# Trigonometric equations

This presentation shows how to solve certain types of trigonometric equations, starting from very basic ones and finishing with ones where some trigonometric identities and algebraic manipulations are required.

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- values of trigonometric functions for standard angles ($0$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$);

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- values of trigonometric functions for standard angles ($0$, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$);
- reduction formulae (eg. $\sin(\pi - x) = \sin x$ or $\sin(\frac{\pi}{2} - x) = \cos x$)

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- graphs of trigonometric functions ($\sin x$, $\cos x$, $\tan x$, $\cot x$), including basic properties of these graphs (domain, range, period, etc.)
- values of trigonometric functions for standard angles ($0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$);
- reduction formulae (eg. $\sin(\pi - x) = \sin x$ or $\sin(\frac{\pi}{2} - x) = \cos x$)
- trigonometric identities: Pythagorean identity, double angle identities, angle sum and difference identities, sum-to-product identities (the last one is not strictly speaking required by IB, but it will be required in my class as it often helps a lot).

# Important note

This presentation is for your use only. Please do not share it publicly. In particular do not post it online anywhere.

# Plan

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- exam questions from Polish matura,
- IB exam questions.

# Basic trigonometric equations - example 1

We will start with the following equation:
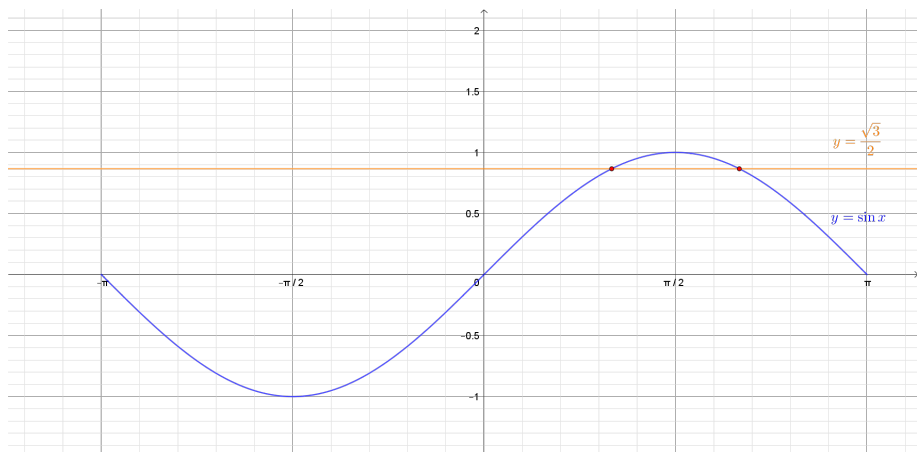
$$\sin x = \frac{\sqrt{3}}{2}$$

# Basic trigonometric equations - example 1

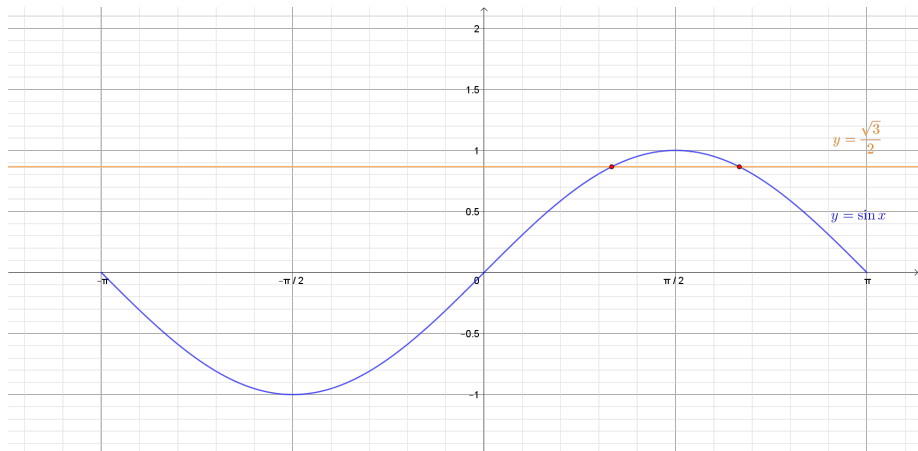We will start with the following equation:

$$\sin x = \frac{\sqrt{3}}{2}$$

We want to draw one period of the sine function (eg. from $-\pi$ to $\pi$) and the line $y = \frac{\sqrt{3}}{2}$.

