for $\sin\beta$, $\cos\beta$, and $\tan\beta$ for β equal to:

a $\frac{\pi}{6}$ **b** $\frac{2\pi}{3}$ **c** $\frac{7\pi}{6}$ **d** $\frac{5\pi}{3}$ **e** $\frac{11\pi}{6}$

3 Find the exact values of:

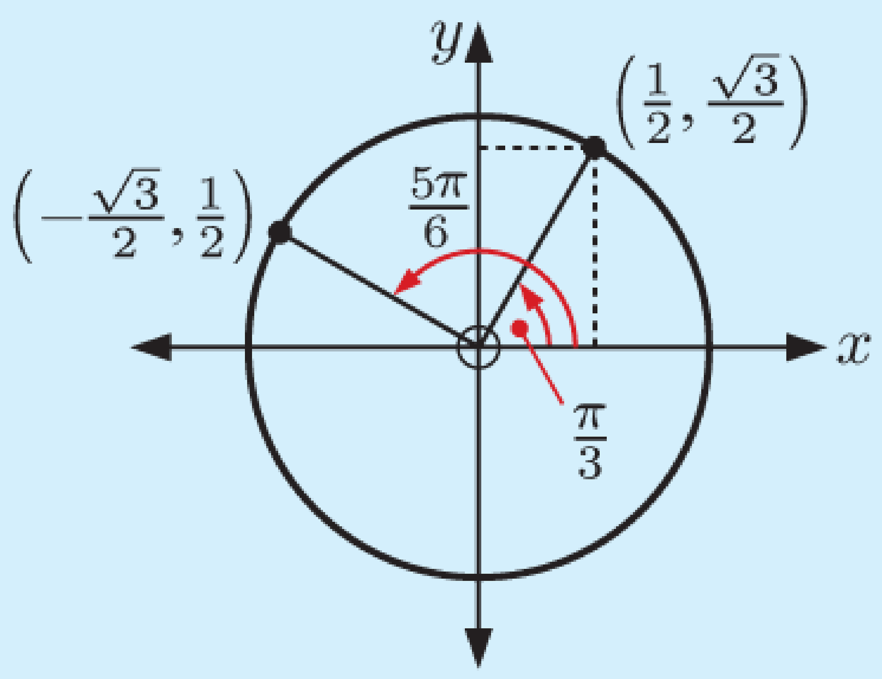
a $\cos\frac{2\pi}{3}$, $\sin\frac{2\pi}{3}$, and $\tan\frac{2\pi}{3}$ **b** $\cos\left(-\frac{\pi}{4}\right)$, $\sin\left(-\frac{\pi}{4}\right)$, and $\tan\left(-\frac{\pi}{4}\right)$

4 **a** Find the exact values of $\cos\frac{\pi}{2}$ and $\sin\frac{\pi}{2}$.

b What can you say about $\tan\frac{\pi}{2}$?

Example 6 **Self Tutor**

Without using a calculator, show that $8\sin\frac{\pi}{3}\cos\frac{5\pi}{6} = -6$.



$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

$$\therefore\ 8\sin\frac{\pi}{3}\cos\frac{5\pi}{6} = 8\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$$
$$= 2(-3)$$
$$= -6$$

5 Without using a calculator, evaluate:

a $\sin^2\left(\frac{\pi}{3}\right)$ **b** $\sin\frac{\pi}{6}\cos\frac{\pi}{3}$ **c** $1-\cos^2\left(\frac{\pi}{6}\right)$

d $\sin^2\left(\frac{2\pi}{3}\right)-1$ **e** $\cos^2\left(\frac{\pi}{4}\right)-\sin\frac{7\pi}{6}$ **f** $\sin\frac{3\pi}{4}-\cos\frac{5\pi}{4}$

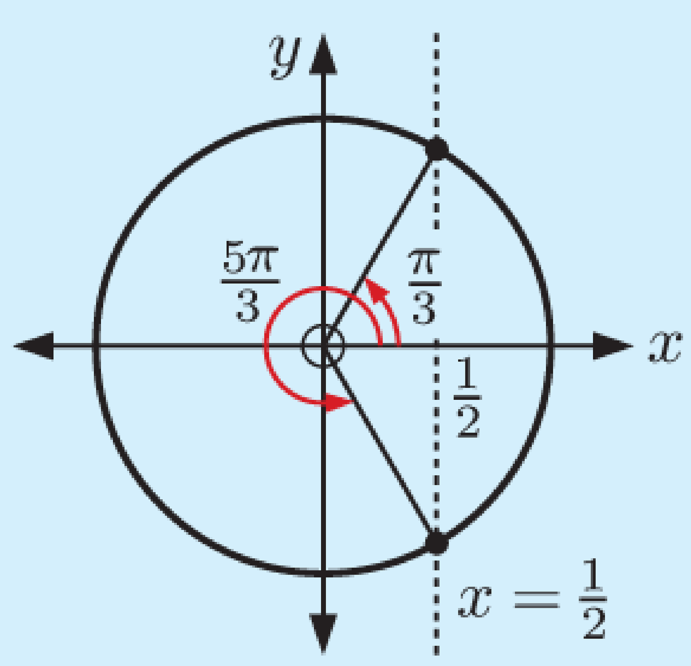
g $1-2\sin^2\left(\frac{7\pi}{6}\right)$ **h** $\cos^2\left(\frac{5\pi}{6}\right)-\sin^2\left(\frac{5\pi}{6}\right)$ **i** $\tan^2\left(\frac{\pi}{3}\right)-2\sin^2\left(\frac{\pi}{4}\right)$

j $2\tan\left(-\frac{5\pi}{4}\right)-\sin\frac{3\pi}{2}$ **k** $\dfrac{2\tan\frac{5\pi}{6}}{1-\tan^2\left(\frac{5\pi}{6}\right)}$ **l** $\dfrac{\cos\frac{\pi}{3}}{\sin\frac{4\pi}{3}+\tan\frac{\pi}{6}}$

Check all answers using your calculator.

Example 7 **Self Tutor**

Find all angles $0 \leqslant \theta \leqslant 2\pi$ with a cosine of $\frac{1}{2}$.



Since the cosine is $\frac{1}{2}$, we draw the vertical line $x = \frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

6 Find all angles between 0 and 2π with:

a a sine of $\frac{1}{2}$ **b** a sine of $\frac{\sqrt{3}}{2}$ **c** a cosine of $\frac{1}{\sqrt{2}}$

d a cosine of $-\frac{1}{2}$ **e** a cosine of $-\frac{1}{\sqrt{2}}$ **f** a sine of $-\frac{\sqrt{3}}{2}$

7 Find all angles between 0 and 2π (inclusive) which have a tangent of:

a 1 **b** -1 **c** $\sqrt{3}$ **d** 0 **e** $\frac{1}{\sqrt{3}}$ **f** $-\sqrt{3}$

8 Find all angles between 0 and 4π with:

a a cosine of $\frac{\sqrt{3}}{2}$ **b** a sine of $-\frac{1}{2}$ **c** a sine of -1

9 Find θ if $0 \leqslant \theta \leqslant 2\pi$ and:

a $\cos\theta = \frac{1}{2}$ **b** $\sin\theta = \frac{\sqrt{3}}{2}$ **c** $\cos\theta = -1$ **d** $\sin\theta = 1$

e $\cos\theta = -\frac{1}{\sqrt{2}}$ **f** $\sin^2\theta = 1$ **g** $\cos^2\theta = 1$ **h** $\cos^2\theta = \frac{1}{2}$

i $\tan\theta = -\frac{1}{\sqrt{3}}$ **j** $\tan^2\theta = 3$

10 Find *all* values of θ for which $\tan\theta$ is:

a zero **b** undefined.

E THE PYTHAGOREAN IDENTITY

From the equation of the unit circle $x^2 + y^2 = 1$, we obtain the **Pythagorean identity**:

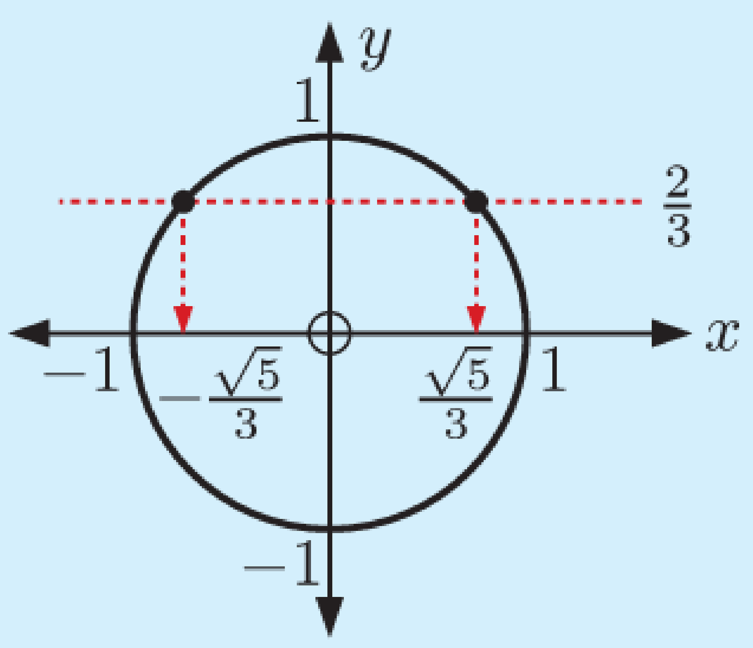
For any angle θ, $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$.

We can use this identity to find one trigonometric ratio from another.

Example 8 **Self Tutor**

Find the possible exact values of $\cos\theta$ for $\sin\theta = \frac{2}{3}$. Illustrate your answers.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \cos^2\theta + \left(\tfrac{2}{3}\right)^2 = 1$$
$$\therefore \cos^2\theta = \tfrac{5}{9}$$
$$\therefore \cos\theta = \pm\tfrac{\sqrt{5}}{3}$$



EXERCISE 8E

1 Find the possible exact values of $\cos\theta$ for:

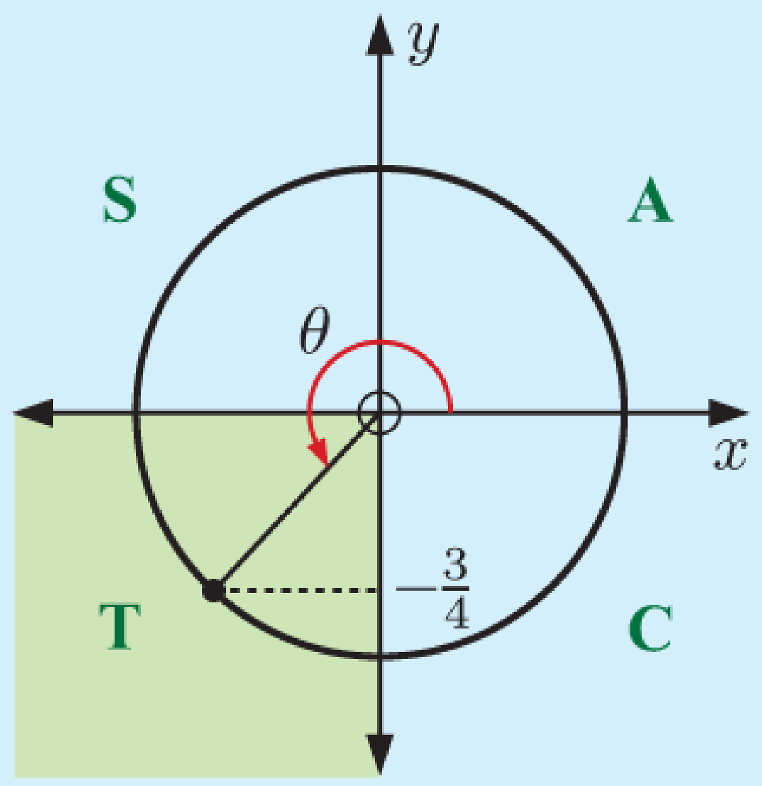
a $\sin\theta = \frac{1}{2}$ **b** $\sin\theta = -\frac{1}{3}$ **c** $\sin\theta = 0$ **d** $\sin\theta = -1$

2 Find the possible exact values of $\sin\theta$ for:

a $\cos\theta = \frac{4}{5}$ **b** $\cos\theta = -\frac{3}{4}$ **c** $\cos\theta = 1$ **d** $\cos\theta = 0$

Example 9 **Self Tutor**

If $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos\theta$ and $\tan\theta$. Give exact values.



Now $\cos^2\theta + \sin^2\theta = 1$

$\therefore \cos^2\theta + \frac{9}{16} = 1$

$\therefore \cos^2\theta = \frac{7}{16}$

$\therefore \cos\theta = \pm\frac{\sqrt{7}}{4}$

But $\pi < \theta < \frac{3\pi}{2}$, so θ is a quadrant 3 angle.

$\therefore$ $\cos\theta$ is negative.

$\therefore$ $\cos\theta = -\frac{\sqrt{7}}{4}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{3}{4}}{-\frac{\sqrt{7}}{4}} = \frac{3}{\sqrt{7}}$

3 Without using a calculator, find:

a $\sin\theta$ if $\cos\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$

b $\cos\theta$ if $\sin\theta = \frac{2}{5}$ and $\frac{\pi}{2} < \theta < \pi$

c $\cos\theta$ if $\sin\theta = -\frac{3}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

d $\sin\theta$ if $\cos\theta = -\frac{5}{13}$ and $\pi < \theta < \frac{3\pi}{2}$.

4 Find $\tan\theta$ exactly given:

a $\sin\theta = \frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$

b $\cos\theta = \frac{1}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

c $\sin\theta = -\frac{1}{\sqrt{3}}$ and $\pi < \theta < \frac{3\pi}{2}$

d $\cos\theta = -\frac{3}{4}$ and $\frac{\pi}{2} < \theta < \pi$.

Example 10 **Self Tutor**

If $\tan\theta = -2$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin\theta$ and $\cos\theta$. Give exact answers.

$\tan\theta = \frac{\sin\theta}{\cos\theta} = -2$

$\therefore \sin\theta = -2\cos\theta$

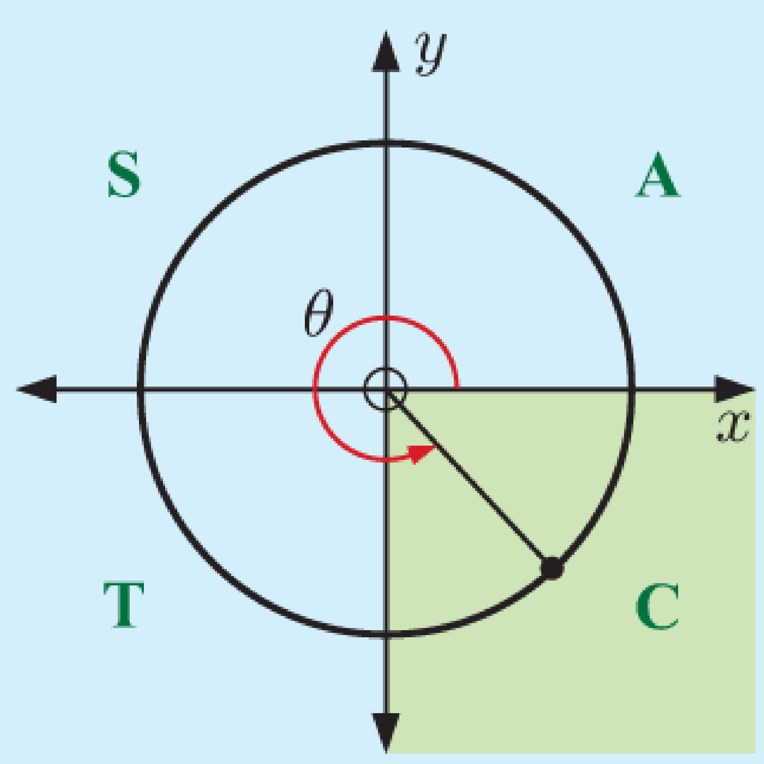
Now $\sin^2\theta + \cos^2\theta = 1$

$\therefore (-2\cos\theta)^2 + \cos^2\theta = 1$

$\therefore 4\cos^2\theta + \cos^2\theta = 1$

$\therefore 5\cos^2\theta = 1$

$\therefore \cos\theta = \pm\frac{1}{\sqrt{5}}$



But $\frac{3\pi}{2} < \theta < 2\pi$, so θ is a quadrant 4 angle. $\cos\theta$ is positive and $\sin\theta$ is negative.

$\therefore$ $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$.

5 Find exact values for $\sin\theta$ and $\cos\theta$ given that:

a $\tan\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$

b $\tan\theta = -\frac{4}{3}$ and $\frac{\pi}{2} < \theta < \pi$

c $\tan\theta = \frac{\sqrt{5}}{3}$ and $\pi < \theta < \frac{3\pi}{2}$

d $\tan\theta = -\frac{12}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

6 Suppose $\tan\theta = k$ where k is a constant and $\pi < \theta < \frac{3\pi}{2}$. Write expressions for $\sin\theta$ and $\cos\theta$ in terms of k.

F FINDING ANGLES

In **Exercise 8C** you should have proven that:

For θ in degrees:

- $\sin(180^\circ - \theta) = \sin\theta$
- $\cos(180^\circ - \theta) = -\cos\theta$
- $\cos(360^\circ - \theta) = \cos\theta$
- $\sin(360^\circ - \theta) = -\sin\theta$

For θ in radians:

- $\sin(\pi - \theta) = \sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$
- $\sin(2\pi - \theta) = -\sin\theta$

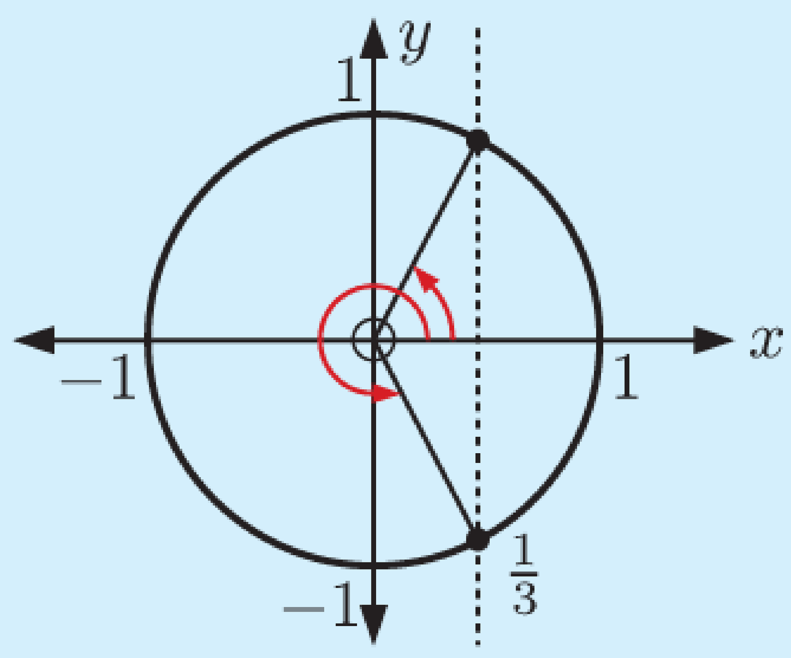
We need results such as these, and also the periodicity of the trigonometric ratios, to find angles which have a particular sine, cosine, or tangent.

Example 11 — Self Tutor

Find the two angles θ on the unit circle, with $0^\circ \leqslant \theta \leqslant 360^\circ$, such that:

a $\cos\theta = \frac{1}{3}$ **b** $\sin\theta = \frac{3}{4}$ **c** $\tan\theta = 2$

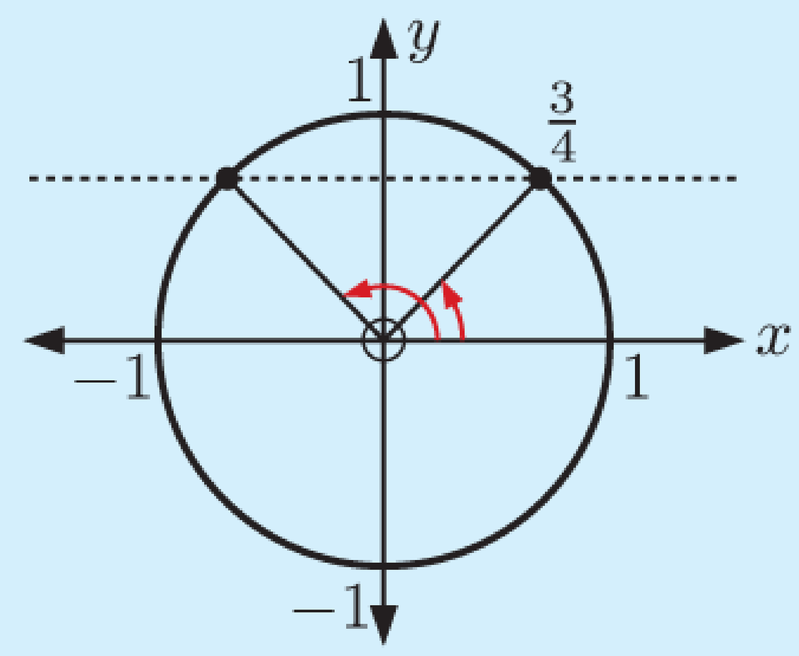
a Using technology, $\cos^{-1}\left(\frac{1}{3}\right) \approx 70.53^\circ$



$\therefore\ \theta \approx 70.53^\circ$ or $360^\circ - 70.53^\circ$

$\therefore\ \theta \approx 70.5^\circ$ or 289.5°

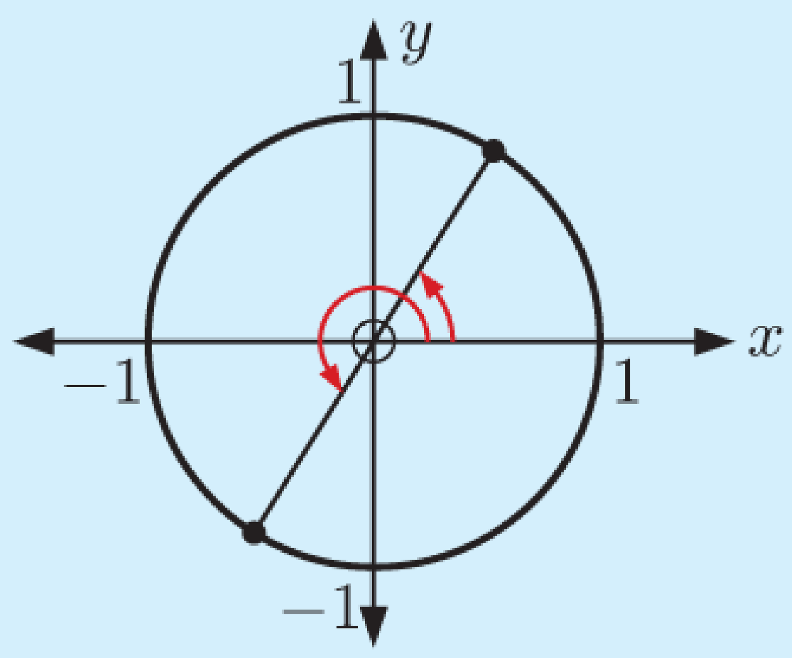
b Using technology, $\sin^{-1}\left(\frac{3}{4}\right) \approx 48.59^\circ$



$\therefore\ \theta \approx 48.59^\circ$ or $180^\circ - 48.59^\circ$

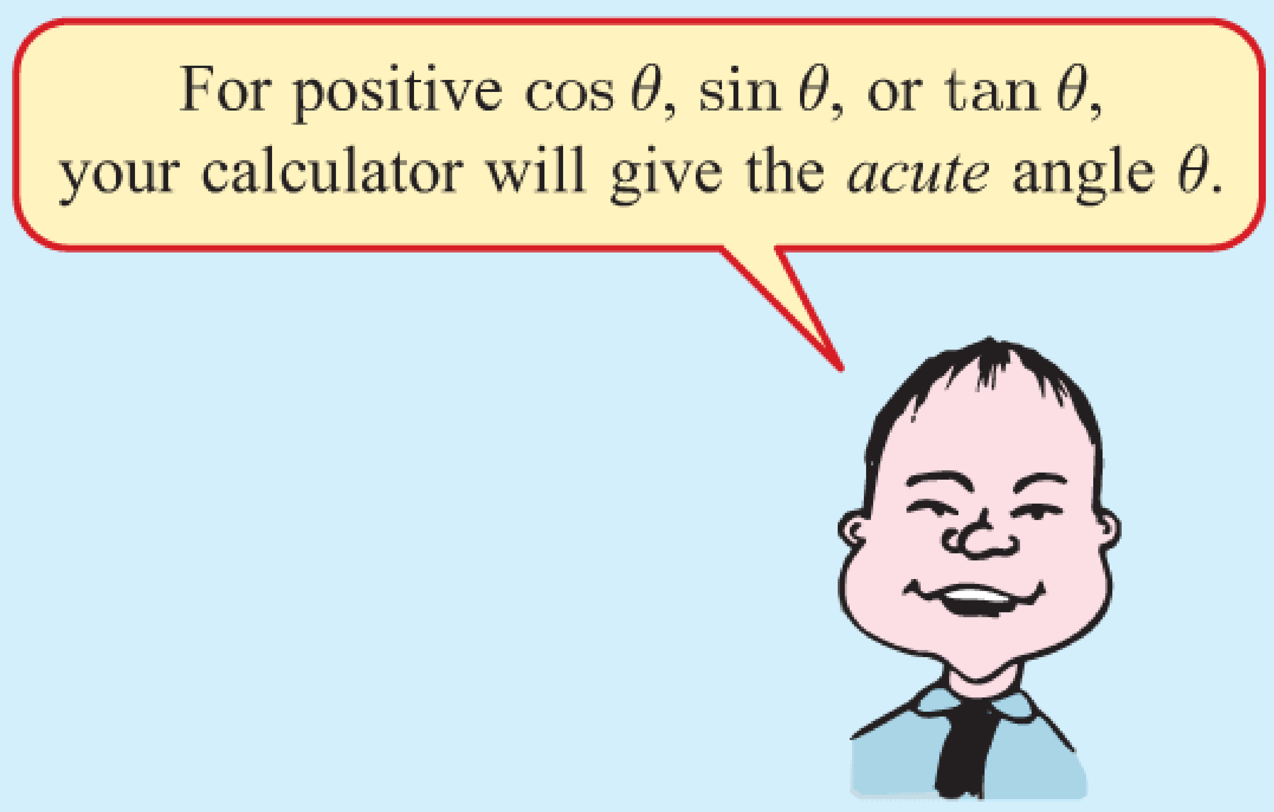
$\therefore\ \theta \approx 48.6^\circ$ or 131.4°

c Using technology, $\tan^{-1}(2) \approx 63.43^\circ$



$\therefore\ \theta \approx 63.43^\circ$ or $180^\circ + 63.43^\circ$

$\therefore\ \theta \approx 63.4^\circ$ or 243.4°



EXERCISE 8F

1 Find two angles θ on the unit circle, with $0^\circ \leqslant \theta \leqslant 360^\circ$, such that:

a $\tan\theta = 4$ **b** $\cos\theta = 0.83$ **c** $\sin\theta = \frac{3}{5}$

d $\cos\theta = 0$ **e** $\tan\theta = 6.67$ **f** $\cos\theta = \frac{2}{17}$

2 Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\tan\theta = \frac{1}{3}$ **b** $\cos\theta = \frac{3}{7}$ **c** $\sin\theta = 0.61$

d $\cos\theta = \frac{1}{4}$ **e** $\tan\theta = 0.114$ **f** $\sin\theta = \frac{1}{6}$

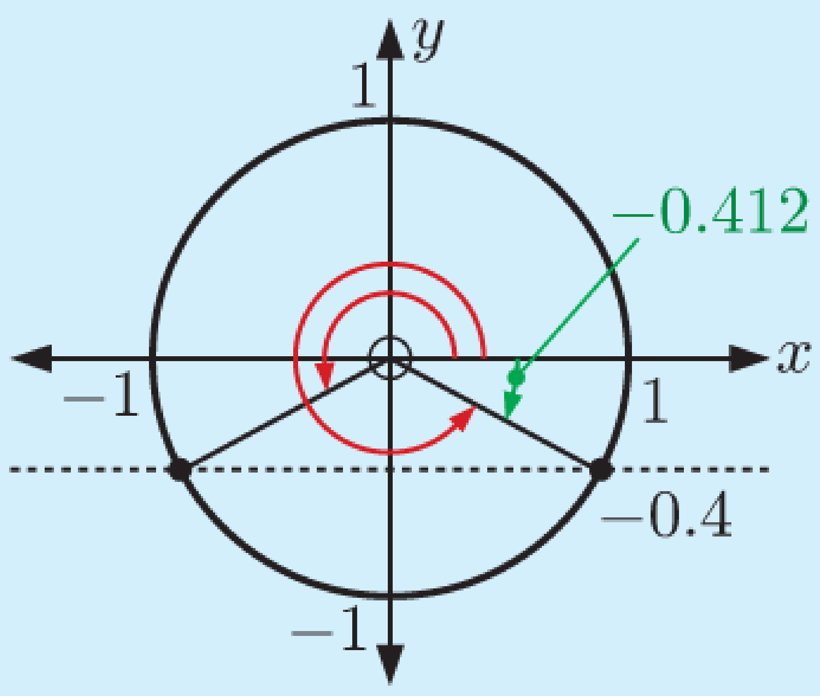
Example 12

Self Tutor

Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\sin\theta = -0.4$ **b** $\cos\theta = -\frac{2}{3}$ **c** $\tan\theta = -\frac{1}{3}$

a Using technology, $\sin^{-1}(-0.4) \approx -0.412$



If $\sin\theta$ or $\tan\theta$ is negative, your calculator will give θ in the domain $-\frac{\pi}{2} < \theta < 0$.

If $\cos\theta$ is negative, your calculator will give the *obtuse* angle θ.

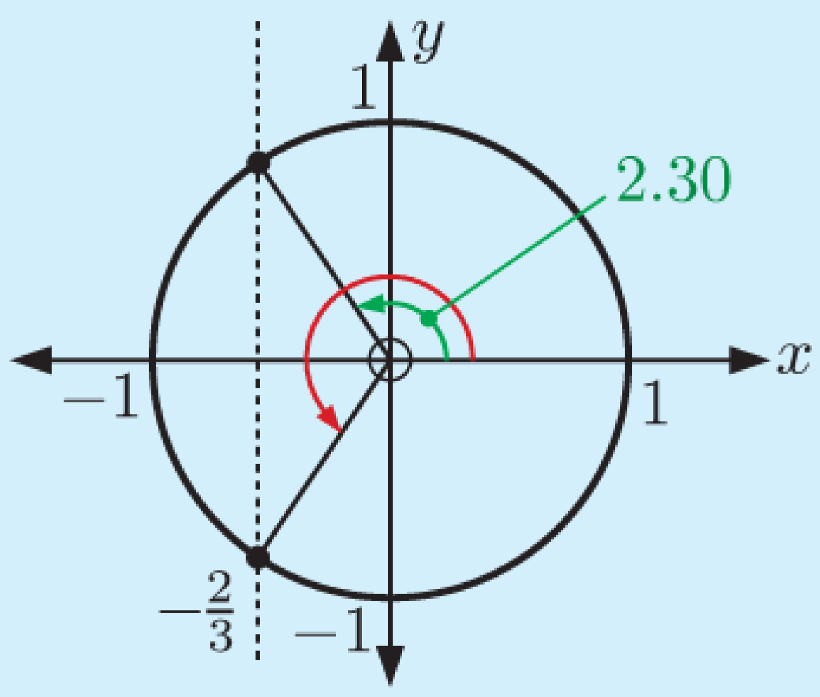
The angles given by your calculator are shown in green.

But $0 \leqslant \theta \leqslant 2\pi$

$\therefore\ \theta \approx \pi + 0.412$ or $2\pi - 0.412$

$\therefore\ \theta \approx 3.55$ or 5.87

b Using technology, $\cos^{-1}\left(-\frac{2}{3}\right) \approx 2.30$

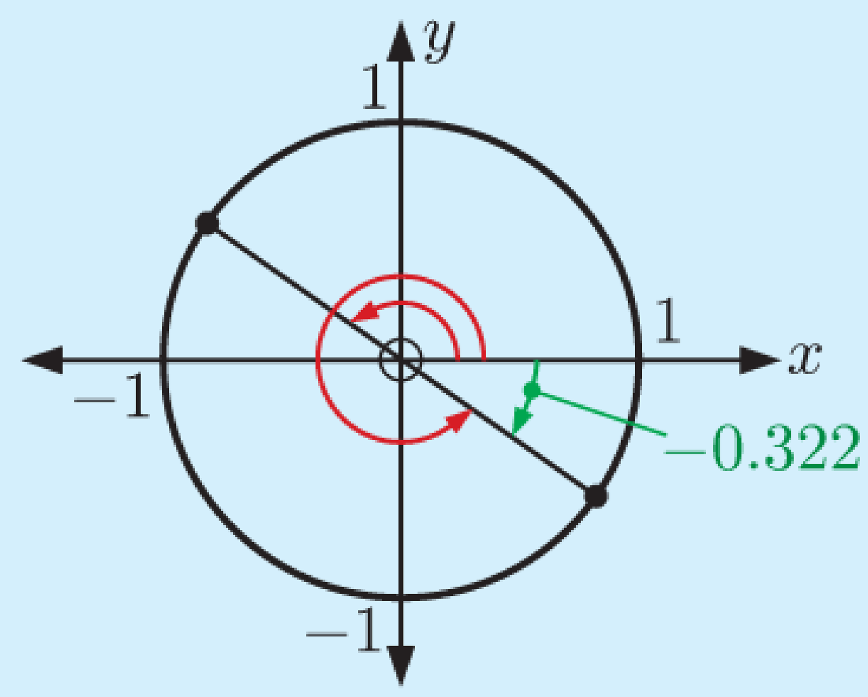


But $0 \leqslant \theta \leqslant 2\pi$

$\therefore\ \theta \approx 2.30$ or $2\pi - 2.30$

$\therefore\ \theta \approx 2.30$ or 3.98

c Using technology, $\tan^{-1}\left(-\frac{1}{3}\right) \approx -0.322$



But $0 \leqslant \theta \leqslant 2\pi$

$\therefore\ \theta \approx \pi - 0.322$ or $2\pi - 0.322$

$\therefore\ \theta \approx 2.82$ or 5.96

3 Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\cos\theta = -\frac{1}{4}$ **b** $\sin\theta = 0$ **c** $\tan\theta = -3.1$ **d** $\sin\theta = -0.421$

e $\tan\theta = 1.2$ **f** $\cos\theta = 0.7816$ **g** $\sin\theta = \frac{1}{11}$ **h** $\cos\theta = -\frac{1}{\sqrt{3}}$

4 Find all θ such that $-180° \leqslant \theta \leqslant 180°$ and:

a $\cos\theta = -\frac{1}{10}$ **b** $\sin\theta = \frac{4}{5}$ **c** $\tan\theta = -\frac{3}{2}$

d $\cos\theta = 0.8$ **e** $\tan\theta = -\frac{5}{6}$ **f** $\sin\theta = -\frac{7}{11}$

5 **a** Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that $\cos\theta = \frac{3}{10}$.

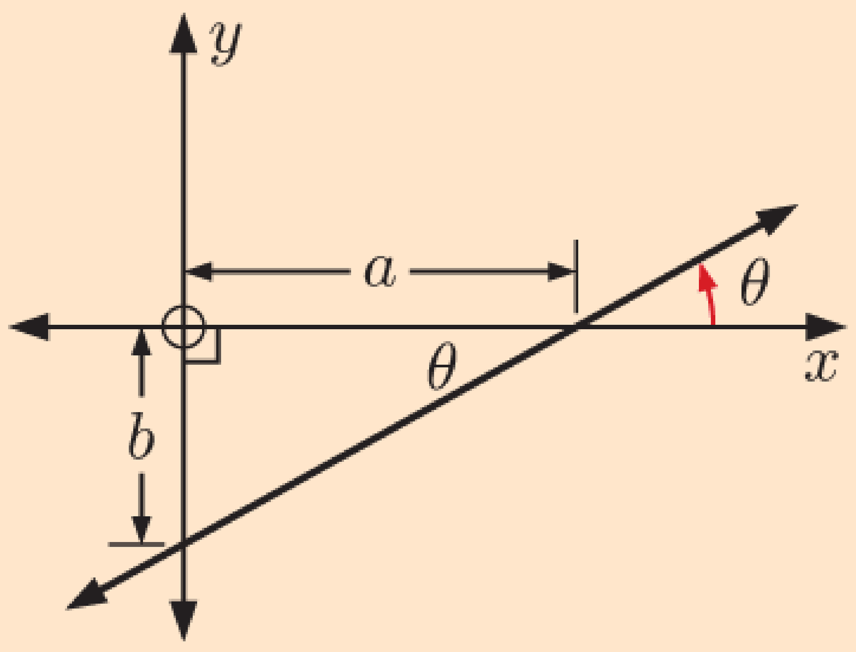
b For each value of θ, find $\sin\theta$ and $\tan\theta$ exactly.

G THE EQUATION OF A STRAIGHT LINE

If a straight line makes an angle of θ with the positive x-axis then its gradient is $m = \tan\theta$.

Proof:

- For $m \geqslant 0$:

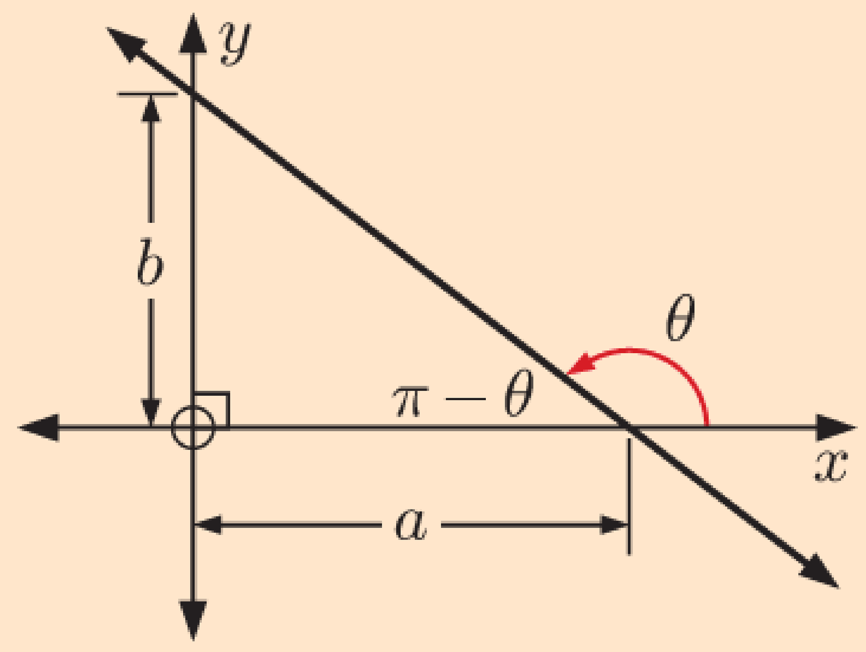


Gradient $m = \dfrac{0 - -b}{a - 0}$

$= \dfrac{b}{a}$

$= \tan\theta$

- For $m < 0$:



Gradient $m = \dfrac{0 - b}{a - 0}$

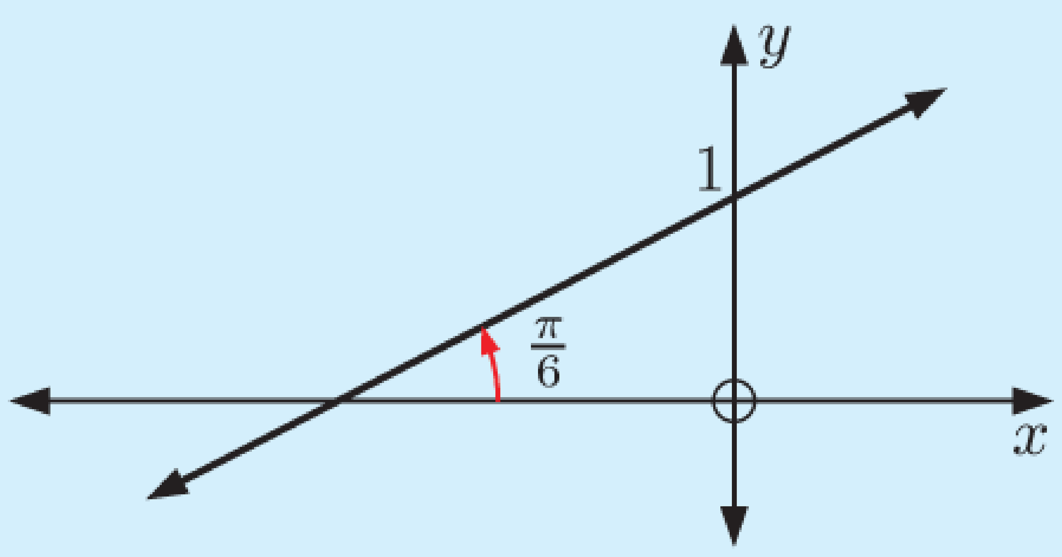
$= -\dfrac{b}{a}$

$= -\tan(\pi - \theta)$

$= \tan\theta$

Example 13 — Self Tutor

Find the equation of the given line:



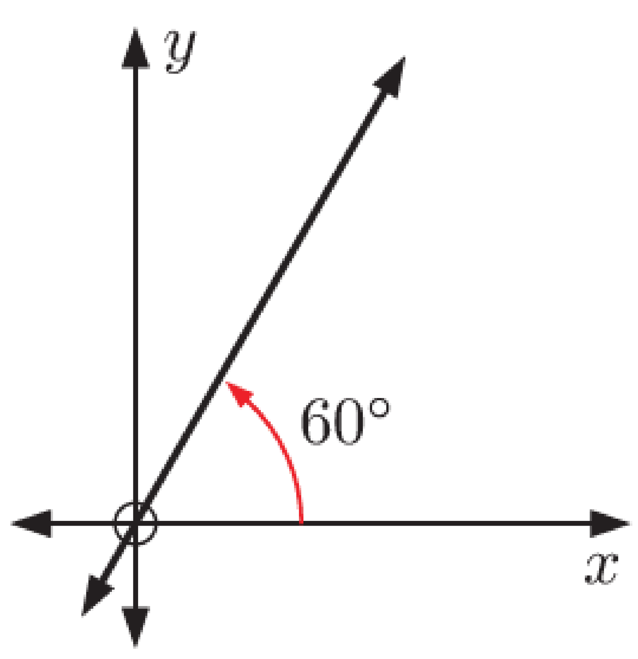
The line has gradient $m = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and y-intercept 1.

$\therefore$ the line has equation $y = \frac{1}{\sqrt{3}}x + 1$.

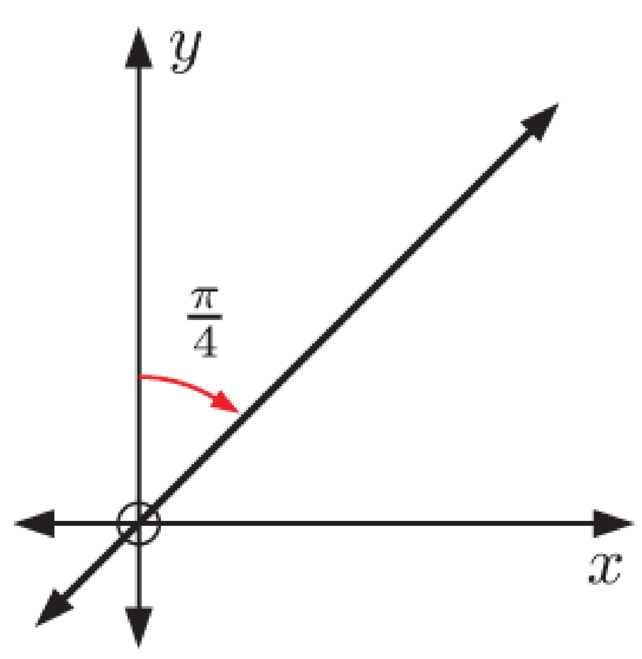
EXERCISE 8G

1 Find the equation of each line:

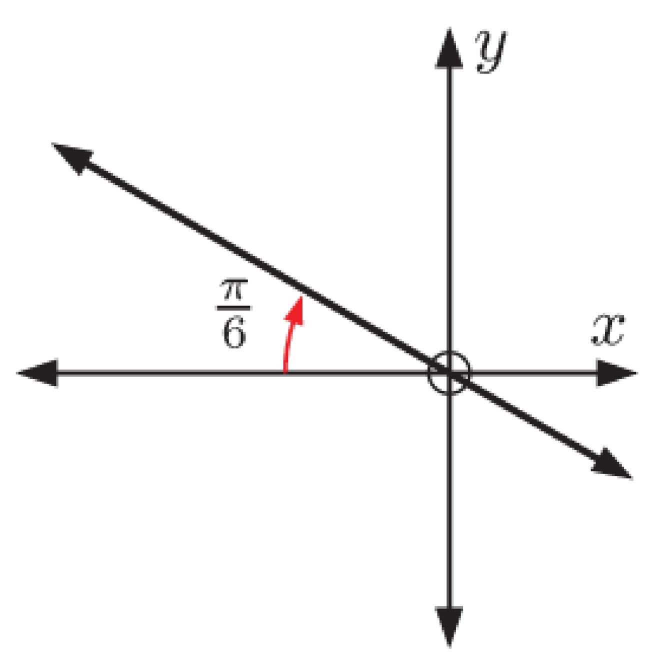
a



b

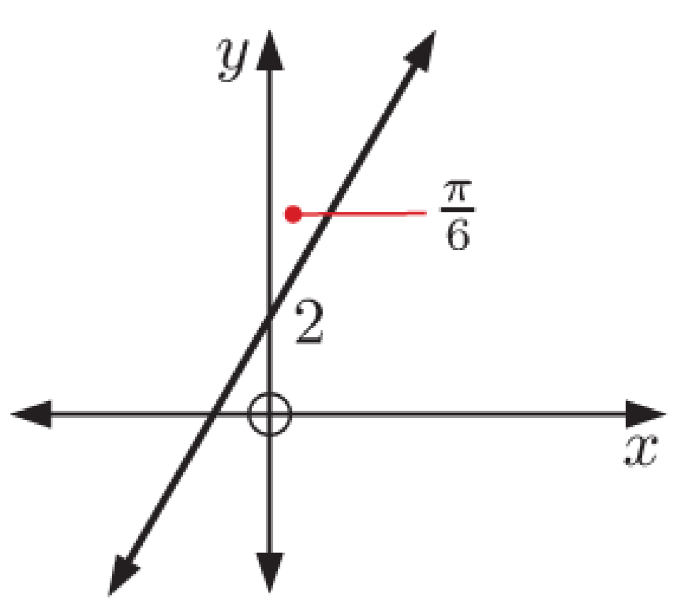


c

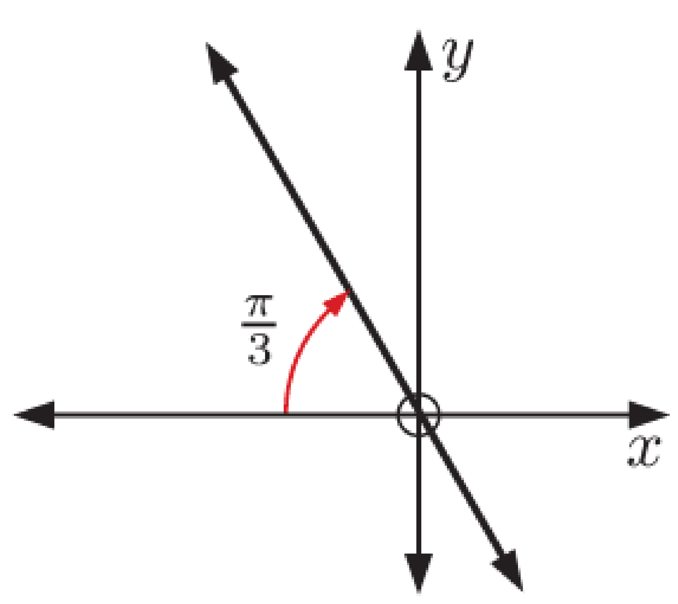


2 Find the equation of each line:

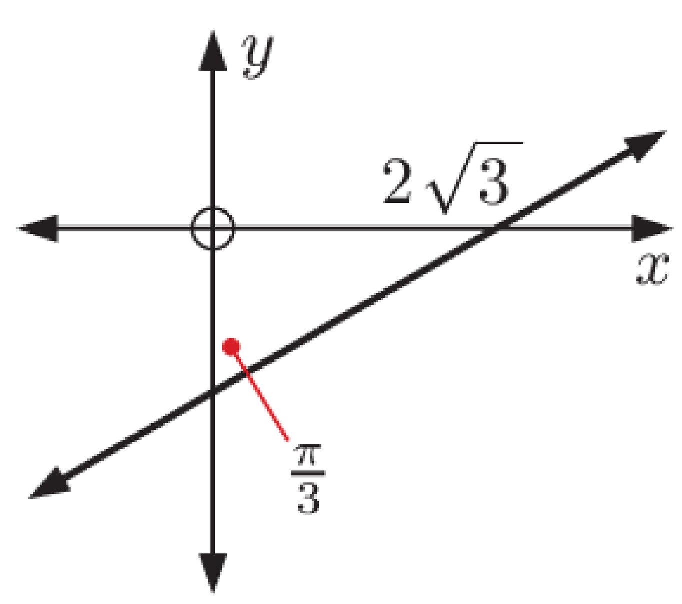
a



b

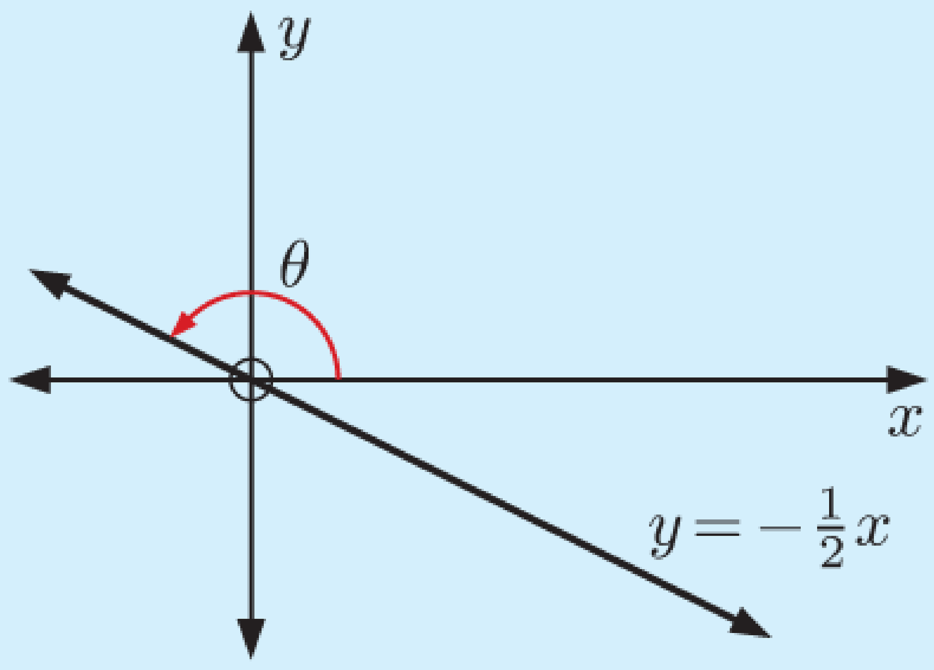


c



Example 14 **Self Tutor**

Find, in radians, the measure of θ:



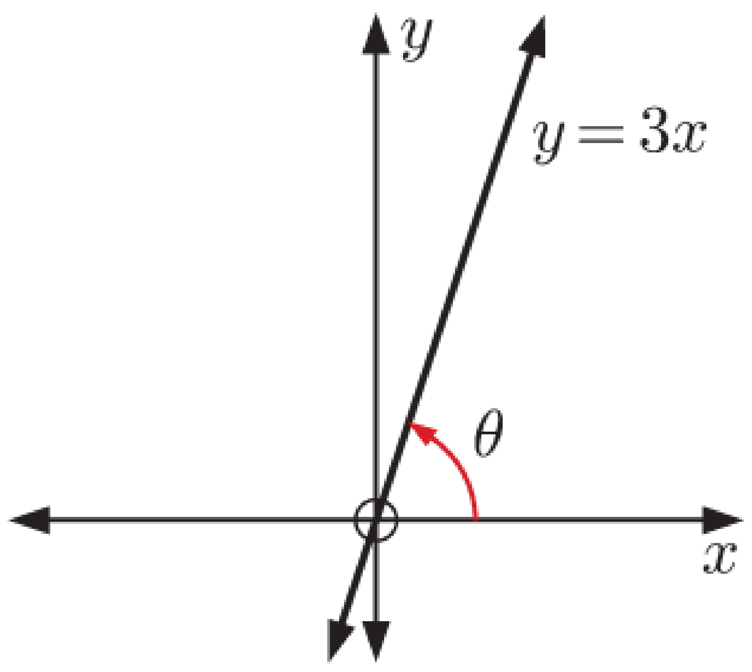
The line has gradient $-\frac{1}{2}$, so $\tan\theta = -\frac{1}{2}$.