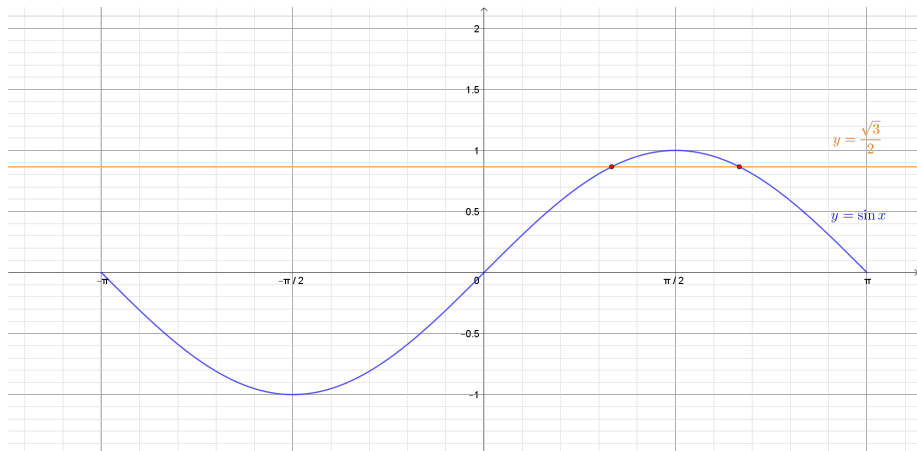# Basic trigonometric equations - example 1

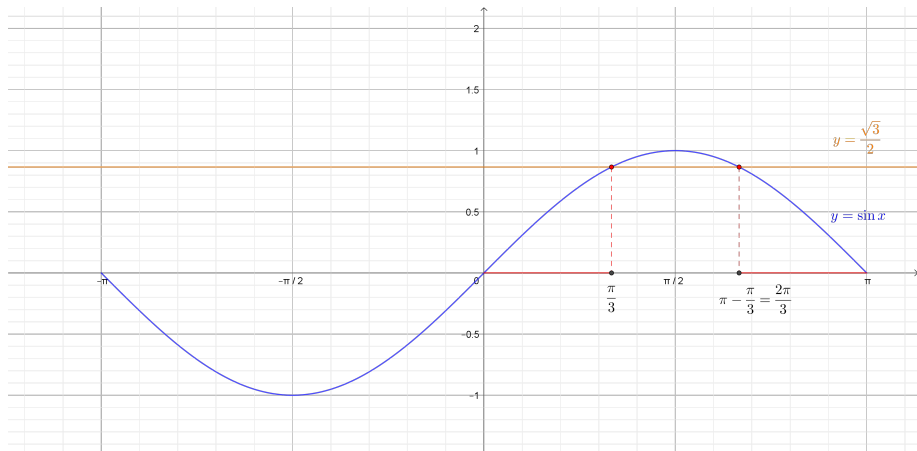# Basic trigonometric equations - example 1



We can see two solutions (red points).

# Basic trigonometric equations - example 1



We can see two solutions (red points). We should know one of those (from tables of values of standard angles), we can find the other one using symmetries of the graph.

# Basic trigonometric equations - example 1



Our solutions are $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$

# Basic trigonometric equations - example 1

So the solutions to

$$\sin x = \frac{\sqrt{3}}{2}$$

are:

# Basic trigonometric equations - example 1

So the solutions to

$$\sin x = \frac{\sqrt{3}}{2}$$

are:

$$x = \frac{\pi}{3} + 2k\pi \qquad \text{or} \qquad x = \frac{2\pi}{3} + 2k\pi$$

where $k \in \mathbb{Z}$, so $k$ is an integer.

# Basic trigonometric equations - example 1

So the solutions to

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are:

$$x = \frac{\pi}{3} + 2k\pi \qquad \text{or} \qquad x = \frac{2\pi}{3} + 2k\pi$$

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Where does the $2k\pi$ come from?

## Basic trigonometric equations - example 1

So the solutions to

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where $k \in \mathbb{Z}$, so $k$ is an integer.

Where does the $2k\pi$ come from? We only drew one period of *sine*, the values repeat themselves every $2\pi$, so adding or subtracting any multiple of $2\pi$ to $x$ will not change the value of the function.

# Basic trigonometric equations - example 2

Now we want to solve:
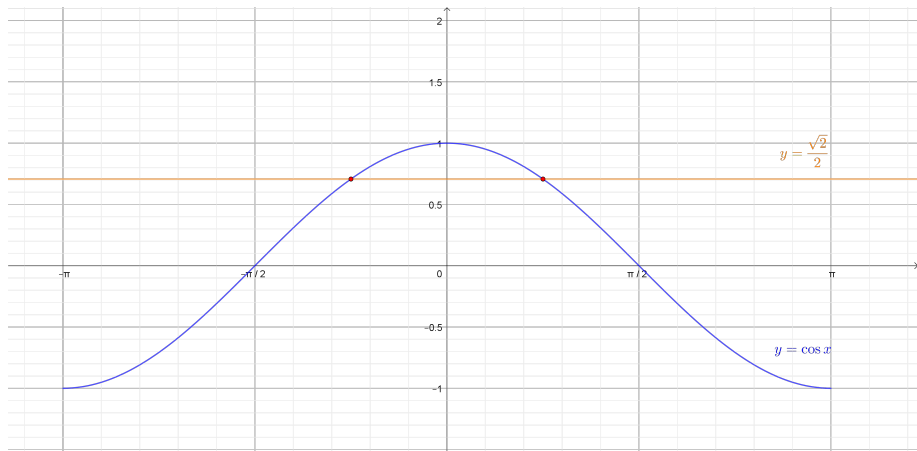
$$\cos x = \frac{\sqrt{2}}{2}$$

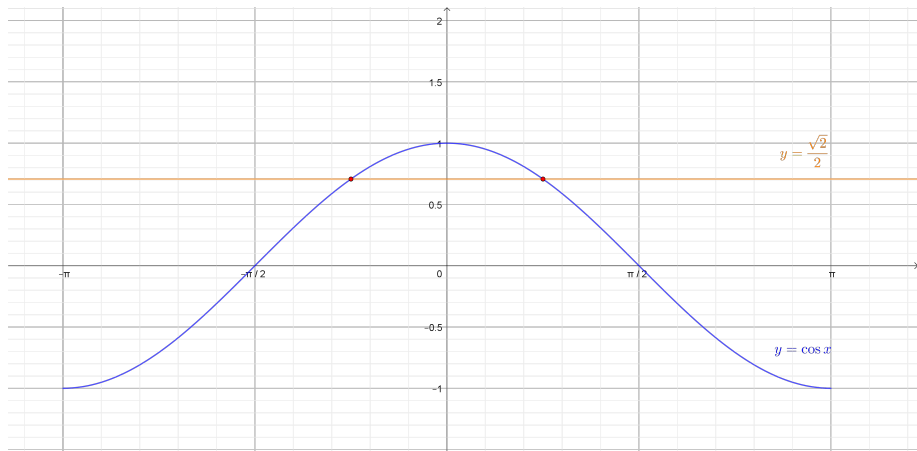## Basic trigonometric equations - example 2

Now we want to solve:

$$\cos x = \frac{\sqrt{2}}{2}$$

We draw one period of the cosine function (again it can be from $-\pi$ to $\pi$) and the line $y = \frac{\sqrt{2}}{2}$.