Using technology, $\tan^{-1}(-\frac{1}{2}) \approx -0.464$

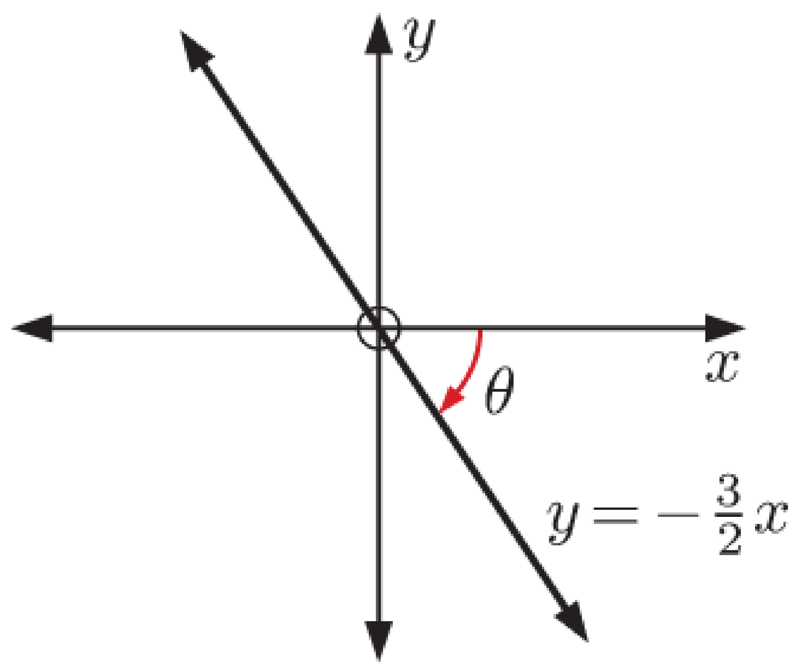
But $0 < \theta < \pi$, so $\theta \approx \pi - 0.464 \approx 2.68$

3 Find, in radians, the measure of θ:

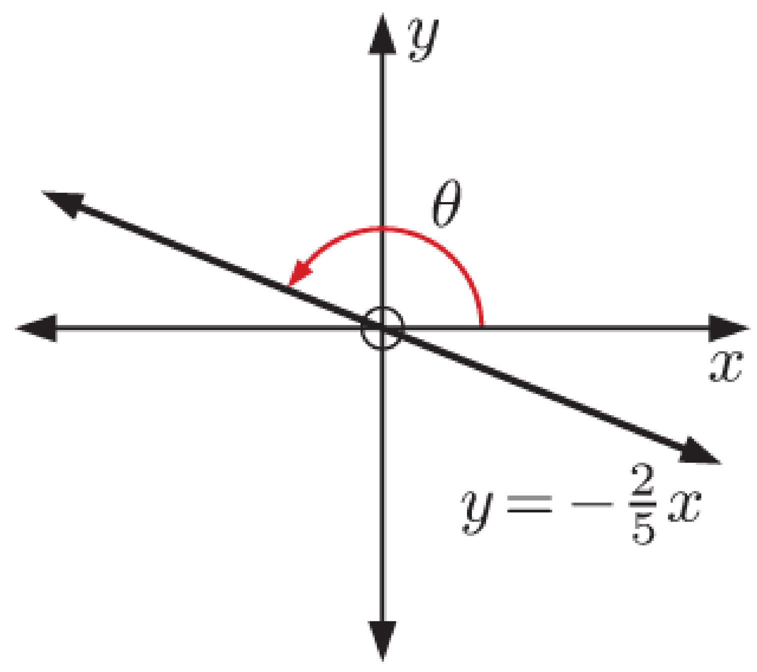
a



b

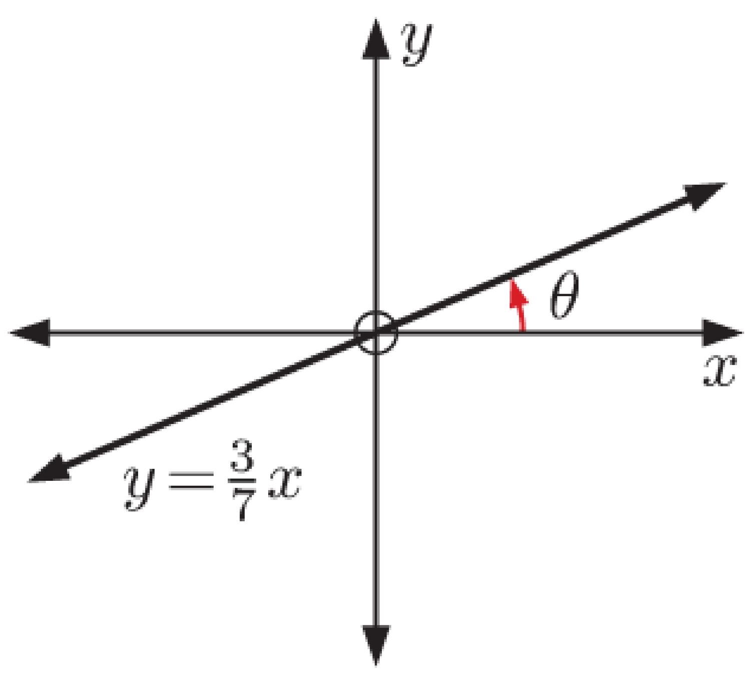


c

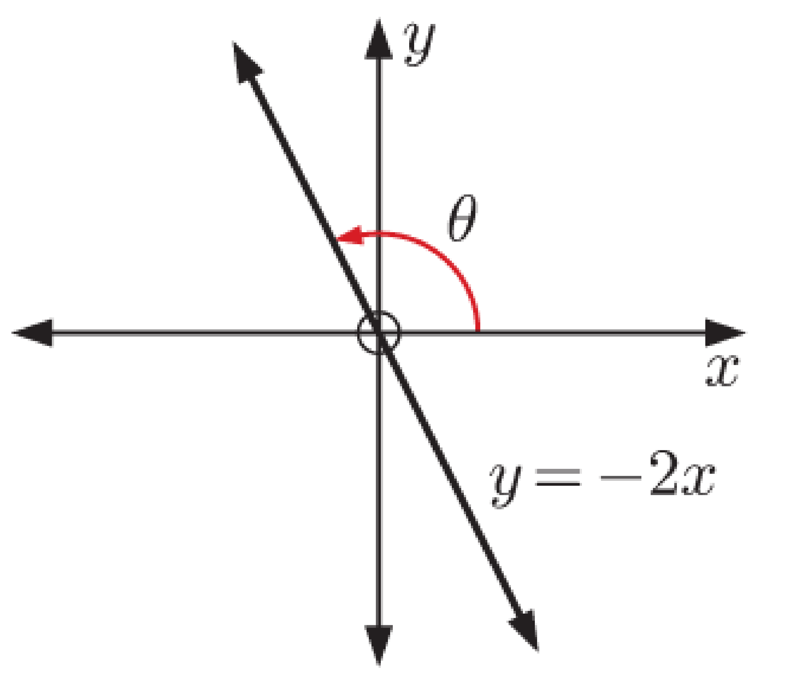


4 Find, in degrees, the measure of θ:

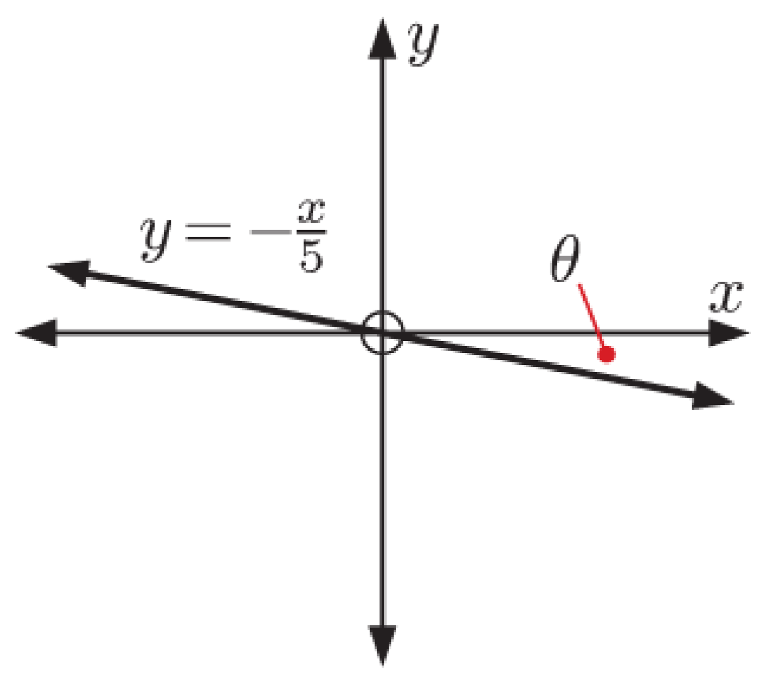
a



b



c



REVIEW SET 8A

1 Convert to radians in terms of π:

a $120°$ **b** $225°$ **c** $150°$ **d** $540°$

2 Illustrate the quadrants where $\sin\theta$ and $\cos\theta$ have the same sign.

3 Determine the coordinates of the point on the unit circle corresponding to an angle of:

a $320°$ **b** $163°$ **c** 0.68^c **d** $\frac{11\pi}{6}$

4 Find the arc length of a sector with angle 1.5 radians and radius 8 cm.

5 Find the acute angles that have the same:

a sine as $\frac{2\pi}{3}$ **b** sine as $165°$ **c** cosine as $276°$.

6 Find:

a $\sin 159°$ if $\sin 21° \approx 0.358$

b $\cos 92°$ if $\cos 88° \approx 0.035$

c $\cos 75°$ if $\cos 105° \approx -0.259$

d $\tan(-133°)$ if $\tan 47° \approx 1.072$.

7 Use a unit circle diagram to find:

a $\cos 360°$ and $\sin 360°$

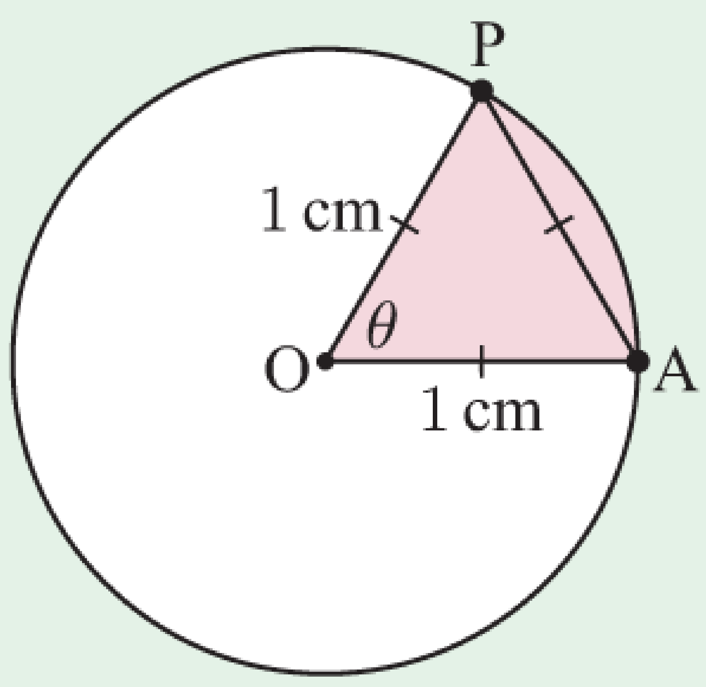
b $\cos(-\pi)$ and $\sin(-\pi)$.

8 Find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ for θ equal to:

a $\frac{2\pi}{3}$

b $\frac{8\pi}{3}$

9



a State the value of θ in:

i degrees

ii radians.

b State the arc length AP.

c State the area of the minor *sector* OAP.

10 If $\sin x = -\frac{1}{4}$ and $\pi < x < \frac{3\pi}{2}$, find $\tan x$ exactly.

11 If $\cos\theta = \frac{3}{4}$, find the possible values of $\sin\theta$.

12 Evaluate:

a $2\sin\frac{\pi}{3}\cos\frac{\pi}{3}$

b $\tan^2\left(\frac{\pi}{4}\right) - 1$

c $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right)$

13 Given $\tan x = -\frac{3}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find:

a $\cos x$

b $\sin x$.

14 Suppose $\cos\theta = \frac{\sqrt{11}}{\sqrt{17}}$ and θ is acute. Find the exact value of $\tan\theta$.

15 Explain how to use the unit circle to find θ when $\cos\theta = -\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$.

16 Find two angles on the unit circle with $0 \leqslant \theta \leqslant 2\pi$, such that:

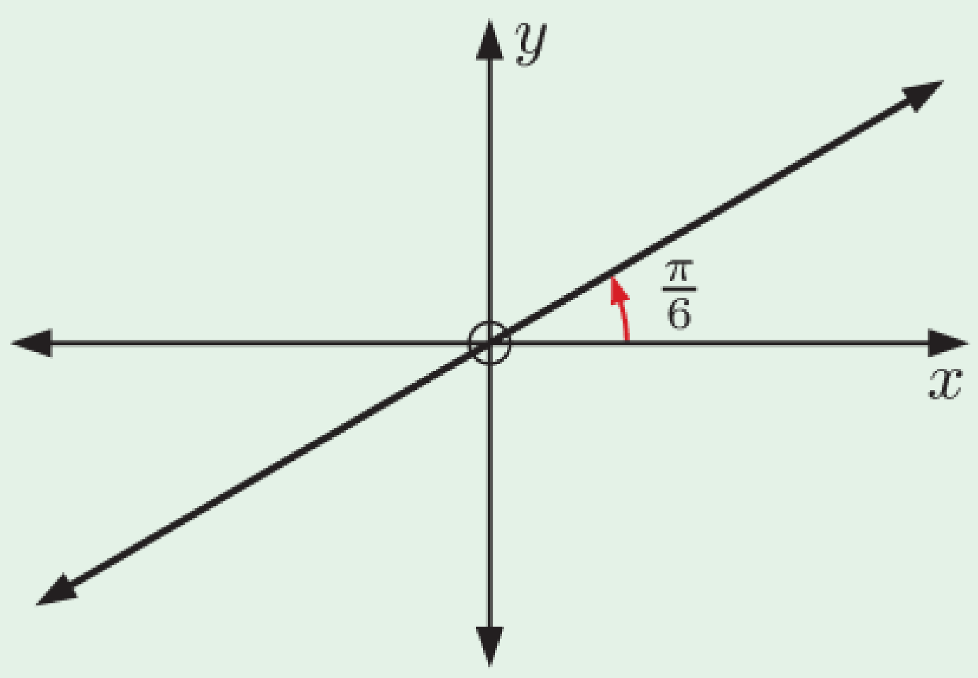
a $\cos\theta = \frac{2}{3}$

b $\sin\theta = -\frac{1}{4}$

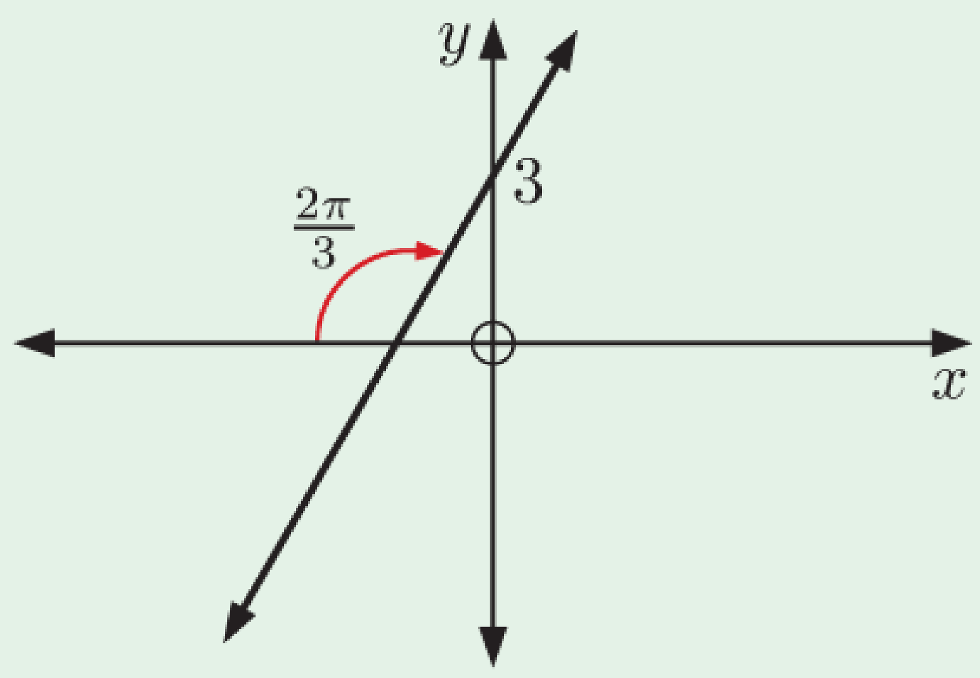
c $\tan\theta = 3$

17 Find the equation of each line:

a



b



REVIEW SET 8B

1 Convert to degrees, to 2 decimal places:

a $\frac{2\pi}{5}$

b 1.46

c 0.435^c

d -5.271

2 Determine the area of a sector with angle $\frac{5\pi}{12}$ and radius 13 cm.

3 Find the angle [OA] makes with the positive x-axis if the x-coordinate of the point A on the unit circle is -0.222.

4 Find the radius and area of a sector of perimeter 36 cm with an angle of $\frac{2\pi}{3}$.

5 A sector has perimeter 21 cm and area 27 cm^2. Find the radius of the sector.

6 Use a unit circle diagram to find:

a $\cos\frac{3\pi}{2}$ and $\sin\frac{3\pi}{2}$ **b** $\cos\left(-\frac{\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$

7 Suppose $m = \sin p$, where p is acute. Write an expression in terms of m for:

a $\sin(\pi - p)$ **b** $\sin(p + 2\pi)$ **c** $\cos p$ **d** $\tan p$

8 Find all angles between 0° and 360° which have:

a a cosine of $-\frac{\sqrt{3}}{2}$ **b** a sine of $\frac{1}{\sqrt{2}}$ **c** a tangent of $-\sqrt{3}$

9 Find θ for $0 \leqslant \theta \leqslant 2\pi$ if:

a $\cos\theta = -1$ **b** $\sin^2\theta = \frac{3}{4}$

10 Find the obtuse angles which have the same:

a sine as 47° **b** sine as $\frac{\pi}{15}$ **c** cosine as 186°

11 Find the perimeter and area of a sector with radius 11 cm and angle 63°.

12 Show that $\cos\frac{3\pi}{4} - \sin\frac{3\pi}{4} = -\sqrt{2}$.

13 If $\cos\theta = -\frac{3}{4}$, $\frac{\pi}{2} < \theta < \pi$ find the exact value of:

a $\sin\theta$ **b** $\tan\theta$ **c** $\cos(\pi - \theta)$

14 Without using a calculator, evaluate:

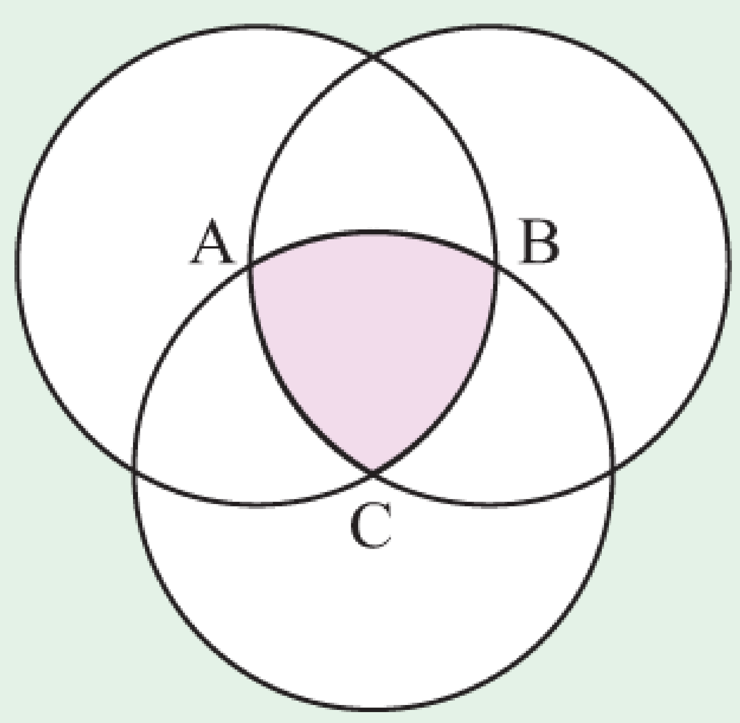
a $\tan^2 60° - \sin^2 45°$ **b** $\cos^2\left(\frac{\pi}{4}\right) + \sin\frac{\pi}{2}$

c $\cos\frac{5\pi}{3} - \tan\frac{5\pi}{4}$ **d** $\tan^2\left(\frac{2\pi}{3}\right)$

15 Use a unit circle diagram to show that $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ for $\frac{\pi}{2} < \theta < \pi$.

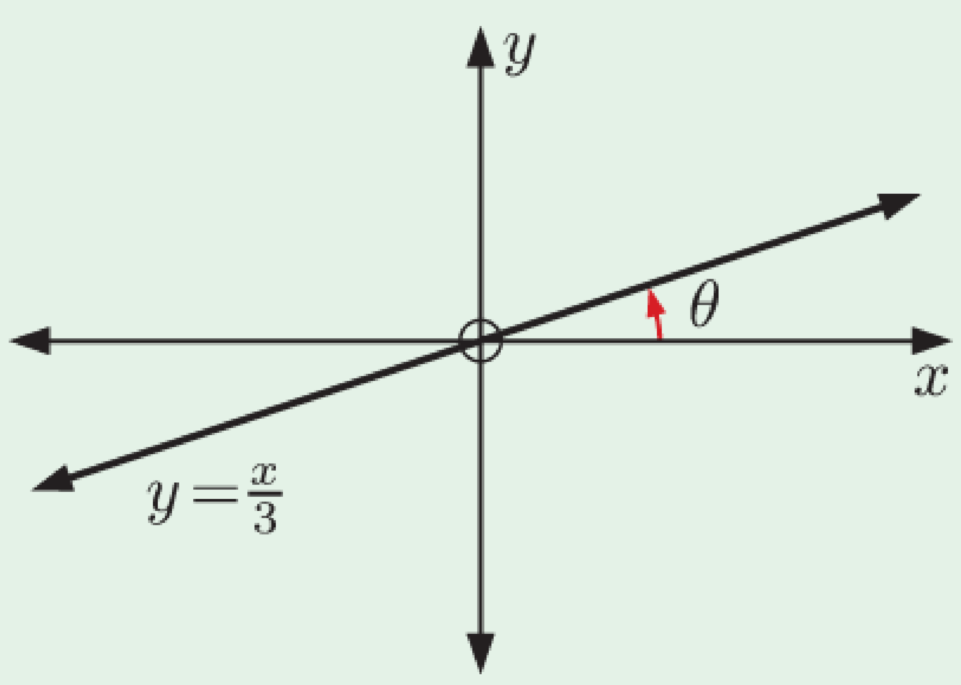
16 Three circles with radius r are drawn as shown, each with its centre on the circumference of the other two circles. A, B, and C are the centres of the three circles.

Prove that an expression for the area of the shaded region is $A = \frac{r^2}{2}(\pi - \sqrt{3})$.

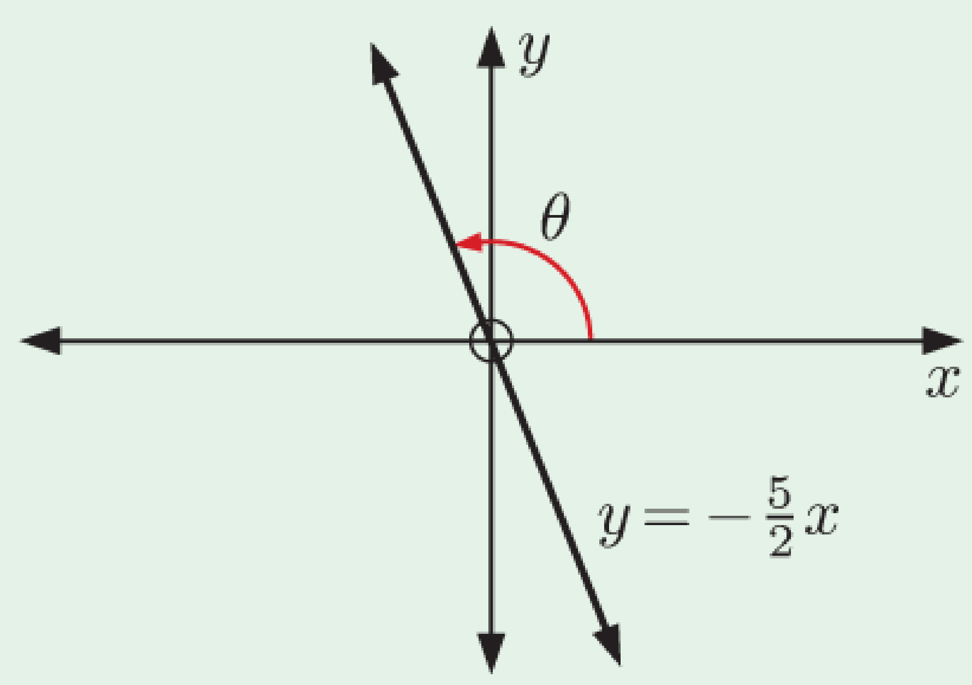


17 Find, in radians, the measure of θ:

a



b



Chapter 9

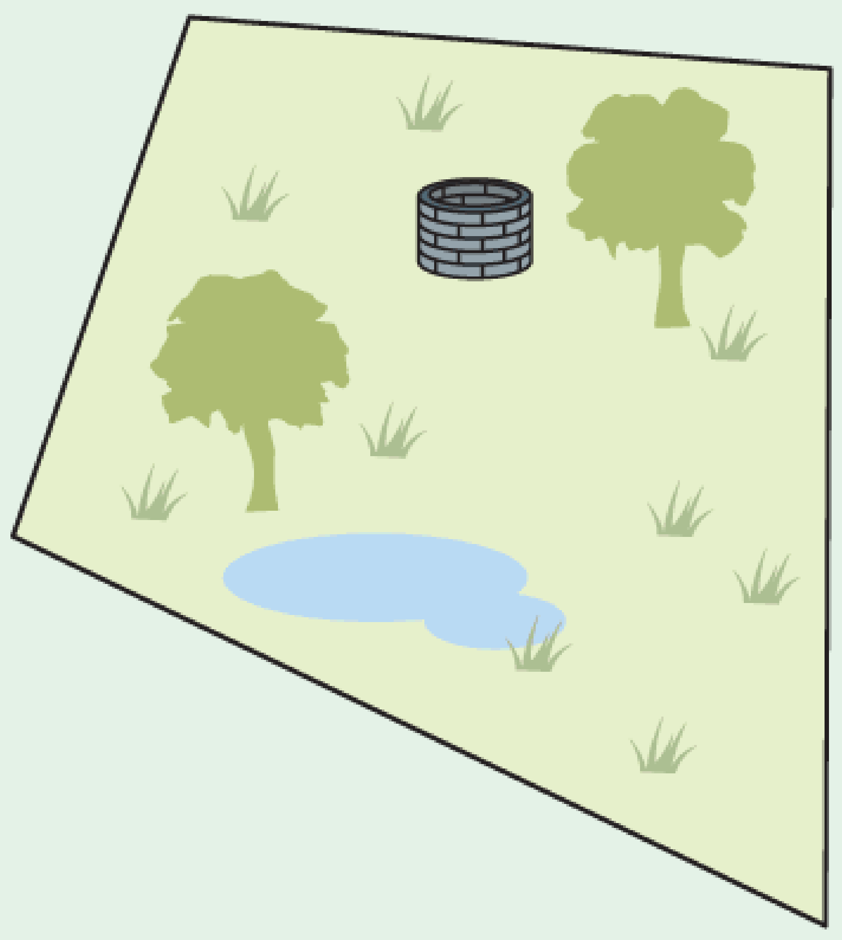
Non-right angled triangle trigonometry

Contents:

OPENING PROBLEM

Robert works for the City Council. He has been asked to find the area of its central park, so he has arrived with a measuring wheel to measure some lengths.

Robert has just realised that while the park is a quadrilateral, it is not a rectangle.



Things to think about:

a Does Robert need to measure angles in order to find the area of the park? If not, what lengths does he need to measure? Are the four side lengths of the quadrilateral sufficient?

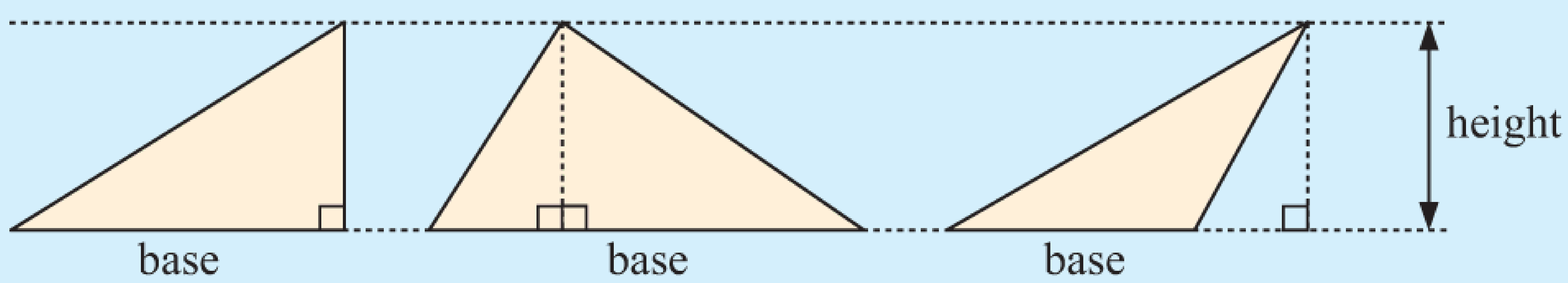
b Will Robert be able to find the angles at the corners of the park using length measurements alone?

In this Chapter we will see how trigonometry can be used to solve problems involving non-right angled triangles.

A THE AREA OF A TRIANGLE

We have seen in previous years that the area of any triangle can be calculated using:

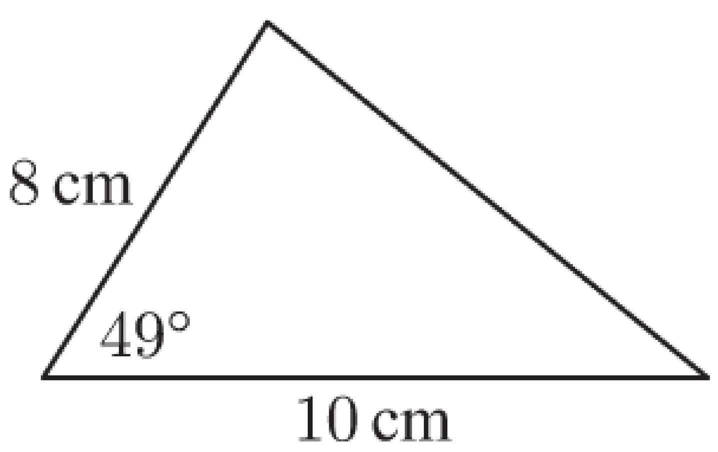
$$\textbf{Area} = \tfrac{1}{2}\ \textbf{base} \times \textbf{height}$$



DEMO

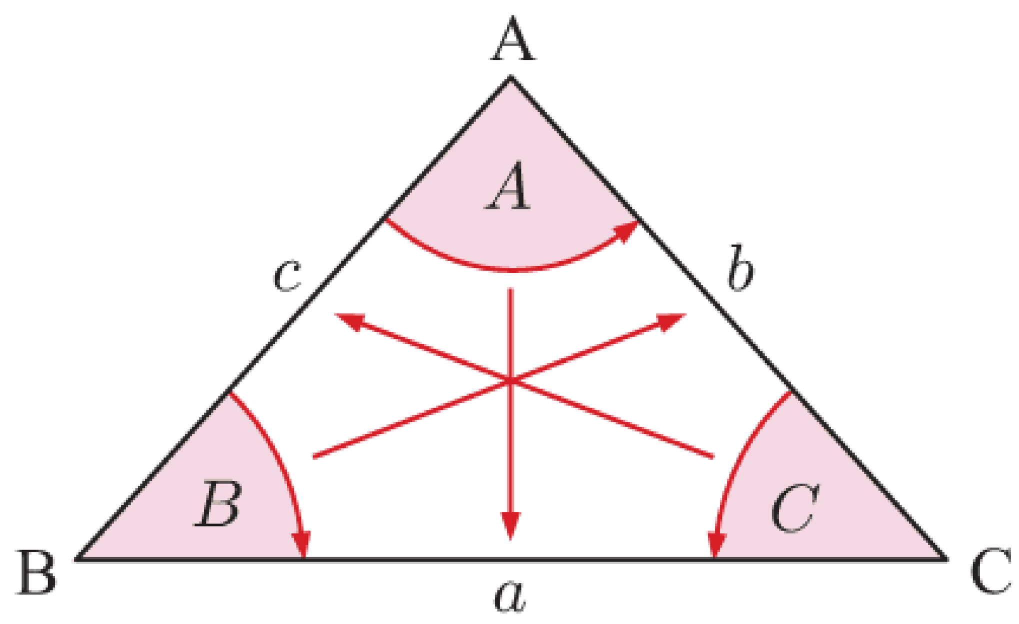
However, if we do not know the perpendicular height of a triangle, we can use trigonometry to calculate the area.

To do this we need to know two sides of the triangle and the **included angle** between them. For example, in the triangle alongside the angle 49° is *included* between the sides of length 8 cm and 10 cm.



CONVENTION FOR LABELLING TRIANGLES

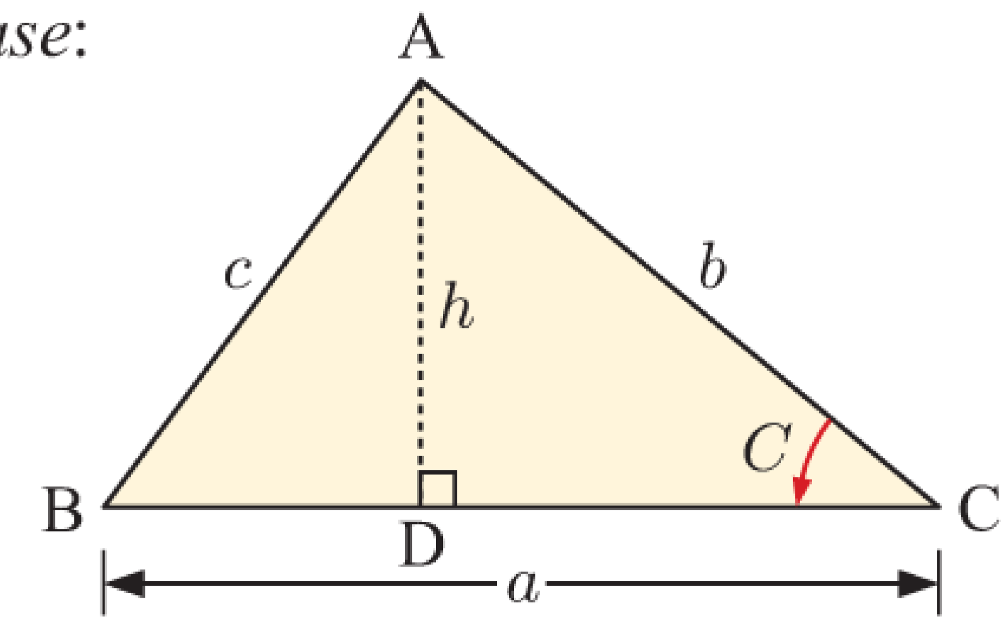
For triangle ABC, the angles at vertices A, B, and C are labelled A, B, and C respectively. The sides opposite these angles are labelled a, b, and c respectively.



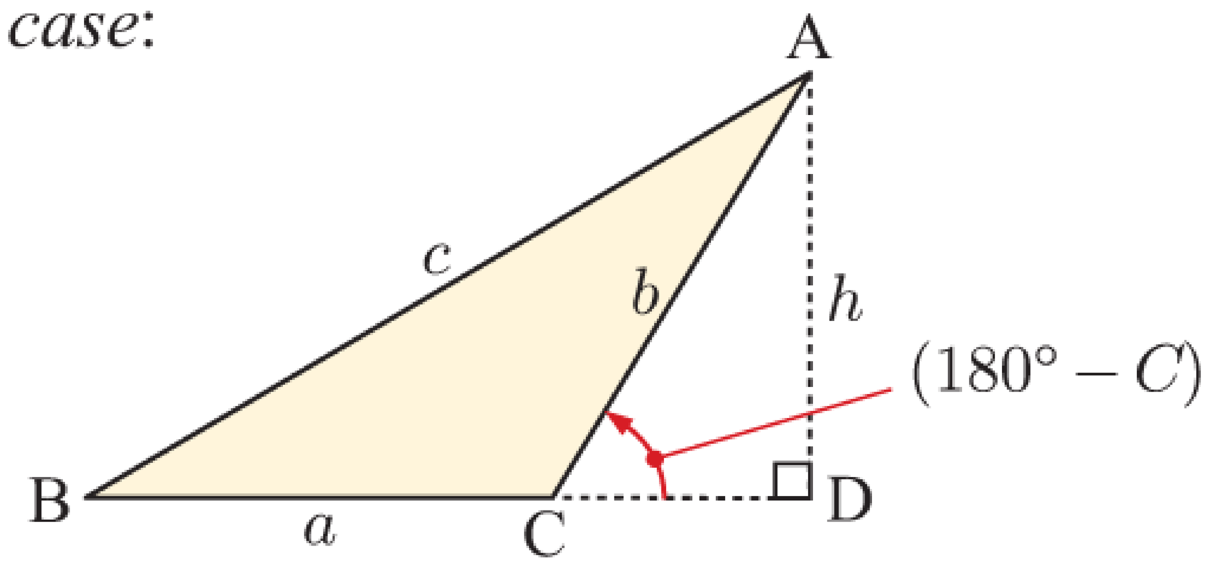
CALCULATING THE AREA OF A TRIANGLE

Any triangle that is not right angled must be either acute or obtuse. We will consider both cases:

Acute case:



Obtuse case:



In both triangles the altitude h is constructed from A to D on [BC] (extended if necessary).

Acute case: $\sin C = \dfrac{h}{b}$

$\therefore\ h = b \sin C$

Obtuse case: $\sin(180^\circ - C) = \dfrac{h}{b}$

$\therefore\ h = b\sin(180^\circ - C)$

But $\sin(180^\circ - C) = \sin C$

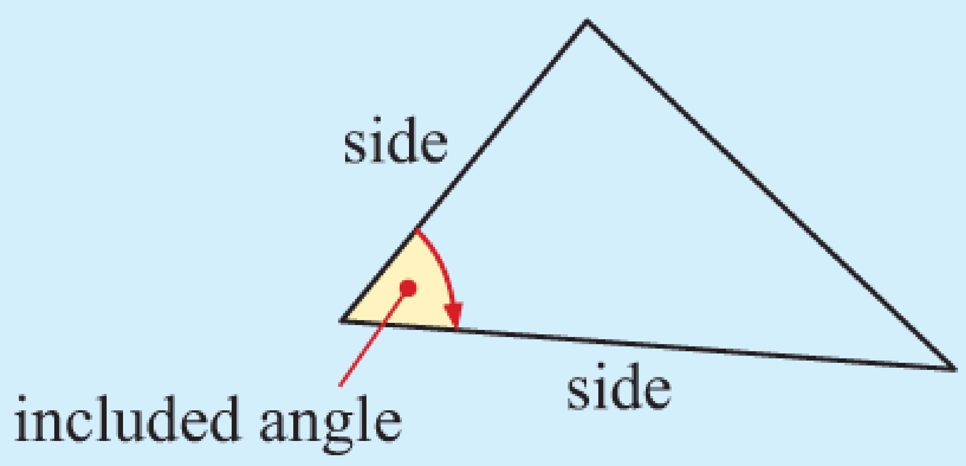
$\therefore\ h = b \sin C$

So, since area $= \frac{1}{2}ah$, we now have **Area $= \frac{1}{2}ab\sin C$.**

Using different altitudes we can show that the area is also $\frac{1}{2}bc\sin A$ or $\frac{1}{2}ac\sin B$.

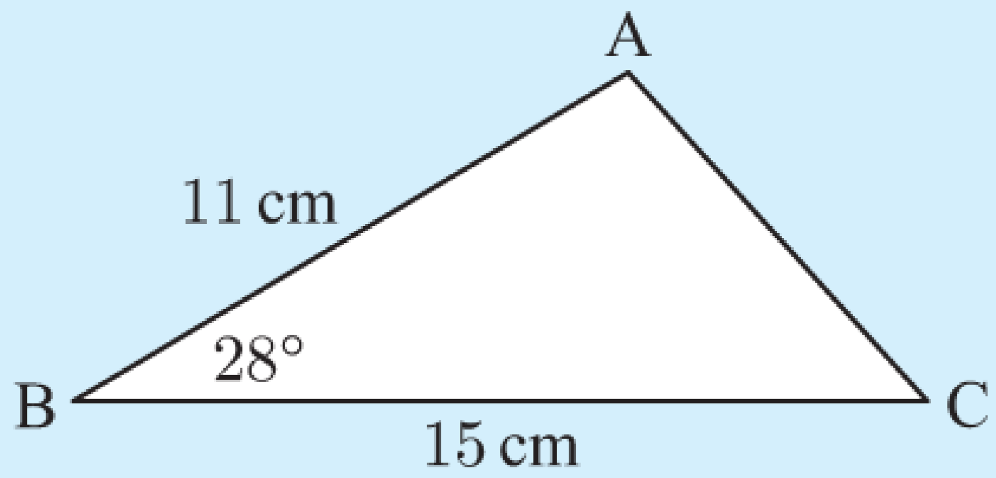
Given the lengths of two sides of a triangle, and the size of the included angle between them, the area of the triangle is

half of the product of two sides and the sine of the included angle.



Example 1 **Self Tutor**

Find the area of triangle ABC.



Area $= \frac{1}{2}ac\sin B$

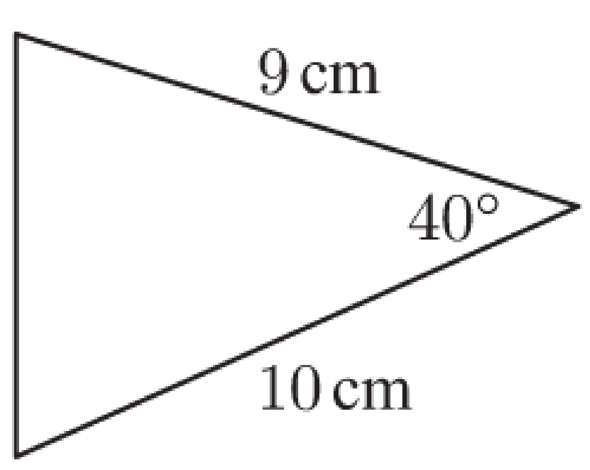
$= \frac{1}{2} \times 15 \times 11 \times \sin 28^\circ$

$\approx 38.7\ \text{cm}^2$

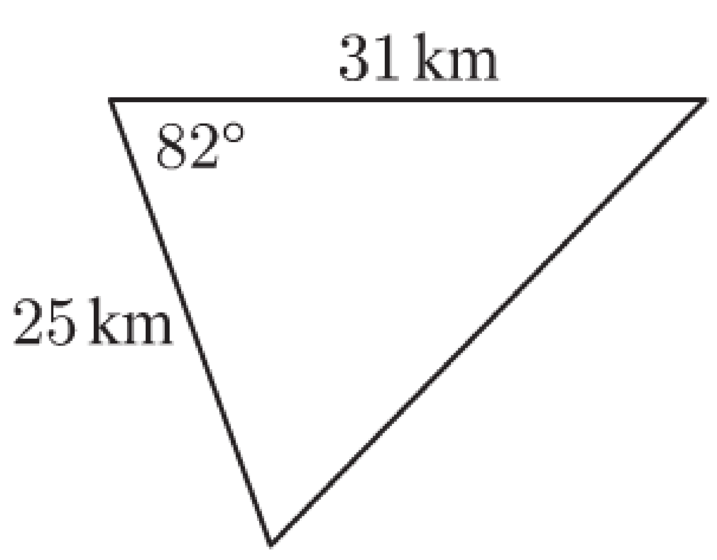
EXERCISE 9A

1 Find the area of:

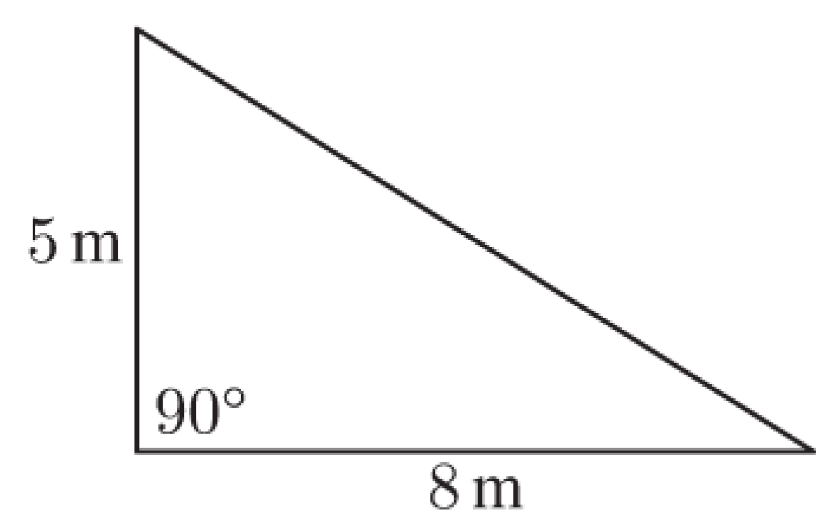
a



b

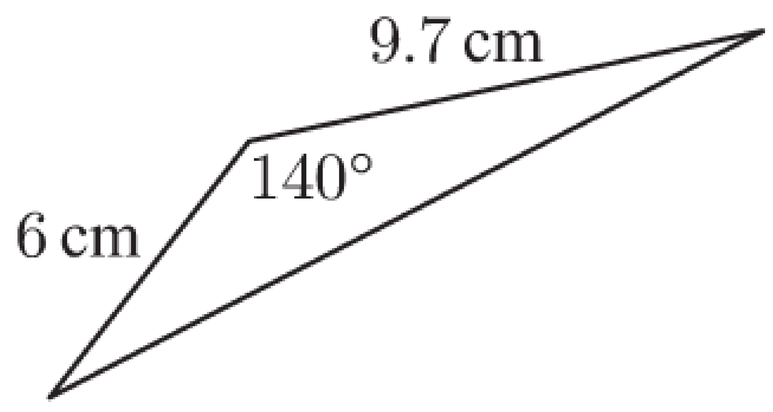


c

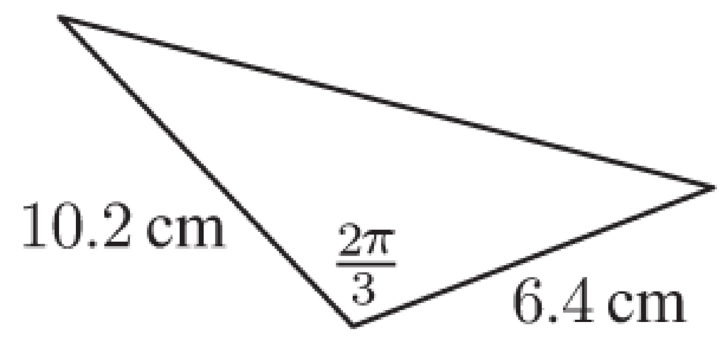


2 Find the area of:

a



b



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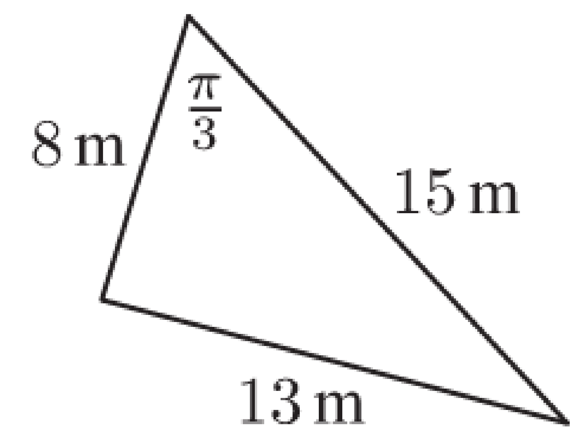
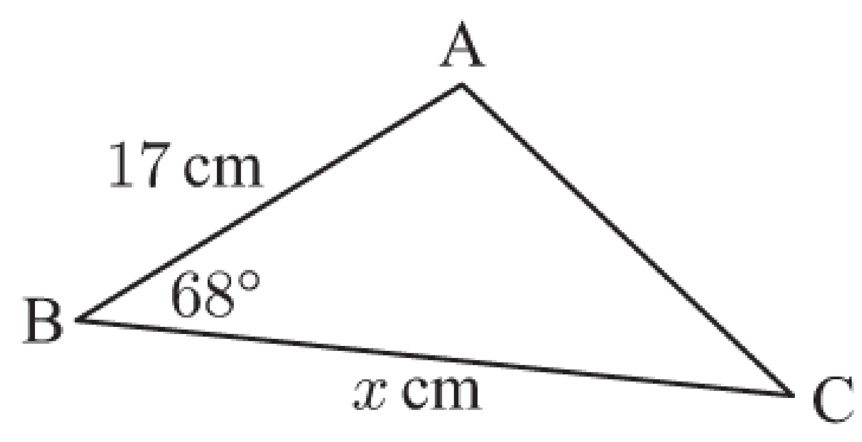


3 Triangle ABC has area 150 cm^2.
Find the value of x.



4 Calculate the area of:

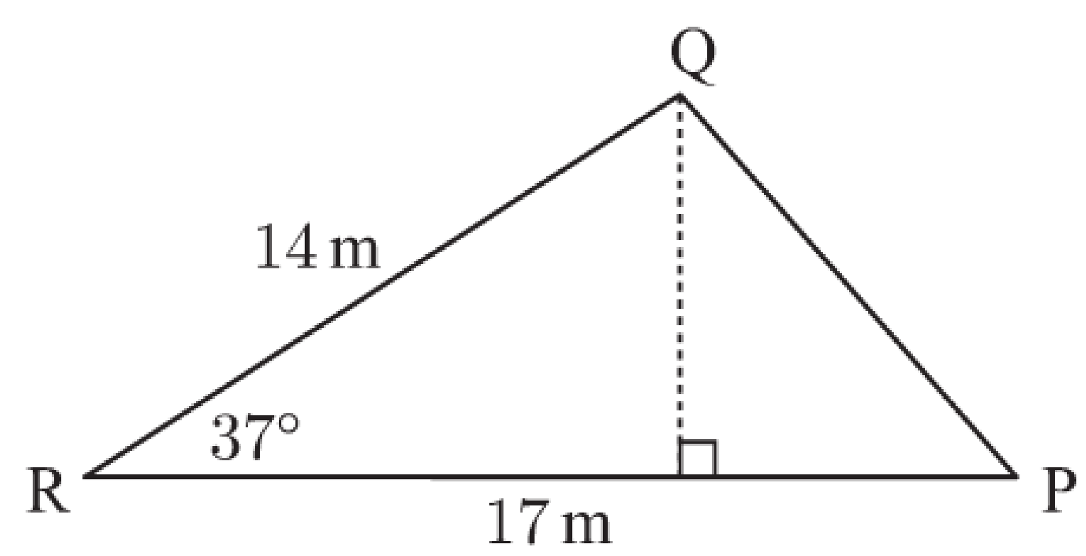
a an isosceles triangle with equal sides of length 21 cm and an included angle of 49°

b an equilateral triangle with sides of length 57 cm.

5 A parallelogram has two adjacent sides with lengths 4 cm and 6 cm respectively. If the included angle measures 52°, find the area of the parallelogram.

6 A rhombus has sides of length 12 cm and an angle of 72°. Find its area.

7



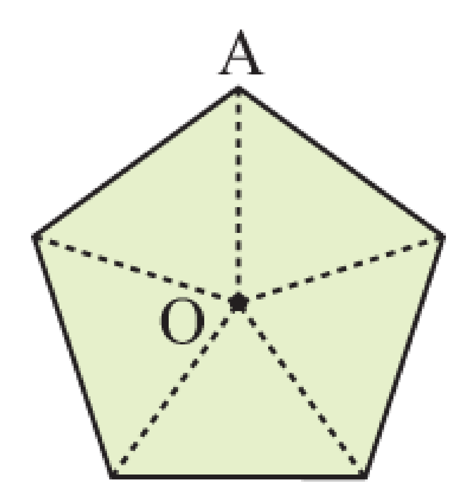
a Find the area of triangle PQR to 3 decimal places.

b Hence find the length of the altitude from Q to [RP].

8 Find the area of a regular hexagon with sides of length 12 cm.

9 A rhombus has area 50 cm^2 and an internal angle of size 63°. Find the length of its sides.

10 A regular pentagonal garden plot has centre of symmetry O and an area of 338 m^2. Find the distance OA.



Example 2 | **Self Tutor**

A triangle has two sides with lengths 10 cm and 11 cm, and an area of 50 cm^2. Determine the possible measures of the included angle. Give your answers accurate to 1 decimal place.

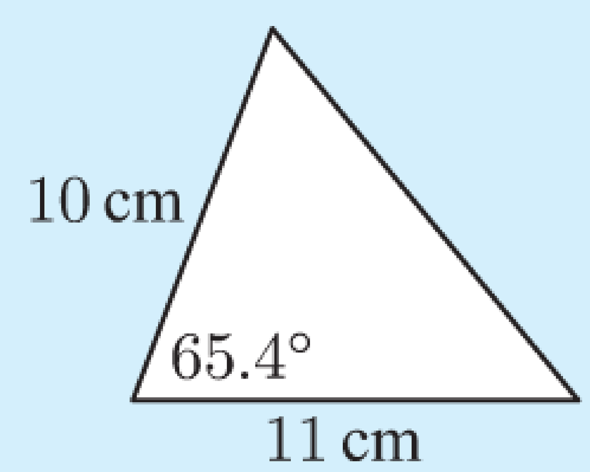
If the included angle is θ, then $\frac{1}{2} \times 10 \times 11 \times \sin\theta = 50$

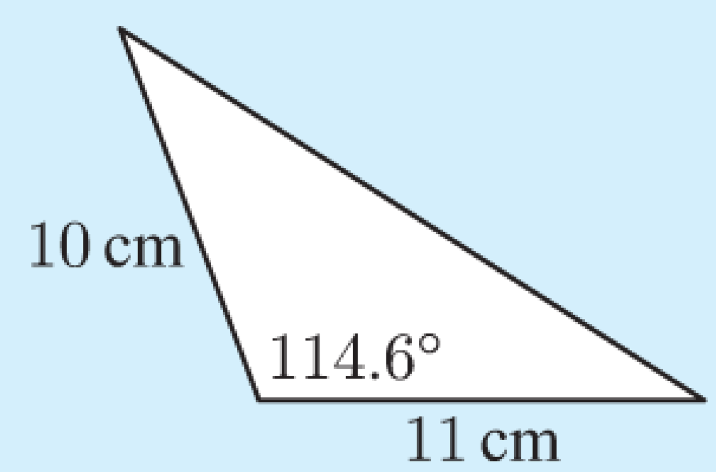
$\therefore \sin\theta = \frac{50}{55}$

Now $\sin^{-1}\left(\frac{50}{55}\right) \approx 65.4°$

$\therefore \theta \approx 65.4°$ or $180° - 65.4°$

$\therefore \theta \approx 65.4°$ or $114.6°$





The two different possible angles are 65.4° and 114.6°.

11 Find the possible values of the included angle of a triangle with:

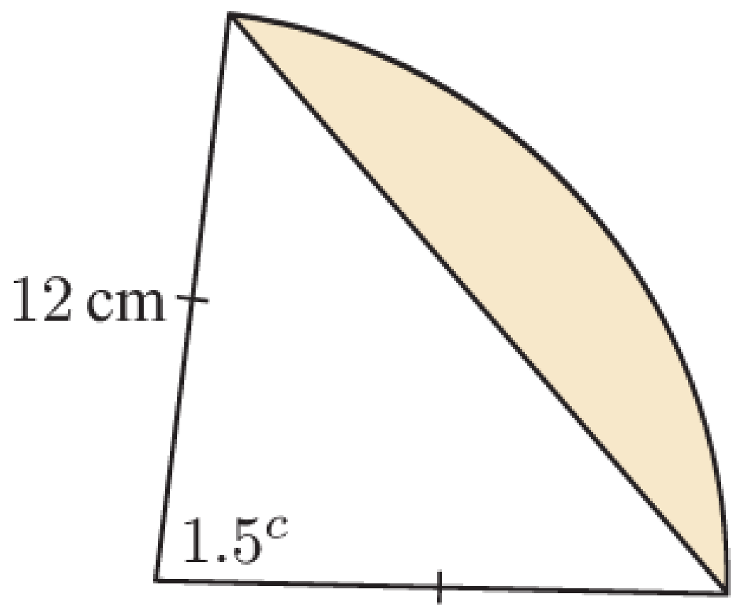
a sides of length 5 cm and 8 cm, and area 15 cm^2

b sides of length 45 km and 53 km, and area 800 km^2.

12 The Australian 50 cent coin has the shape of a regular dodecagon, which is a polygon with 12 sides.
Eight of these 50 cent coins will fit exactly on an Australian \$5 note as shown. What fraction of the \$5 note is *not* covered?

13 Find the shaded area:

a



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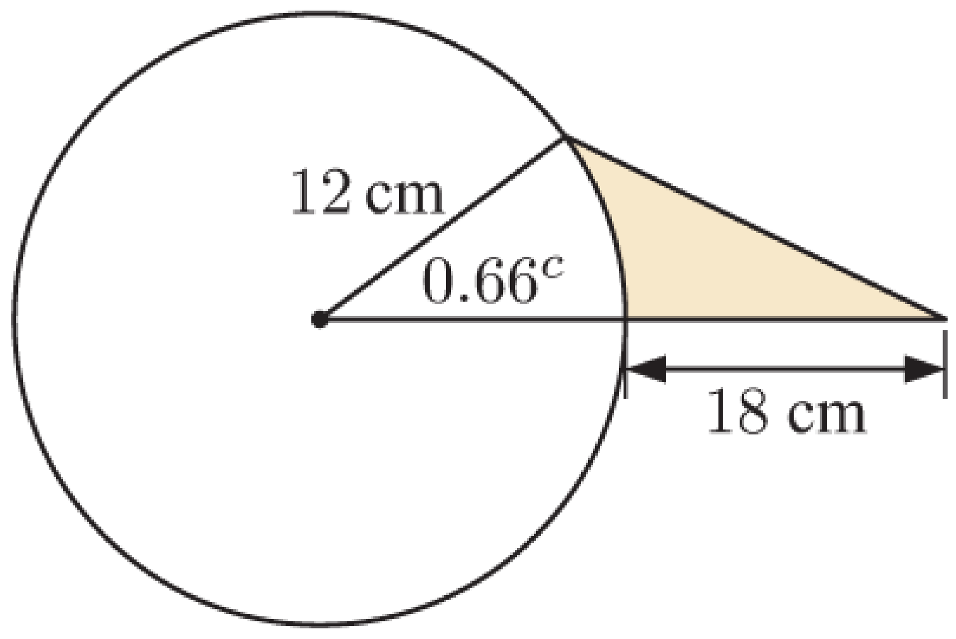

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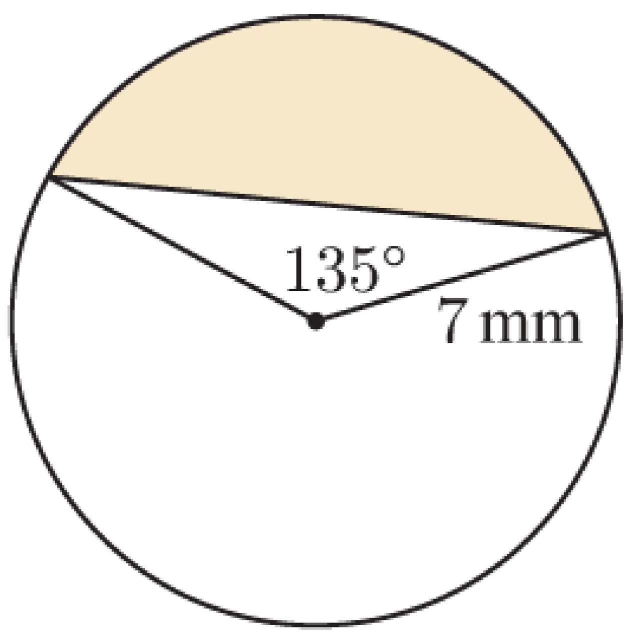

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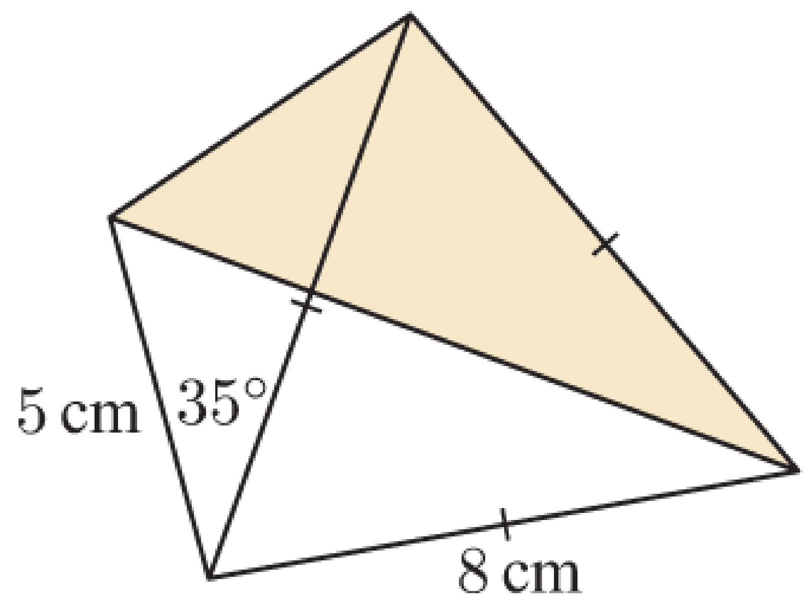
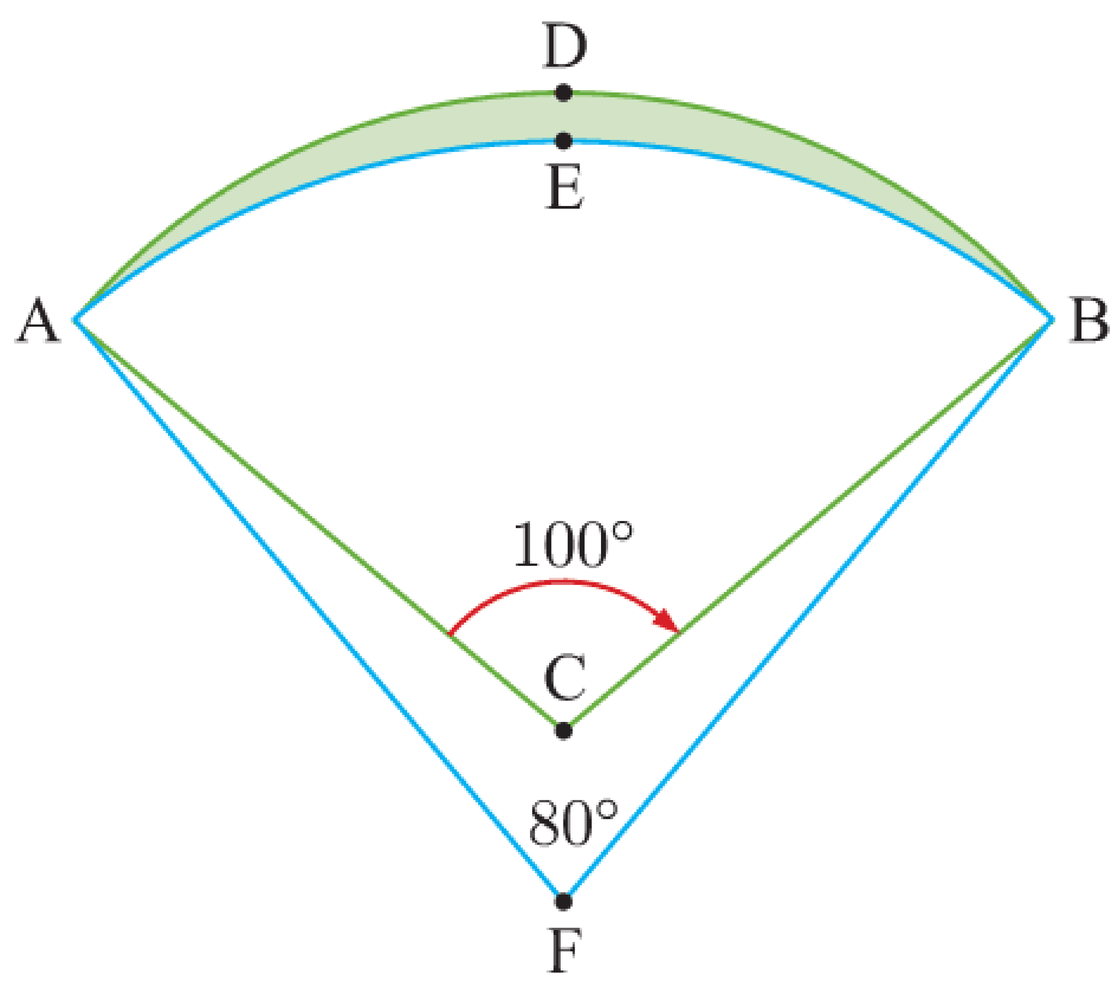
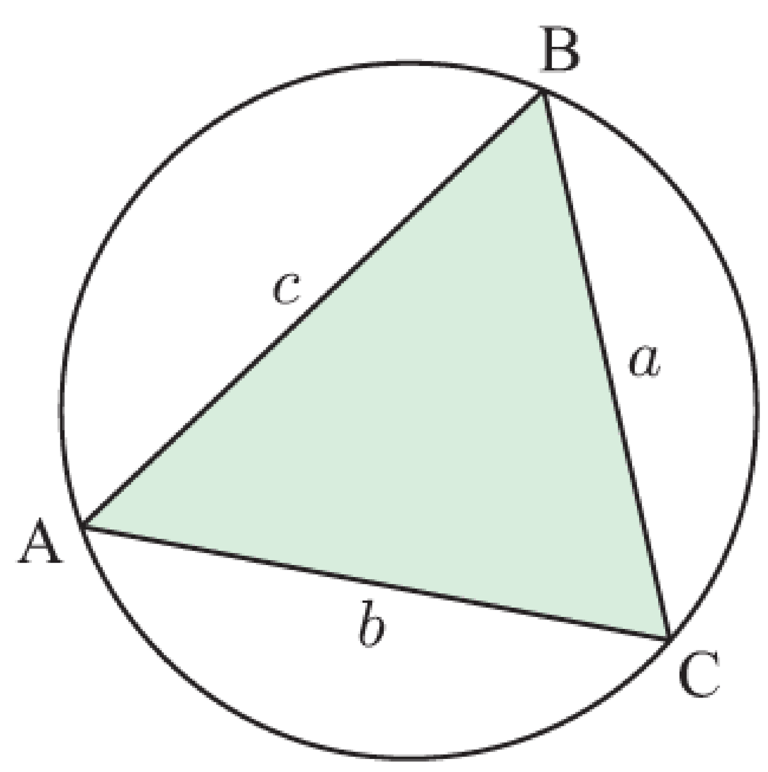


14 ADB is an arc of the circle with centre C and radius 7.3 cm. AEB is an arc of the circle with centre F and radius 8.7 cm.
Find the shaded area.



15 The acute angled triangle ABC has vertices on a circle with radius r.
Show that the area of the triangle is $\frac{abc}{4r}$.

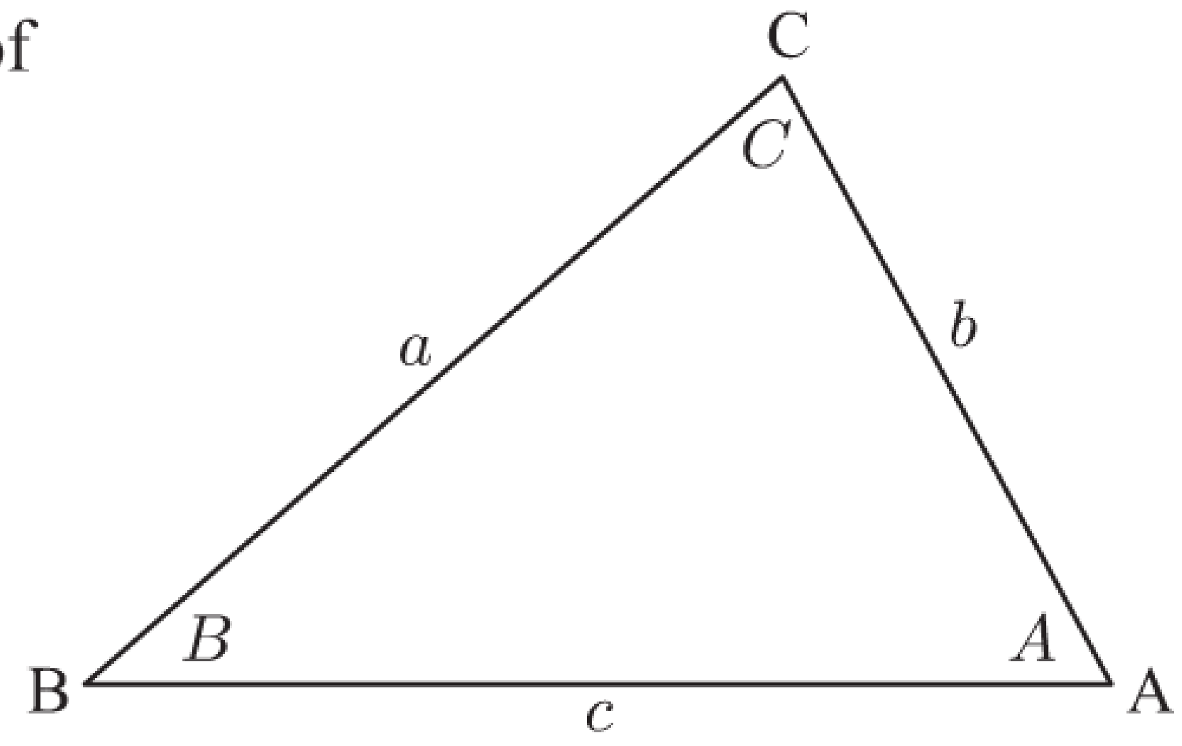


B THE COSINE RULE

The **cosine rule** relates the three sides of a triangle and one of the angles.

In any $\triangle$ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
or $$b^2 = a^2 + c^2 - 2ac\cos B$$
or $$c^2 = a^2 + b^2 - 2ab\cos C$$



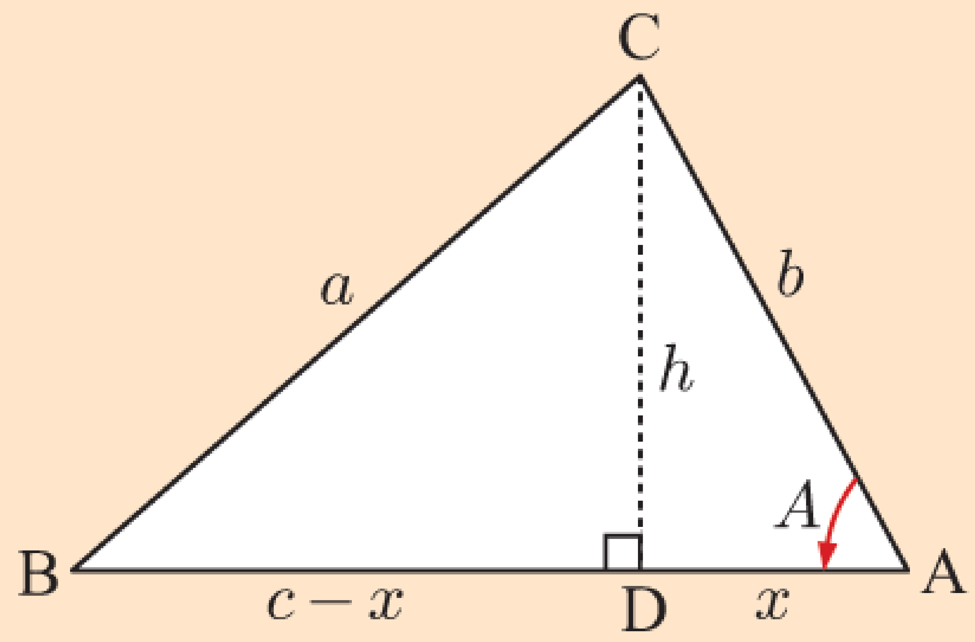
Proof: First, consider an **acute angled triangle** ABC.

Draw the altitude from C to [AB].

Let AD $= x$ and let CD $= h$.

Applying Pythagoras in $\triangle$BCD,

$$a^2 = h^2 + (c-x)^2$$
$$\therefore\ a^2 = h^2 + c^2 - 2cx + x^2$$



Applying Pythagoras in $\triangle$ADC gives $h^2 + x^2 = b^2$

$$\therefore\ h^2 = b^2 - x^2$$
$$\therefore\ a^2 = b^2 + c^2 - 2cx$$

In $\triangle$ADC, $\cos A = \dfrac{x}{b}$

$$\therefore\ x = b\cos A$$
$$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A$$

Now consider an **obtuse angled triangle** ABC with the obtuse angle at A.

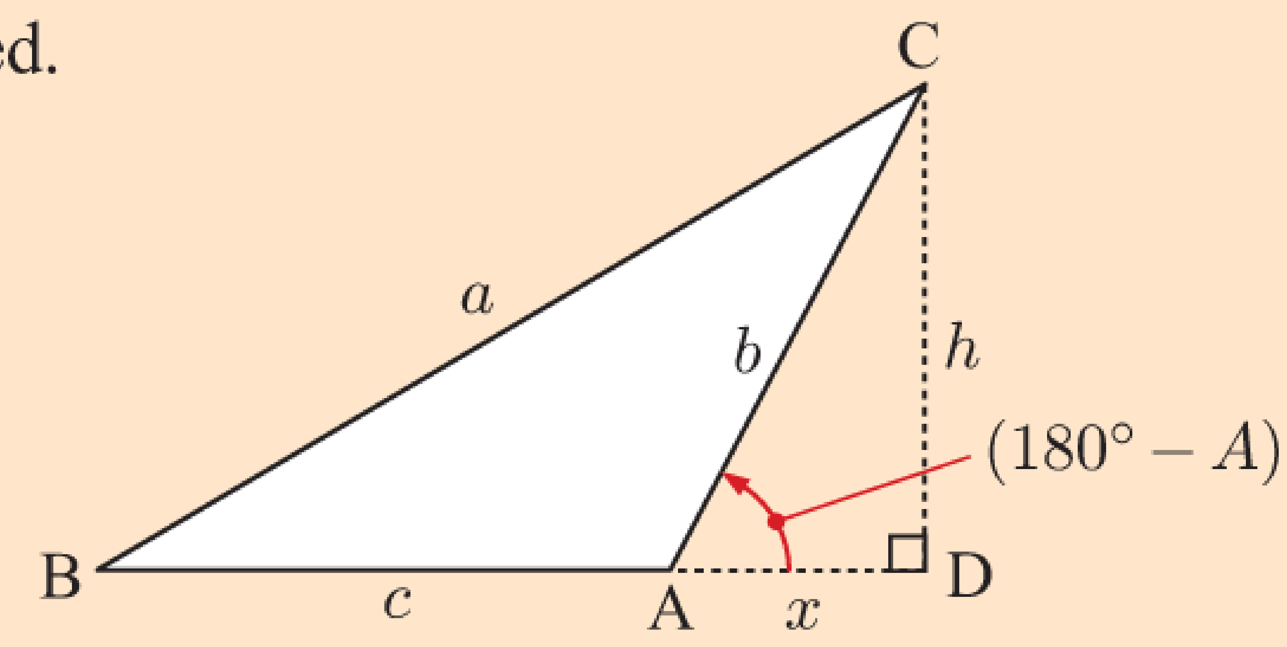
It still has two acute angles at B and C for which the proof above is valid.

Draw the altitude from C to [AB] extended.

Let AD $= x$ and let CD $= h$.

Applying Pythagoras in $\triangle$BCD,

$$a^2 = h^2 + (c+x)^2$$
$$\therefore\ a^2 = h^2 + c^2 + 2cx + x^2$$

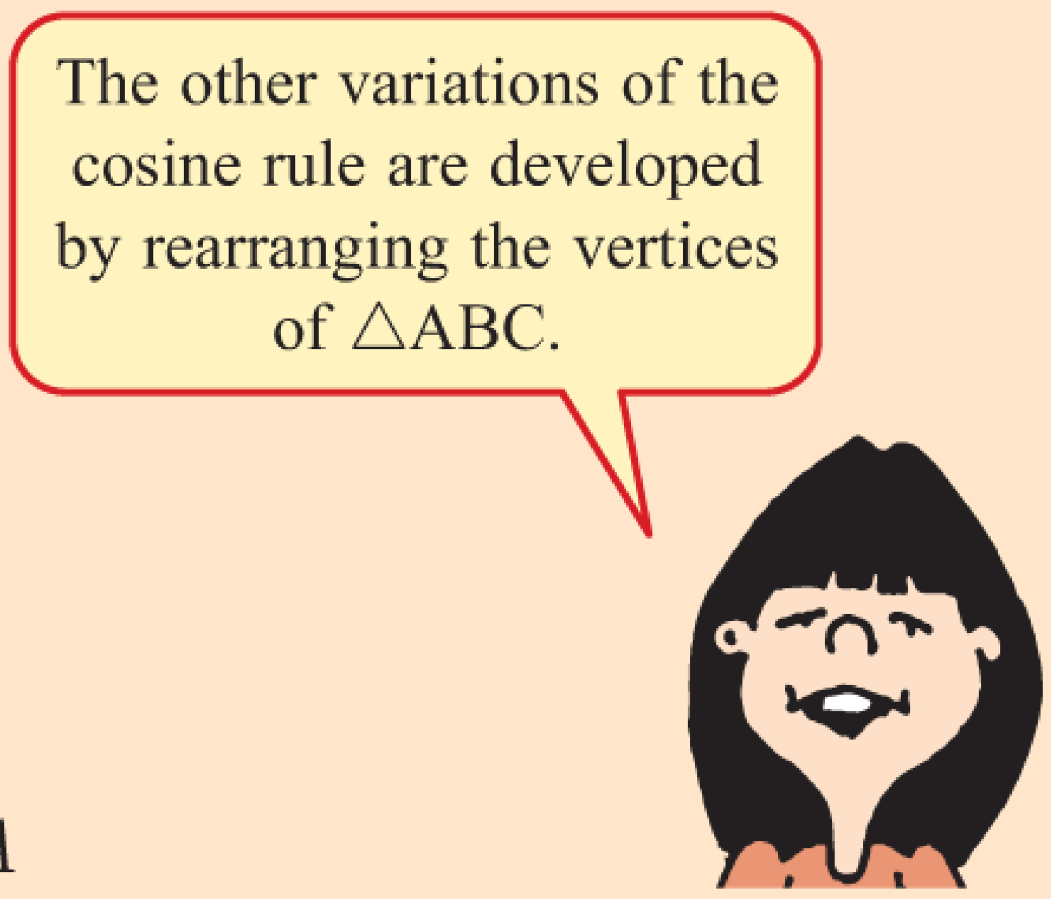


Applying Pythagoras in $\triangle$ADC gives

$$h^2 = b^2 - x^2$$
$$\therefore\ a^2 = b^2 + c^2 + 2cx$$

In $\triangle$ADC, $\cos(180° - A) = \dfrac{x}{b}$

$$\therefore\ -\cos A = \frac{x}{b}$$
$$\therefore\ x = -b\cos A$$
$$\therefore\ a^2 = b^2 + c^2 - 2bc\cos A$$



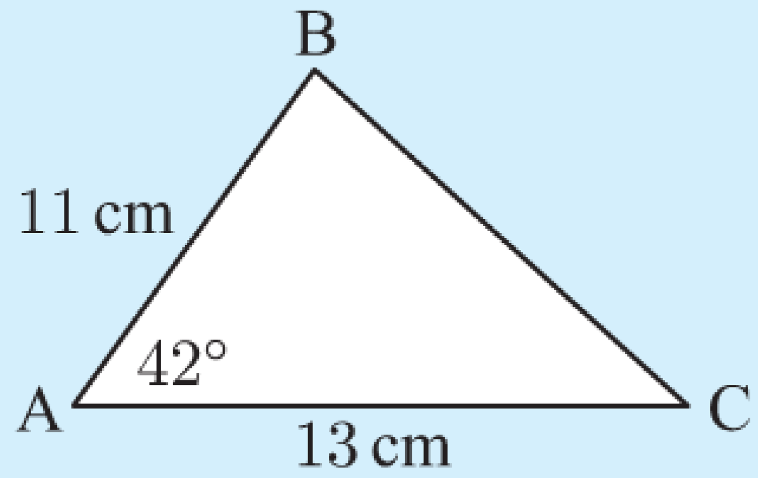
Note that if $A = 90°$ then $\cos A = 0$, and $a^2 = b^2 + c^2 - 2bc\cos A$ reduces to $a^2 = b^2 + c^2$, which is the Pythagorean Rule.

There are two situations in which the cosine rule can be used.

- If we are given **two sides** and an **included angle**, the cosine rule can be used to find the length of the third side.

Example 3 Self Tutor

Find, correct to 2 decimal places, the length of [BC].



By the cosine rule:

$BC^2 = 11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42°$

$\therefore$ $BC = \sqrt{(11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42°)}$

$\therefore$ $BC \approx 8.80$

$\therefore$ [BC] is about 8.80 cm in length.

- If we are given **all three sides** of a triangle, the cosine rule can be used to find any of the angles. To do this, we rearrange the original cosine rule formulae:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

We then use the inverse cosine function $\cos^{-1}$ to evaluate the angle.

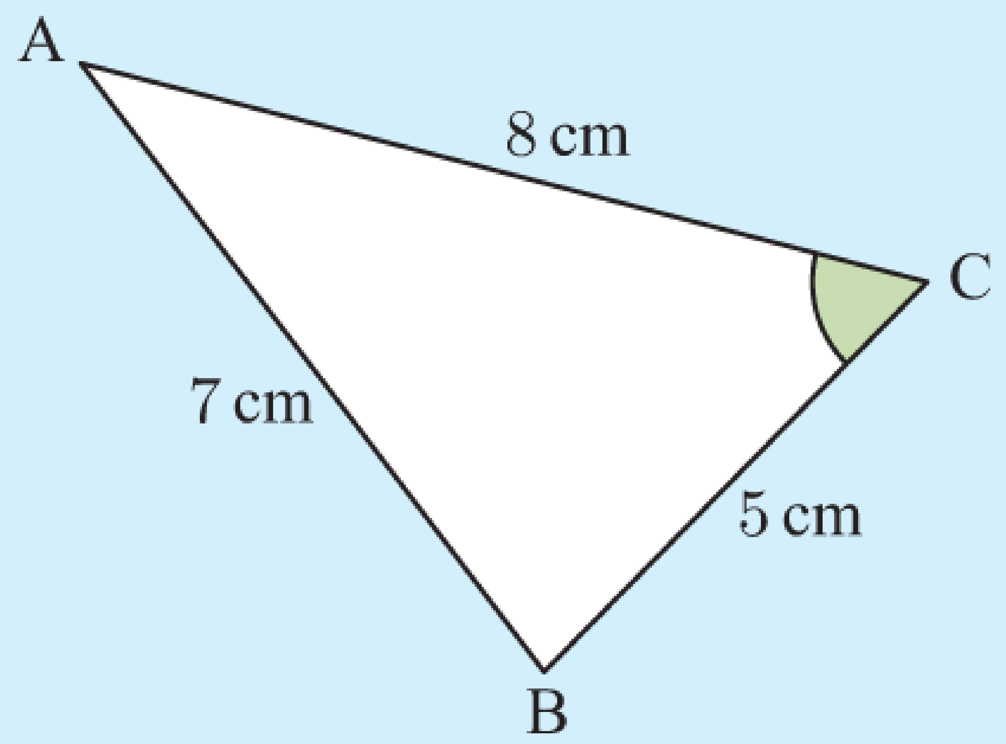
Example 4 Self Tutor

In triangle ABC, AB = 7 cm, BC = 5 cm, and CA = 8 cm.

a Find the measure of $B\widehat{C}A$.

b Find the area of triangle ABC.

a



By the cosine rule:

$\cos C = \dfrac{(5^2 + 8^2 - 7^2)}{(2 \times 5 \times 8)}$

$\therefore$ $C = \cos^{-1}\left(\dfrac{5^2 + 8^2 - 7^2}{2 \times 5 \times 8}\right)$

$\therefore$ $C = \cos^{-1}\left(\frac{1}{2}\right)$

$\therefore$ $C = 60°$ So, $B\widehat{C}A$ measures 60°.

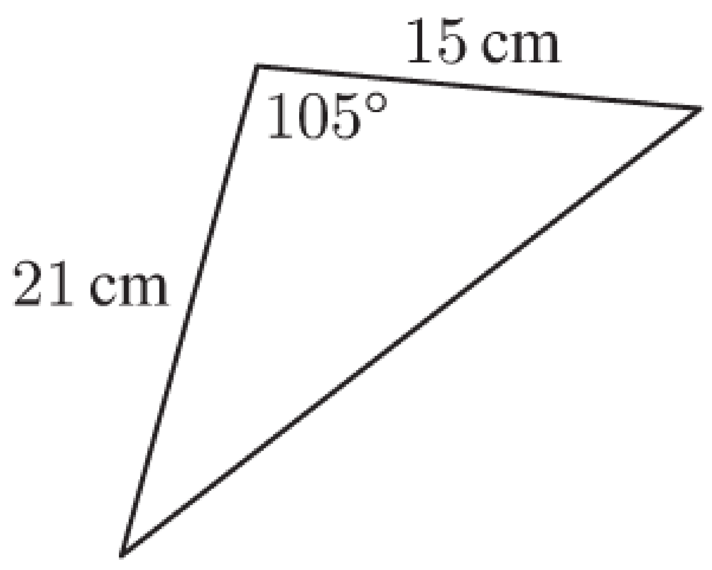
b The area of $\triangle ABC = \frac{1}{2} \times 8 \times 5 \times \sin 60°$

$\approx 17.3\ \text{cm}^2$

EXERCISE 9B

1 Find the length of the remaining side in each triangle:

a



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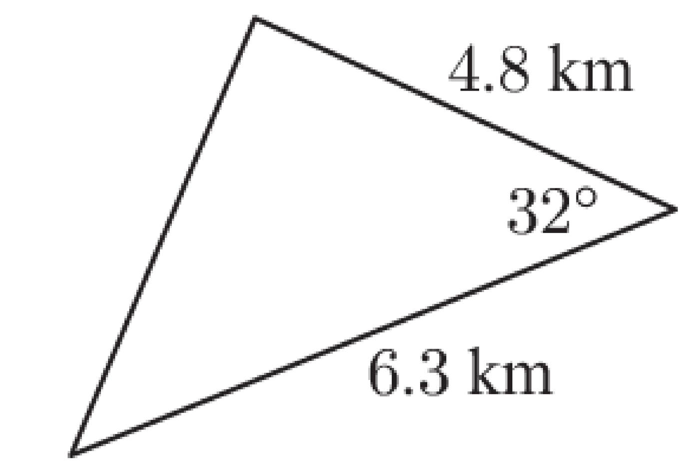
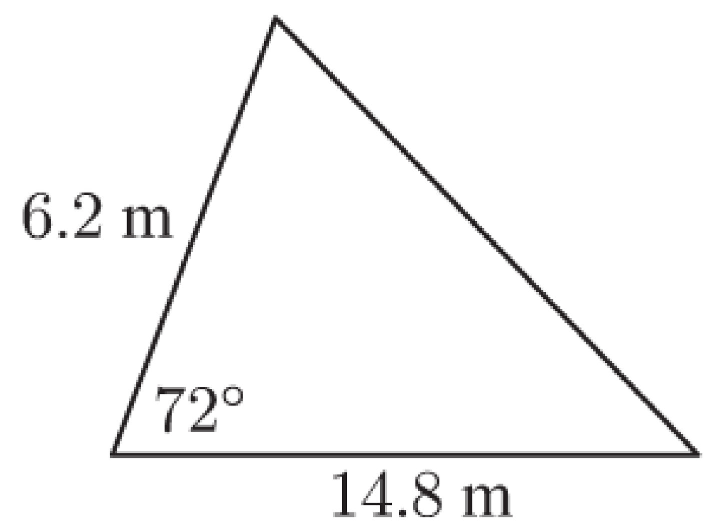
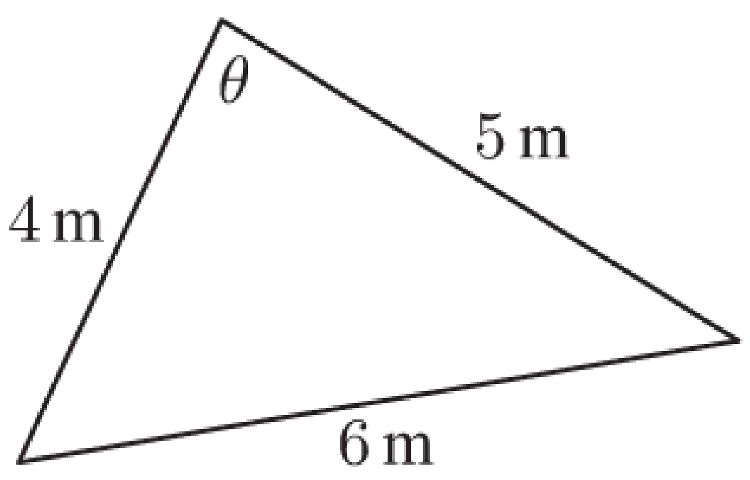


c

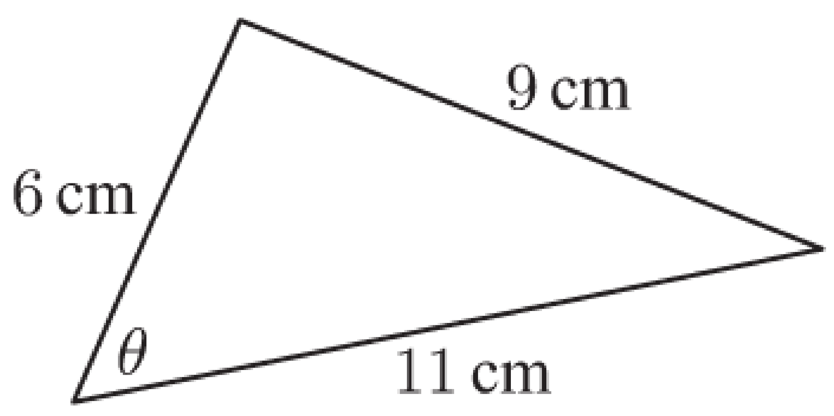


2 Find the measure of the angle marked θ:

a



b



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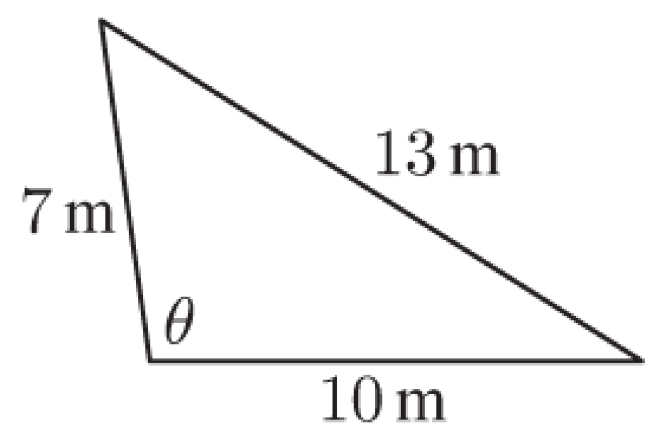
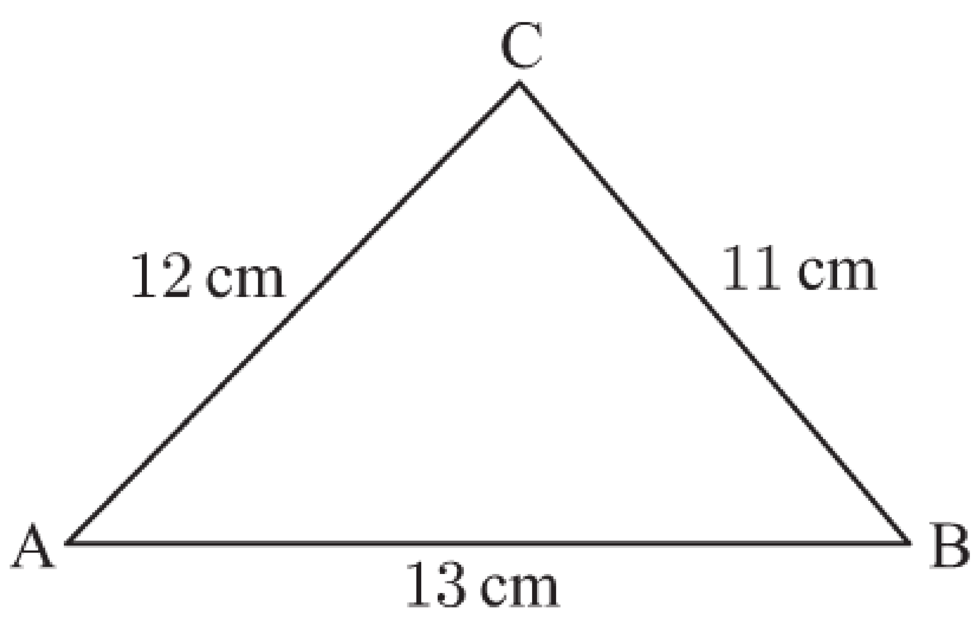
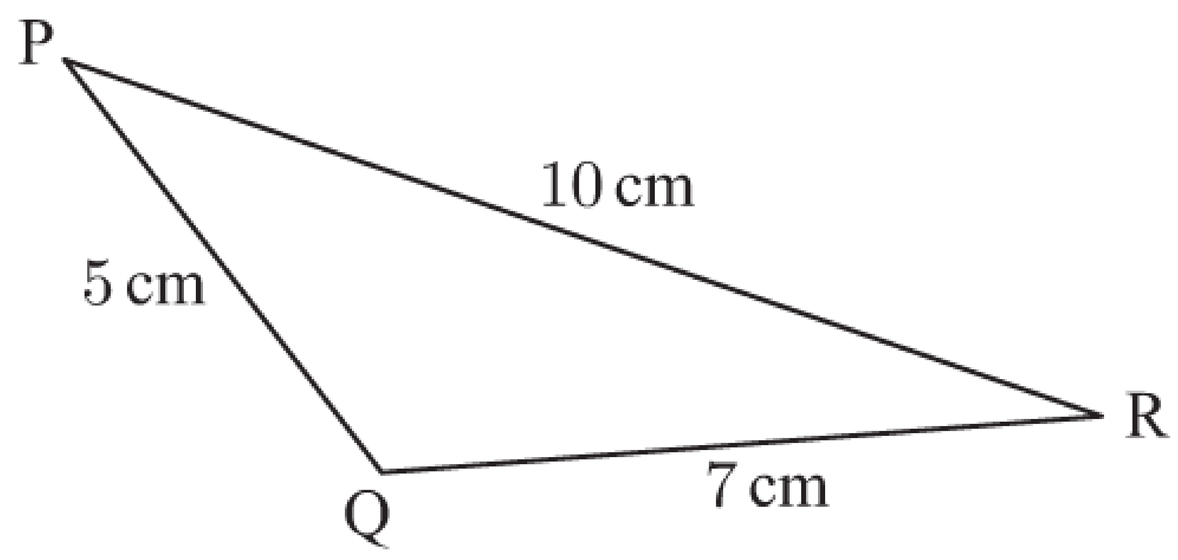


3 Find the measure of all angles of:



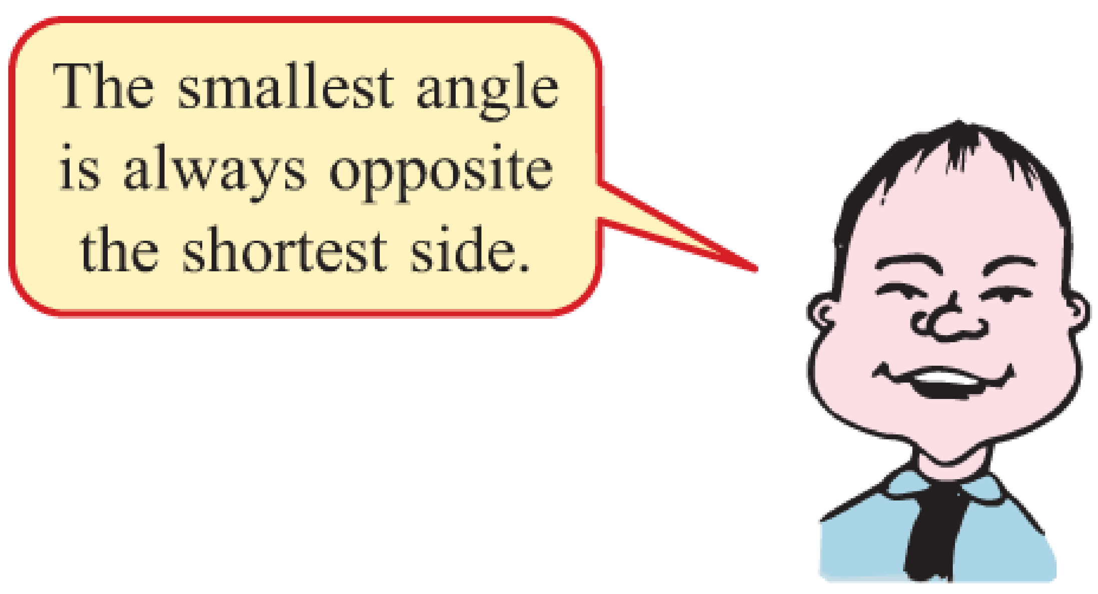
4 **a** Find the measure of obtuse $P\widehat{Q}R$.

b Hence find the area of $\triangle PQR$.