# Basic trigonometric equations - example 2

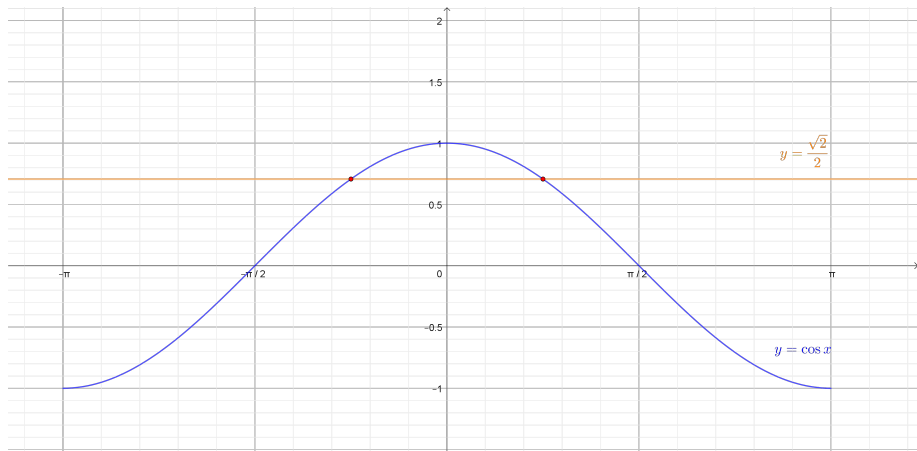# Basic trigonometric equations - example 2



We can see two solutions (red points).

# Basic trigonometric equations - example 2



We can see two solutions (red points). We should know one of those solutions and we can find the other one using symmetries of the graph.

# Basic trigonometric equations - example 2



One solution is $x = \frac{\pi}{4}$, the other is of course $x = -\frac{\pi}{4}$.

## Basic trigonometric equations - example 2

Finally we get that the solutions to the equation

$$\cos x = \frac{\sqrt{2}}{2}$$

are:

# Basic trigonometric equations - example 2

Finally we get that the solutions to the equation

$$\cos x = \frac{\sqrt{2}}{2}$$

are:

$$x = \frac{\pi}{4} + 2k\pi \qquad \text{or} \qquad x = -\frac{\pi}{4} + 2k\pi$$

where $k \in \mathbb{Z}$.

# Basic trigonometric equations - example 3

Solve:

$$\tan x = \frac{\sqrt{3}}{3}$$

# Basic trigonometric equations - example 3

Solve:

$$\tan x = \frac{\sqrt{3}}{3}$$

We draw one period of tangent function (remember that the period of $\tan x$ is $\pi$, it's best to draw it from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$) and the line $y = \frac{\sqrt{3}}{3}$.

# Basic trigonometric equations - example 3
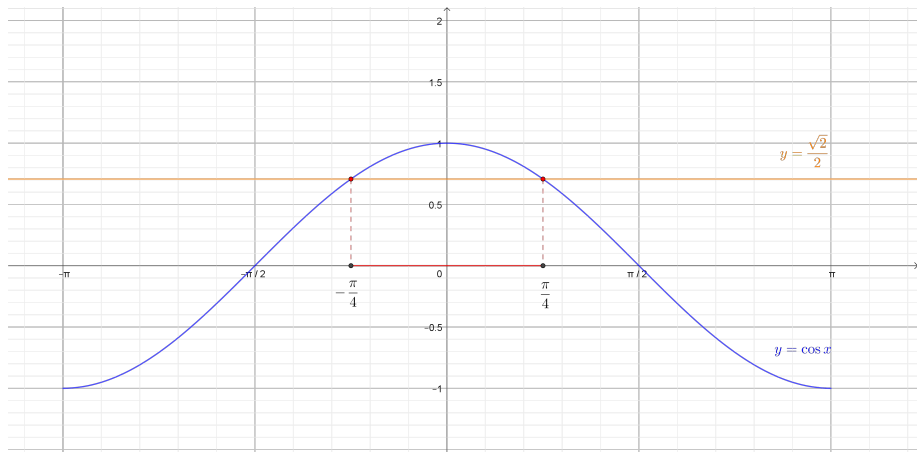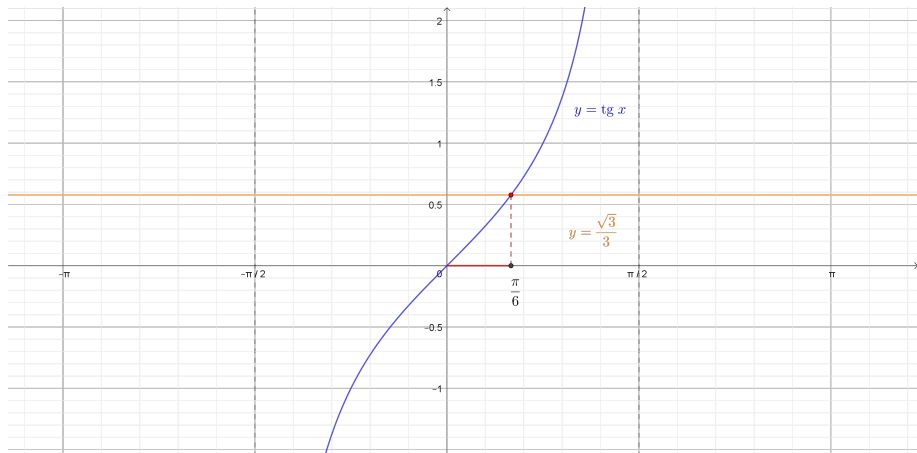
# Basic trigonometric equations - example 3



There's one solution (red point).

# Basic trigonometric equations - example 3



There's one solution (red point). We know it from the table of standard angles, $x = \frac{\pi}{6}$.

# Basic trigonometric equations - example 3

We get that the solutions to the equations

$$\tan x = \frac{\sqrt{3}}{3}$$

are:

# Basic trigonometric equations - example 3

We get that the solutions to the equations

$$\tan x = \frac{\sqrt{3}}{3}$$

are:

$$x = \frac{\pi}{6} + k\pi$$

where $k \in \mathbb{Z}$.

# Basic trigonometric equations - example 4

Solve:

$$\cot x = 1$$
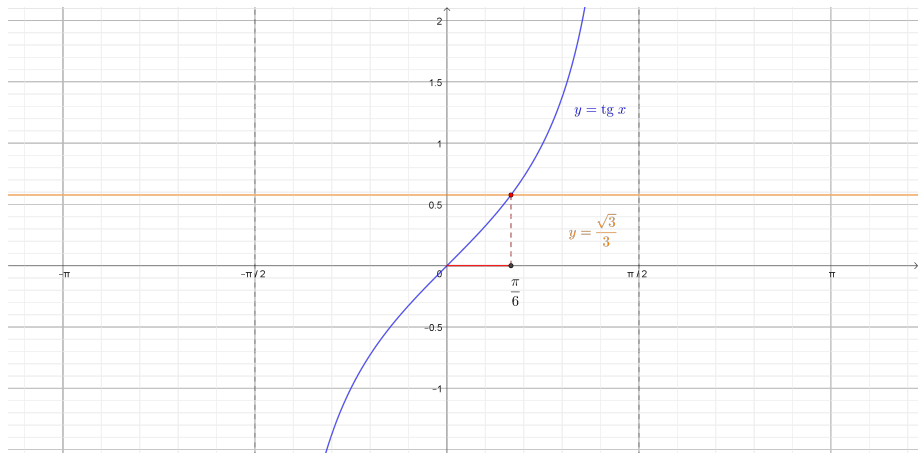
# Basic trigonometric equations - example 4

Solve:

$$\cot x = 1$$

We draw one period of cotangent function (the period is $\pi$, we'll draw it between 0 and $\pi$) and the line $y = 1$.