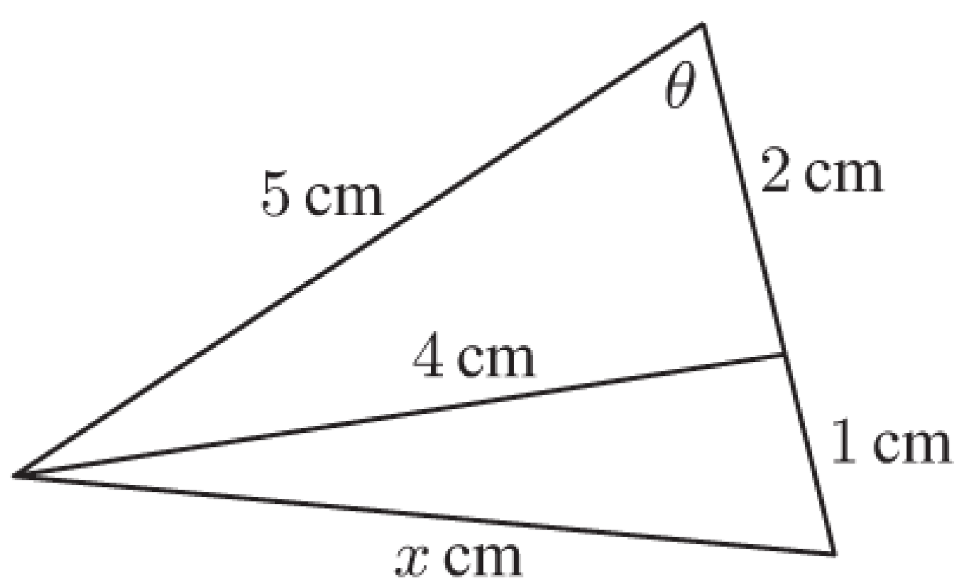
5 **a** Find the smallest angle of a triangle with sides 11 cm, 13 cm, and 17 cm.

b Find the largest angle of a triangle with sides 4 cm, 7 cm, and 9 cm.



6 **a** Find $\cos\theta$ but not θ.

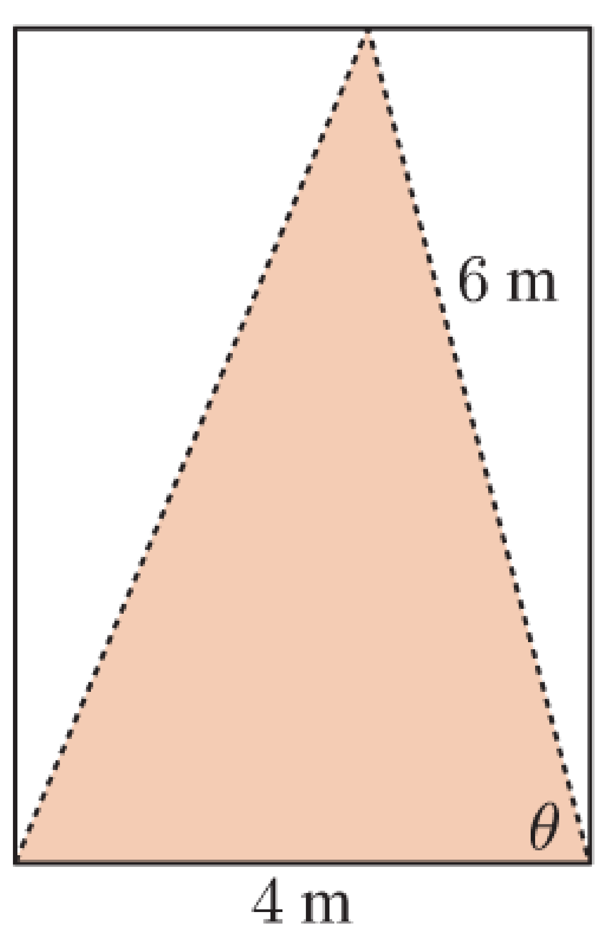
b Hence find the value of x.

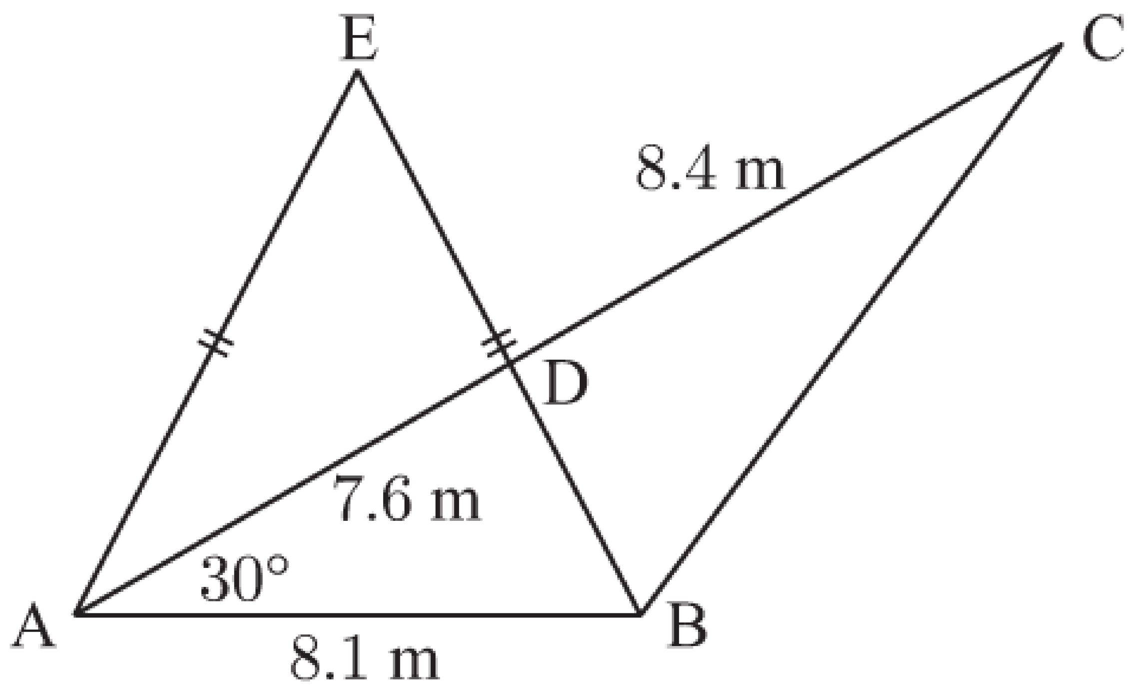


7 A triangular sail is to be cut from a section of cloth. Two of the sides must have lengths 4 m and 6 m as illustrated. The total area for the sail must be 11.6 m^2, the maximum allowed for the boat to race in its class.

a Find the angle θ between the two sides of given length.

b Find the length of the third side of the sail.



8 

Consider the figure shown.

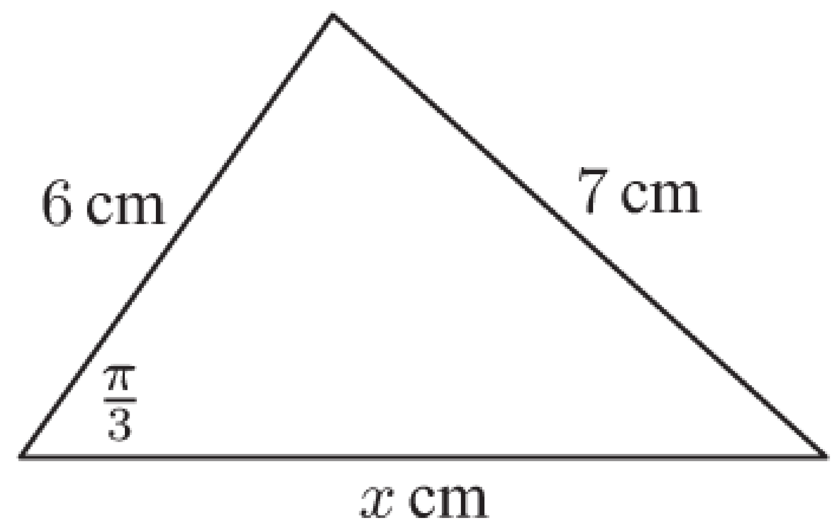
a Find the lengths of [DB] and [BC].

b Calculate the measures of $A\widehat{B}E$ and $D\widehat{B}C$.

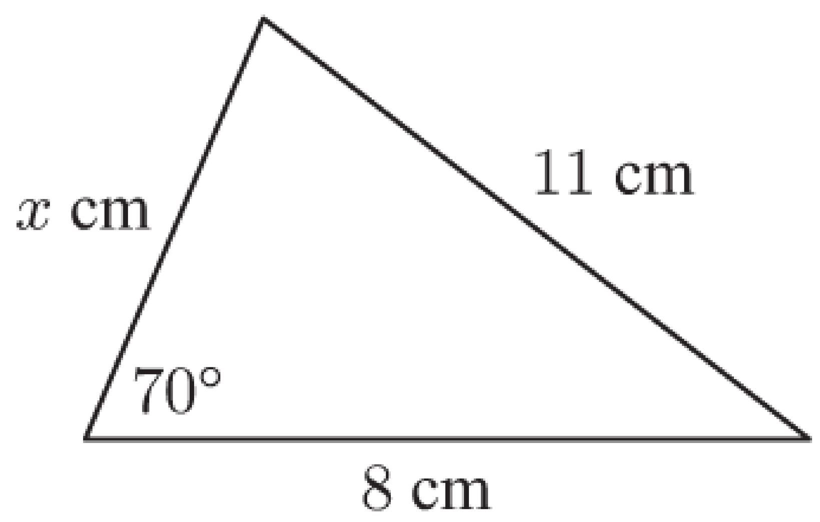
c Find the area of $\triangle$BCD.

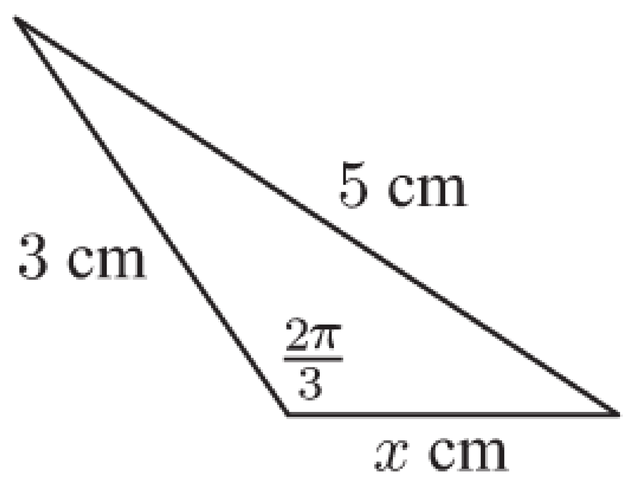
9 **a** Show that $x^2 - 6x - 13 = 0$.

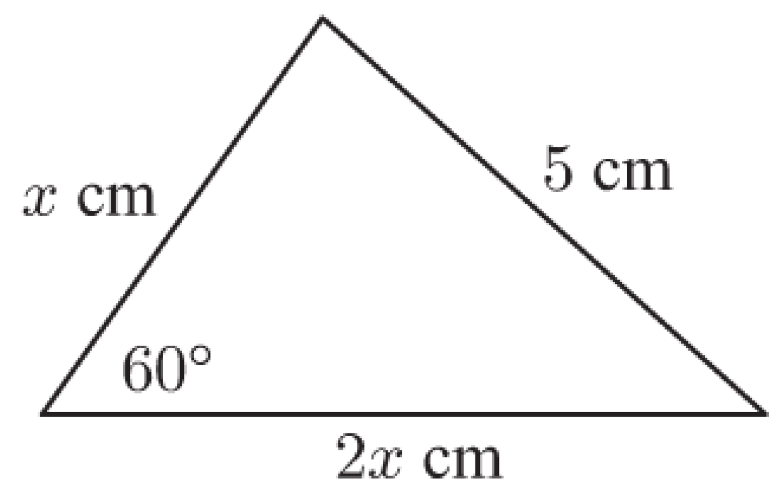
b Hence find the exact value of x.



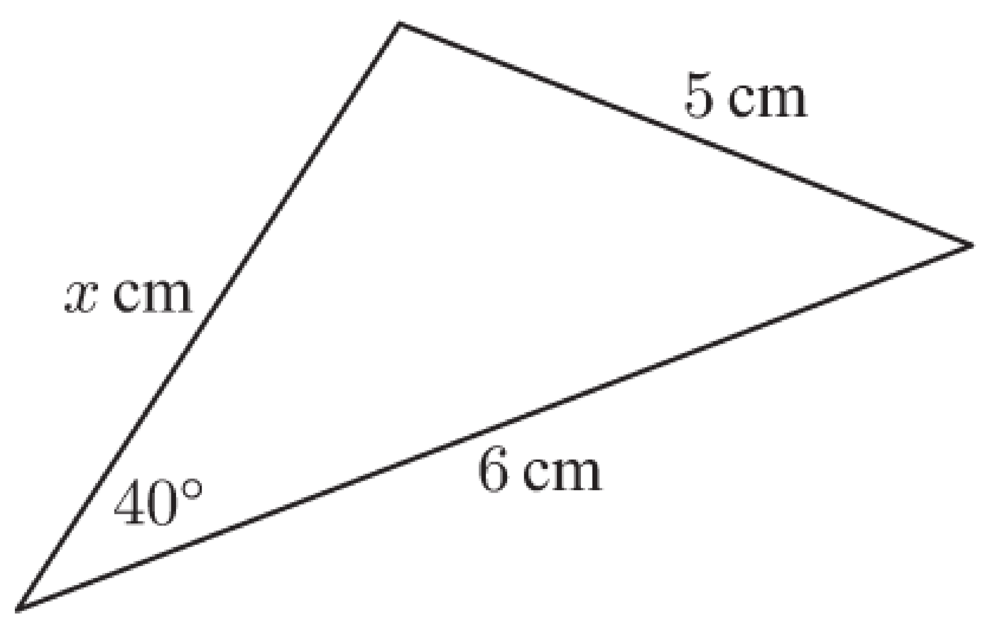
10 Find the value of x:

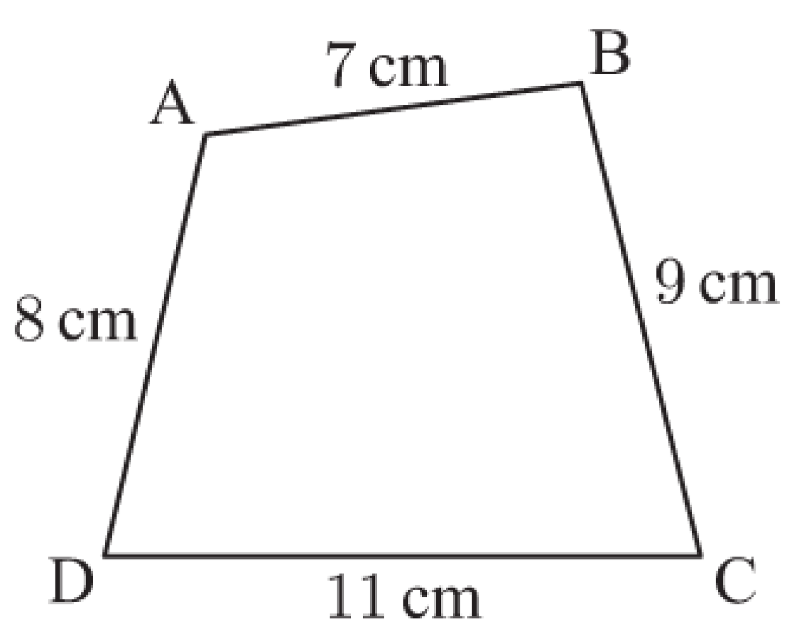
a 

b 

c 

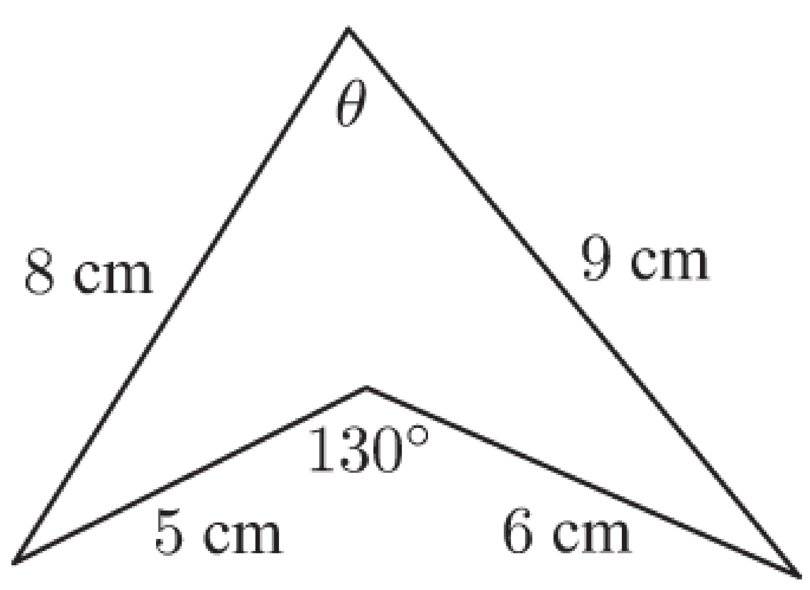
11 Find the possible values for x.



12 

In quadrilateral ABCD, the diagonal [AC] has length 12 cm. Find the length of the other diagonal [BD].

13 Find the angle θ:

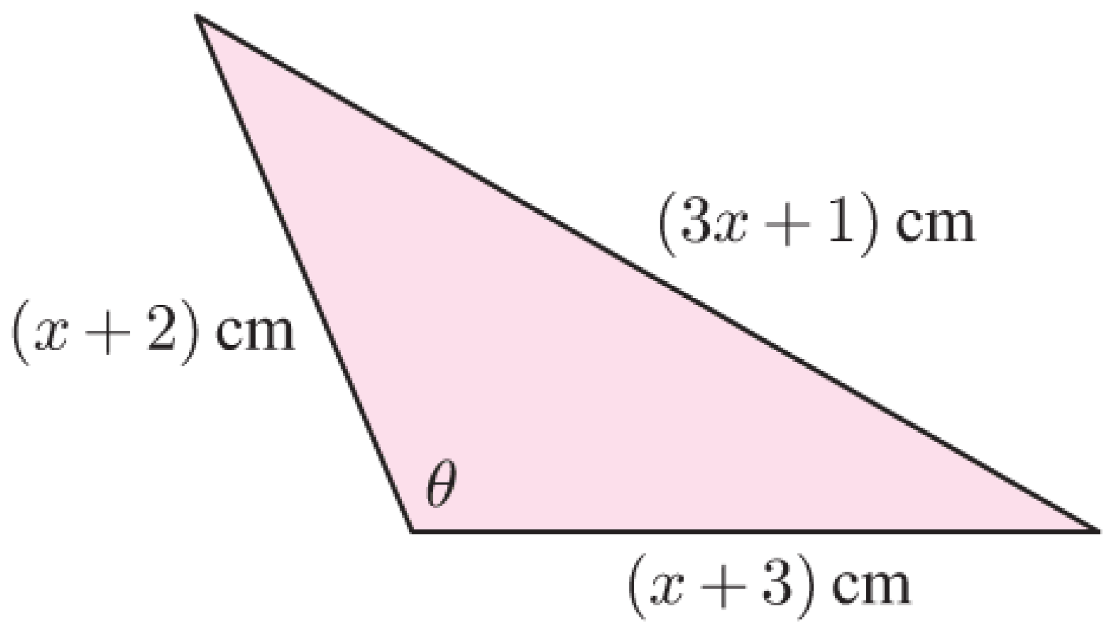


14 ABC is an equilateral triangle with sides 10 cm long. P is a point within the triangle which is 5 cm from A and 6 cm from B. How far is P from C?

15 In the diagram alongside, $\cos\theta = -\frac{1}{5}$.

a Find x.

b Hence find the exact area of the triangle.



16 The parallel sides of a trapezium have lengths 5 cm and 8 cm. The other two sides have lengths 6 cm and 4 cm. Find the angles of the trapezium, to the nearest degree.

C THE SINE RULE

The **sine rule** is a set of equations which connects the lengths of the sides of any triangle with the sines of the angles of the triangle. The triangle does not have to be right angled for the sine rule to be used.

INVESTIGATION 1 THE SINE RULE

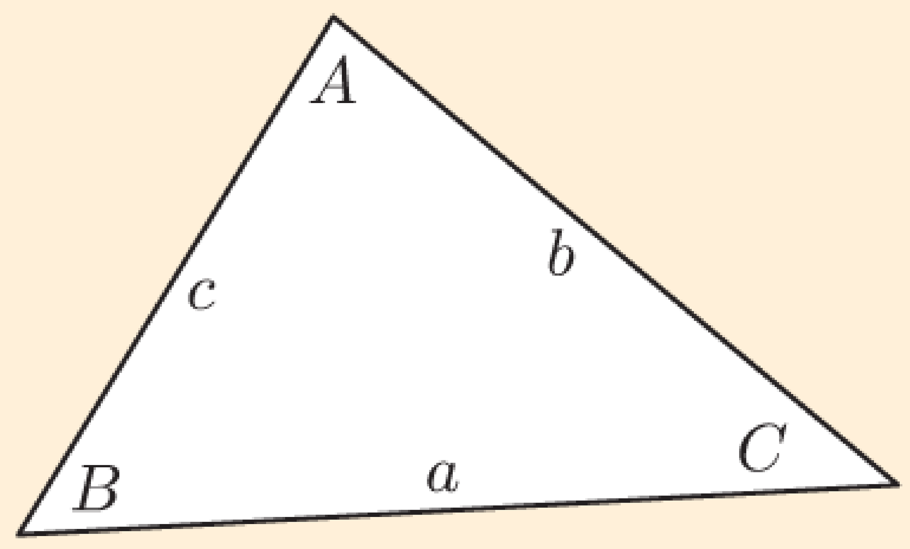
You will need: Paper, scissors, ruler, protractor

What to do:

1 Cut out a large triangle. Label the sides a, b, and c, and the opposite angles A, B, and C.

2 Use your ruler to measure the length of each side.

3 Use your protractor to measure the size of each angle.



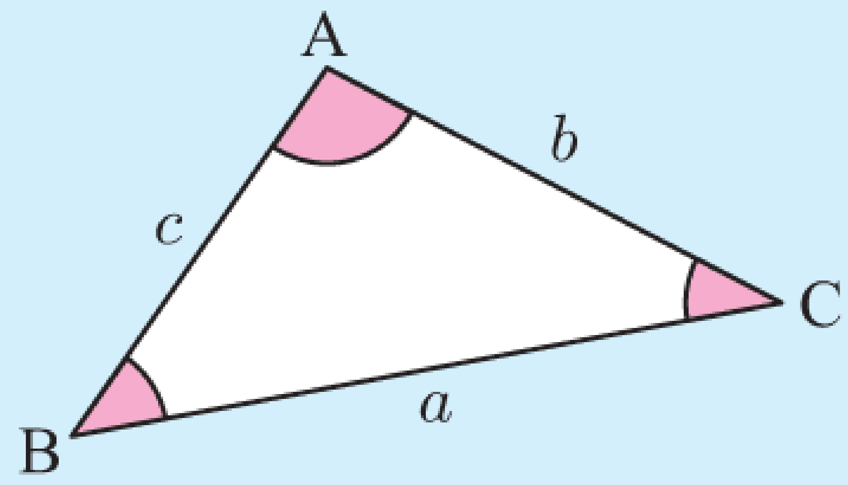
4 Copy and complete this table:

a	b	c	A	B	C	$\frac{\sin A}{a}$	$\frac{\sin B}{b}$	$\frac{\sin C}{c}$

5 Comment on your results.

In any triangle ABC with sides a, b, and c units in length, and opposite angles A, B, and C respectively,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad or \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



Proof: The area of any triangle ABC is given by $\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$.

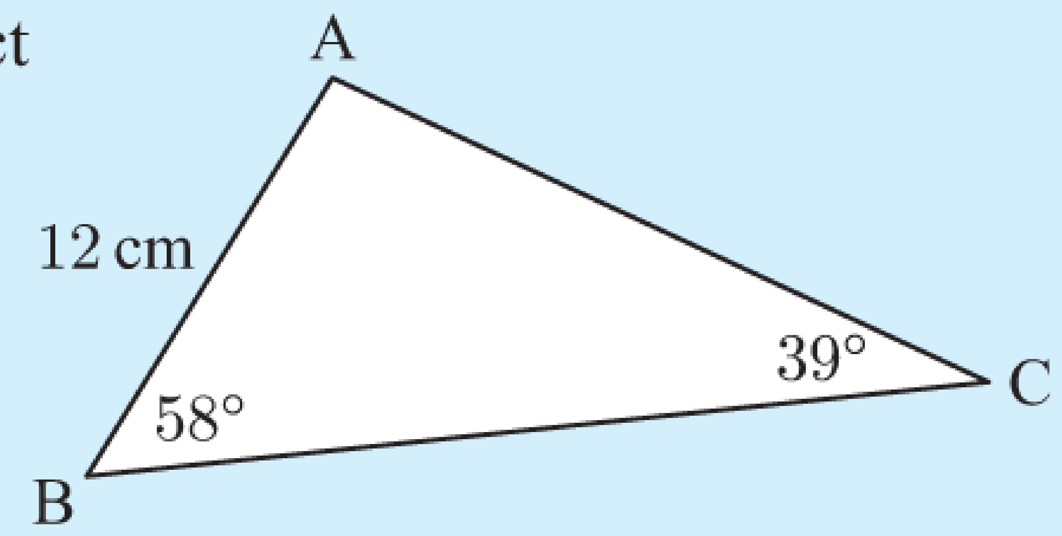
Dividing each expression by $\frac{1}{2}abc$ gives $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

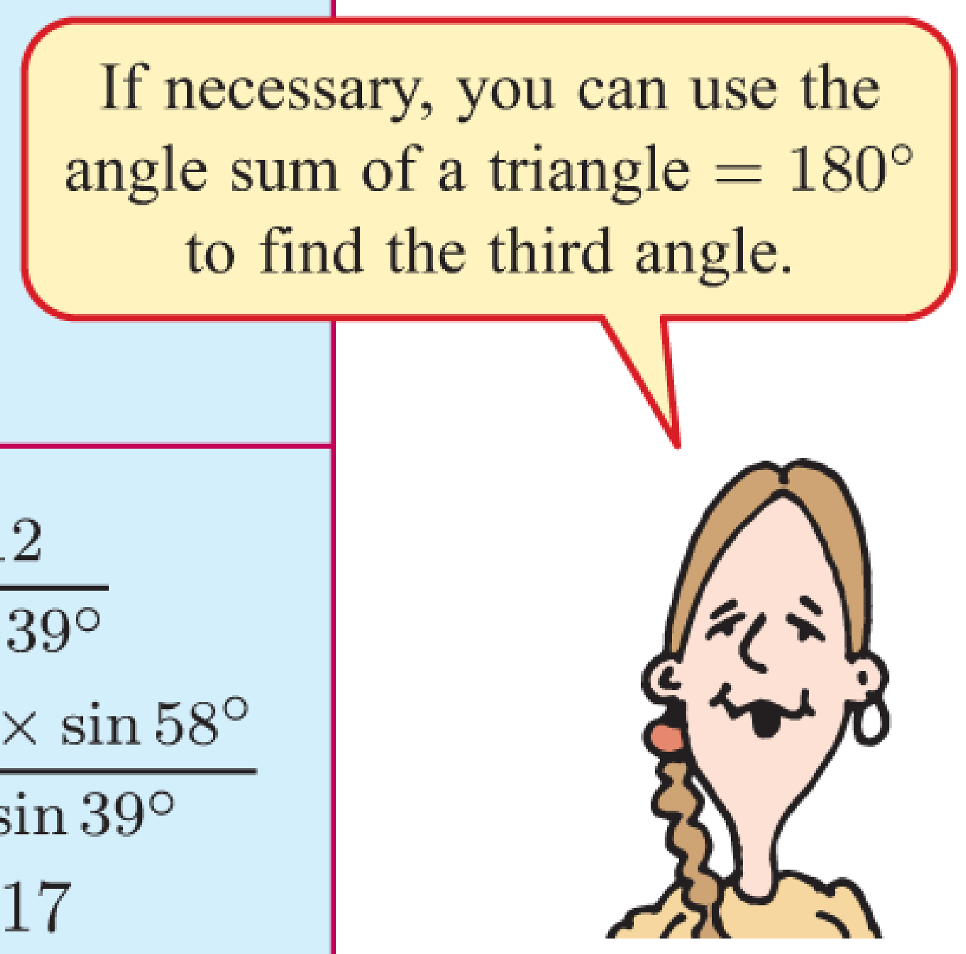
FINDING SIDE LENGTHS

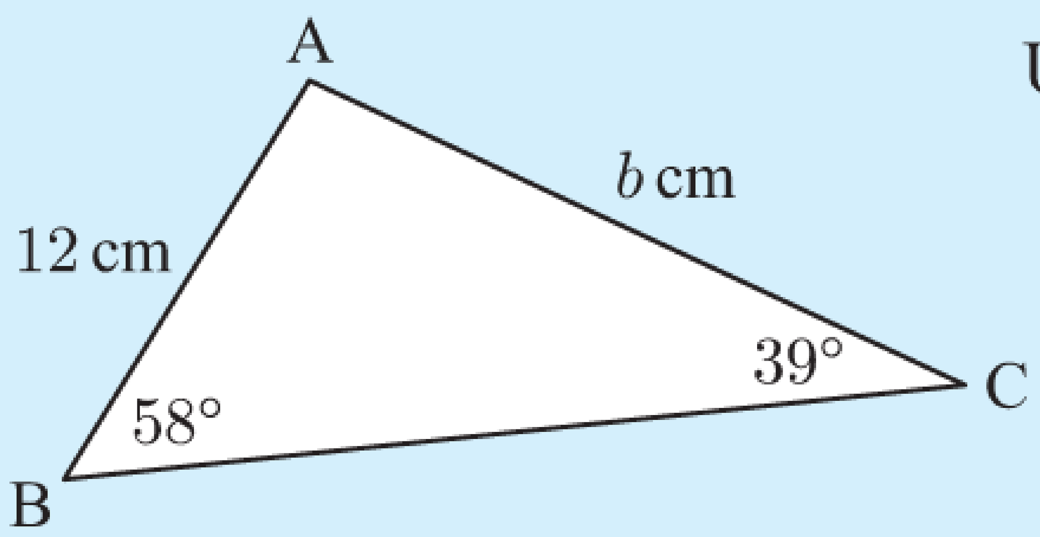
If we are given two angles and one side of a triangle we can use the sine rule to find another side length.

Example 5 **Self Tutor**

Find the length of [AC] correct to 2 decimal places.







Using the sine rule, $\dfrac{b}{\sin 58°} = \dfrac{12}{\sin 39°}$

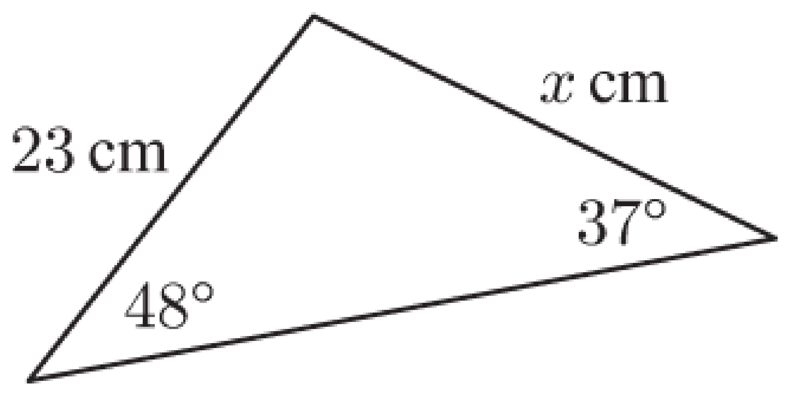
$\therefore \quad b = \dfrac{12 \times \sin 58°}{\sin 39°}$

$\therefore \quad b \approx 16.17$

$\therefore$ [AC] is about 16.17 cm long.

EXERCISE 9C.1

1 Find the value of x:

a


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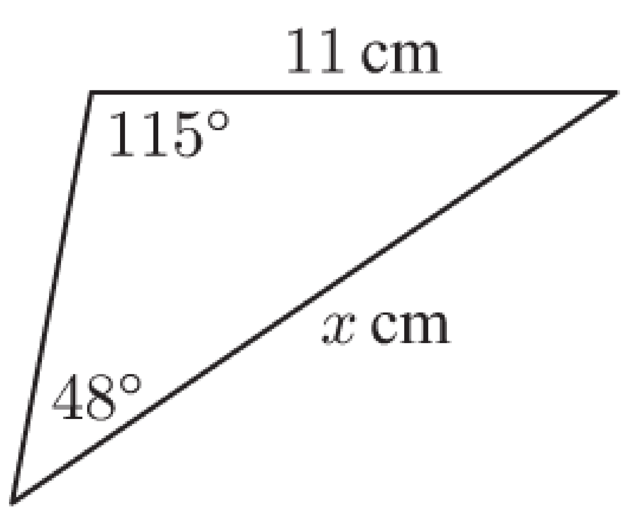

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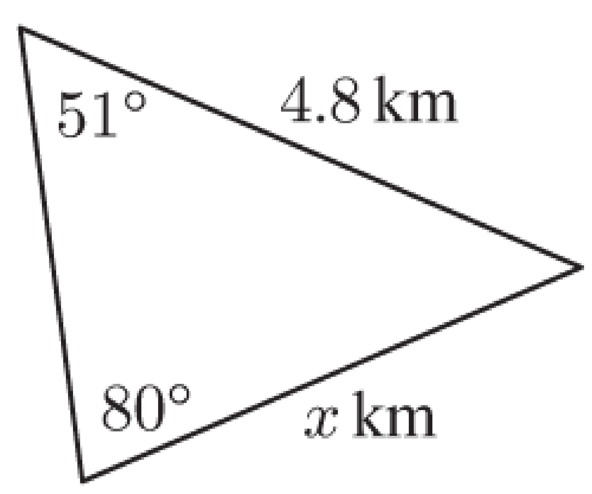
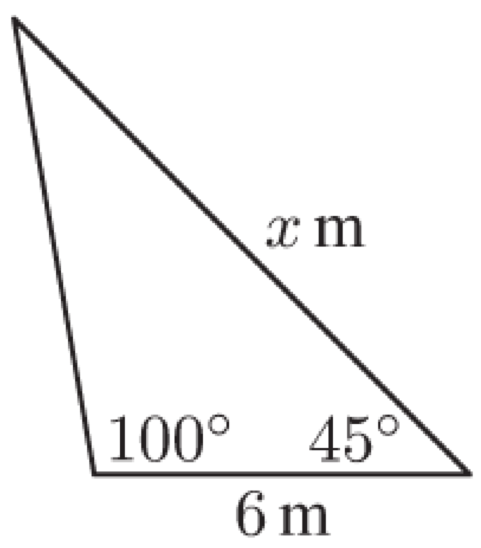
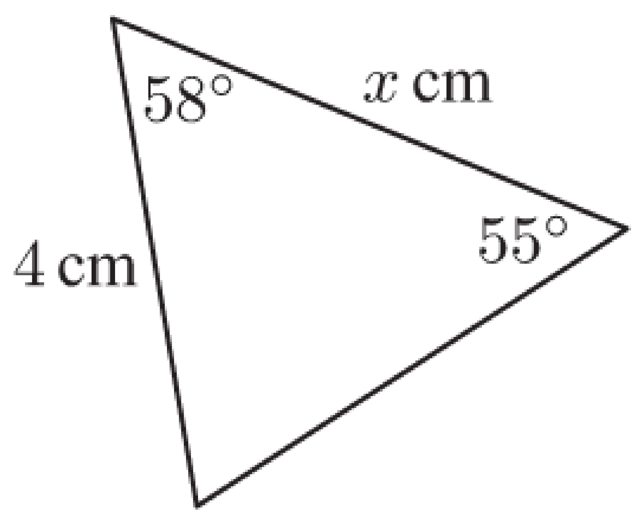
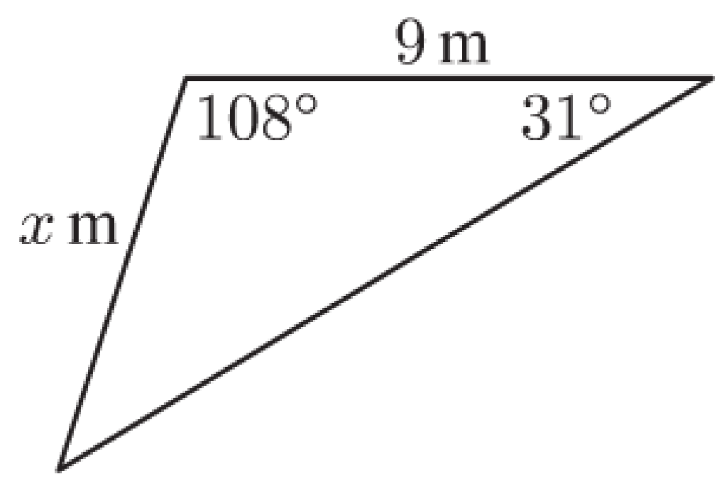


d


e


f


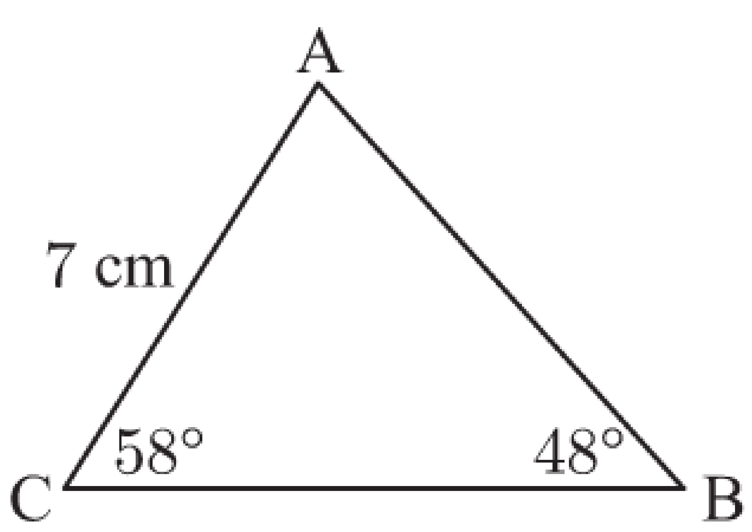
2 Consider triangle ABC.

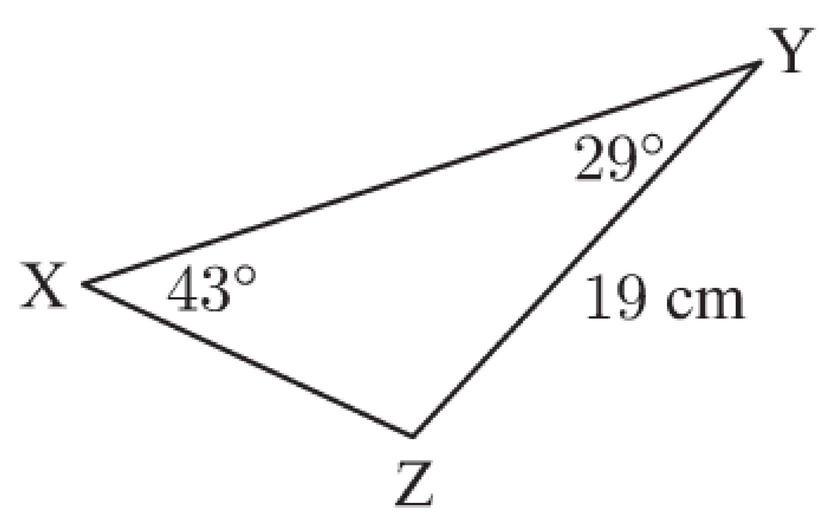
a Given $A = 63°$, $B = 49°$, and $b = 18$ cm, find a.

b Given $A = 82°$, $C = 25°$, and $c = 34$ cm, find b.

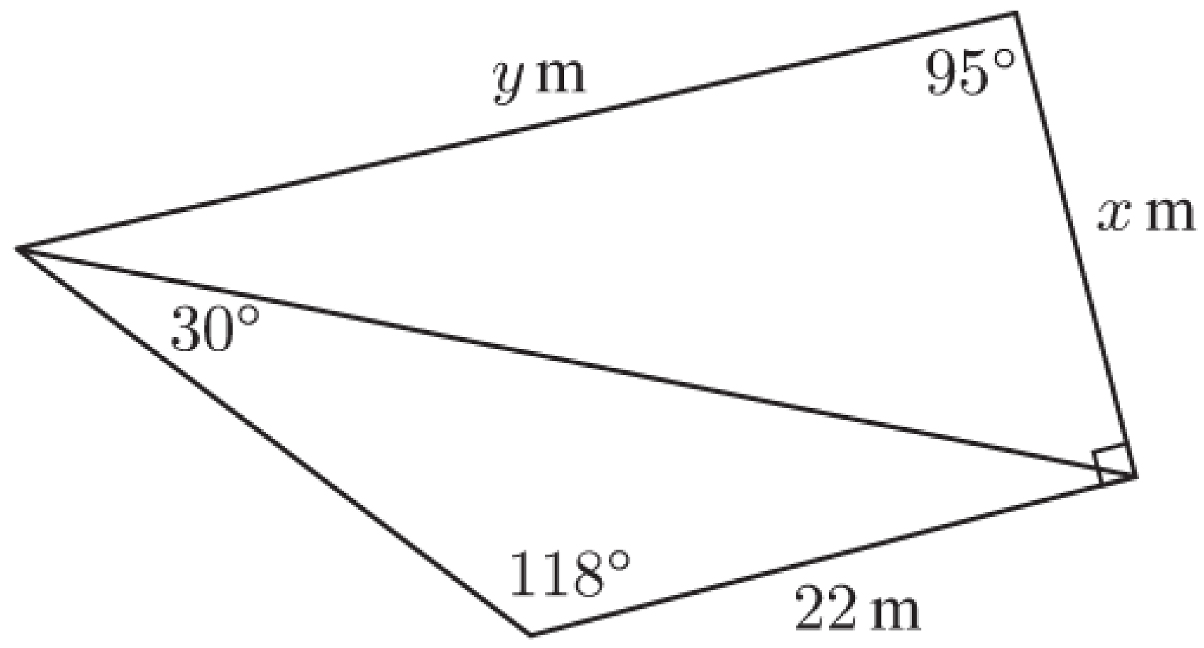
c Given $B = 21°$, $C = 48°$, and $a = 6.4$ cm, find c.

3 Find *all* unknown sides and angles of:

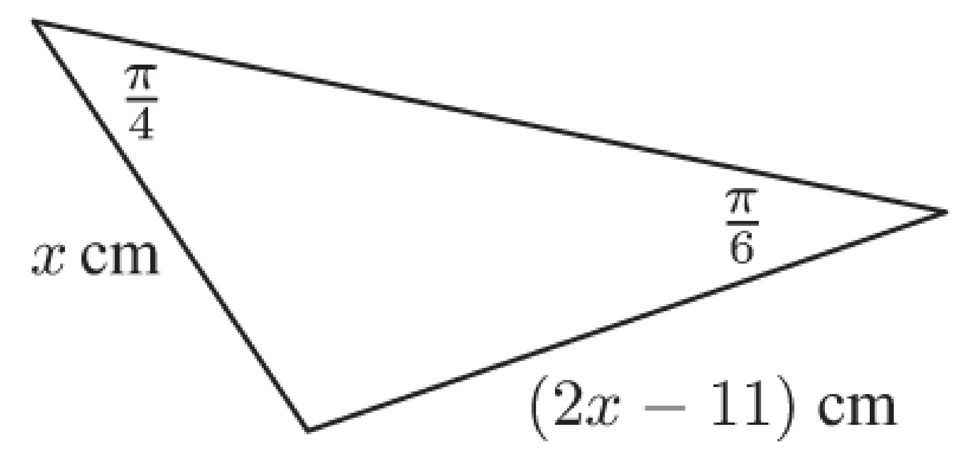
a


b


4 Find x and y in the given figure.



5 Find the exact value of x, giving your answer in the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$.



FINDING ANGLES

Finding angles using the sine rule is complicated because there may be two possible answers. For example, if $\sin\theta = \frac{1}{2}$ then θ could be 30° or 150°. We call this situation an **ambiguous case**.

You can click on the icon to obtain an interactive demonstration of the ambiguous case, or else you can work through the following **Investigation**.

DEMO

INVESTIGATION 2 THE AMBIGUOUS CASE

You will need a blank sheet of paper, a ruler, a protractor, and a compass for the tasks that follow. In each task you will be required to construct triangles from given information.

What to do:

1 Draw AB = 10 cm. Construct an angle of 30° at point A. Using B as the centre, draw an arc of a circle with radius 6 cm. Let C denote the point where the arc intersects the ray from A. How many different possible points C are there, and therefore how many different triangles ABC may be constructed?

2 Repeat the procedure from **1** three times, starting with AB = 10 cm and constructing an angle of 30° at point A. When you draw the arc with centre B, use the radius:

a 5 cm **b** 3 cm **c** 12 cm

3 Using your results from **1** and **2**, discuss the possible number of triangles you can obtain given two sides and a non-included angle.

You should have discovered that when you are given two sides and a non-included angle, you could get two triangles, one triangle, or it may be impossible to draw any triangles at all.

Now consider the calculations involved in each of the cases in the **Investigation**.