# Basic trigonometric equations - example 4
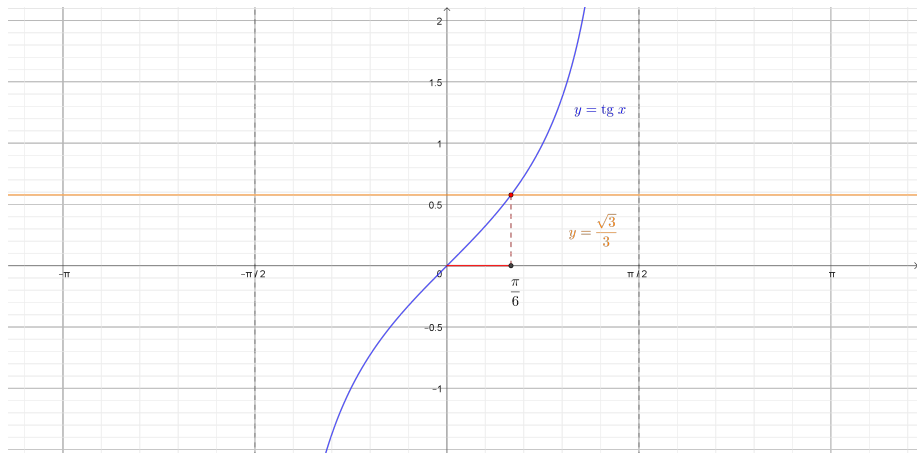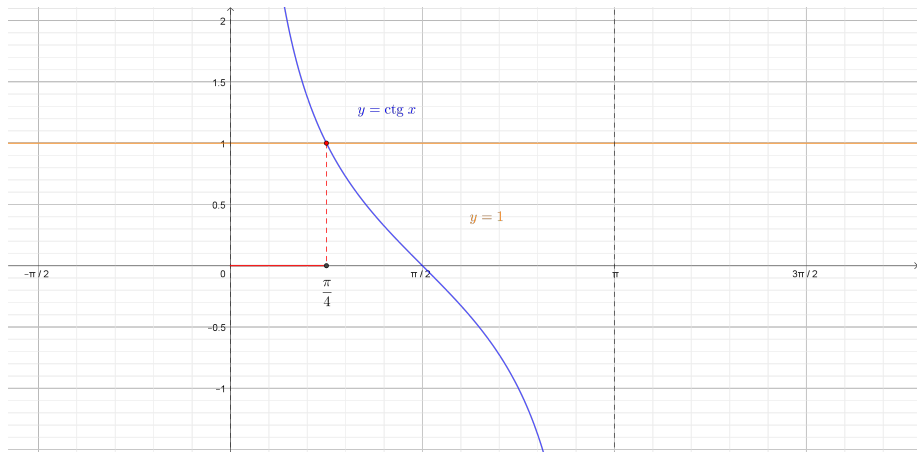
# Basic trigonometric equations - example 4
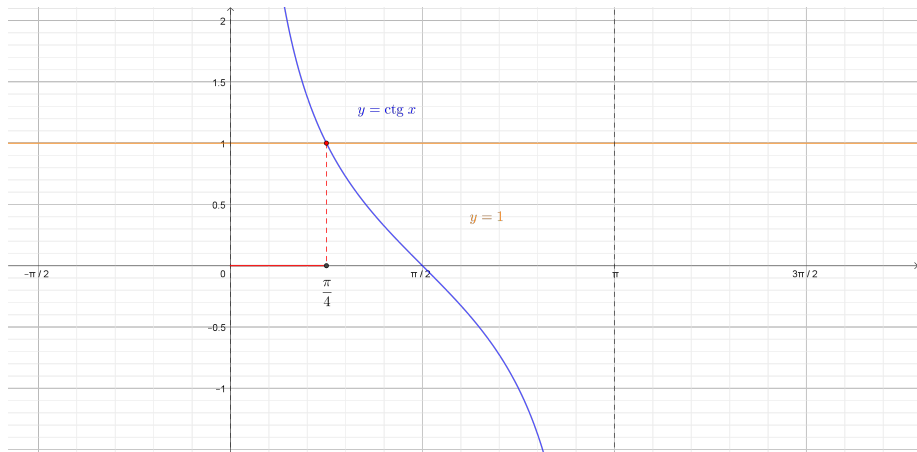


We can see one solution (red point).

# Basic trigonometric equations - example 4
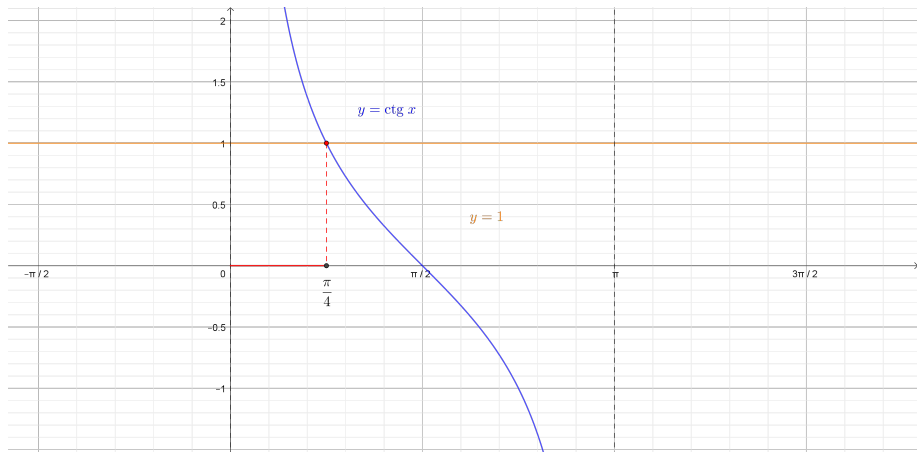


We can see one solution (red point). It is $x = \frac{\pi}{4}$.

# Basic trigonometric equations - example 4

Therefore the solutions to

$$\cot x = 1$$

are:

# Basic trigonometric equations - example 4

Therefore the solutions to

$$\cot x = 1$$

are:

$$x = \frac{\pi}{4} + k\pi$$

where $k \in \mathbb{Z}$.

# Basic trigonometric equations - exercises

Solve the following equations:

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Solve the following equations:

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$$\sin x = \frac{1}{2}$$

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Solution:

# Basic trigonometric equations - exercises

Solve the following equations:

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  Solution:

$$x = \frac{\pi}{6} + 2k\pi \qquad \text{or} \qquad x = \frac{5\pi}{6} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

## Basic trigonometric equations - exercises

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- Equation:

$$\cos x = 0$$

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- Equation:
$$\tan x = \sqrt{3}$$

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# Basic trigonometric equations - exercises

- Equation:

$$\tan x = \sqrt{3}$$

  Solution:

$$x = \frac{\pi}{3} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

- Equation:

$$\cot x = 0$$

# Basic trigonometric equations - exercises

- Equation:

$$\tan x = \sqrt{3}$$

  Solution:

$$x = \frac{\pi}{3} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

- Equation:

$$\cot x = 0$$

  Solution:

# Basic trigonometric equations - exercises

- Equation:

$$\tan x = \sqrt{3}$$

  Solution:

$$x = \frac{\pi}{3} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

- Equation:

$$\cot x = 0$$

  Solution:

$$x = \frac{\pi}{2} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

# Basic trigonometric equations - example 5

Solve the equation:

$$\sin x = -1$$

# Basic trigonometric equations - example 5

Solve the equation:

$$\sin x = -1$$

We draw one period of sine function and the line $y = -1$.

# Basic trigonometric equations - example 5

# Basic trigonometric equations - example 5



We can see one solution, it's of course $x = -\frac{\pi}{2}$.

# Basic trigonometric equations - example 5

The solutions to

$$\sin x = -1$$

are:

# Basic trigonometric equations - example 5

The solutions to

$$\sin x = -1$$

are:

$$x = -\frac{\pi}{2} + 2k\pi$$

where $k \in \mathbb{Z}$.

# Basic trigonometric equations - example 6

Solve:

$$\cos x = -\frac{1}{2}$$

Solve:

$$\cos x = -\frac{1}{2}$$

We draw one period of cosine function and the line $y = -\frac{1}{2}$.

# Basic trigonometric equations - example 6

# Basic trigonometric equations - example 6



We can see two solutions. If we were to solve $\cos x = \frac{1}{2}$, then we would know that $x = \frac{\pi}{3}$ is one of the solutions, here we can use the symmetry to get $x = \frac{2\pi}{3}$ as a solution and then we also get $x = -\frac{2\pi}{3}$.

# Basic trigonometric equations - example 6

We get that the solutions to

$$\cos x = -\frac{1}{2}$$

are:

## Basic trigonometric equations - example 6

We get that the solutions to

$$\cos x = -\frac{1}{2}$$

are:

$$x = -\frac{2\pi}{3} + 2k\pi \qquad \text{or} \qquad x = \frac{2\pi}{3} + 2k\pi$$

where $k \in \mathbb{Z}$.