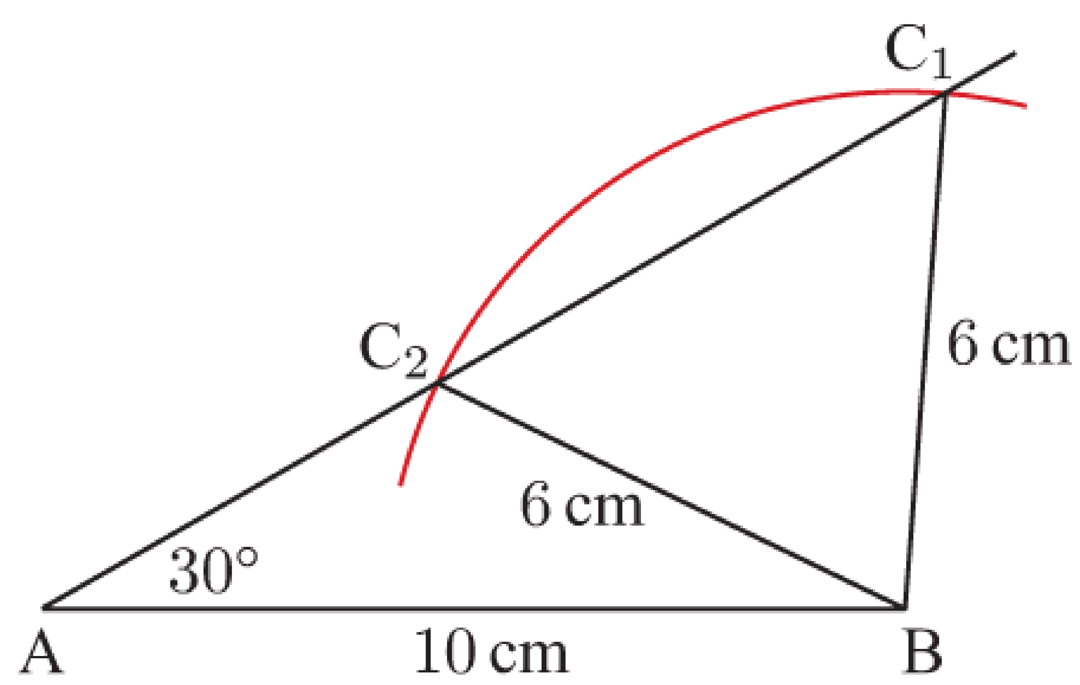
Case 1: Given: $c = 10$ cm, $a = 6$ cm, $A = 30°$

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

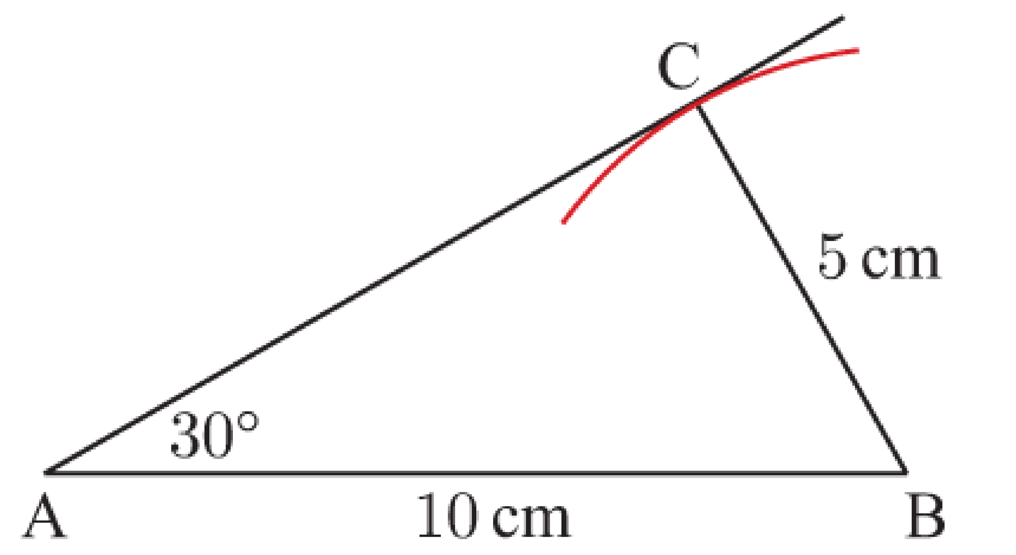
$$\therefore \sin C = \frac{c \sin A}{a}$$

$$\therefore \sin C = \frac{10 \times \sin 30°}{6} \approx 0.8333$$

$$\therefore C \approx 56.44° \text{ or } 180° - 56.44° = 123.56°$$



Case 2: Given: $c = 10$ cm, $a = 5$ cm, $A = 30°$



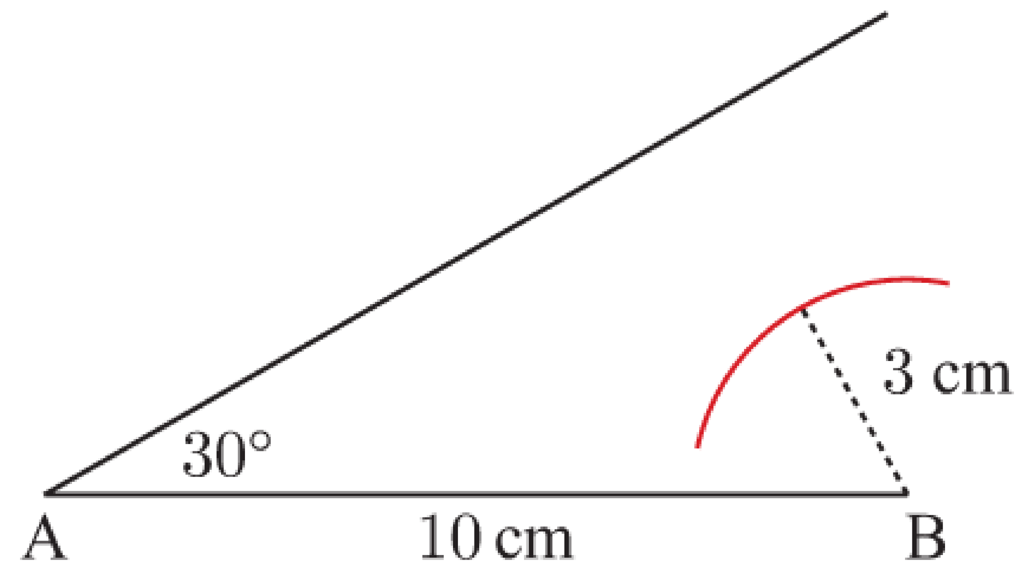
$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \sin C = \frac{c \sin A}{a}$$

$$\therefore \sin C = \frac{10 \times \sin 30°}{5} = 1$$

There is only one possible solution for C in the range from $0°$ to $180°$, and that is $C = 90°$. Only one triangle is therefore possible. Complete the solution of the triangle yourself.

Case 3: Given: $c = 10$ cm, $a = 3$ cm, $A = 30°$



$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \sin C = \frac{c \sin A}{a}$$

$$\therefore \sin C = \frac{10 \times \sin 30°}{3} \approx 1.6667$$

There is no angle that has a sine value > 1, so no triangles can be drawn to match the information given.

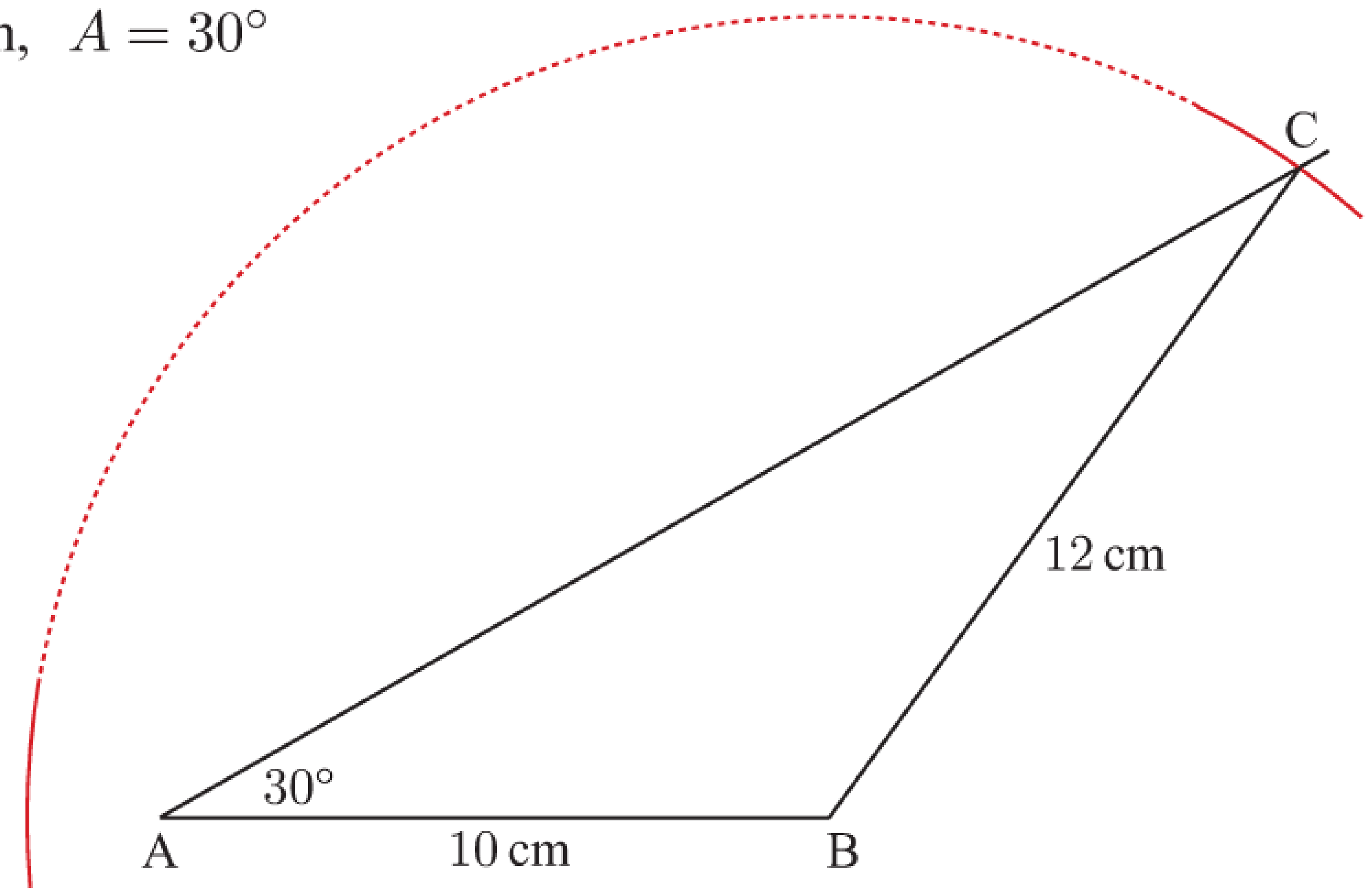
Case 4: Given: $c = 10$ cm, $a = 12$ cm, $A = 30°$

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \sin C = \frac{c \sin A}{a}$$

$$\therefore \sin C = \frac{10 \times \sin 30°}{12} \approx 0.4167$$

$\therefore C \approx 24.62°$ *or*
$180° - 24.62° = 155.38°$



However, in this case only one of these two angles is valid. Since $A = 30°$, C cannot possibly equal $155.38°$ because $30° + 155.38° > 180°$.

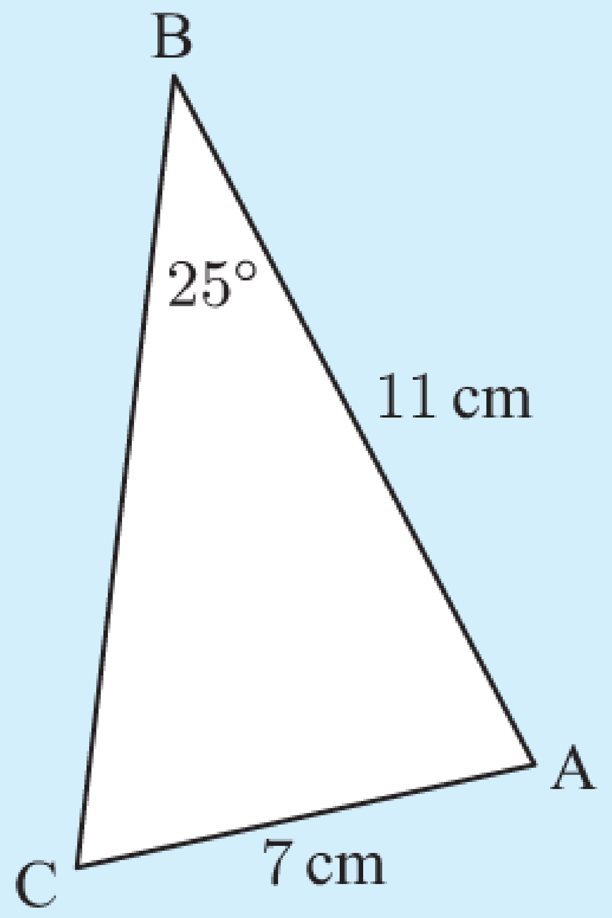
Therefore, there is only one possible solution, $C \approx 24.62°$.

Conclusion: Each situation using the sine rule with two sides and a non-included angle must be examined very carefully.

In particular, if the given angle is acute and opposite the shorter of the two given sides, then two different triangles are possible.

Example 6 Self Tutor

Find the measure of angle C in triangle ABC if AC = 7 cm, AB = 11 cm, and angle B measures 25°.



$$\frac{\sin C}{c} = \frac{\sin B}{b} \quad \{\text{sine rule}\}$$

$$\therefore \frac{\sin C}{11} = \frac{\sin 25^\circ}{7}$$

$$\therefore \sin C = \frac{11 \times \sin 25^\circ}{7}$$

$$\therefore C = \sin^{-1}\left(\frac{11 \times \sin 25^\circ}{7}\right) \text{ or its supplement } \{\text{as } C \text{ may be obtuse}\}$$

$$\therefore C \approx 41.6^\circ \text{ or } 180^\circ - 41.6^\circ$$

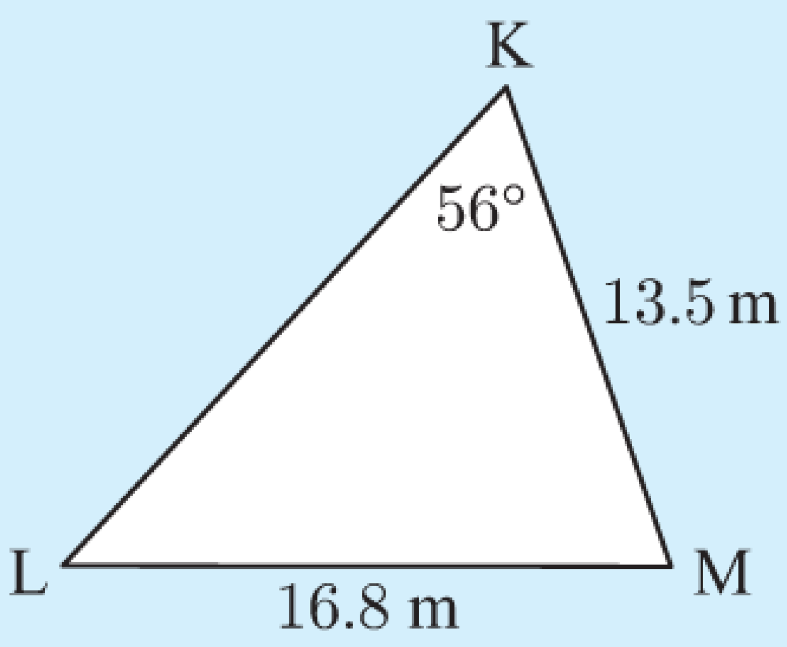
$$\therefore C \approx 41.6^\circ \text{ or } 138.4^\circ$$

$\therefore$ C measures 41.6° if angle C is acute, or 138.4° if angle C is obtuse.

In this case there is insufficient information to determine the actual shape of the triangle. There are two possible triangles.

Example 7 Self Tutor

Find the measure of angle L in triangle KLM given that angle K measures 56°, LM = 16.8 m, and KM = 13.5 m.



$$\frac{\sin L}{13.5} = \frac{\sin 56^\circ}{16.8} \quad \{\text{sine rule}\}$$

$$\therefore \sin L = \frac{13.5 \times \sin 56^\circ}{16.8}$$

$$\therefore L = \sin^{-1}\left(\frac{13.5 \times \sin 56^\circ}{16.8}\right) \text{ or its supplement}$$

$$\therefore L \approx 41.8^\circ \text{ or } 180^\circ - 41.8^\circ$$

$$\therefore L \approx 41.8^\circ \text{ or } 138.2^\circ$$

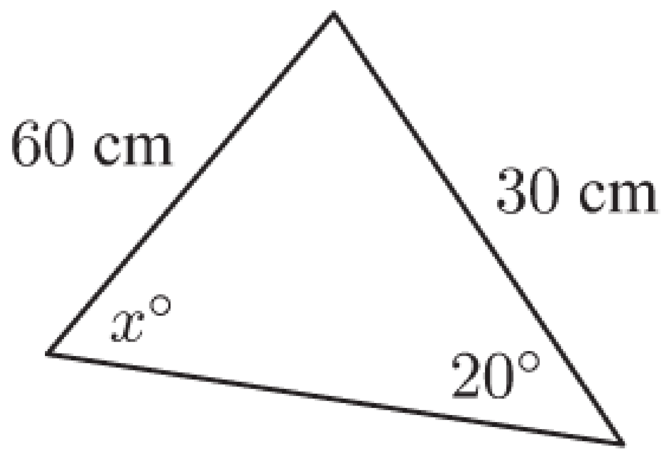
We reject $L \approx 138.2^\circ$, since $138.2^\circ + 56^\circ > 180^\circ$ which is impossible in a triangle.

$\therefore$ $L \approx 41.8^\circ$, a unique solution in this case.

EXERCISE 9C.2

1 Find the value of x:

a



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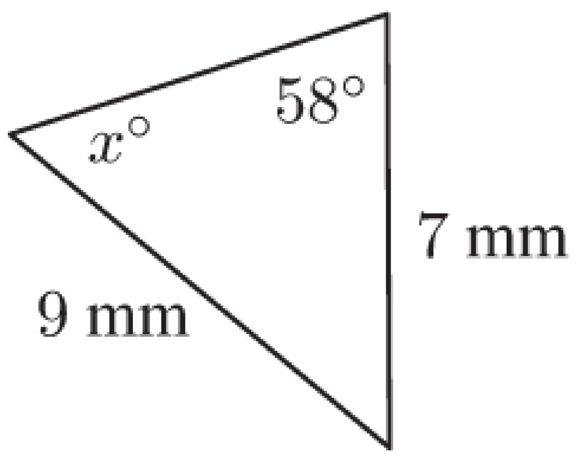
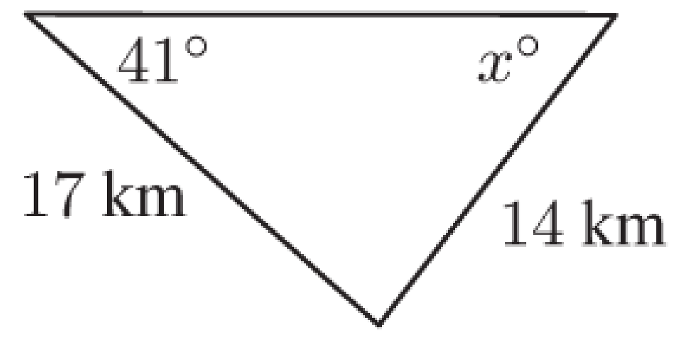


c



2 Triangle ABC has angle $B = 40^\circ$ and side lengths $b = 8$ cm and $c = 11$ cm. Find the two possible measures of angle C.

3 Consider triangle ABC.

a Given $a = 8$ cm, $b = 11$ cm, and $A\widehat{B}C = 45°$, find the measure of $B\widehat{A}C$.

b Given $a = 32$ cm, $b = 23$ cm, and $B\widehat{A}C = 42°$, find the measure of $A\widehat{B}C$.

c Given $c = 30$ m, $b = 36$ m, and $A\widehat{B}C = 37°$, find the measure of $A\widehat{C}B$.

d Given $a = 8.4$ cm, $b = 10.3$ cm, and $A\widehat{B}C = 63°$, find the measure of $B\widehat{A}C$.

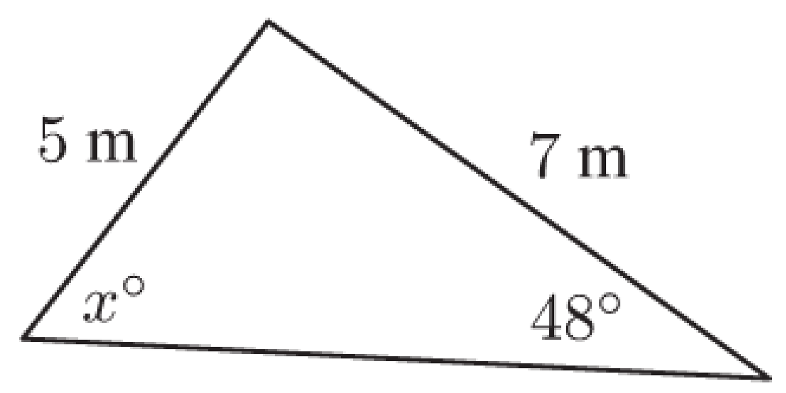
e Given $b = 22.1$ cm, $c = 16.5$ cm, and $A\widehat{C}B = 38°$, find the measure of $A\widehat{B}C$.

f Given $a = 3.1$ km, $c = 4.3$ km, and $B\widehat{A}C = 18°$, find the measure of $A\widehat{C}B$.

4 Unprepared for class, Mr Whiffen asks his students to find the value of x in the diagram shown.

a Show that Mr Whiffen's question cannot be solved.

b Explain what this means about the triangle Mr Whiffen created.

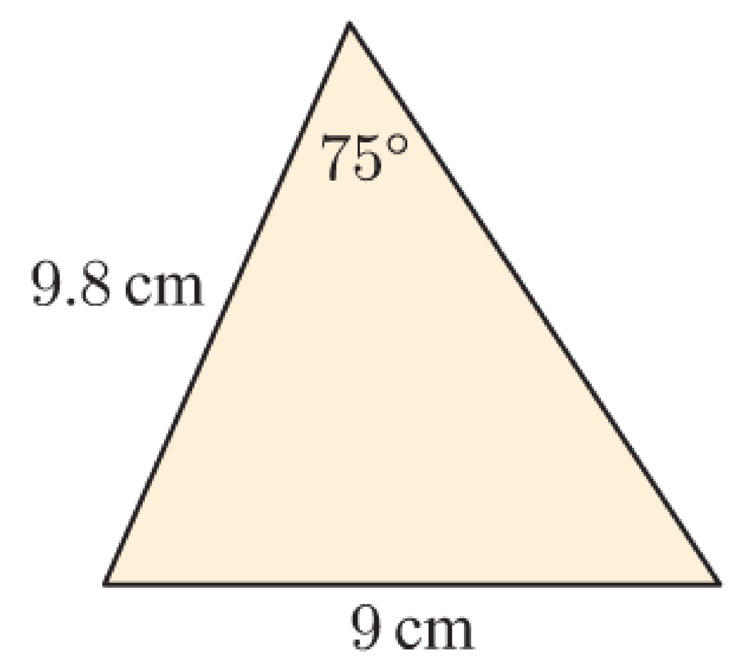


5 In triangle ABC, $A\widehat{B}C = 30°$, AC = 9 cm, and AB = 7 cm.

a Find the measure of: **i** $A\widehat{C}B$ **ii** $B\widehat{A}C$

b Hence find the area of the triangle.

6 Is it possible to have a triangle with the measurements shown? Explain your answer.



7 In triangle PQR, $P\widehat{R}Q = 50°$, PR = 11 m, and PQ = 9 m.

a Show that there are two possible measures of $P\widehat{Q}R$.

b Sketch triangle PQR for each case.

c For each case, find:

i the measure of $Q\widehat{P}R$ **ii** the area of the triangle **iii** the perimeter of the triangle.

D PROBLEM SOLVING WITH TRIGONOMETRY

If we are given a problem involving a triangle, we must first decide which rule is best to use.

If the triangle is right angled then the trigonometric ratios or Pythagoras' theorem can be used. For some problems we can add an extra line or two to the diagram to create a right angled triangle.

However, if we do not have a right angled triangle then we usually have to choose between the sine and cosine rules. In these cases the following checklist may be helpful:

Use the **cosine rule** when given:
- three sides
- two sides and an included angle.

Use the **sine rule** when given:
- one side and two angles
- two sides and a non-included angle.

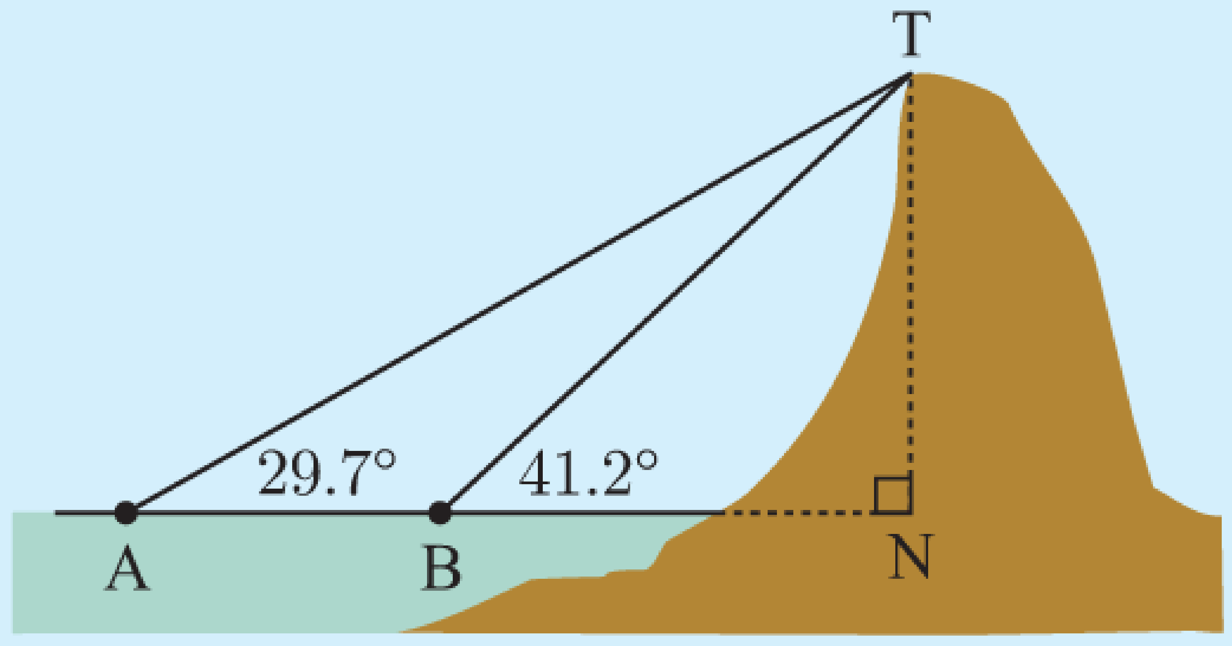
In situations where either rule could be used to find an angle, the cosine rule should be used to avoid the ambiguous case.

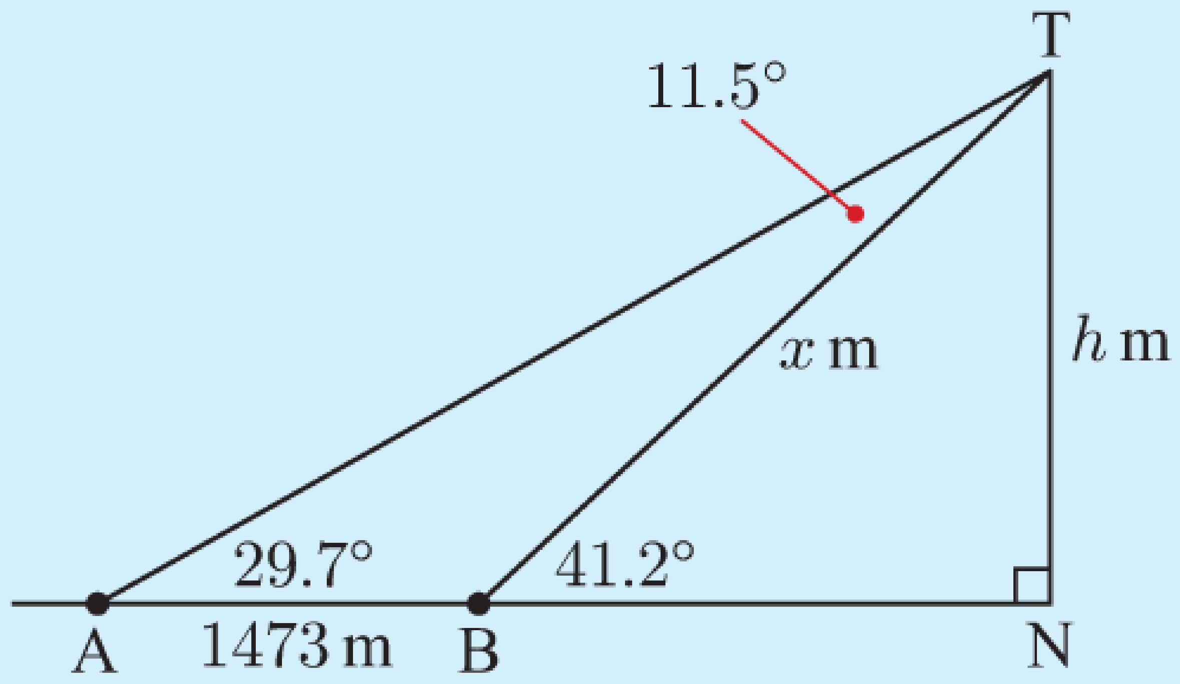
Example 8 **Self Tutor**

The angles of elevation to the top of a mountain are measured from two beacons A and B at sea.

The measurements are shown on the diagram.

If the beacons are 1473 m apart, how high is the mountain?





Let BT be x m and NT be h m.

$\widehat{ATB} = 41.2° - 29.7°$ {exterior angle of $\triangle$BNT}
$= 11.5°$

We find x in $\triangle$ABT using the sine rule:

$$\frac{x}{\sin 29.7°} = \frac{1473}{\sin 11.5°}$$

$$\therefore \; x = \frac{1473}{\sin 11.5°} \times \sin 29.7° \approx 3660.62$$

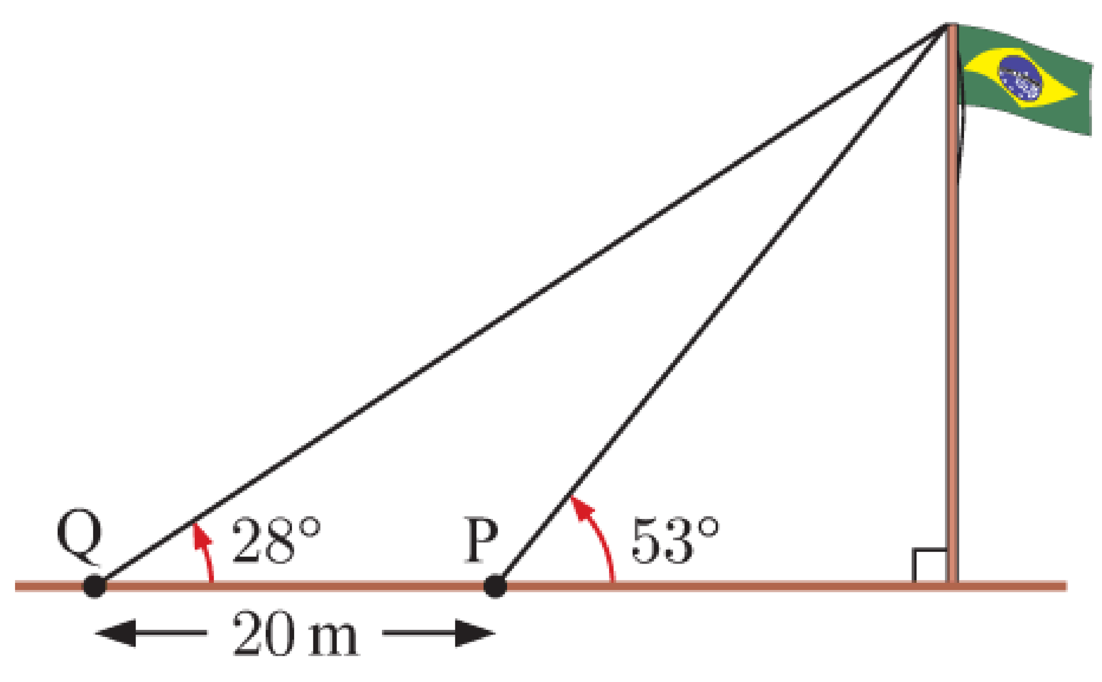
Now, in $\triangle$BNT, $\sin 41.2° = \frac{h}{x} \approx \frac{h}{3660.62}$

$$\therefore \; h \approx \sin 41.2° \times 3660.62 \approx 2410$$

The mountain is about 2410 m high.

EXERCISE 9D

1 Rodrigo wishes to determine the height of a flagpole. He takes a sighting to the top of the flagpole from point P. He then moves 20 metres further away from the flagpole to point Q, and takes a second sighting. The information is shown in the diagram alongside. How high is the flagpole?



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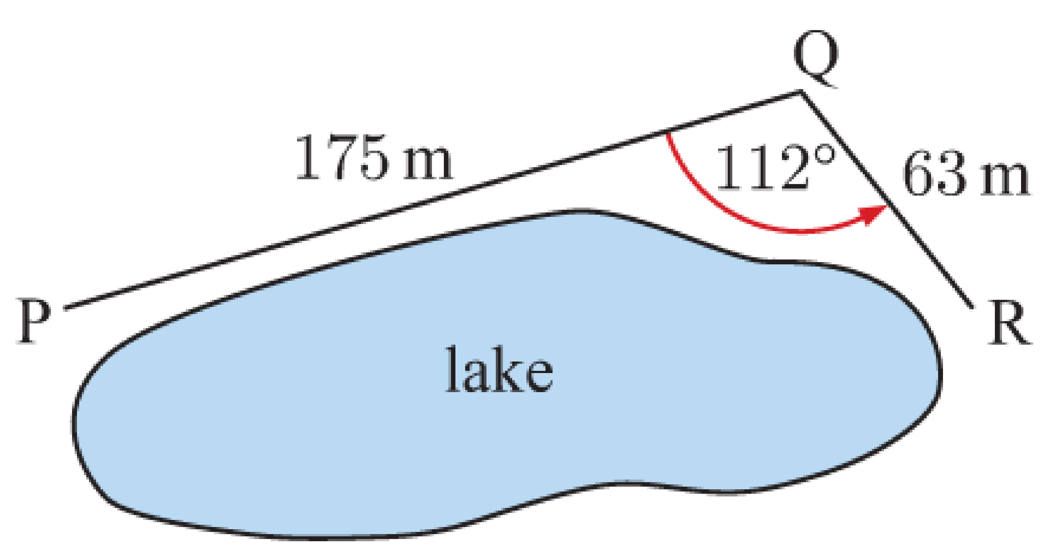
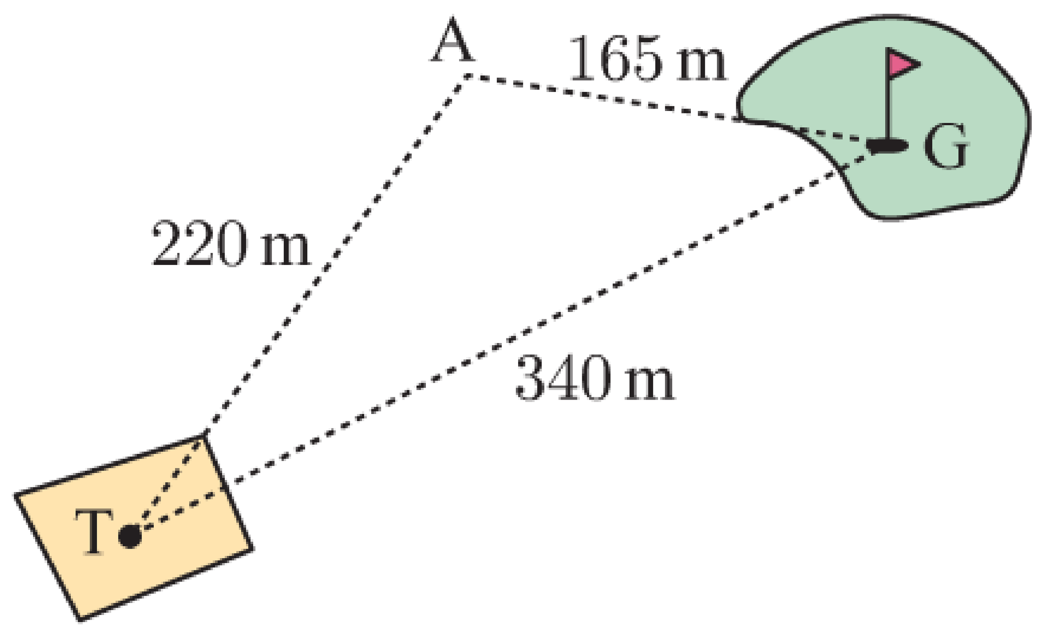


To get from P to R, a park ranger has to walk along a path to Q and then to R.
What is the distance in a straight line from P to R?

3 An orienteer runs for $4\frac{1}{2}$ km, then turns through an angle of 32° and runs for another 6 km. How far is she from her starting point?

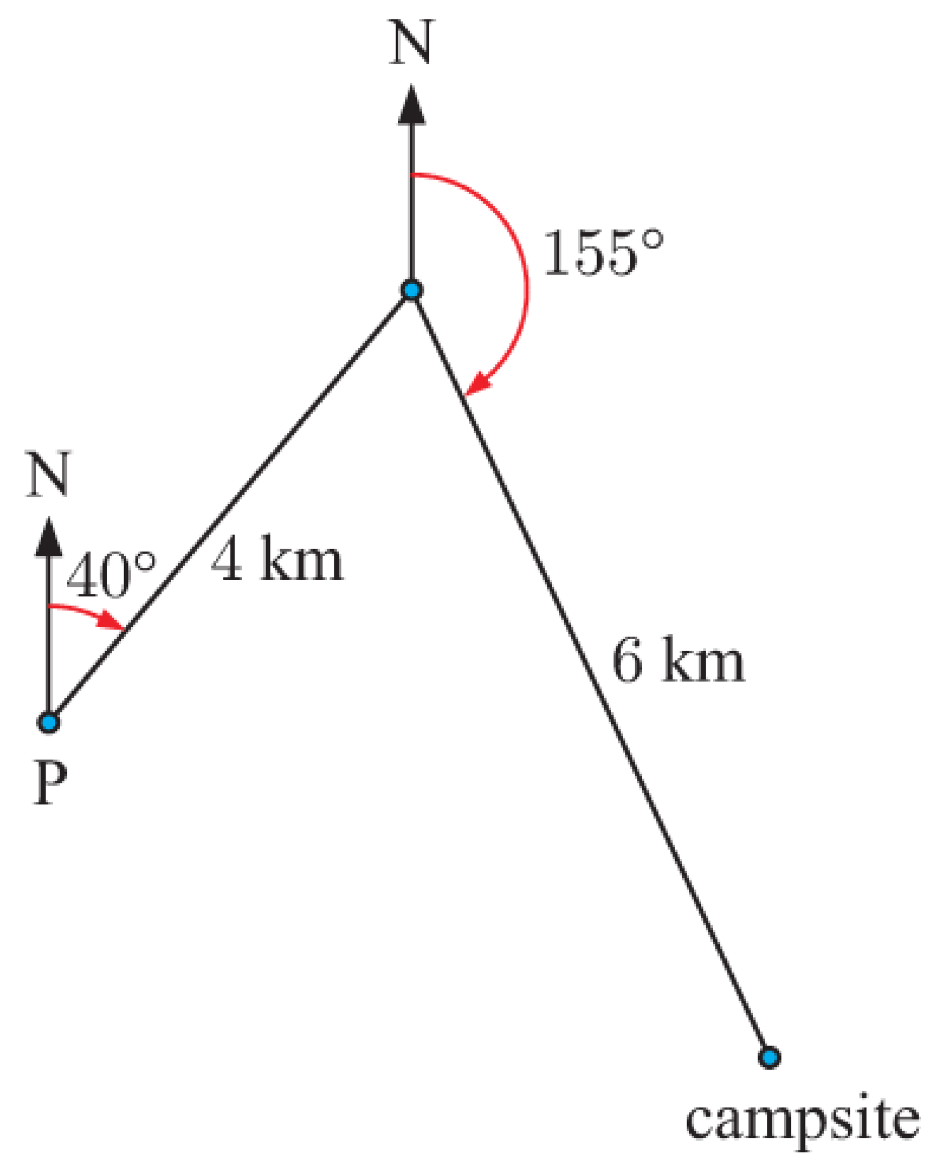
4 A golfer played his tee shot a distance of 220 m to point A. He then played a 165 m six iron to the green. If the distance from tee to green is 340 m, determine the angle the golfer was off line with his tee shot.



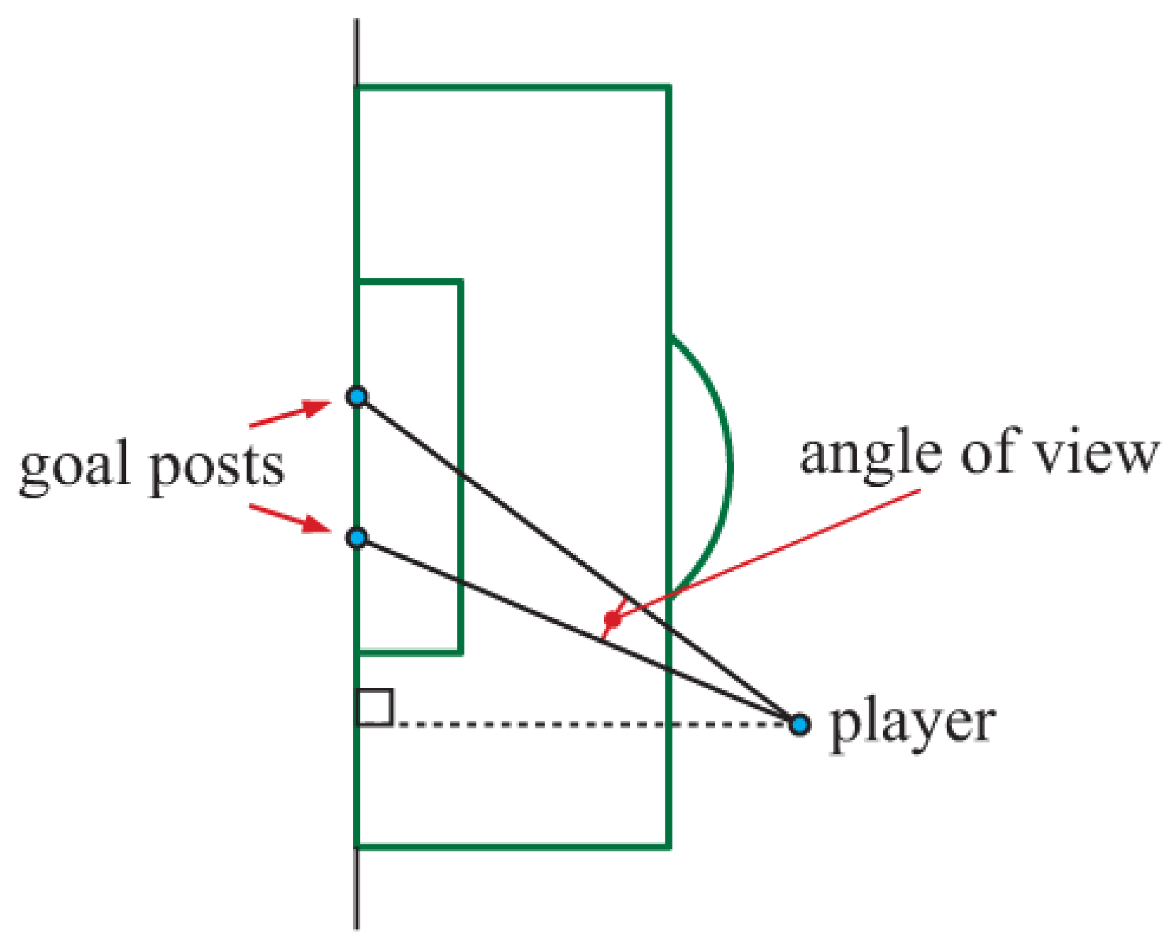
5 A helicopter A observes two ships B and C. B is 23.8 km from the helicopter and C is 31.9 km from it. The angle of view $B\widehat{A}C$ from the helicopter to B and C, is 83.6°. How far are the ships apart?

6 Hikers Ritva and Esko leave point P at the same time. Ritva walks 4 km on the bearing 040°, then a further 6 km on the bearing 155° to get to their campsite.
Esko hikes to the camp site directly from P.

- **a** How far does Esko hike?
- **b** In which direction does Esko hike?
- **c** Ritva hikes at 5 km h^{-1} and Esko hikes at 3 km h^{-1}.
  - **i** Who will arrive at the camp site first?
  - **ii** How long will this person need to wait before the other person arrives?
- **d** On what bearing should the hikers walk from the camp site to return directly to P?



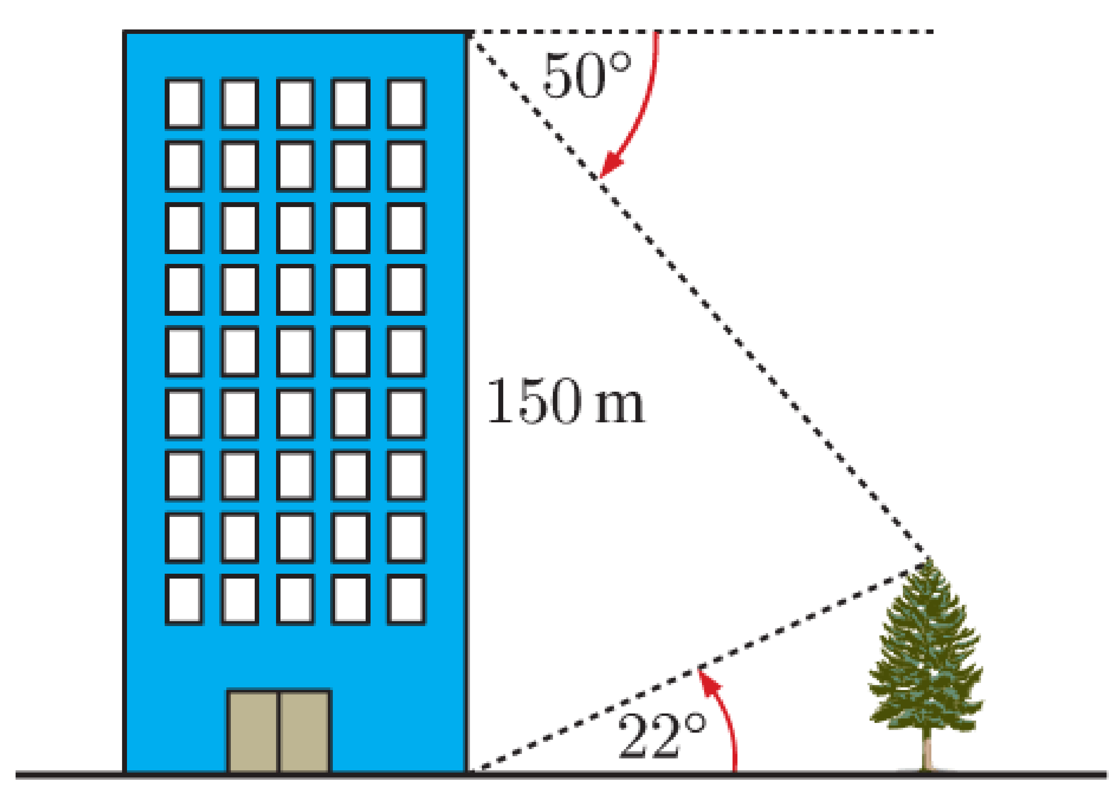
7 A football goal is 5 metres wide. When a player is 26 metres from one goal post and 23 metres from the other, he shoots for goal. What is the angle of view of the goal that the player sees?



8 A tower 42 metres high stands on top of a hill. From a point some distance from the base of the hill, the angle of elevation to the top of the tower is 13.2°, and the angle of elevation to the bottom of the tower is 8.3°. Find the height of the hill.

9 From the foot of a building I have to look 22° upwards to sight the top of a tree. From the top of the building, 150 metres above ground level, I have to look down at an angle of 50° below the horizontal to sight the tree top.

- **a** How high is the tree?
- **b** How far from the building is this tree?

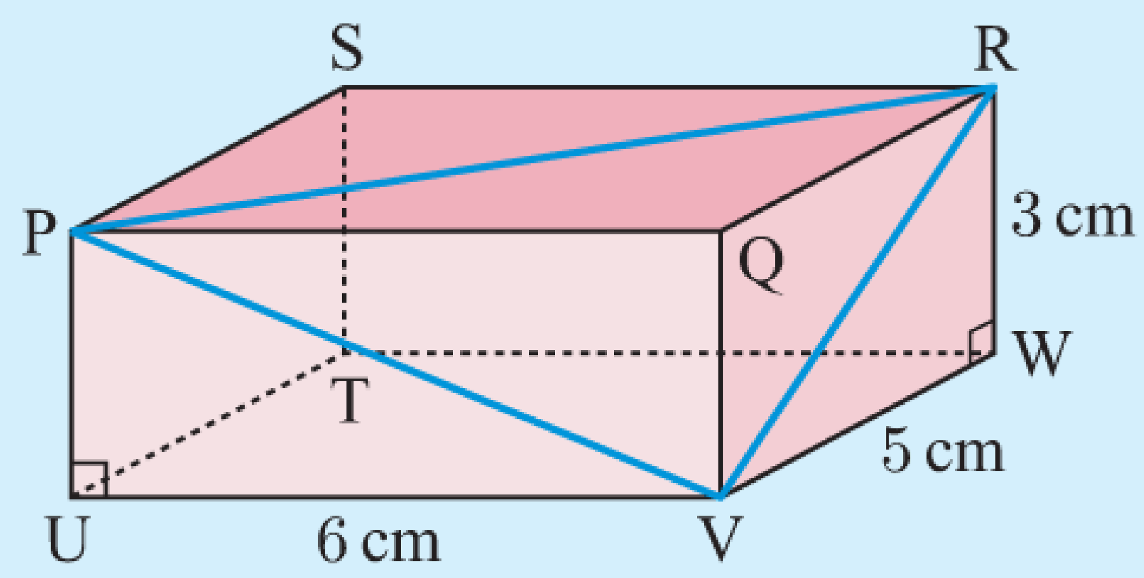


10 Two yachts are sailing close to a dangerous reef. The captain of the *Porpoise* notices a lighthouse 2.4 km away on the bearing 223°. The captain of the *Queen Maria* measures the lighthouse as 2.1 km away. He also observes the *Porpoise* to the right of the lighthouse, with an angle of 53° between them.

- **a** Display this information on a diagram.
- **b** Find the distance between the yachts.
- **c** Find the bearing of the *Queen Maria* from the *Porpoise*.

Example 9 **Self Tutor**

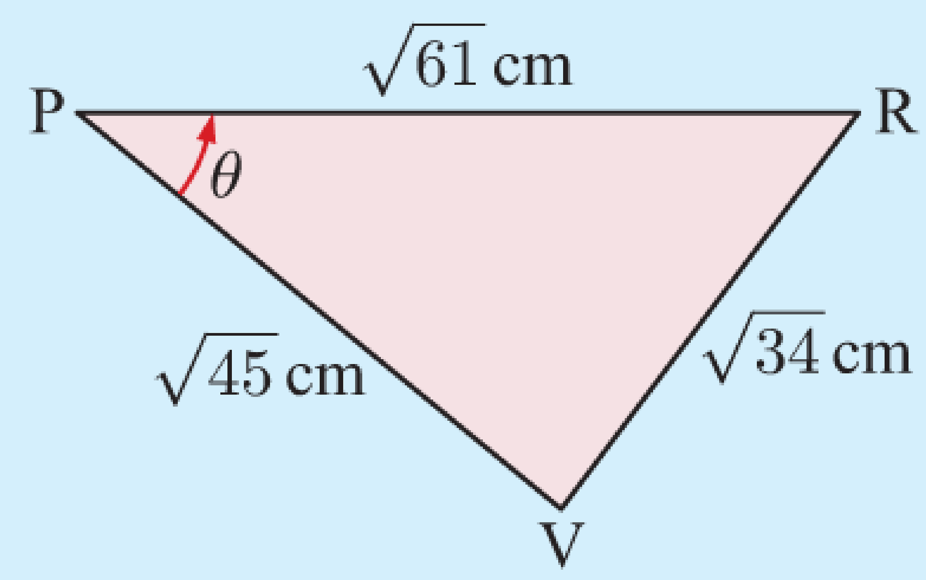
Find the measure of $R\widehat{P}V$.



In $\triangle$RVW, $RV = \sqrt{5^2 + 3^2} = \sqrt{34}$ cm. {Pythagoras}

In $\triangle$PUV, $PV = \sqrt{6^2 + 3^2} = \sqrt{45}$ cm. {Pythagoras}

In $\triangle$PQR, $PR = \sqrt{6^2 + 5^2} = \sqrt{61}$ cm. {Pythagoras}



By rearrangement of the cosine rule,

$$\cos\theta = \frac{(\sqrt{61})^2 + (\sqrt{45})^2 - (\sqrt{34})^2}{2\sqrt{61}\sqrt{45}}$$

$$= \frac{61 + 45 - 34}{2\sqrt{61}\sqrt{45}}$$

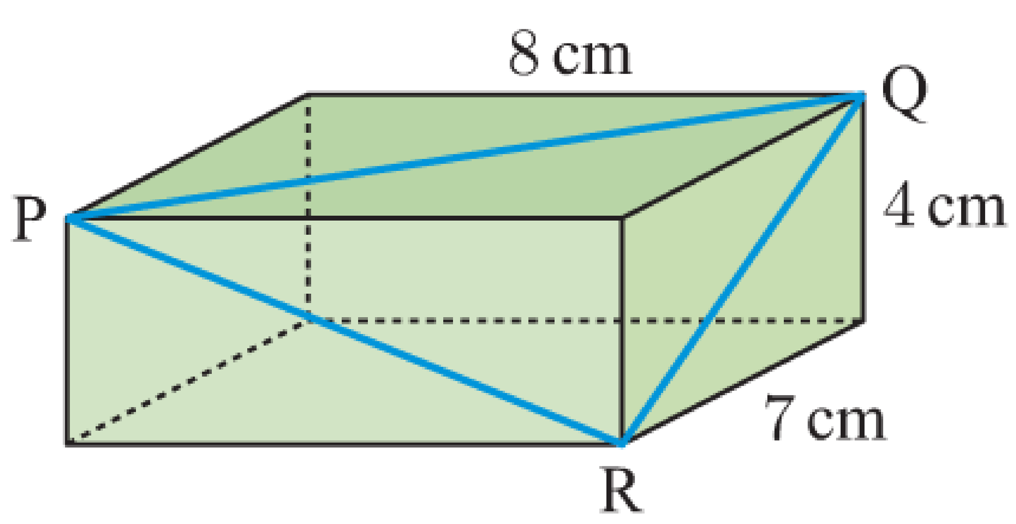
$$= \frac{72}{2\sqrt{61}\sqrt{45}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{36}{\sqrt{61}\sqrt{45}}\right) \approx 46.6°$$

$\therefore$ $R\widehat{P}V$ measures about 46.6°.

11 Find the measure of $P\widehat{Q}R$:

a



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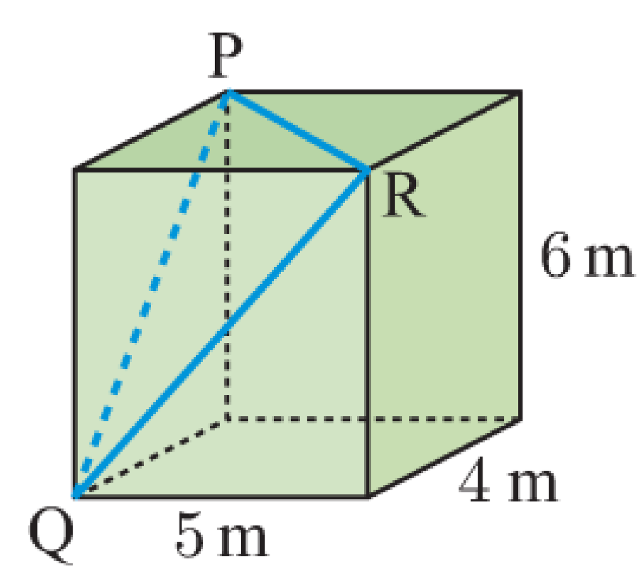
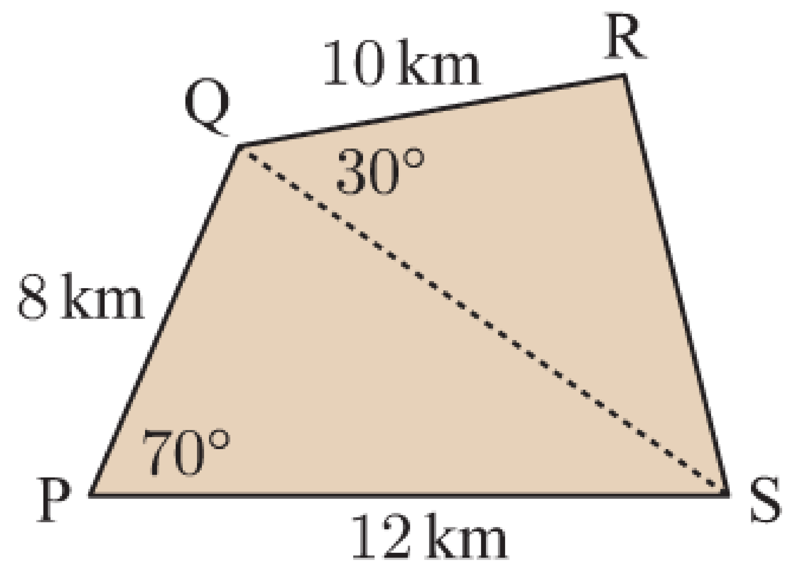


12 Two observation posts A and B are 12 km apart. A third observation post C is located 15 km from A such that $C\widehat{B}A$ is 67°. Find the measure of $C\widehat{A}B$.

13 Thabo and Palesa start at point A. They each walk in a straight line at an angle of 120° to one another. Thabo walks at 6 km h^{-1} and Palesa walks at 8 km h^{-1}. How far apart are they after 45 minutes?

14 Stan and Olga are considering buying a sheep farm. A surveyor has supplied them with the given accurate sketch. Find the area of the property, giving your answer in:

a km^2 **b** hectares.



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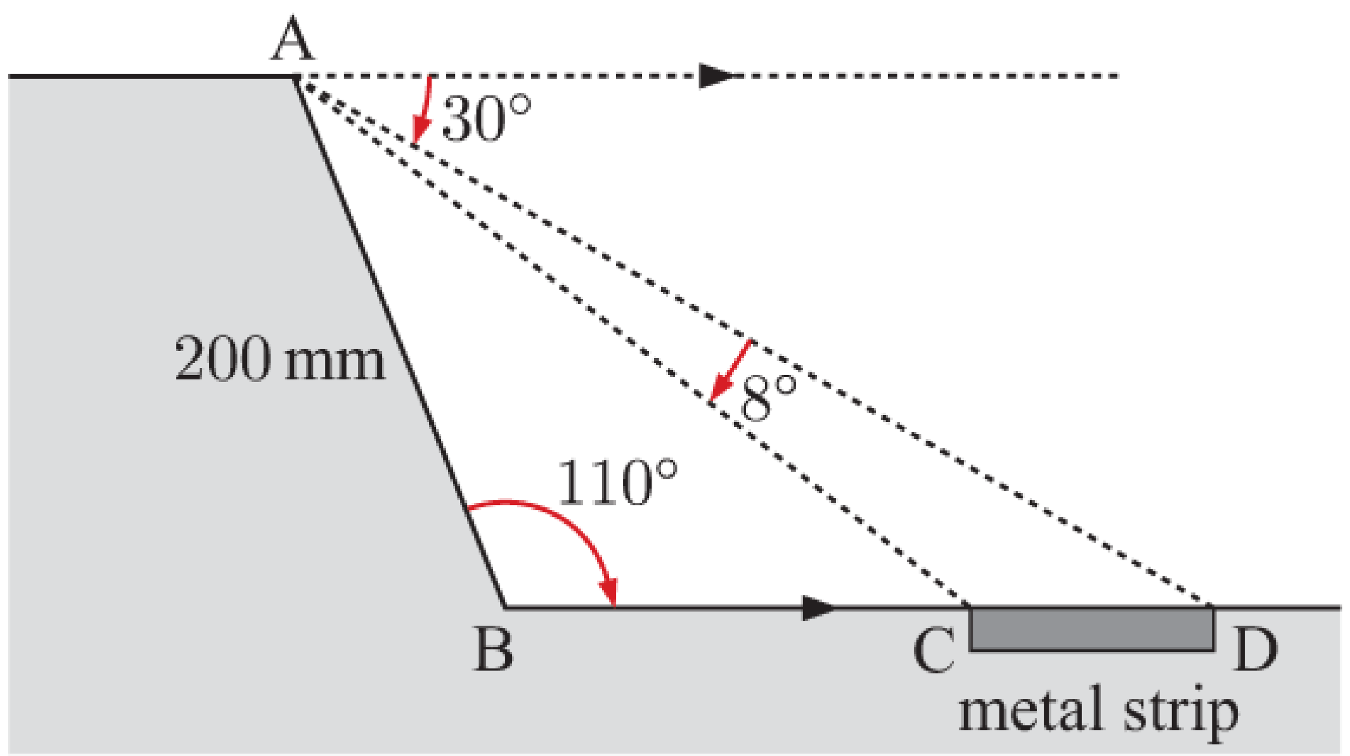
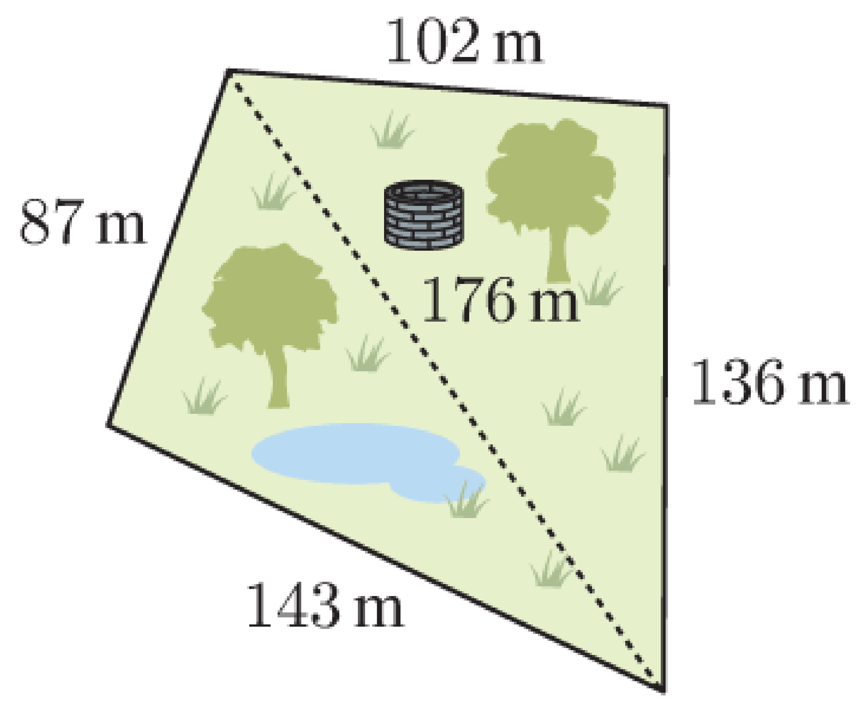


The cross-section design of the kerbing for a driverless-bus roadway is shown opposite. The metal strip is inlaid into the concrete and is used to control the direction and speed of the bus. Find the width of the metal strip.

16 Robert from the City Council has made some measurements of the park in the **Opening Problem**. They are summarised in the diagram.
Find the area of the park and the angle at each corner.



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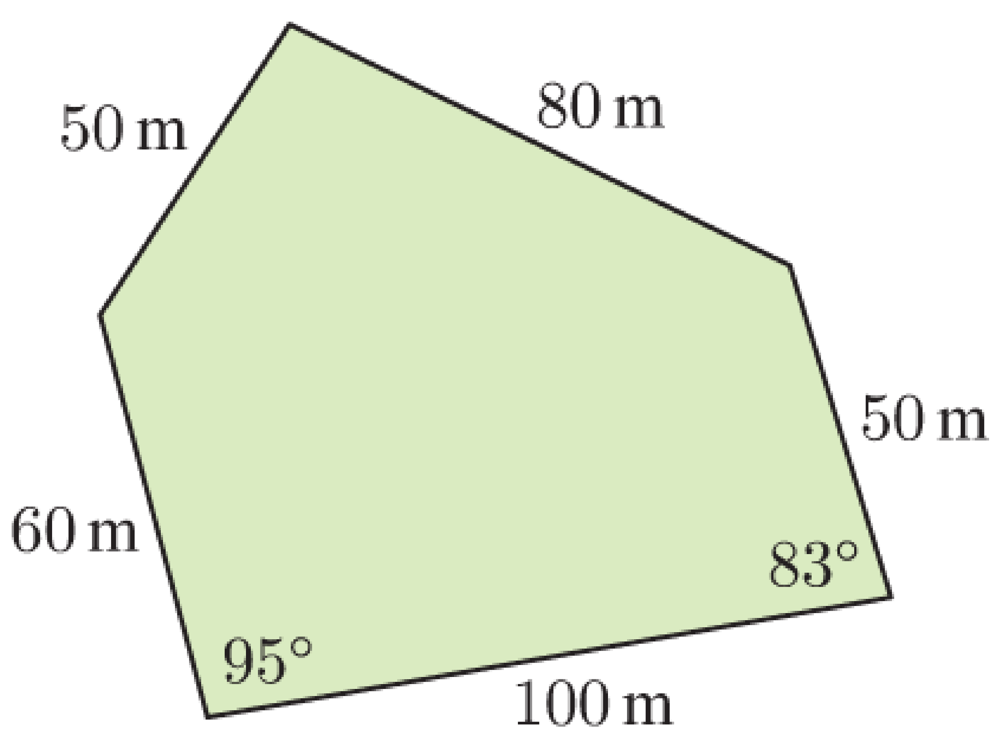
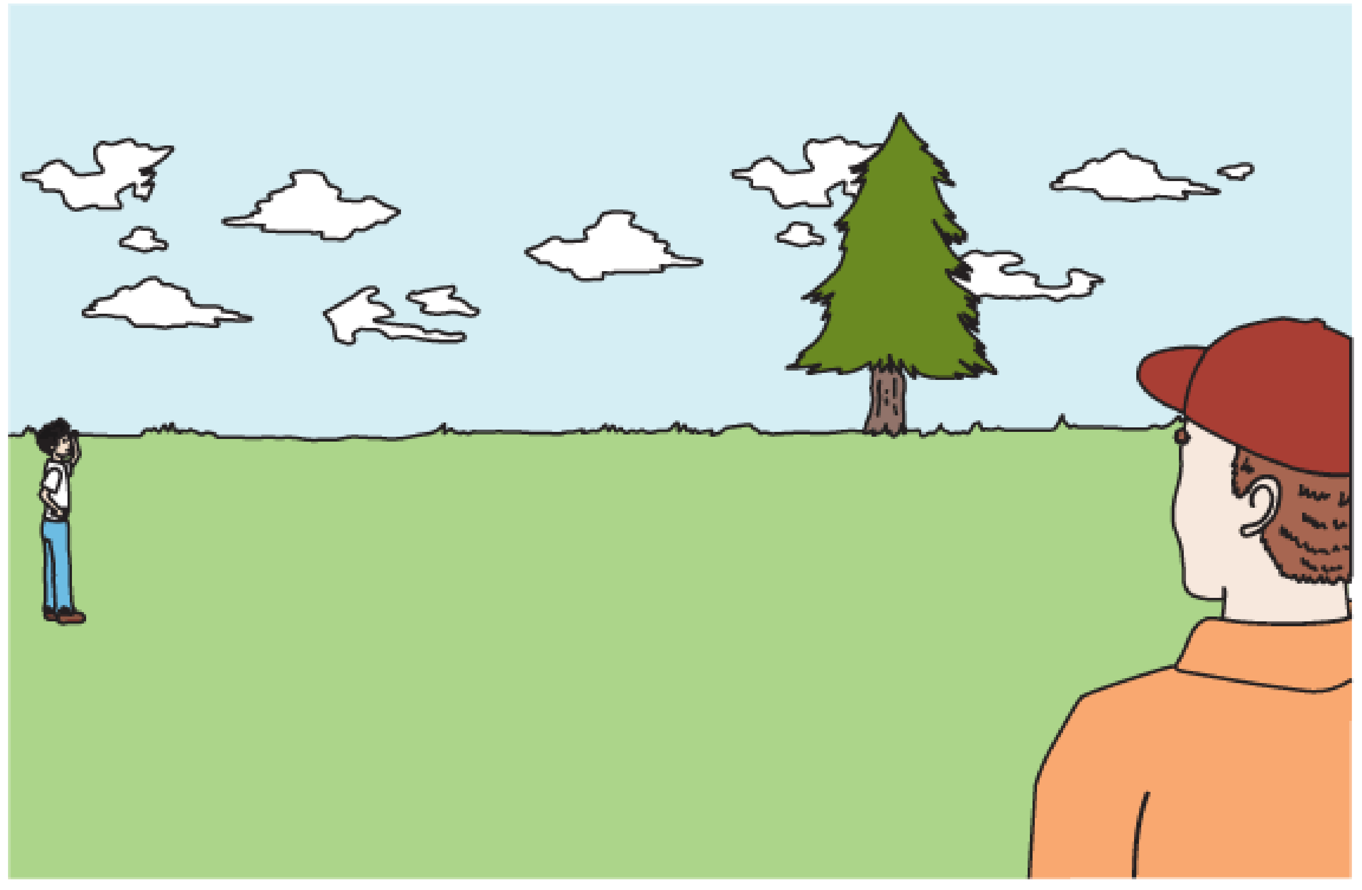


A surveyor has produced this plan of a property.
Find its area.

18 A Chinese restaurant and a fish and chip shop are 7 km apart. The Chinese restaurant offers free delivery within 5 km, and the fish and chip shop offers free delivery within 3 km. Find the area of the region which receives free delivery from both locations.

19 One angle of a triangular garden measures 60°. The garden has perimeter 36 m and area $30\sqrt{3}$ m^2. Find the measure of the remaining two angles of the garden.

20 Sam and Markus are standing on level ground 100 metres apart. A large tree is due north of Markus and on the bearing 065° from Sam. The top of the tree appears at an angle of elevation of 25° to Sam and 15° to Markus. Find the height of the tree.



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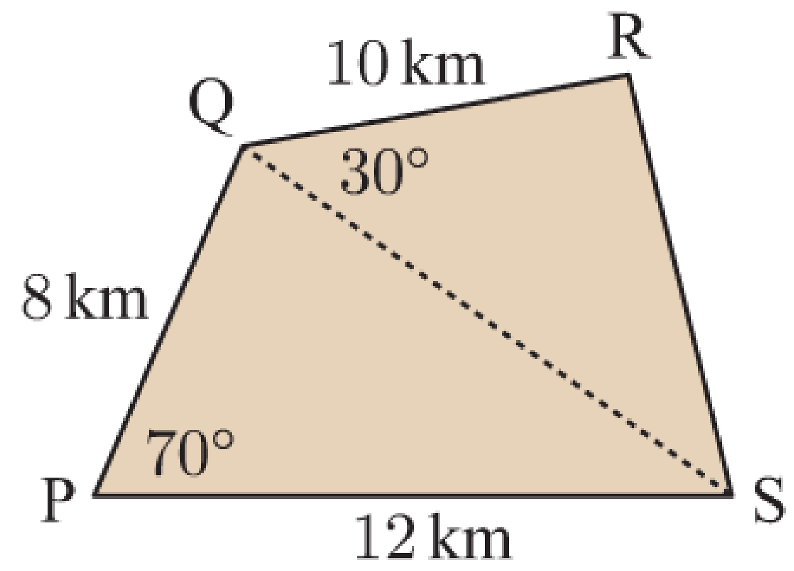
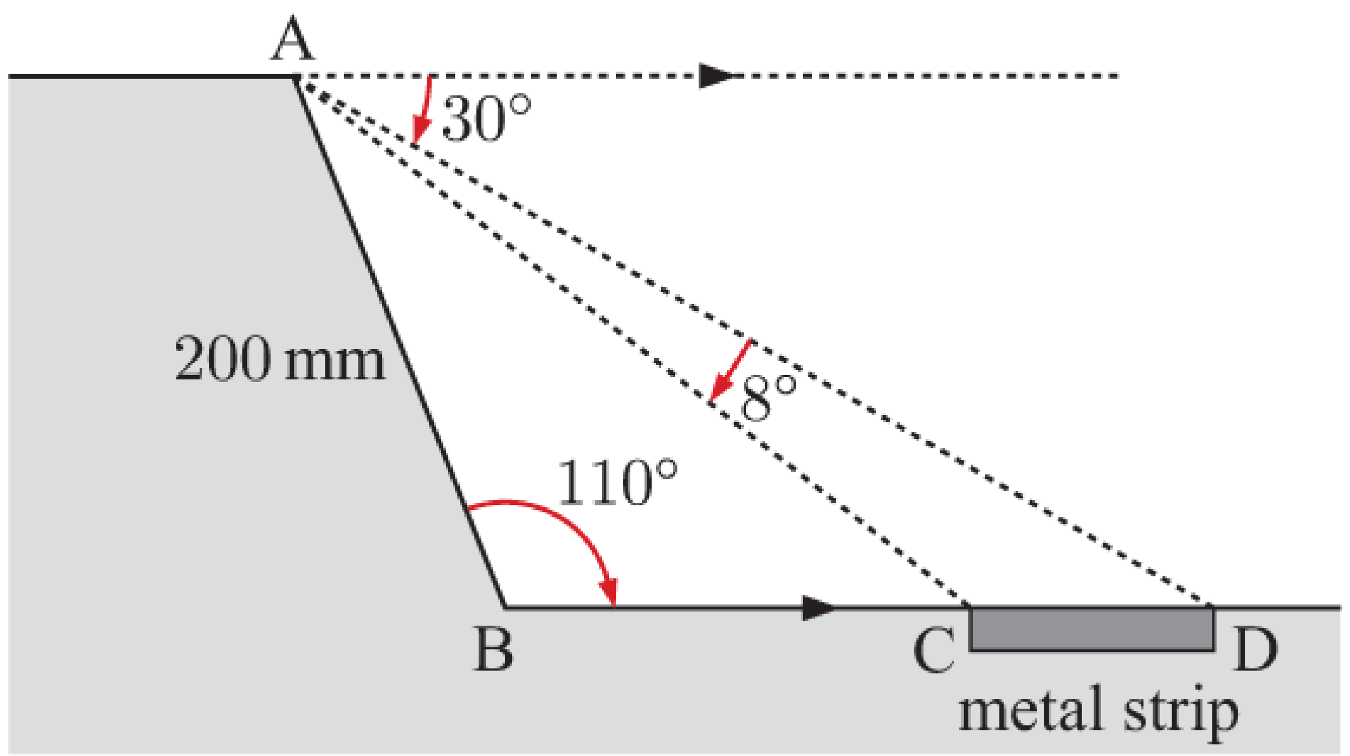
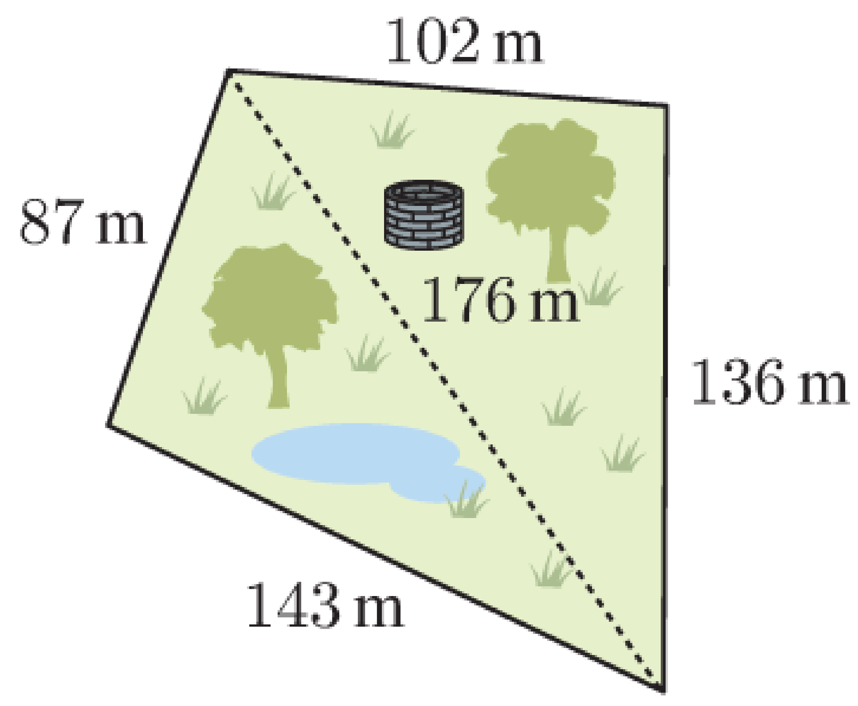
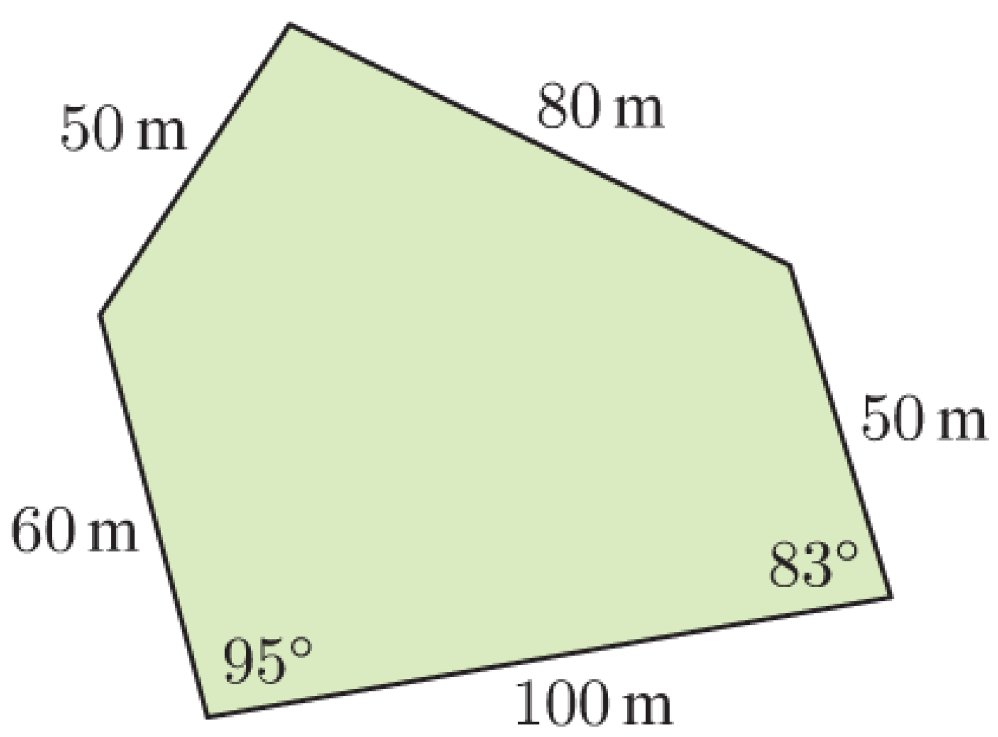


15 

The cross-section design of the kerbing for a driverless-bus roadway is shown opposite. The metal strip is inlaid into the concrete and is used to control the direction and speed of the bus. Find the width of the metal strip.

16 Robert from the City Council has made some measurements of the park in the **Opening Problem**. They are summarised in the diagram.
Find the area of the park and the angle at each corner.



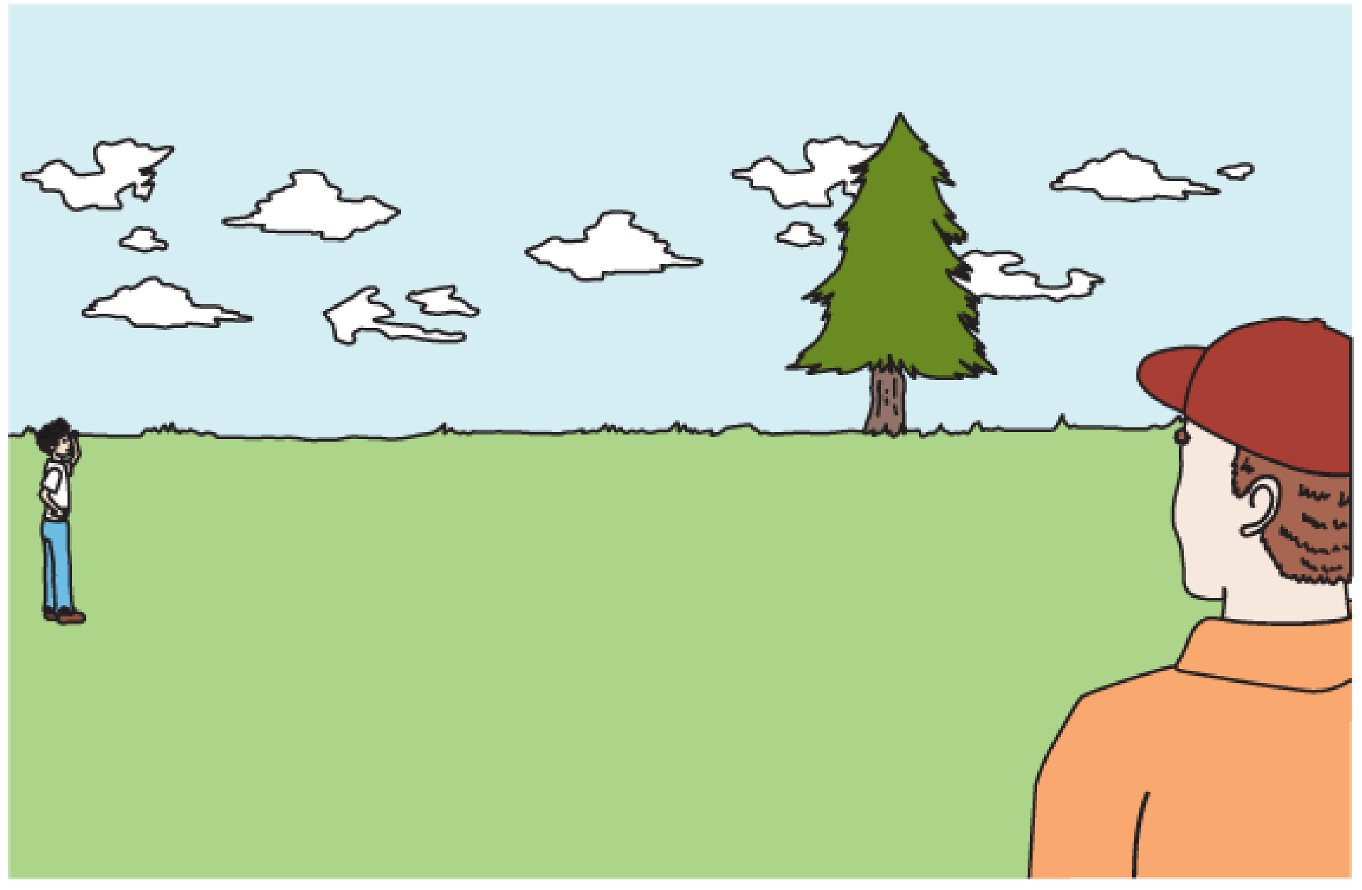
17 

A surveyor has produced this plan of a property.
Find its area.

18 A Chinese restaurant and a fish and chip shop are 7 km apart. The Chinese restaurant offers free delivery within 5 km, and the fish and chip shop offers free delivery within 3 km. Find the area of the region which receives free delivery from both locations.

19 One angle of a triangular garden measures 60°. The garden has perimeter 36 m and area $30\sqrt{3}$ m^2. Find the measure of the remaining two angles of the garden.

20 Sam and Markus are standing on level ground 100 metres apart. A large tree is due north of Markus and on the bearing 065° from Sam. The top of the tree appears at an angle of elevation of 25° to Sam and 15° to Markus. Find the height of the tree.



3 A man walks south from the North Pole and turns 90° left when he reaches the equator. He walks for a while and then turns 90° left to walk back to the North Pole.

  a Has the man walked in a triangle?

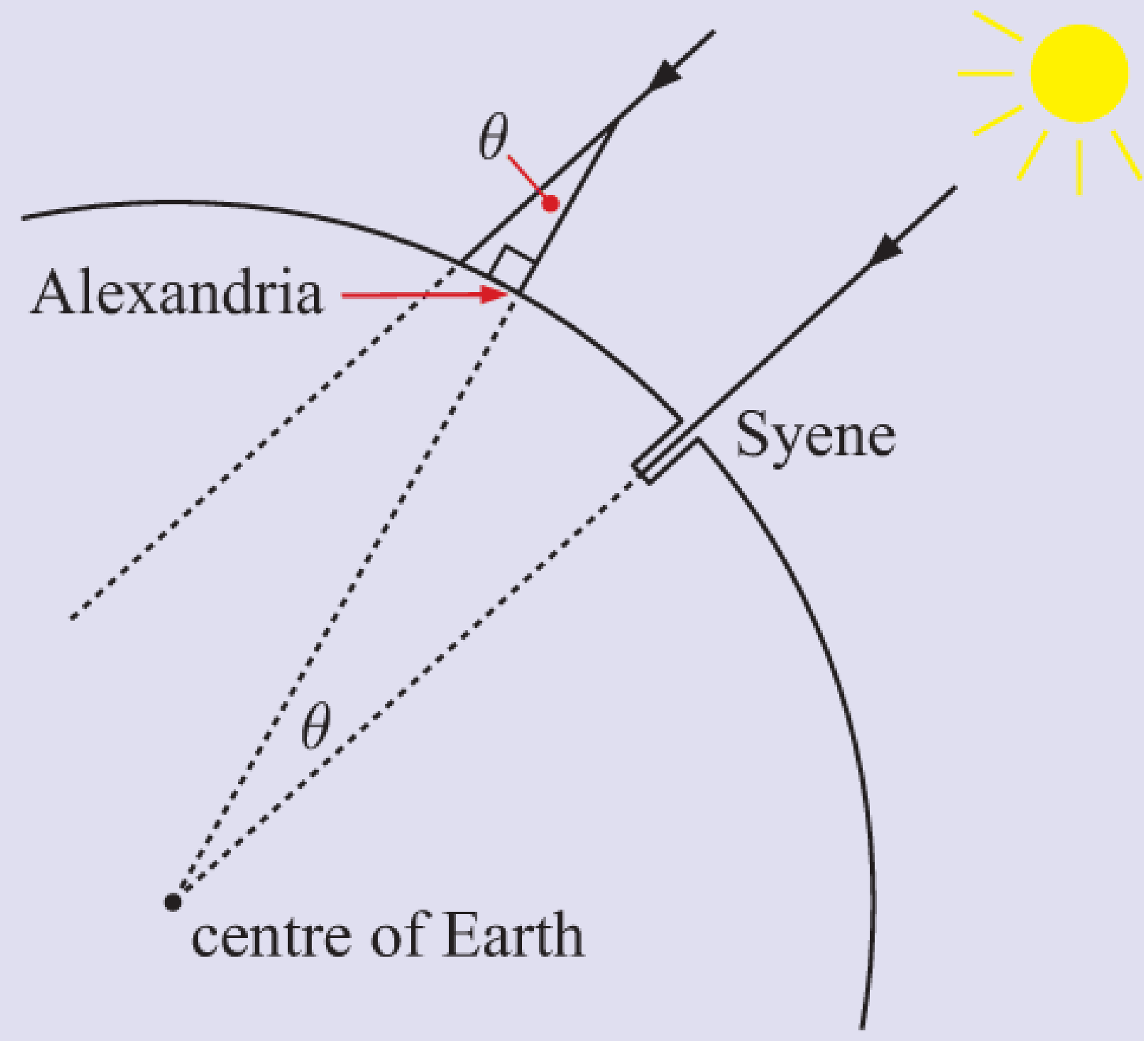
  b Is the angle sum of a triangle always equal to 180°?

4 Is a spherical triangle more or less complicated than a triangle in a plane?

5 How does the research of the ancient astronomers relate to modern problems of satellites, telecommunications, and GPS navigation?

Erathosthenes of Cyrene (276 BC - 194 BC) deduced that the Earth is round using two observations made at noon on the summer solstice:

- In Syene (now Aswan, Egypt), sunlight illuminated the bottom of a well.
- In Alexandria, 5000 stadia ($\approx$ 880 km) away, a perfectly vertical rod cast a shadow.

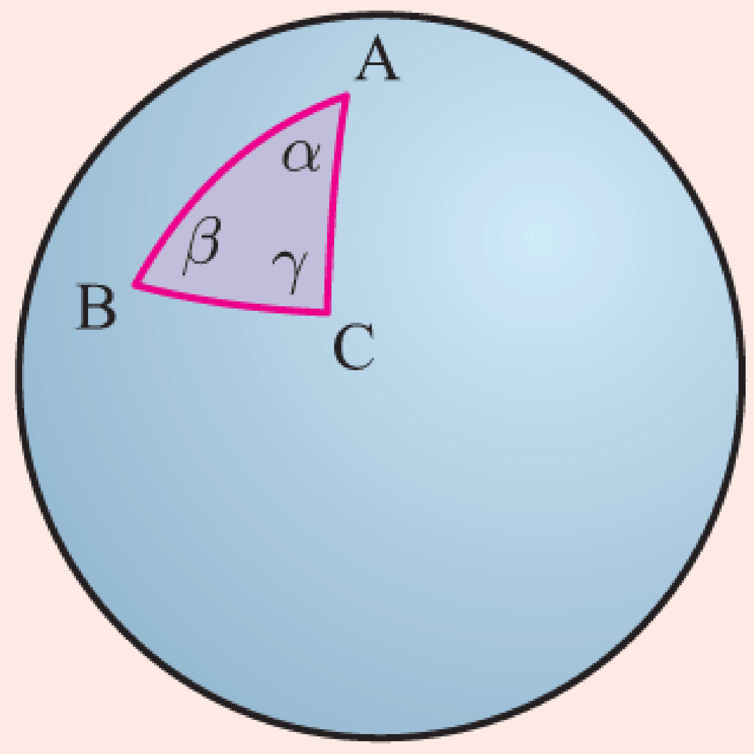


Using the lengths of the rod and its shadow, Erathosthenes calculated the angular separation of Syene and Alexandria. He hence calculated the circumference of the Earth within 10% of its actual value.

6 Why did a "flat Earth" theory persist for so long, despite ancient astronomers knowing the Earth was round? What lessons can be learned from this in protecting and promoting knowledge?

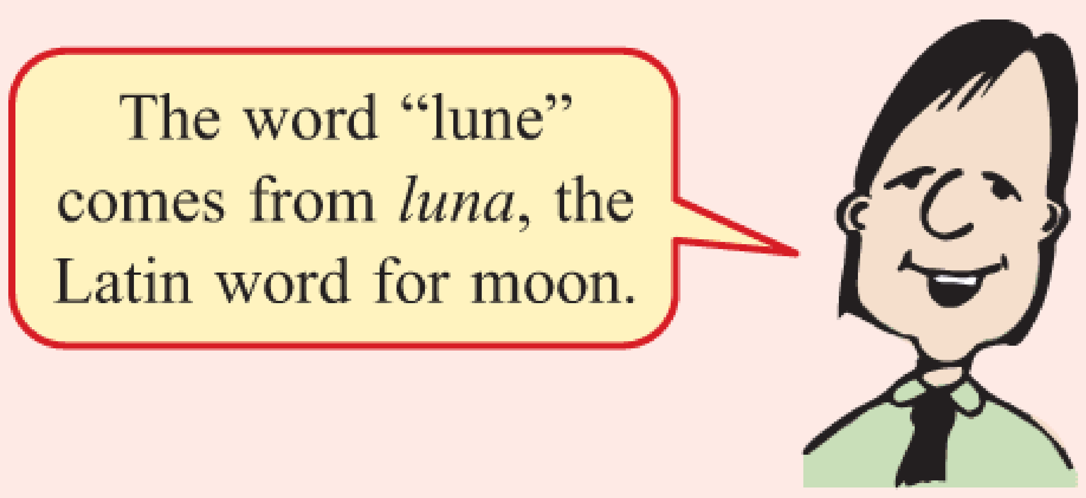
ACTIVITY THE AREA OF A SPHERICAL TRIANGLE

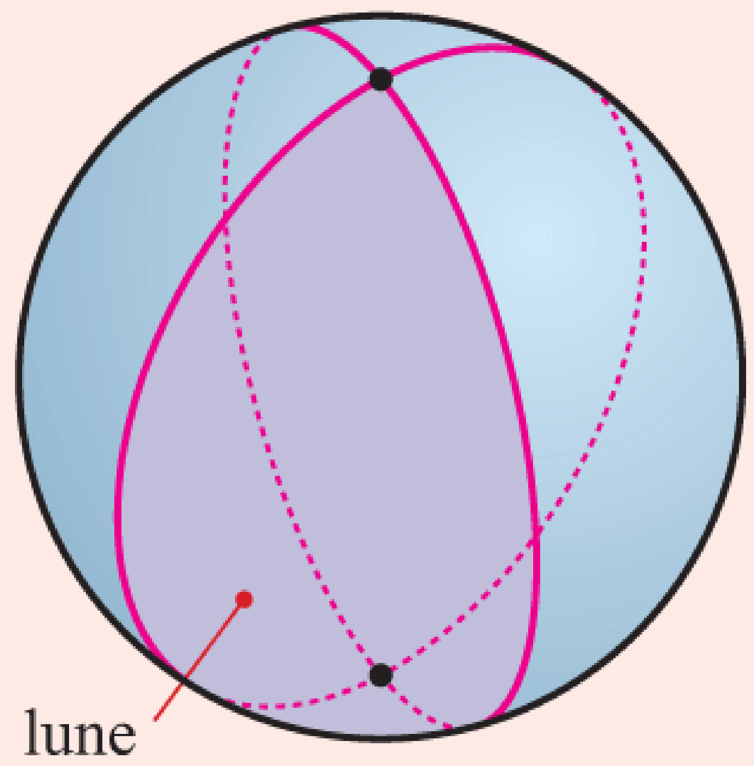
In this Activity we derive a formula for the area of a spherical triangle ABC with angles α, β, and γ.



To achieve this we need some definitions:

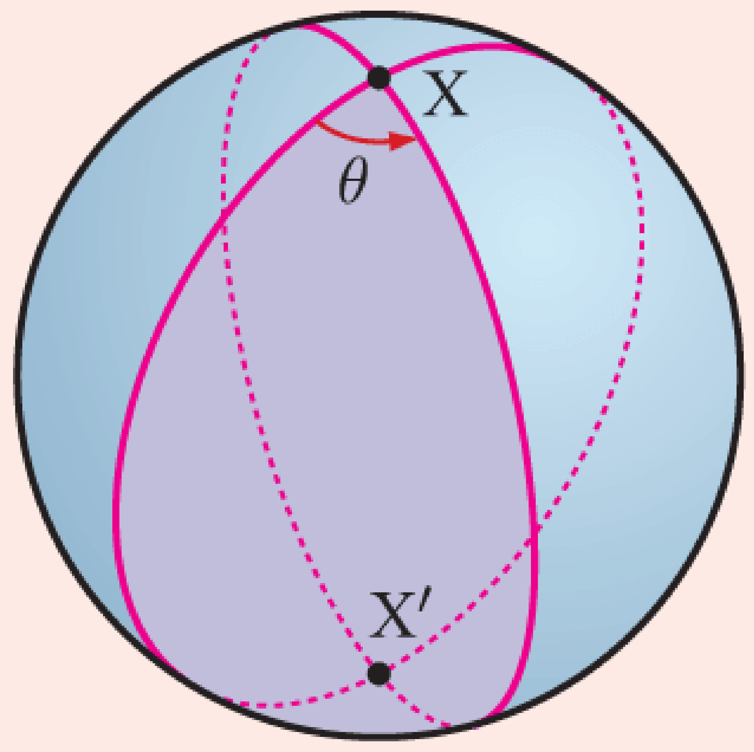
- A **great circle** is any circle drawn on a sphere whose centre is the centre of the sphere.
- A **spherical lune** is an area on a sphere bounded by two half great circles which meet at diametrically opposite points.





What to do:

1 Two great circles on a sphere with radius r meet at X and X′. They form a lune with angle θ as shown. Explain why the surface area $S_{X,\,\theta}$ of the lune is given by $S_{X,\,\theta} = 2r^2\theta$.



2 Consider a spherical triangle ABC with angles α, β, and γ. Suppose the arcs AB, AC, and BC are extended to form great circles.

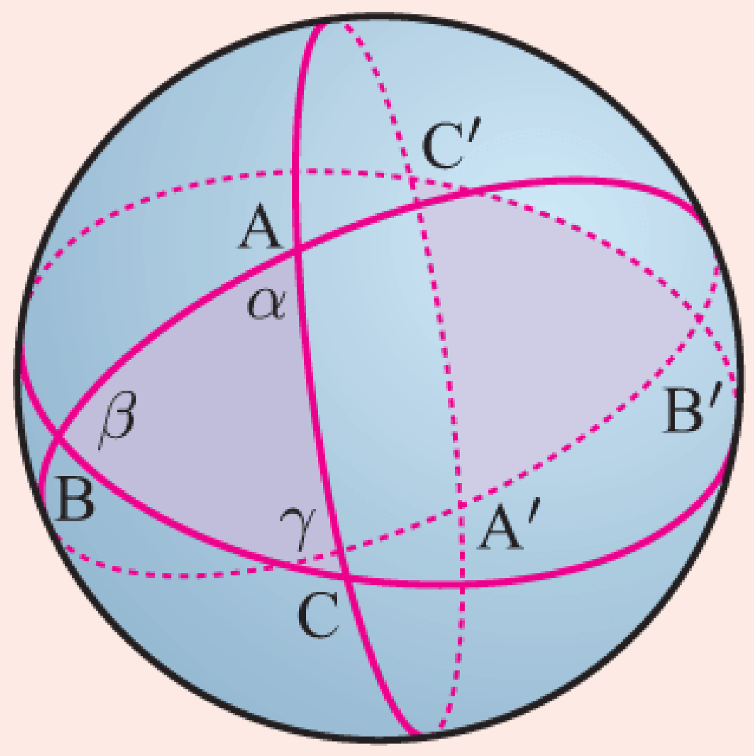
a Explain why the great circles will form a congruent spherical triangle A′B′C′ on the other side of the sphere.

b By considering the areas of lunes, explain why

$$4\pi r^2 = 2S_{A,\,\alpha} + 2S_{B,\,\beta} + 2S_{C,\,\gamma} - 4A$$

where A is the area of the spherical triangle ABC.

c Hence show that the area of the spherical triangle is given by $A = (\alpha + \beta + \gamma - \pi)r^2$.



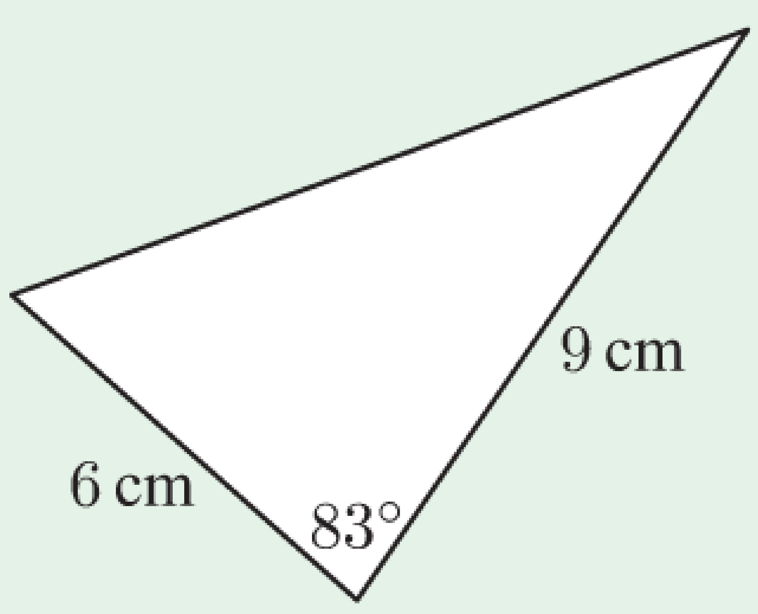
3 Explain why the area formula verifies that the angle sum of a spherical triangle is greater than 180°.

4 Is it possible for two spherical triangles on a given sphere to be similar but not congruent? Explain your answer.

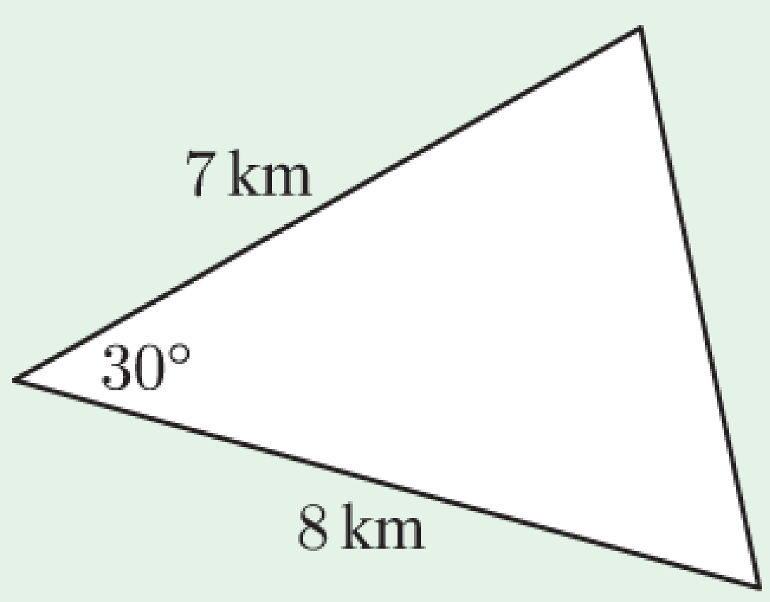
REVIEW SET 9A

1 Find the area of:

a



b



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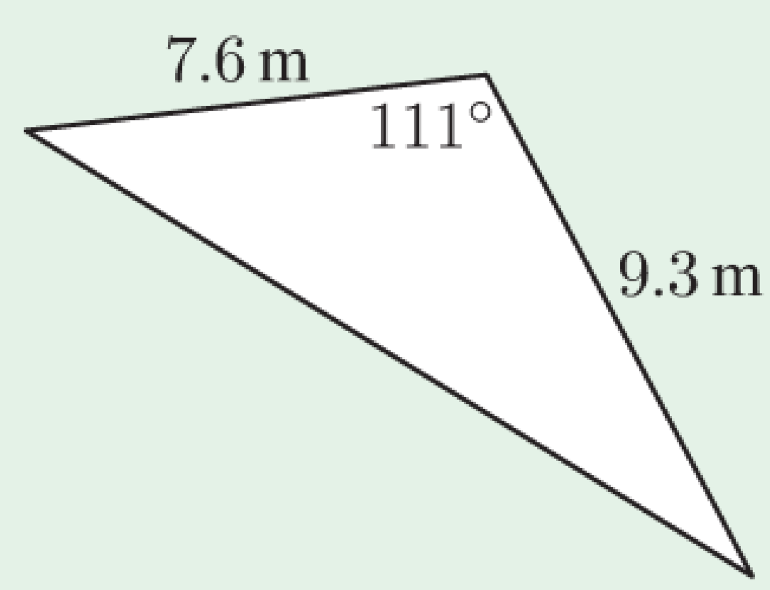
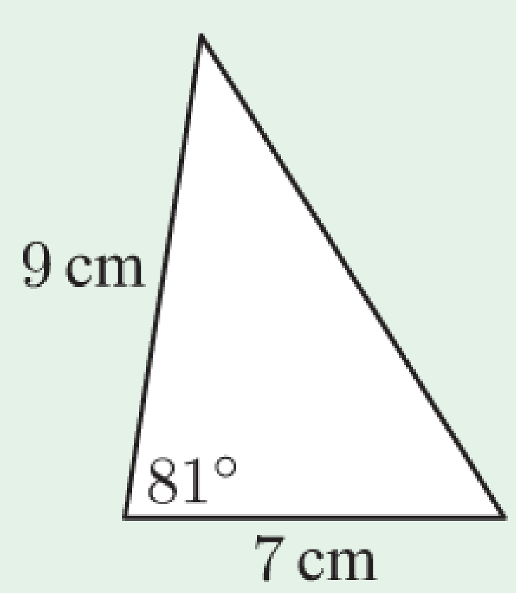


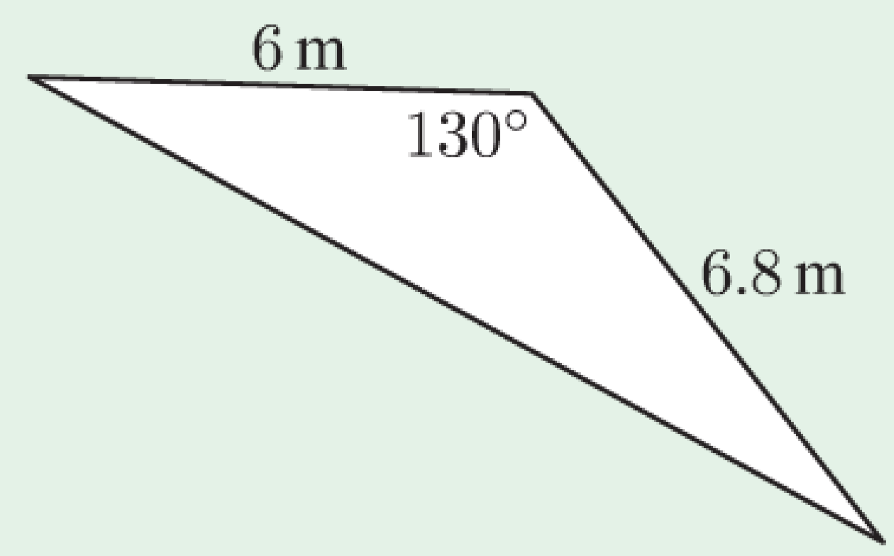
2 A rhombus has sides of length 5 cm and an angle of 65°. Find its area.

3 Find the length of the remaining side in each triangle:

a

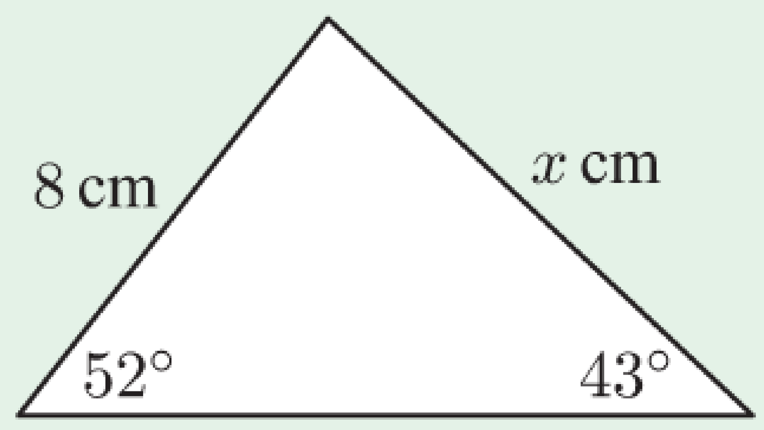


b

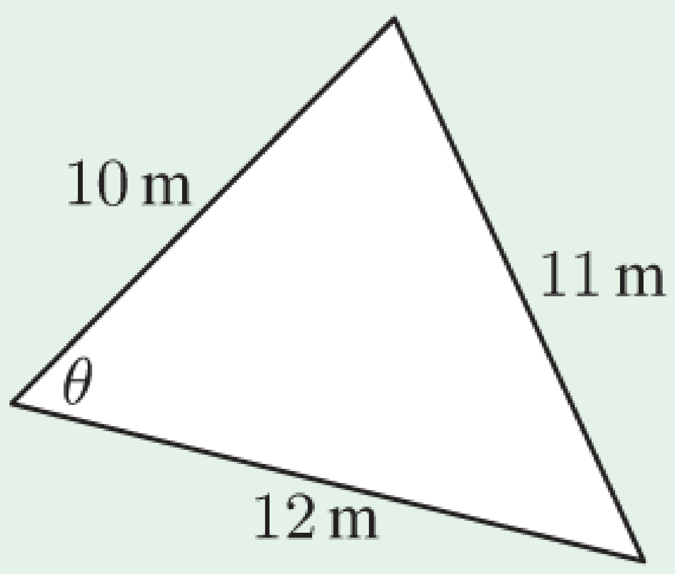


4 Find the unknown in:

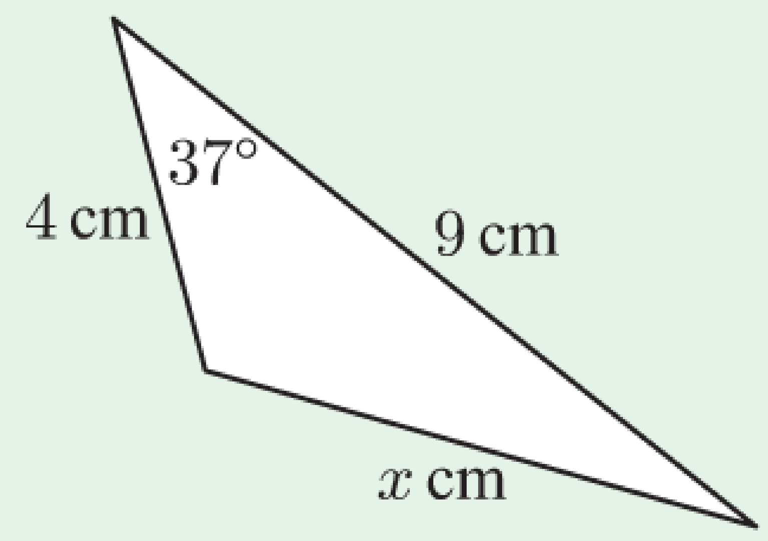
a



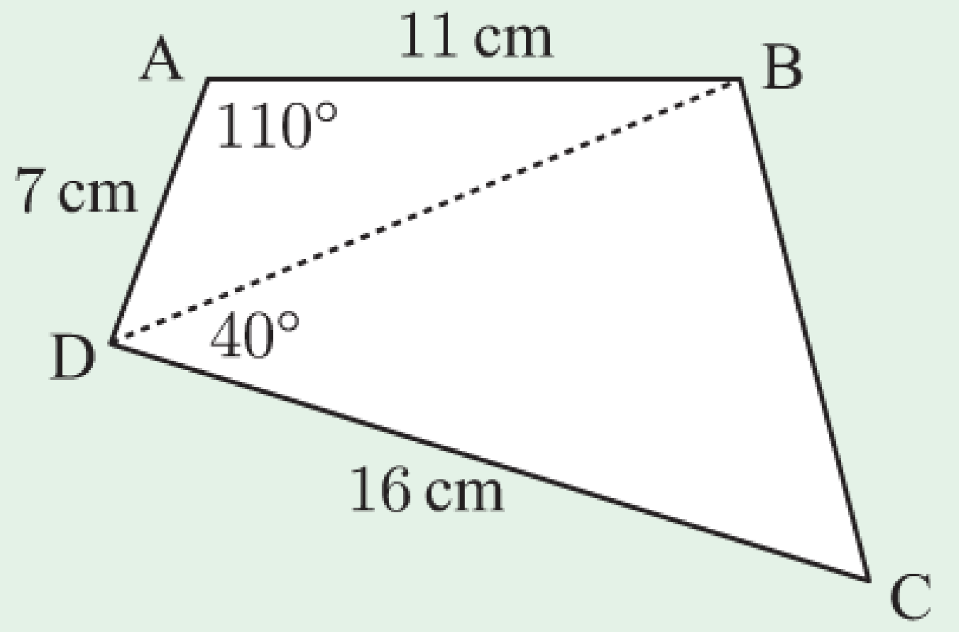
b



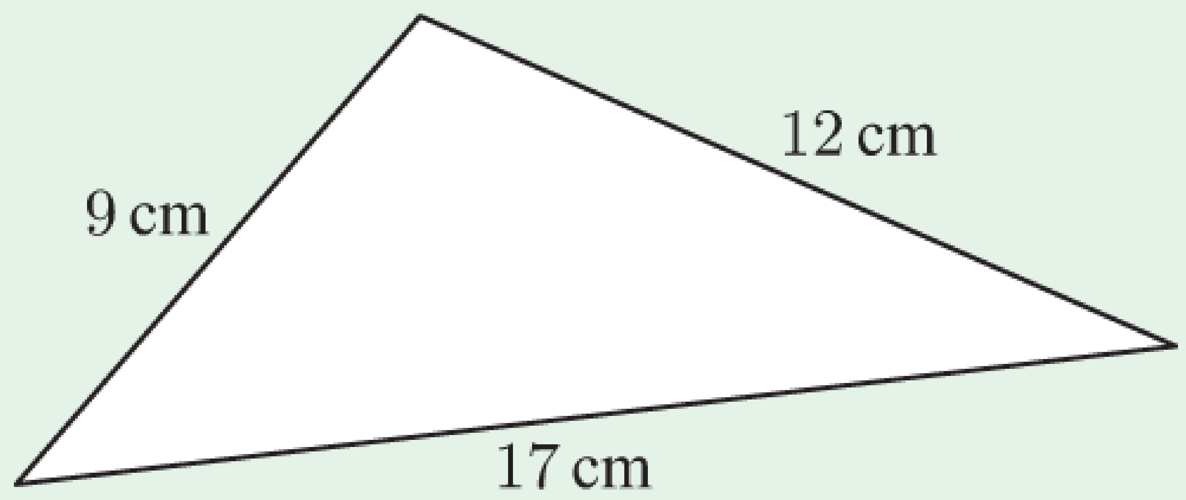
c



5 Find the area of quadrilateral ABCD:



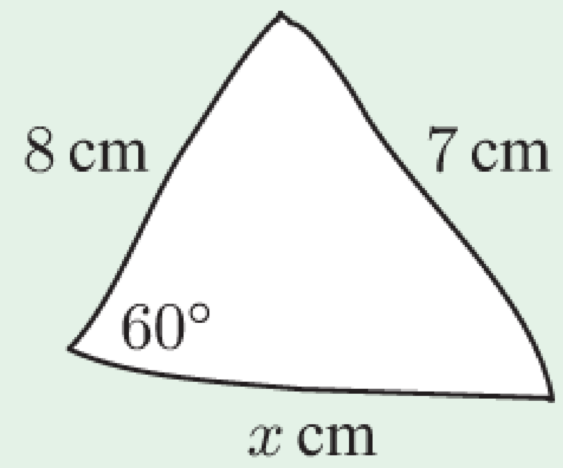
6 Find the area of this triangle.



7 Kady was asked to draw the illustrated triangle exactly.

a Use the cosine rule to find x.

b What should Kady's response be?

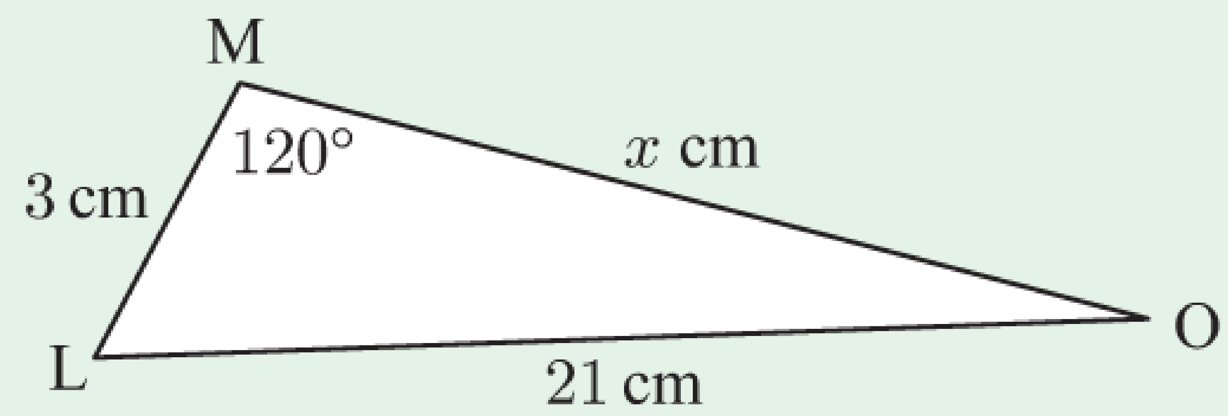


8 Triangle LMO has $L\widehat{M}O = 120°$, $LM = 3$ cm, $LO = 21$ cm, and $MO = x$ cm.

a Show that $x^2 + 3x - 432 = 0$.

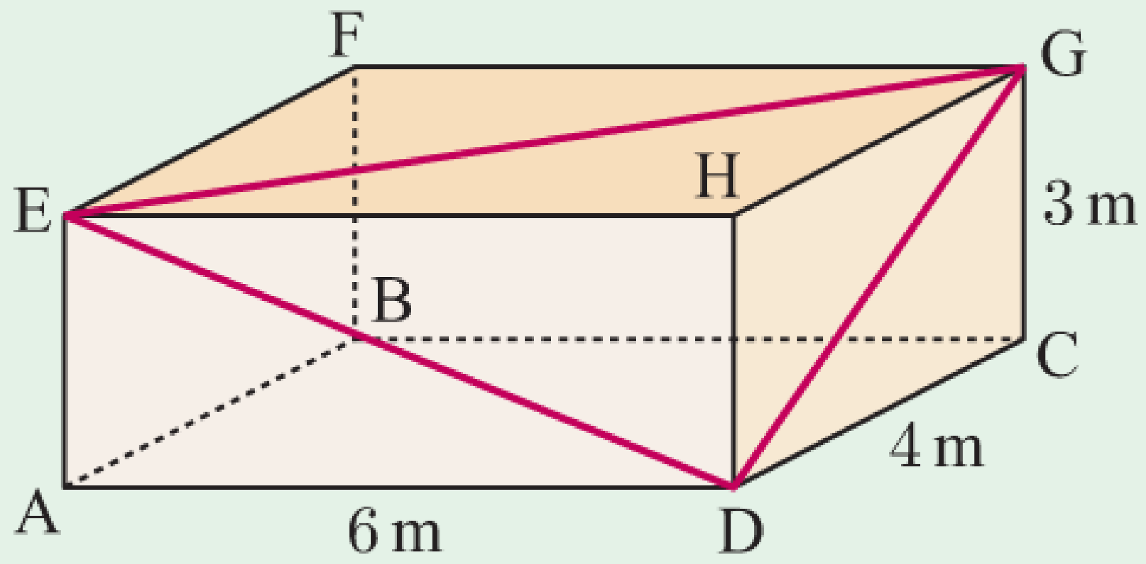
b Find x correct to 3 significant figures.

c Find the perimeter of triangle LMO.

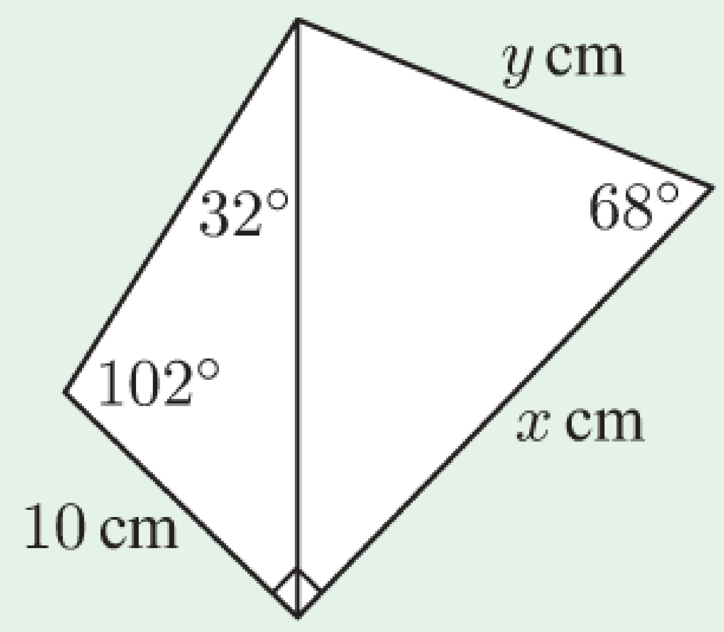


9 Two angles of a triangle have size 35° and 82°, and the area of the triangle is 40 cm^2. Find the length of each side of the triangle.

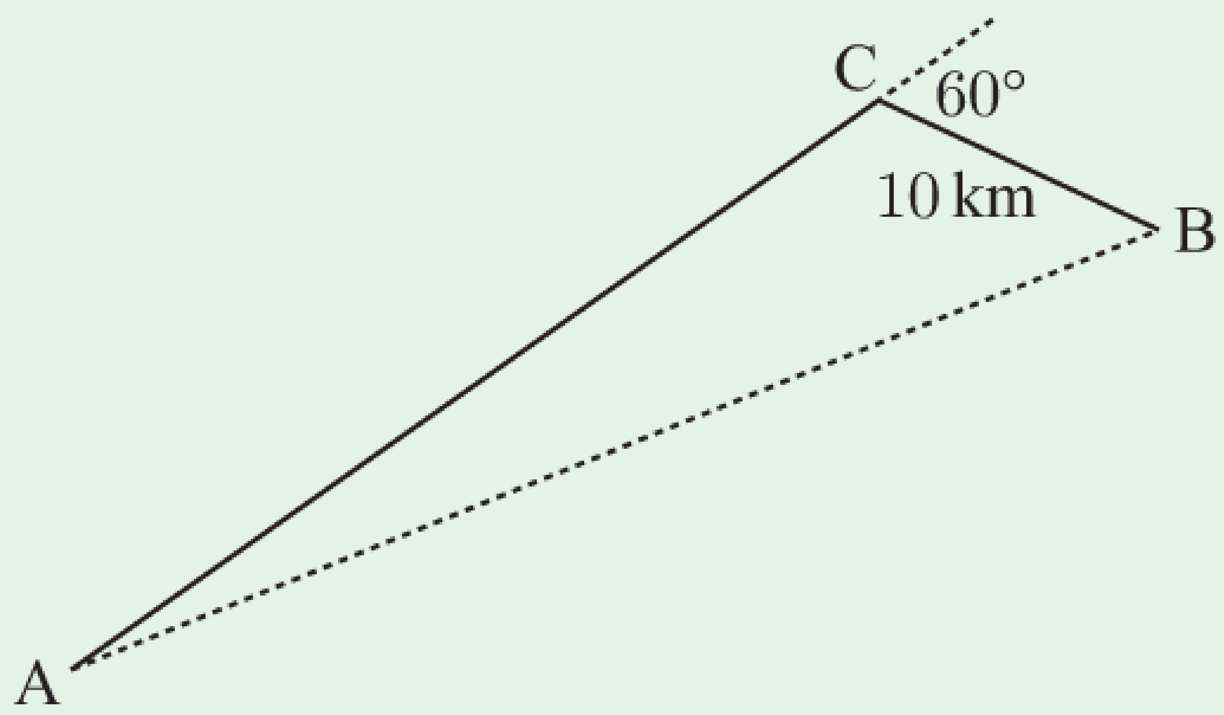
10 Find the measure of $E\widehat{D}G$:



11 Find x and y in this figure.



12



A boat was supposed to be sailing directly from A to B. However, it travelled in a straight line to C before the captain realised he was off course. He turned the boat through an angle of 60°, then travelled another 10 km to B. The trip would have been 4 km shorter if the boat had gone straight from A to B. How far did the boat travel?

13 In triangle ABC, $\widehat{ACB} = 42°$, AB = 5 cm, and AC = 7 cm.

a Find the two possible measures of $\widehat{ABC}$.

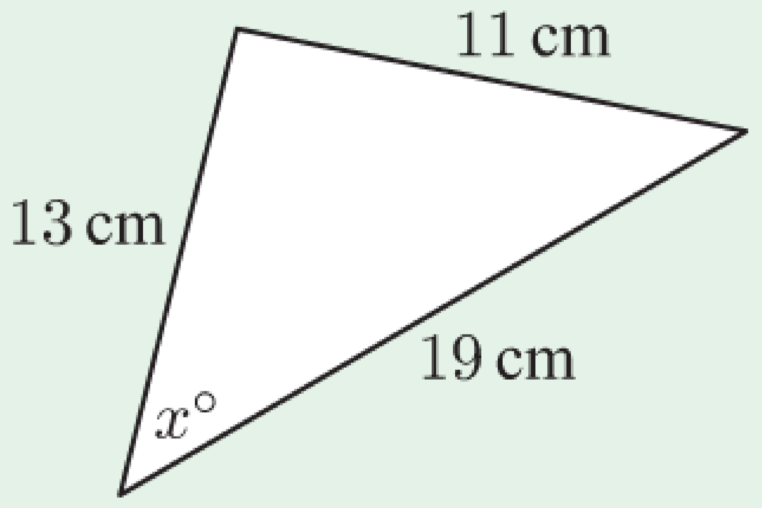
b Find the area of triangle ABC in each case.

14 Dune buggies X and Y are 500 m apart, and the bearing of Y from X is 215°. X travels 200 m due east and Y travels 100 m due north. How far apart are the dune buggies now?

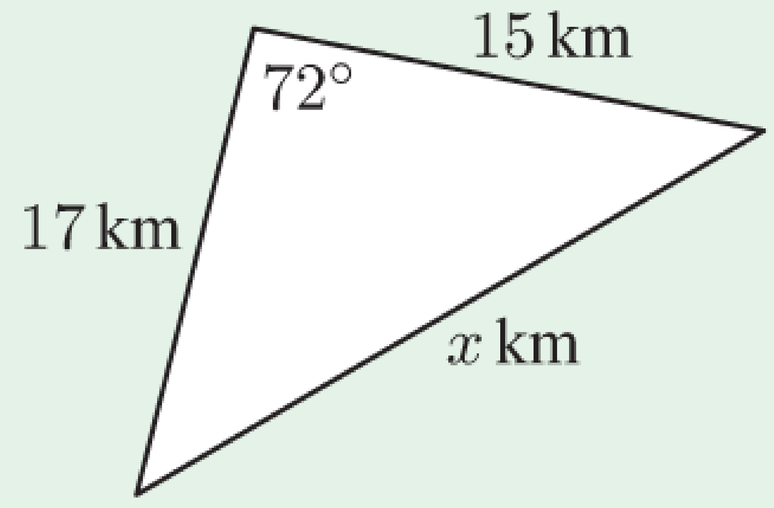
REVIEW SET 9B

1 Find the value of x:

a



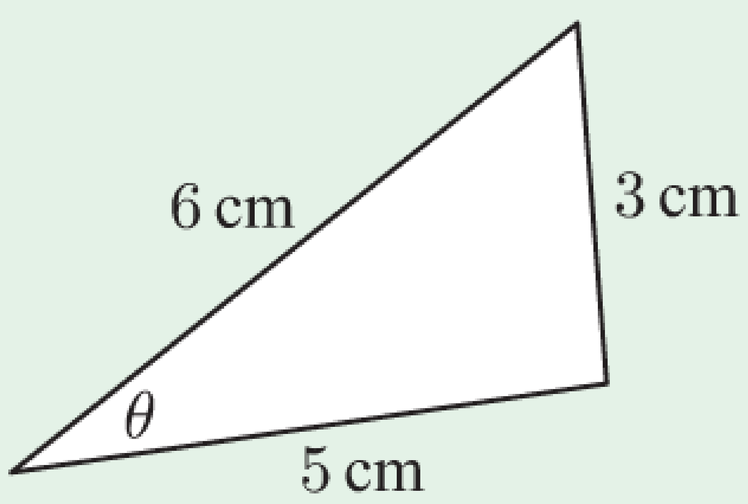
b



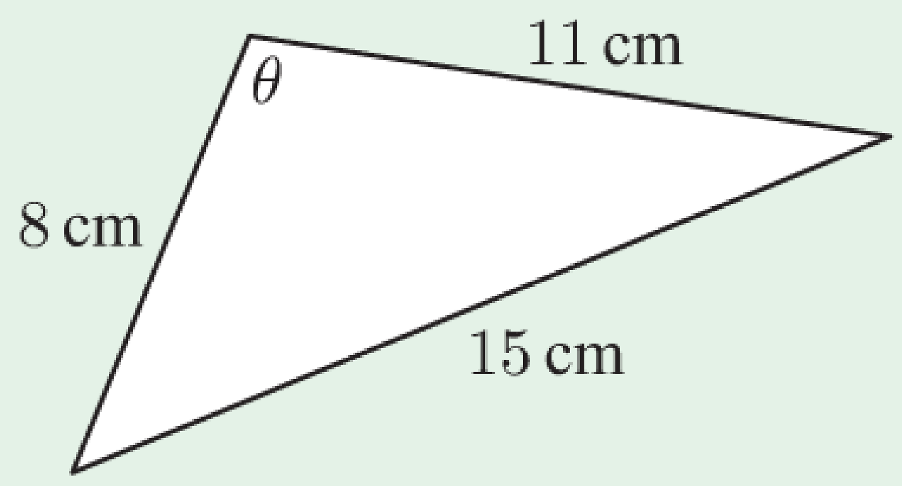
2 A triangle has two sides with lengths 11.3 cm and 19.2 cm, and an area of 80 cm^2. Find the possible measures of the included angle.

3 Find the measure of the angle marked θ:

a

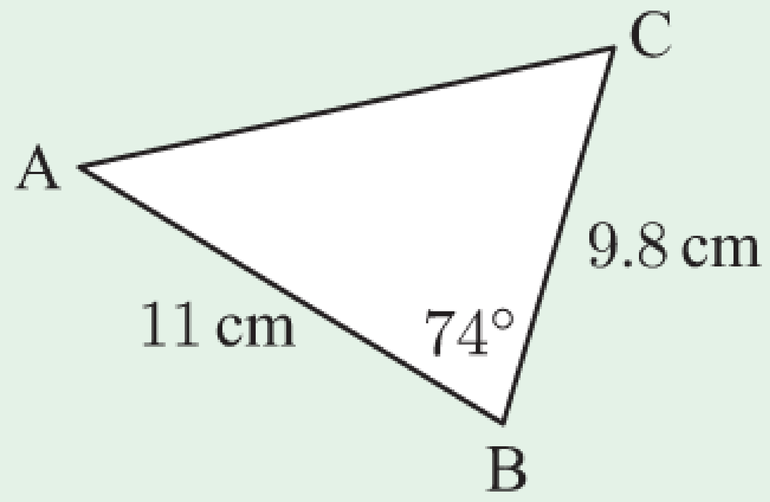


b



4 Find any unknown sides and angles in:

a



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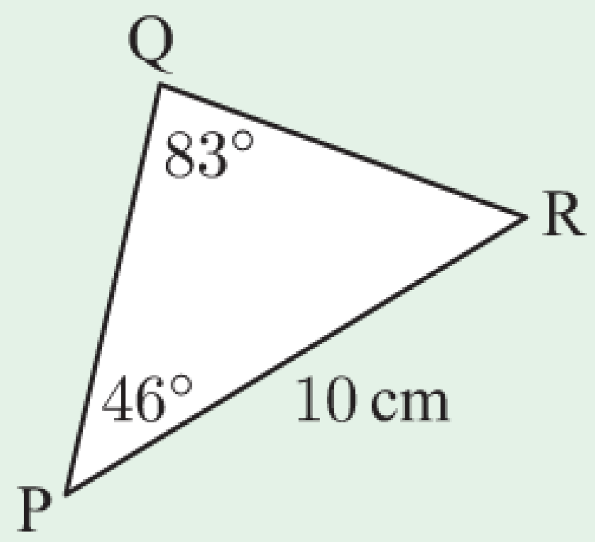
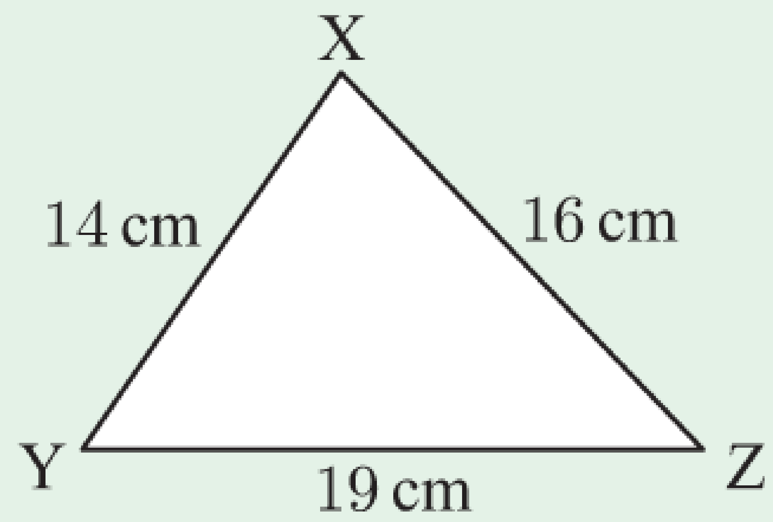
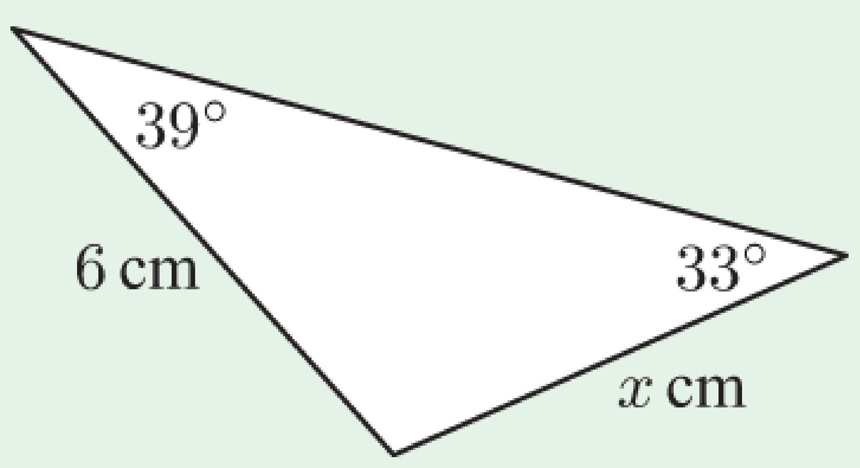


c

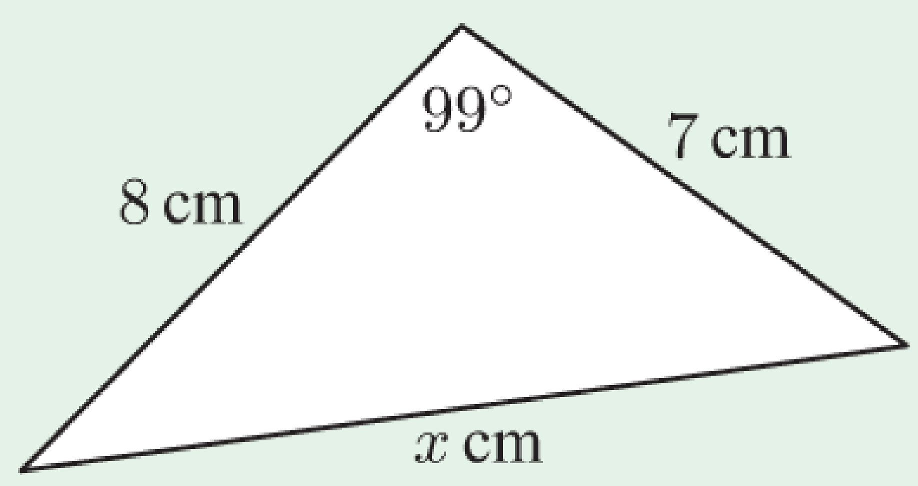


5 Find x in each triangle:

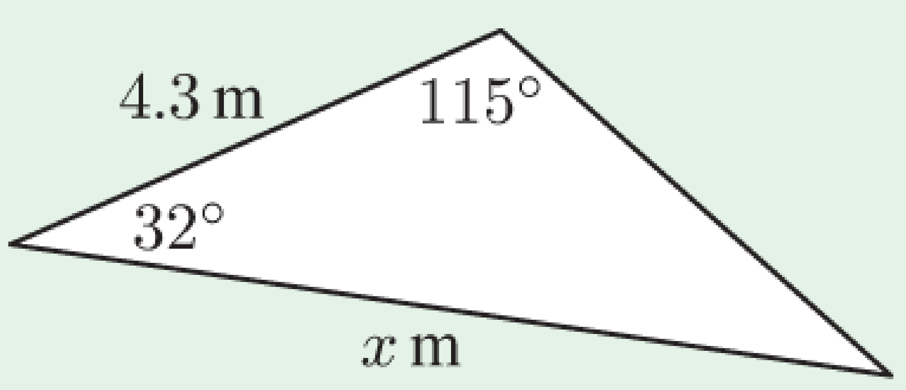
a



b



c



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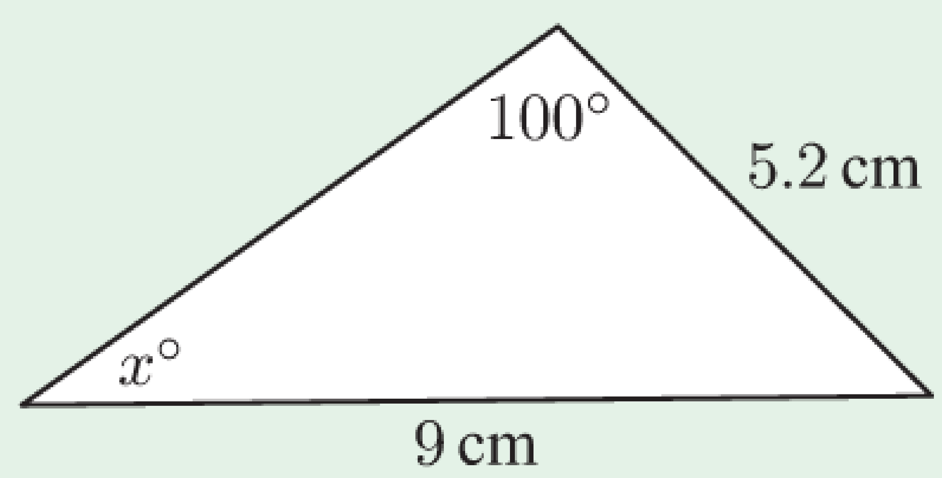
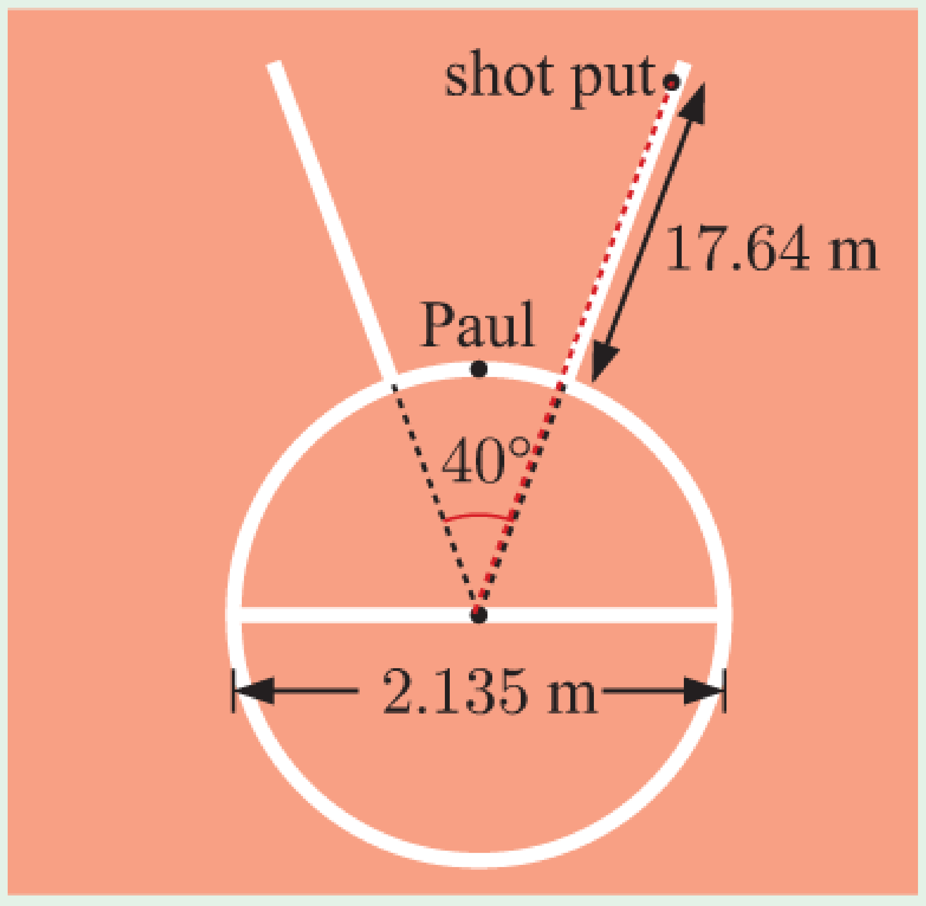
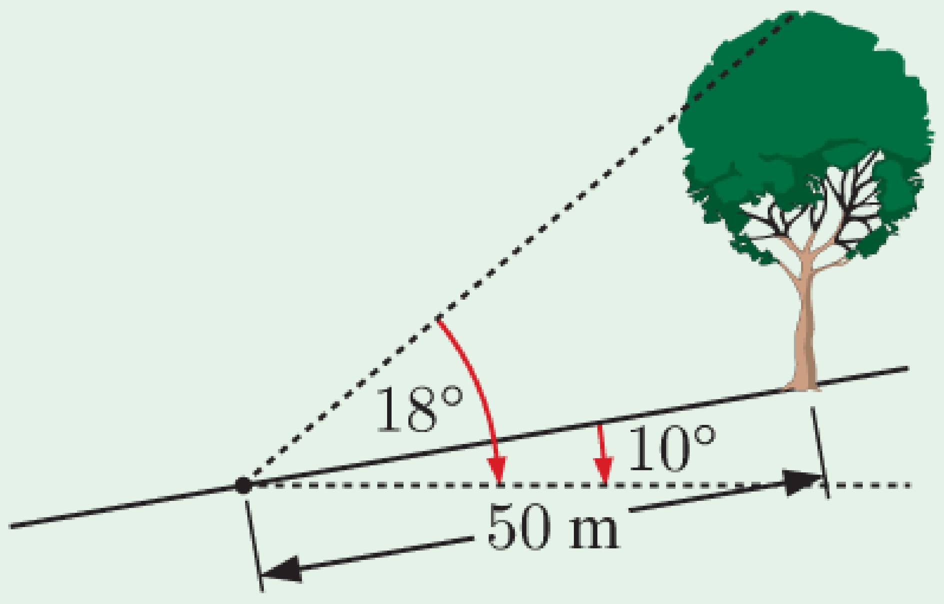


6 Paul "puts" a shot put from the front of a throwing circle with diameter 2.135 m. It only just lands inside the 40° throwing boundaries.
The official measurement goes from the shot to the nearest point of the throwing circle, and reads 17.64 m.
How far did Paul actually put the shot?

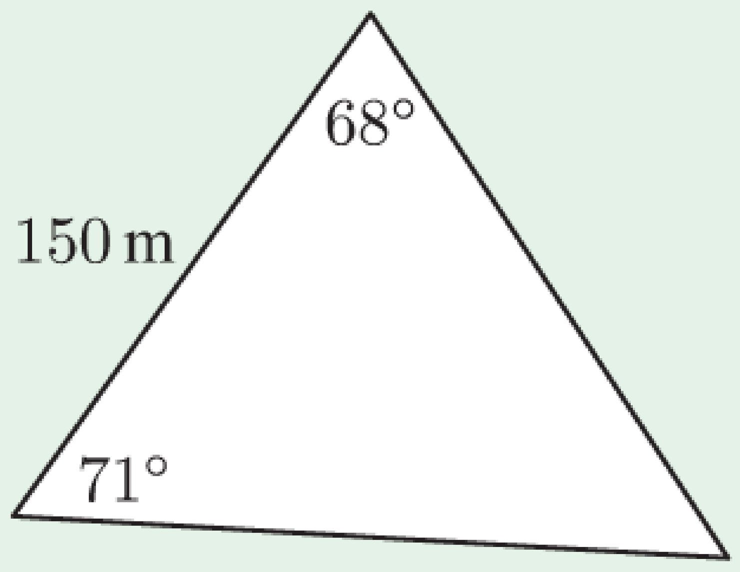


7



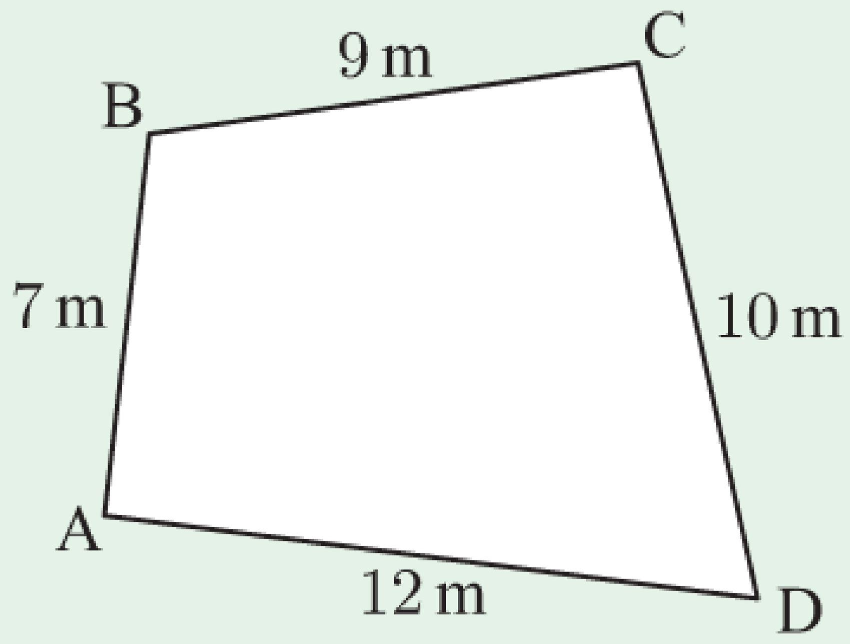
A vertical tree is growing on the side of a hill with gradient 10° to the horizontal. From a point 50 m downhill from the tree, the angle of elevation to the top of the tree is 18°. Find the height of the tree.

8 Find the perimeter and area of this triangle.



9 Peter, Sue, and Alix are sea-kayaking. Peter is 430 m from Sue on the bearing 113°. Alix is on the bearing 210° and is 310 m from Sue. Find the distance and bearing of Peter from Alix.

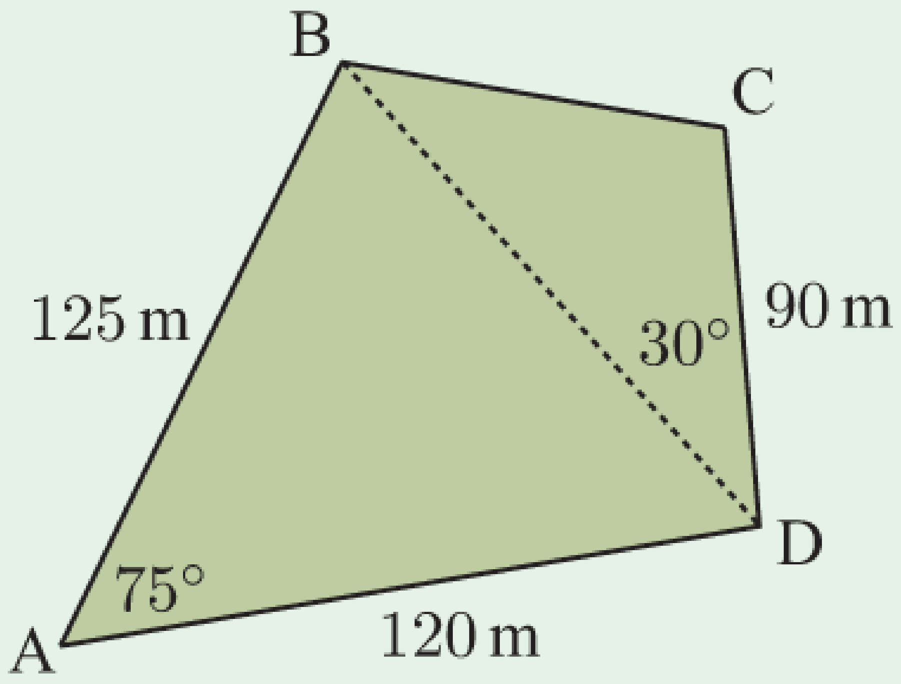
10 In quadrilateral ABCD, $A\widehat{B}C = 105°$. Find the measure of the other three angles.



11 Find the measure of angle Q in triangle PQR given that $Q\widehat{P}R = 47°$, QR = 11 m, and PR = 9.6 m.

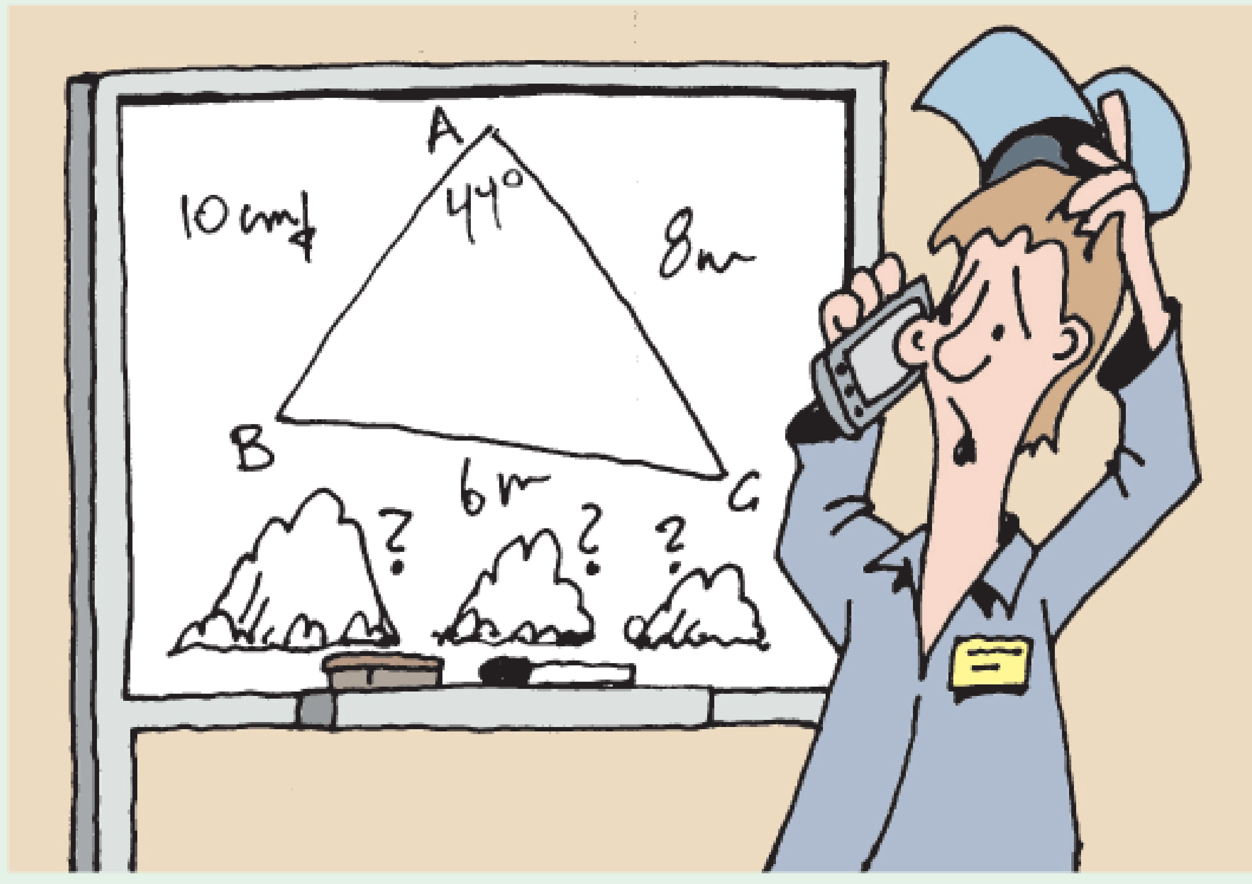
12 Anke and Lucas are considering buying a block of land. The land agent supplies them with the given accurate sketch. Find the area of the property, giving your answer in:

a m^2 **b** hectares.



13 Soil contractor Frank was given the following dimensions over the telephone:

The triangular garden plot ABC has $C\widehat{A}B$ measuring 44°, [AC] is 8 m long, and [BC] is 6 m long. Frank needs to supply soil for the plot to a depth of 10 cm.



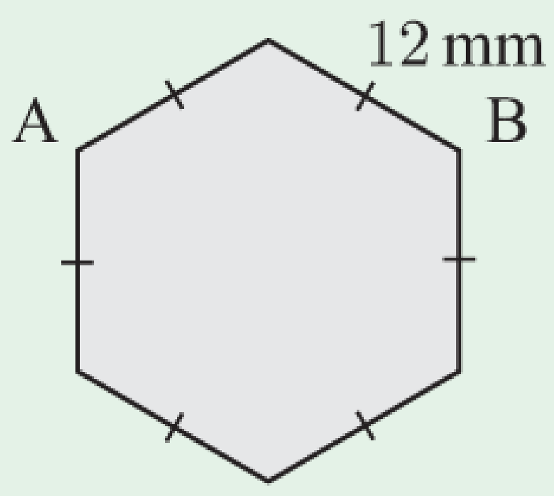
a Explain why Frank needs extra information from his client.

b What is the maximum volume of soil that could be needed if his client is unable to supply the necessary information?

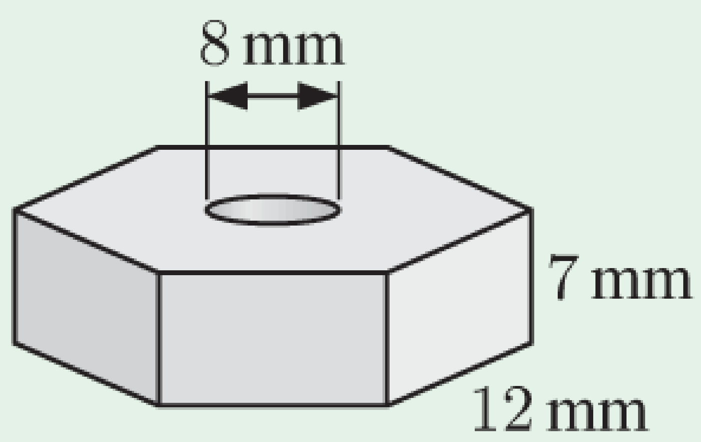
14 **a** Consider the regular hexagon shown. Find:

i the length AB

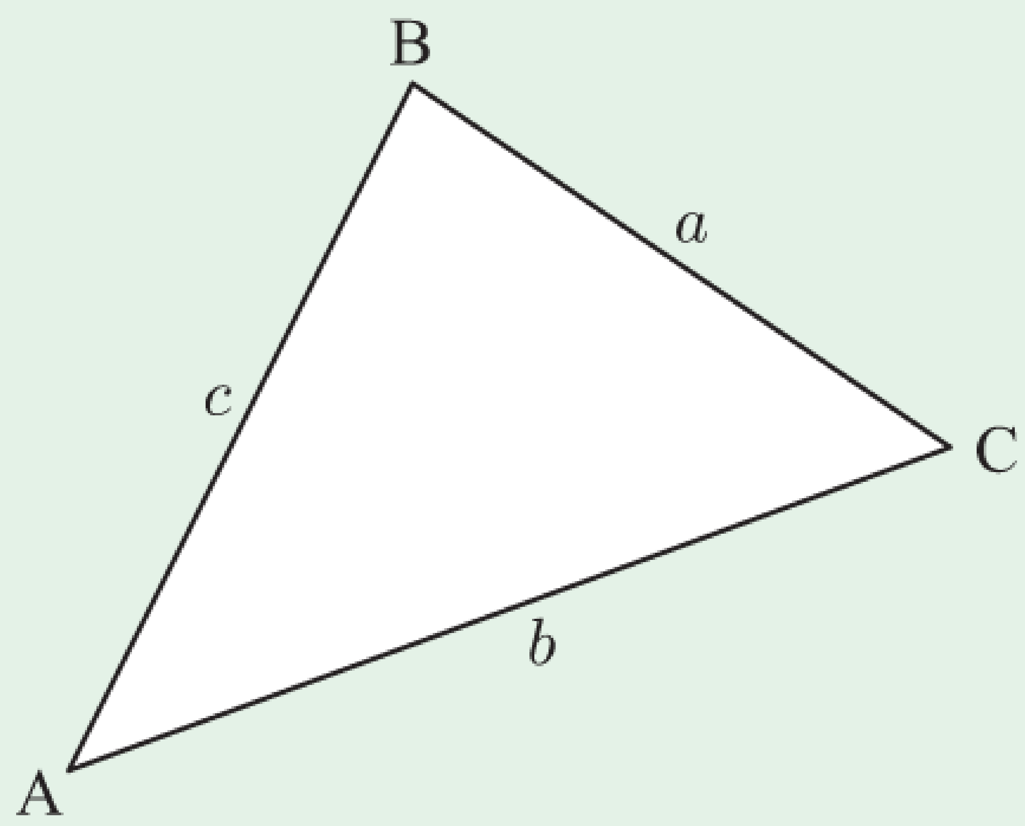
ii the area of the hexagon.



b Find the volume of this metal nut.



15 Over 2000 years ago, the Greek mathematician **Heron** or **Hero** discovered a formula for finding the area of a triangle with sides a, b, and c. Heron's formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{a+b+c}{2}$ is the **semi-perimeter** of the triangle.

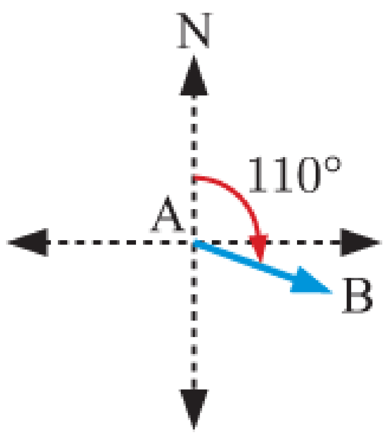


a Show that

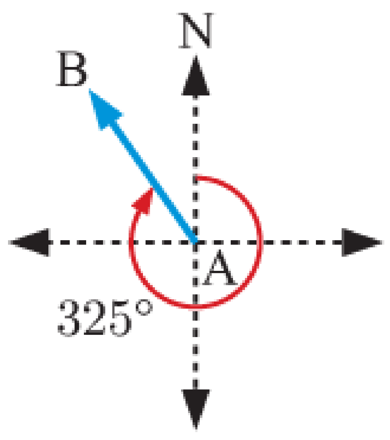
$$A^2 = \frac{b^2c^2}{4}\left(1 + \frac{b^2+c^2-a^2}{2bc}\right)\left(1 - \frac{b^2+c^2-a^2}{2bc}\right).$$

b Hence prove Heron's formula.

c



d

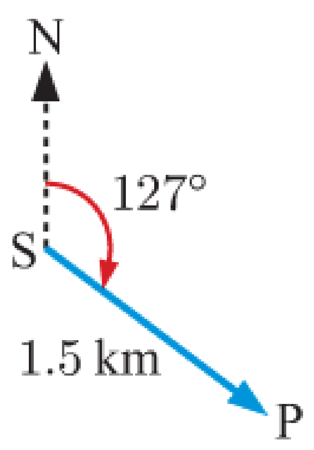


2 **a** 126° **b** 245° **c** 152° **d** 308°

3 **a** 072° **b** 252° **c** 162° **d** 342°
e 113° **f** 293°

4 $\approx 125°$ **5** **a** ≈ 224 m **b** $\approx 333°$ **c** $\approx 153°$

6 **a**



b ≈ 1.20 km **c** ≈ 0.903 km

7 ≈ 2.41 km **8** ≈ 12.6 km

9 **a** ≈ 854 m **b** $\approx 203°$

10 ≈ 73.3 km on the bearing $\approx 191°$

11 ≈ 17.8 km on the bearing $\approx 162°$

12 **a** $\approx 046.6°$ **b** ≈ 4.22 km

EXERCISE 7F

1 **a** **i** [EH] **ii** [EF] **iii** [EG] **iv** [FH]
b **i** [MR] **ii** [MN]

2 **a** **i** $\widehat{AFE}$ **ii** $\widehat{BMF}$ **iii** $\widehat{ADE}$ **iv** $\widehat{BNF}$
b **i** $\widehat{BAM}$ **ii** $\widehat{BNM}$ **iii** $\widehat{EAN}$

3 **a** **i** $\approx 36.9°$ **ii** $\approx 25.1°$ **iii** $\approx 56.3°$ **iv** $\approx 29.1°$
b **i** $\approx 33.7°$ **ii** $\approx 33.7°$ **iii** $\approx 25.2°$ **iv** $\approx 30.8°$
c **i** $\approx 59.0°$ **ii** $\approx 22.0°$ **iii** $\approx 22.6°$
d **i** $\approx 64.9°$ **ii** $\approx 71.7°$

4 $\approx 31.7°$

REVIEW SET 7A

1 **a** 10 cm **b** $\frac{6}{10} = \frac{3}{5}$ **c** $\frac{8}{10} = \frac{4}{5}$ **d** $\frac{6}{8} = \frac{3}{4}$

2 **a** $x \approx 3.51$ **b** $x \approx 51.1$ **c** $x \approx 5.62$

3 ≈ 43.4 cm^2 **4** $\theta = 33°$, $x \approx 3.90$, $y \approx 7.15$

5 $\theta \approx 8.19°$ **6** $\approx 124°$

7 **a** $x \approx 2.8$ **b** $x \approx 4.2$ **c** $x \approx 5.2$

8 ≈ 13.5 m **9** **a** 118° **b** 231° **c** 329°

10 13 km on the bearing $\approx 203°$ from the helipad.

11 $\approx 8.74°$ **12** ≈ 0.607 L **13** **a** $\approx 53.1°$ **b** $\approx 62.1°$

REVIEW SET 7B

1 **a** AB ≈ 4.5 cm, AC ≈ 2.2 cm, BC ≈ 5.0 cm
b **i** ≈ 0.44 **ii** ≈ 0.90 **iii** ≈ 0.49

2 **a** $\theta \approx 34.8°$ **b** $\theta \approx 39.7°$ **c** $\theta \approx 36.0°$

3 AB ≈ 120 mm, AC ≈ 111 mm

4 $x \approx 25.7$, $\theta \approx 53.6°$, $\alpha \approx 36.4°$

5 **a** ≈ 200 cm **b** ≈ 1500 cm^2 **6** ≈ 2.54 cm

7 ≈ 204 m **8** **a** 90° **b** $\approx 33.9°$

9 ≈ 3.91 km on the bearing $\approx 253°$ from his starting point.

10 ≈ 5.46 km **11** ≈ 485 m^3

12 **a** $\approx 14.4°$ **b** $\approx 18.9°$ **c** $\approx 21.8°$

13 **a** **i** ≈ 27.6 cm **ii** ≈ 23.3 cm **b** ≈ 6010 cm^3

EXERCISE 8A

1 **a** $\frac{\pi}{2}$ **b** $\frac{\pi}{3}$ **c** $\frac{\pi}{6}$ **d** $\frac{\pi}{10}$ **e** $\frac{\pi}{20}$
f $\frac{3\pi}{4}$ **g** $\frac{5\pi}{4}$ **h** $\frac{3\pi}{2}$ **i** 2π **j** 4π
k $\frac{7\pi}{4}$ **l** 3π **m** $\frac{\pi}{5}$ **n** $\frac{4\pi}{9}$ **o** $\frac{23\pi}{18}$

2 **a** $\approx 0.641^c$ **b** $\approx 2.39^c$ **c** $\approx 5.55^c$ **d** $\approx 3.83^c$
e $\approx 6.92^c$

3 **a** 36° **b** 108° **c** 135° **d** 10° **e** 20°
f 140° **g** 18° **h** 27° **i** 210° **j** 22.5°

4 **a** $\approx 114.59°$ **b** $\approx 87.66°$ **c** $\approx 49.68°$
d $\approx 182.14°$ **e** $\approx 301.78°$

5 **a**

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Deg.	0	30	60	90	120	150	180	210	240	270	300	330	360
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

EXERCISE 8B

1 **a** 7 cm **b** 12 cm **c** ≈ 13.0 m

2 **a** 6 cm^2 **b** 48 cm^2 **c** ≈ 8.21 cm^2

3 **a** arc length ≈ 49.5 cm, area ≈ 223 cm^2
b arc length ≈ 23.0 cm, area ≈ 56.8 cm^2

4 **a** $\approx 0.686^c$ **b** 0.6^c

5 **a** $\theta = 0.75^c$, area $= 24$ cm^2
b $\theta = 1.68^c$, area $= 21$ cm^2
c $\theta \approx 2.32^c$, area $= 126.8$ cm^2

6 **a** ≈ 3.15 m **b** ≈ 9.32 m^2

7 **a** ≈ 5.91 cm **b** ≈ 18.9 cm

8 **a** $\alpha \approx 0.3218^c$ **b** $\theta \approx 2.498^c$ **c** ≈ 387 m^2

9 **a** ≈ 11.7 cm **b** $r \approx 11.7$ **c** ≈ 37.7 cm **d** $\theta \approx 3.23^c$

10 ≈ 25.9 cm **11** **b** ≈ 2 h 24 min **12** ≈ 227 m^2

13 **a** $\alpha \approx 5.739$ **b** $\theta \approx 168.5$ **c** $\phi \approx 191.5$
d ≈ 71.62 cm

14 **a** 4 cm **b** **i** ≈ 2.16 cm^2 **ii** ≈ 29.3 cm^2

15 **a** **Hint:** Let the largest circle have radius r_1, and use a right angled triangle to show that $\sin\frac{\pi}{6} = \dfrac{r_1}{10 - r_1}$.
b $\frac{25\pi}{2}$ units^2 **c** $\frac{3}{4}$

EXERCISE 8C

1

θ (degrees)	0°	90°	180°	270°	360°	450°
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
sine	0	1	0	-1	0	1
cosine	1	0	-1	0	1	0
tangent	0	undef.	0	undef.	0	undef.

2 **a** **i** A(cos 26°, sin 26°), B(cos 146°, sin 146°), C(cos 199°, sin 199°)
ii A(0.899, 0.438), B(-0.829, 0.559), C(-0.946, -0.326)
b **i** A(cos 123°, sin 123°), B(cos 251°, sin 251°), C(cos($-35°$), sin($-35°$))
ii A(-0.545, 0.839), B(-0.326, -0.946), C(0.819, -0.574)

3 **a** **i** $\frac{1}{\sqrt{2}} \approx 0.707$ **ii** $\frac{\sqrt{3}}{2} \approx 0.866$

b

θ (degrees)	$30°$	$45°$	$60°$	$135°$	$150°$	$240°$	$315°$
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{7\pi}{4}$
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tangent	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$	-1

4

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve	+ve
2	$90° < \theta < 180°$	$\frac{\pi}{2} < \theta < \pi$	−ve	+ve	−ve
3	$180° < \theta < 270°$	$\pi < \theta < \frac{3\pi}{2}$	−ve	−ve	+ve
4	$270° < \theta < 360°$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	−ve	−ve

5 **a** 1 and 4 **b** 2 and 3 **c** 3 **d** 2

6 **a** $\cos 400° = \cos(360 + 40)° = \cos 40°$

b $\sin\frac{5\pi}{7} = \sin\left(\frac{5\pi}{7} + 2\pi\right) = \sin\frac{19\pi}{7}$

c $\tan\frac{13\pi}{8} = \tan\left(\frac{13\pi}{8} - 3\pi\right) = \tan\left(-\frac{11\pi}{8}\right)$

7 **B** and **D** **8** **B** and **E**

9 **a** **i** ≈ 0.985 **ii** ≈ 0.985 **iii** ≈ 0.866 **iv** ≈ 0.866
v 0.5 **vi** 0.5 **vii** ≈ 0.707 **viii** ≈ 0.707

b $\sin(180° - \theta) = \sin\theta$ **c** $\sin(\pi - \theta) = \sin\theta$

d The points have the same y-coordinate.

e **i** $135°$ **ii** $129°$ **iii** $\frac{2\pi}{3}$ **iv** $\frac{5\pi}{6}$

10 **a** **i** ≈ 0.342 **ii** ≈ -0.342 **iii** 0.5
iv -0.5 **v** ≈ 0.906 **vi** ≈ -0.906
vii ≈ 0.174 **viii** ≈ -0.174

b $\cos(180° - \theta) = -\cos\theta$ **c** $\cos(\pi - \theta) = -\cos\theta$

d The x-coordinates of the points have the same magnitude but are opposite in sign.

e **i** $140°$ **ii** $161°$ **iii** $\frac{4\pi}{5}$ **iv** $\frac{3\pi}{5}$

11 $\tan(\pi - \theta) = -\tan\theta$

12 **a** ≈ 0.6820 **b** ≈ 0.8572 **c** ≈ -0.7986
d ≈ 0.9135 **e** ≈ 0.9063 **f** ≈ -0.6691

13 **a**

θ^c	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75	≈ 0.682	≈ -0.682	≈ 0.732	≈ 0.732
1.772	≈ 0.980	≈ -0.980	≈ -0.200	≈ -0.200
3.414	≈ -0.269	≈ 0.269	≈ -0.963	≈ -0.963
6.25	≈ -0.0332	≈ 0.0332	≈ 0.999	≈ 0.999
−1.17	≈ -0.921	≈ 0.921	≈ 0.390	≈ 0.390

b $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$

c Q has coordinates $(\cos(-\theta), \sin(-\theta))$ or
$(\cos\theta, -\sin\theta)$ (since it is the reflection of P in the x-axis)
$\therefore$ $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

d $\cos(2\pi - \theta) = \cos(-\theta) = \cos\theta$
$\sin(2\pi - \theta) = \sin(-\theta) = -\sin\theta$

e $\tan(2\pi - \theta) = -\tan\theta$

14 **a** The angle between [OP] and the positive x-axis is $\left(\frac{\pi}{2} - \theta\right)$.
$\therefore$ P is $\left(\cos\left(\frac{\pi}{2} - \theta\right), \sin\left(\frac{\pi}{2} - \theta\right)\right)$

b **i** In $\triangle$OXP, $\sin\theta = \frac{\text{XP}}{\text{OP}} = \frac{\text{XP}}{1}$
$\therefore$ XP $= \sin\theta$

ii In $\triangle$OXP, $\cos\theta = \frac{\text{OX}}{\text{OP}} = \frac{\text{OX}}{1}$
$\therefore$ OX $= \cos\theta$

c **i** $\cos\left(\frac{\pi}{2} - \theta\right) = \text{XP} = \sin\theta$

ii $\sin\left(\frac{\pi}{2} - \theta\right) = \text{OX} = \cos\theta$

d **i** $\cos\frac{\pi}{5} = \sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right) = \sin\frac{3\pi}{10} \approx 0.809$

ii $\sin\frac{\pi}{8} = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos\frac{3\pi}{8} \approx 0.383$

e $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\theta}$

EXERCISE 8D

1

	a	**b**	**c**	**d**	**e**
$\sin\theta$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$
$\tan\theta$	1	-1	-1	0	1

2

	a	**b**	**c**	**d**	**e**
$\sin\beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos\beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\tan\beta$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$

3 **a** $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$

b $\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, $\tan\left(-\frac{\pi}{4}\right) = -1$

4 **a** $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$ **b** $\tan\frac{\pi}{2}$ is undefined

5 **a** $\frac{3}{4}$ **b** $\frac{1}{4}$ **c** $\frac{1}{4}$ **d** $-\frac{1}{4}$ **e** 1 **f** $\sqrt{2}$
g $\frac{1}{2}$ **h** $\frac{1}{2}$ **i** 2 **j** -1 **k** $-\sqrt{3}$ **l** $-\sqrt{3}$

6 **a** $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\frac{\pi}{4}, \frac{7\pi}{4}$ **d** $\frac{2\pi}{3}, \frac{4\pi}{3}$
e $\frac{3\pi}{4}, \frac{5\pi}{4}$ **f** $\frac{4\pi}{3}, \frac{5\pi}{3}$

7 **a** $\frac{\pi}{4}, \frac{5\pi}{4}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{3}, \frac{4\pi}{3}$ **d** $0, \pi, 2\pi$
e $\frac{\pi}{6}, \frac{7\pi}{6}$ **f** $\frac{2\pi}{3}, \frac{5\pi}{3}$

8 **a** $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$ **b** $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$ **c** $\frac{3\pi}{2}, \frac{7\pi}{2}$

9 **a** $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\theta = \pi$
d $\theta = \frac{\pi}{2}$ **e** $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$ **f** $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$
g $\theta = 0, \pi, 2\pi$ **h** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
i $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$ **j** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

10 **a** $\theta = k\pi$, $k \in \mathbb{Z}$ **b** $\theta = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

EXERCISE 8E

1 **a** $\cos\theta = \pm\frac{\sqrt{3}}{2}$ **b** $\cos\theta = \pm\frac{2\sqrt{2}}{3}$ **c** $\cos\theta = \pm 1$
d $\cos\theta = 0$

2 **a** $\sin\theta = \pm\frac{3}{5}$ **b** $\sin\theta = \pm\frac{\sqrt{7}}{4}$ **c** $\sin\theta = 0$
d $\sin\theta = \pm 1$

3 **a** $\sin\theta = \frac{\sqrt{5}}{3}$ **b** $\cos\theta = -\frac{\sqrt{21}}{5}$ **c** $\cos\theta = \frac{4}{5}$
d $\sin\theta = -\frac{12}{13}$

4 **a** $\tan\theta = -\frac{1}{2\sqrt{2}}$ **b** $\tan\theta = -2\sqrt{6}$ **c** $\tan\theta = \frac{1}{\sqrt{2}}$
d $\tan\theta = -\frac{\sqrt{7}}{3}$

5 **a** $\sin\theta = \frac{2}{\sqrt{13}}$, $\cos\theta = \frac{3}{\sqrt{13}}$ **b** $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$
c $\sin\theta = -\sqrt{\frac{5}{14}}$, $\cos\theta = -\frac{3}{\sqrt{14}}$
d $\sin\theta = -\frac{12}{13}$, $\cos\theta = \frac{5}{13}$

6 $\sin\theta = \dfrac{-k}{\sqrt{k^2+1}}$, $\cos\theta = \dfrac{-1}{\sqrt{k^2+1}}$

EXERCISE 8F

1 **a** $\theta \approx 76.0°$ or $256°$ **b** $\theta \approx 33.9°$ or $326.1°$
c $\theta \approx 36.9°$ or $143.1°$ **d** $\theta = 90°$ or $270°$
e $\theta \approx 81.5°$ or $261.5°$ **f** $\theta \approx 83.2°$ or $276.8°$

2 **a** $\theta \approx 0.322$ or 3.46 **b** $\theta \approx 1.13$ or 5.16
c $\theta \approx 0.656$ or 2.49 **d** $\theta \approx 1.32$ or 4.97
e $\theta \approx 0.114$ or 3.26 **f** $\theta \approx 0.167$ or 2.97

3 **a** $\theta \approx 1.82$ or 4.46 **b** $\theta = 0$, π, or 2π
c $\theta \approx 1.88$ or 5.02 **d** $\theta \approx 3.58$ or 5.85
e $\theta \approx 0.876$ or 4.02 **f** $\theta \approx 0.674$ or 5.61
g $\theta \approx 0.0910$ or 3.05 **h** $\theta \approx 2.19$ or 4.10

4 **a** $\theta \approx -95.7°$ or $95.7°$ **b** $\theta \approx 53.1°$ or $126.9°$
c $\theta \approx -56.3°$ or $123.7°$ **d** $\theta \approx -36.9°$ or $36.9°$
e $\theta \approx -39.8°$ or $140.2°$ **f** $\theta \approx -140.5°$ or $-39.5°$

5 **a** $\theta \approx 1.27$ or 5.02
b For $\theta \approx 1.27$: $\sin\theta = \frac{\sqrt{91}}{10}$, $\tan\theta = \frac{\sqrt{91}}{3}$
For $\theta \approx 5.02$: $\sin\theta = -\frac{\sqrt{91}}{10}$, $\tan\theta = -\frac{\sqrt{91}}{3}$

EXERCISE 8G

1 **a** $y = \sqrt{3}\,x$ **b** $y = x$ **c** $y = -\frac{1}{\sqrt{3}}x$

2 **a** $y = \sqrt{3}\,x + 2$ **b** $y = -\sqrt{3}\,x$ **c** $y = \frac{1}{\sqrt{3}}x - 2$

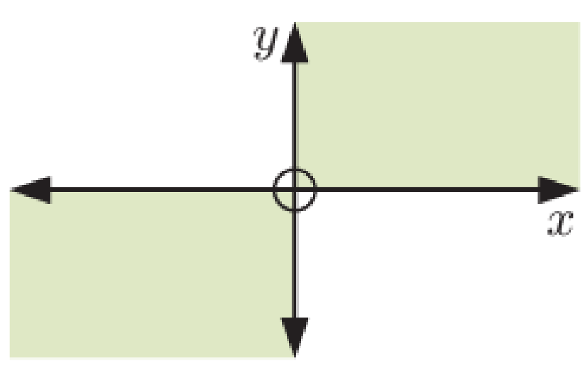
3 **a** $\theta \approx 1.25$ **b** $\theta \approx -0.983$ **c** $\theta \approx -0.381$

4 **a** $\theta \approx 23.2°$ **b** $\theta \approx 117°$ **c** $\theta \approx -11.3°$

REVIEW SET 8A

1 **a** $\frac{2\pi}{3}$ **b** $\frac{5\pi}{4}$ **c** $\frac{5\pi}{6}$ **d** 3π

2



3 **a** $(0.766, -0.643)$ **b** $(-0.956, 0.292)$
c $(0.778, 0.629)$ **d** $(0.866, -0.5)$

4 12 cm **5** **a** $\frac{\pi}{3}$ **b** $15°$ **c** $84°$

6 **a** ≈ 0.358 **b** ≈ -0.035 **c** ≈ 0.259 **d** ≈ 1.072

7 **a** $\cos 360° = 1$, $\sin 360° = 0$
b $\cos(-\pi) = -1$, $\sin(-\pi) = 0$

8 **a** $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$
b $\sin\frac{8\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{8\pi}{3} = -\frac{1}{2}$, $\tan\frac{8\pi}{3} = -\sqrt{3}$

9 **a** **i** $60°$ **ii** $\frac{\pi}{3}$ **b** $\frac{\pi}{3}$ cm **c** $\frac{\pi}{6}$ cm^2

10 $\tan x = \frac{1}{\sqrt{15}}$ **11** $\sin\theta = \pm\frac{\sqrt{7}}{4}$

12 **a** $\frac{\sqrt{3}}{2}$ **b** 0 **c** $\frac{1}{2}$ **13** **a** $\frac{2}{\sqrt{13}}$ **b** $-\frac{3}{\sqrt{13}}$

14 $\tan\theta = \frac{\sqrt{6}}{\sqrt{11}}$

16 **a** $\theta \approx 0.841$ or 5.44 **b** $\theta \approx 3.39$ or 6.03
c $\theta \approx 1.25$ or 4.39

17 **a** $y = \frac{1}{\sqrt{3}}x$ **b** $y = \sqrt{3}\,x + 3$

REVIEW SET 8B

1 **a** $72°$ **b** $\approx 83.65°$ **c** $\approx 24.92°$ **d** $\approx -302.01°$

2 ≈ 111 cm^2 **3** $\approx 103°$

4 radius ≈ 8.79 cm, area ≈ 81.0 cm^2 **5** 4.5 cm or 6 cm

6 **a** $\cos\frac{3\pi}{2} = 0$, $\sin\frac{3\pi}{2} = -1$
b $\cos\left(-\frac{\pi}{2}\right) = 0$, $\sin\left(-\frac{\pi}{2}\right) = -1$

7 **a** $\sin(\pi - p) = m$ **b** $\sin(p + 2\pi) = m$
c $\cos p = \sqrt{1 - m^2}$ **d** $\tan p = \dfrac{m}{\sqrt{1-m^2}}$

8 **a** $150°, 210°$ **b** $45°, 135°$ **c** $120°, 300°$

9 **a** $\theta = \pi$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

10 **a** $133°$ **b** $\frac{14\pi}{15}$ **c** $174°$

11 perimeter ≈ 34.1 cm, area ≈ 66.5 cm^2

13 **a** $\frac{\sqrt{7}}{4}$ **b** $-\frac{\sqrt{7}}{3}$ **c** $\frac{3}{4}$

14 **a** $2\frac{1}{2}$ **b** $1\frac{1}{2}$ **c** $-\frac{1}{2}$ **d** 3

17 **a** $\theta \approx 0.322$ **b** $\theta \approx 1.95$

EXERCISE 9A

1 **a** ≈ 28.9 cm^2 **b** ≈ 384 km^2 **c** 20 m^2

2 **a** ≈ 18.7 cm^2 **b** ≈ 28.3 cm^2 **c** ≈ 52.0 m^2

3 $x \approx 19.0$ **4** **a** ≈ 166 cm^2 **b** ≈ 1410 cm^2

5 ≈ 18.9 cm^2 **6** ≈ 137 cm^2

7 **a** ≈ 71.616 m^2 **b** ≈ 8.43 m

8 ≈ 374 cm^2 **9** ≈ 7.49 cm **10** ≈ 11.9 m

11 **a** $\approx 48.6°$ or $\approx 131.4°$ **b** $\approx 42.1°$ or $\approx 137.9°$

12 $\frac{1}{4}$ is not covered

13 **a** ≈ 36.2 cm^2 **b** ≈ 62.8 cm^2 **c** ≈ 40.4 mm^2
d ≈ 19.3 cm^2

14 ≈ 4.69 cm^2

EXERCISE 9B

1 **a** ≈ 28.8 cm **b** ≈ 3.38 km **c** ≈ 14.2 m

2 **a** $\theta \approx 82.8°$ **b** $\theta \approx 54.8°$ **c** $\theta \approx 98.2°$

3 $\widehat{BAC} \approx 52.0°$, $\widehat{ABC} \approx 59.3°$, $\widehat{ACB} \approx 68.7°$

4 **a** $\approx 112°$ **b** ≈ 16.2 cm^2

5 **a** $\approx 40.3°$ **b** $\approx 107°$

6 **a** $\cos\theta = 0.65$ **b** $x \approx 3.81$

7 **a** $\theta \approx 75.2°$ **b** ≈ 6.30 m

8 **a** DB ≈ 4.09 m, BC ≈ 9.86 m
b $\widehat{ABE} \approx 68.2°$, $\widehat{DBC} \approx 57.5°$ **c** ≈ 17.0 m^2

9 **b** $x = 3 + \sqrt{22}$

10 **a** $x \approx 10.8$ **b** $x \approx 2.77$ **c** $x \approx 2.89$

11 $x \approx 1.41$ or 7.78 **12** BD ≈ 12.4 cm

13 $\theta \approx 71.6°$ **14** ≈ 6.40 cm

15 **a** $x = 2$ **b** $4\sqrt{6}$ cm^2 **16** $\approx 63°, 117°, 36°, 144°$

EXERCISE 9C.1

1 **a** $x \approx 28.4$ **b** $x \approx 13.4$ **c** $x \approx 3.79$
d $x \approx 10.3$ **e** $x \approx 4.49$ **f** $x \approx 7.07$

2 **a** $a \approx 21.3$ cm **b** $b \approx 76.9$ cm **c** $c \approx 5.09$ cm

3 **a** $\widehat{BAC} = 74°$, $AB \approx 7.99$ cm, $BC \approx 9.05$ cm

b $\widehat{XZY} = 108°$, $XZ \approx 13.5$ cm, $XY \approx 26.5$ cm

4 $x \approx 17.7$, $y \approx 33.1$ **5** $x = 11 + \frac{11}{2}\sqrt{2}$

EXERCISE 9C.2

1 **a** $x \approx 9.85$ **b** $x \approx 41.3$ **c** $x \approx 52.8$

2 $C \approx 62.1°$ or $C \approx 117.9°$

3 **a** $\widehat{BAC} \approx 30.9°$ **b** $\widehat{ABC} \approx 28.7°$

c $\widehat{ACB} \approx 30.1°$ **d** $\widehat{BAC} \approx 46.6°$

e $\widehat{ABC} \approx 55.5°$ or $124.5°$ **f** $\widehat{ACB} \approx 25.4°$ or $154.6°$

4 **a** We find that $\sin x \approx 1.04$ which has no solutions.

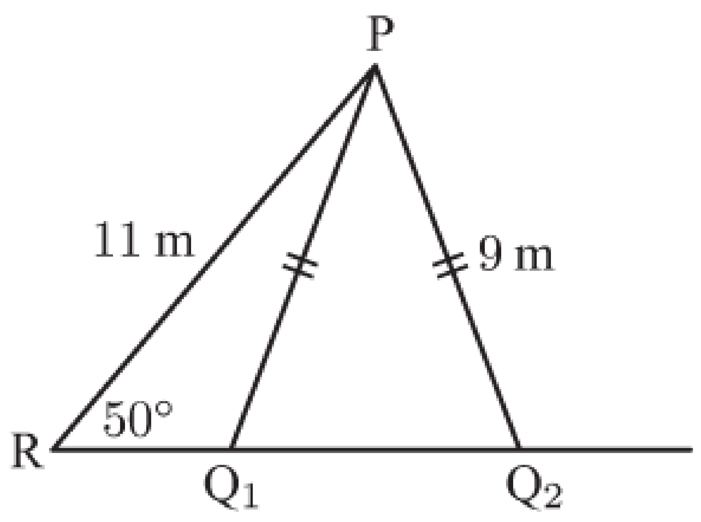
b The triangle cannot be drawn with the given dimensions.

5 **a** **i** $\widehat{ACB} \approx 22.9°$ **ii** $\widehat{BAC} \approx 127.1°$ **b** ≈ 25.1 cm^2

6 No, the angle opposite the 9.8 cm side has a sine of 1.05, which is impossible.

7 **a** $\approx 69.4°$ or $\approx 110.6°$

b



c For $\widehat{PQR} \approx 69.4°$:

i $\approx 60.6°$

ii ≈ 43.1 m^2

iii ≈ 30.2 m

For $\widehat{PQR} \approx 110.6°$:

i $\approx 19.4°$

ii ≈ 16.5 m^2

iii ≈ 23.9 m

EXERCISE 9D

1 ≈ 17.7 m **2** ≈ 207 m **3** ≈ 10.1 km

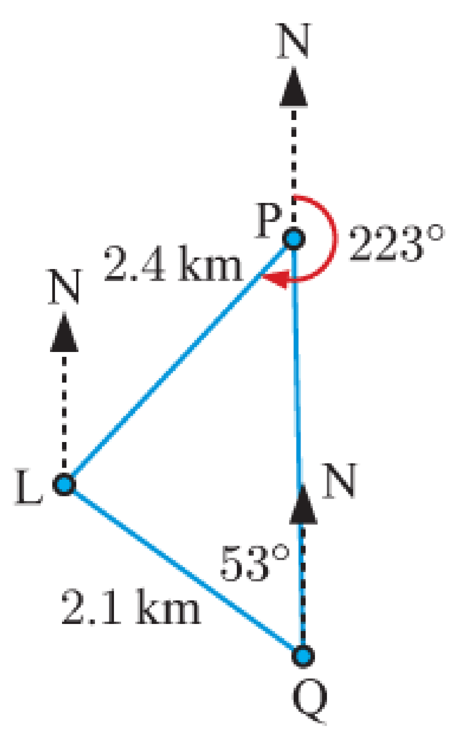
4 $\approx 23.9°$ **5** ≈ 37.6 km

6 **a** ≈ 5.63 km **b** on the bearing $\approx 115°$

c **i** Esko **ii** ≈ 7.37 min ($\approx$ 7 min 22 s) **d** $\approx 295°$

7 $\approx 9.38°$ **8** ≈ 69.1 m **9** **a** ≈ 38.0 m **b** ≈ 94.0 m

10 **a**



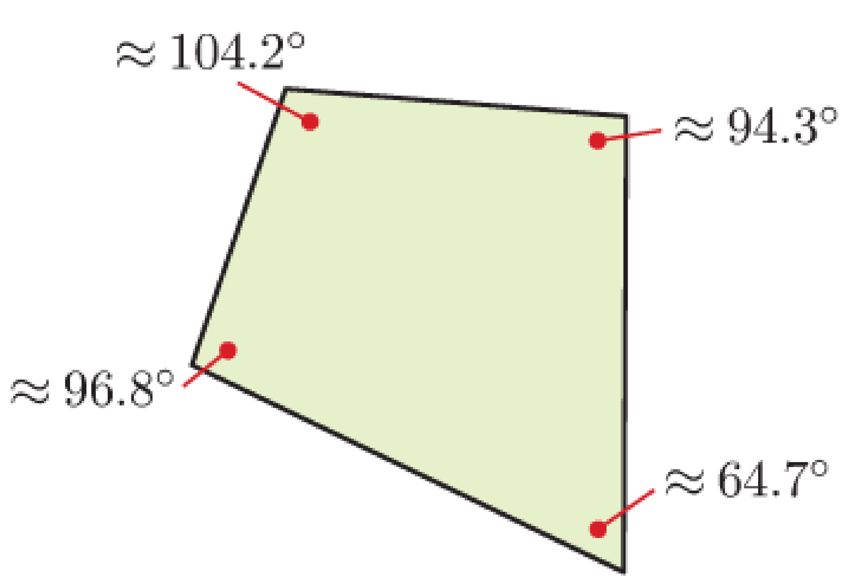
b ≈ 2.98 km

c $\approx 179°$

11 **a** $\approx 55.1°$ **b** $\approx 50.3°$ **12** $\approx 65.6°$ **13** ≈ 9.12 km

14 **a** ≈ 74.9 km^2 **b** ≈ 7490 ha **15** ≈ 85.0 mm

16



Area $\approx 13\,100$ m^2

17 ≈ 7400 m^2 **18** ≈ 2.52 km^2 **19** $\approx 32.2°$ and $\approx 87.8°$

20 ≈ 29.2 m **21** **a** ≈ 3.97 km **b** ≈ 1.13 km

22 **b** ≈ 1.467 cm

REVIEW SET 9A

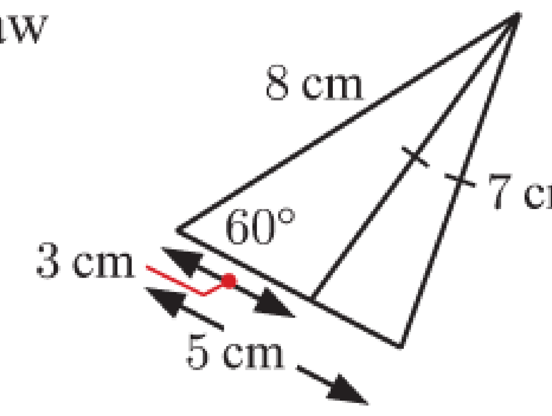
1 **a** ≈ 26.8 cm^2 **b** 14 km^2 **c** ≈ 33.0 m^2

2 ≈ 22.7 cm^2 **3** **a** ≈ 10.5 cm **b** ≈ 11.6 m

4 **a** $x \approx 9.24$ **b** $\theta \approx 59.2°$ **c** $x \approx 6.28$

5 ≈ 113 cm^2 **6** ≈ 51.6 cm^2

7 **a** $x = 3$ or 5 **b** Kady can draw 2 triangles:



8 **b** $x \approx 19.3$ **c** ≈ 43.3 cm

9 ≈ 7.21 cm, ≈ 11.2 cm, ≈ 12.5 cm

10 $\approx 74.4°$ **11** $x \approx 18.5$, $y \approx 13.8$ **12** 42 km

13 **a** $\approx 69.5°$ or $\approx 110.5°$

b For $\widehat{ABC} \approx 69.5°$, area ≈ 16.3 cm^2.

For $\widehat{ABC} \approx 110.5°$, area ≈ 8.09 cm^2.

14 ≈ 577 m

REVIEW SET 9B

1 **a** $x \approx 34.1$ **b** $x \approx 18.9$ **2** $\approx 47.5°$ or $132.5°$

3 **a** $\theta \approx 29.9°$ **b** $\theta \approx 103°$

4 **a** $AC \approx 12.6$ cm, $\widehat{BAC} \approx 48.6°$, $\widehat{ACB} \approx 57.4°$

b $\widehat{PRQ} = 51°$, $PQ \approx 7.83$ cm, $QR \approx 7.25$ cm

c $\widehat{YXZ} \approx 78.3°$, $\widehat{XYZ} \approx 55.5°$, $\widehat{XZY} \approx 46.2°$

5 **a** $x \approx 6.93$ **b** $x \approx 11.4$ **c** $x \approx 7.16$ **d** $x \approx 34.7$

6 ≈ 17.7 m **7** ≈ 7.32 m

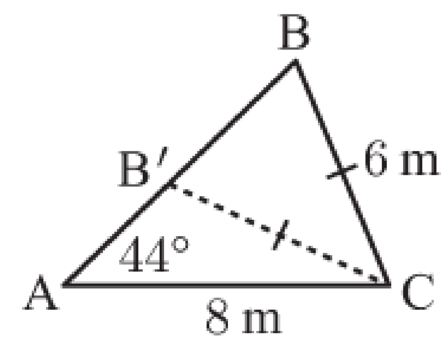
8 perimeter ≈ 578 m, area $\approx 15\,000$ m^2

9 ≈ 560 m on the bearing $\approx 079.7°$

10 $\widehat{BAD} \approx 90.5°$, $\widehat{BCD} \approx 94.3°$, $\widehat{ADC} \approx 70.2°$

11 $Q \approx 39.7°$ **12** **a** $\approx 10\,600$ m^2 **b** ≈ 1.06 ha

13 **a** The information given could give two triangles:



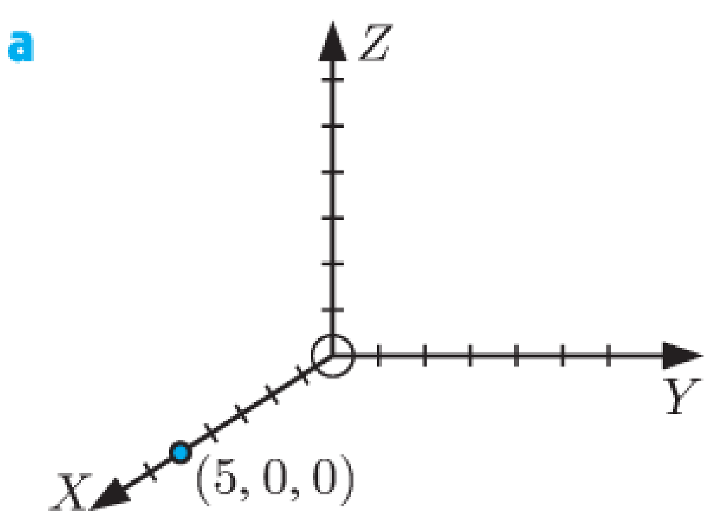
b ≈ 2.23 m^3

14 **a** **i** ≈ 20.8 mm **ii** ≈ 374 mm^2 **b** ≈ 2270 mm^3

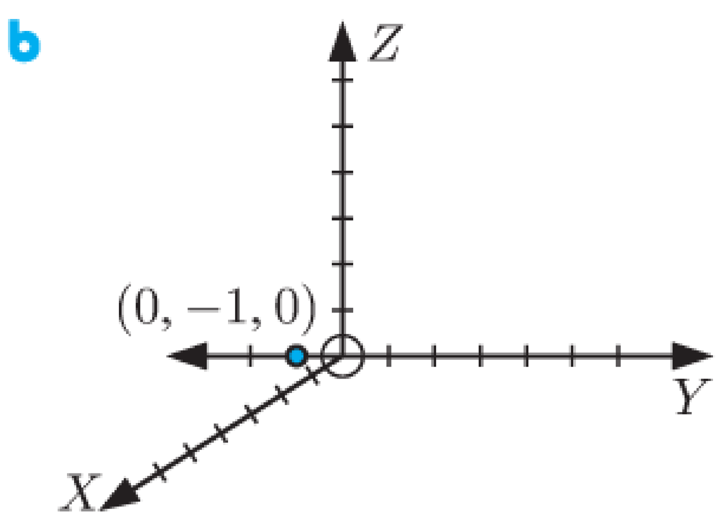
15 **a** **Hint:** Let $\widehat{BAC} = \theta$, so $A = \frac{1}{2}bc\sin\theta$.

EXERCISE 10A

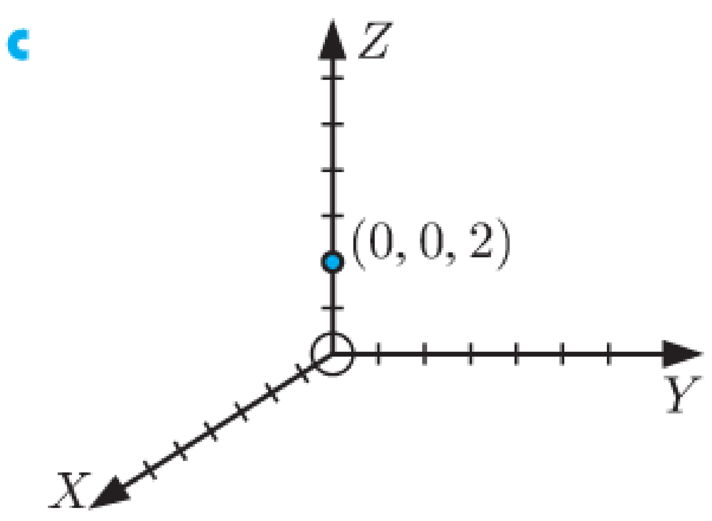
1 **a**



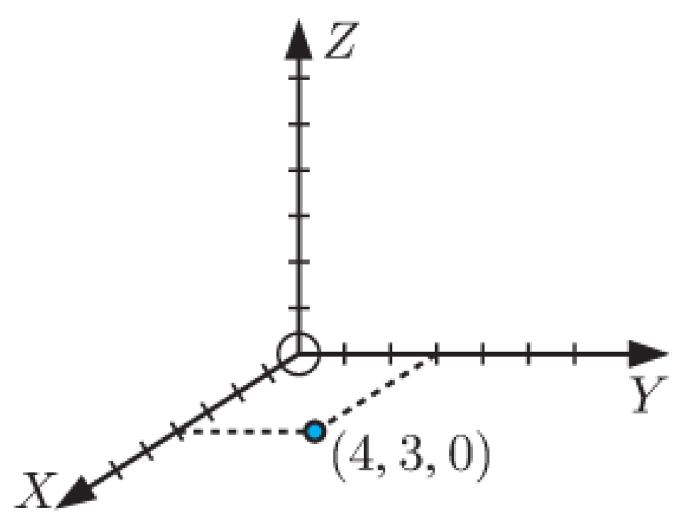
b



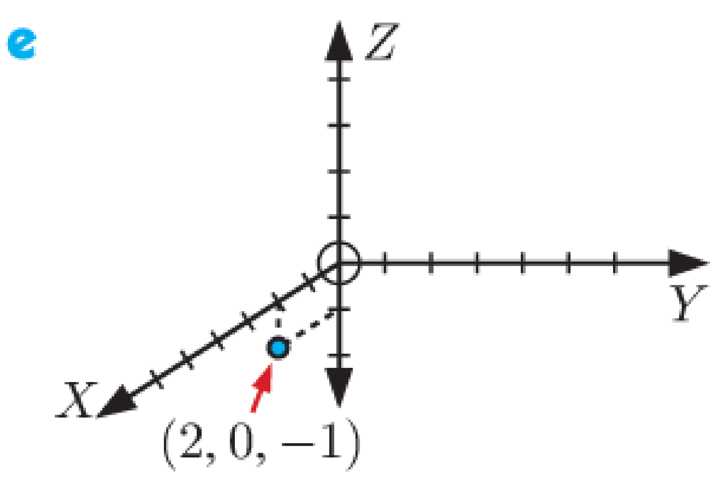
c



d



e



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