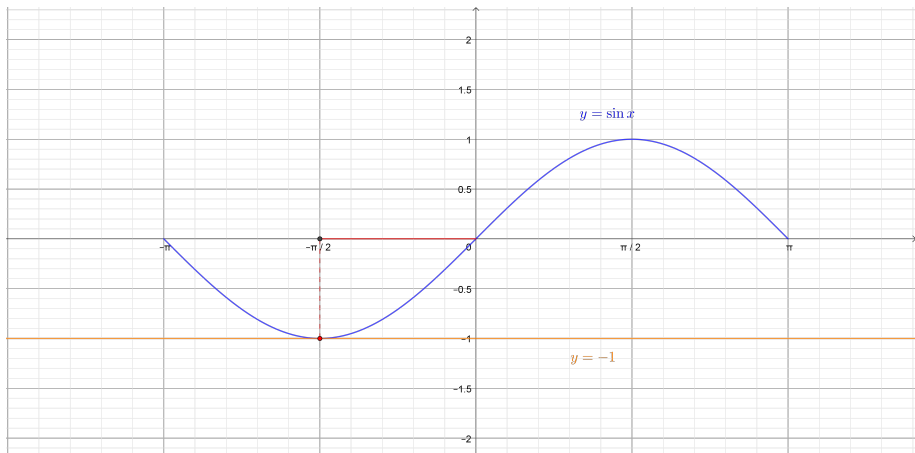# Basic trigonometric equations - example 7

Solve:

$$\tan x = -1$$

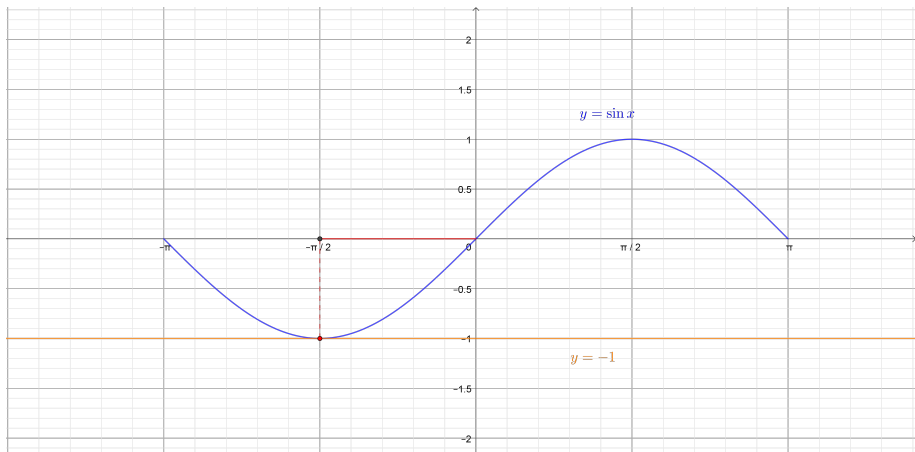# Basic trigonometric equations - example 7

Solve:

$$\tan x = -1$$

As always we draw one period of tangent function and the line $y = -1$.

# Basic trigonometric equations - example 7
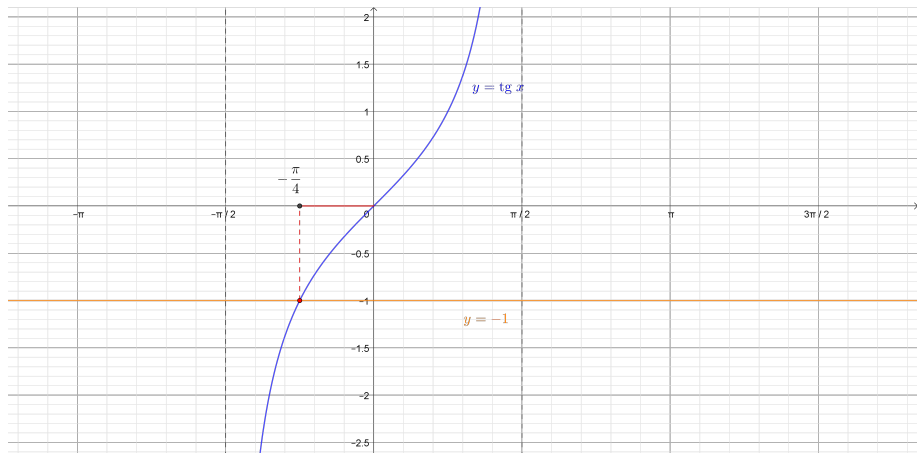
# Basic trigonometric equations - example 7
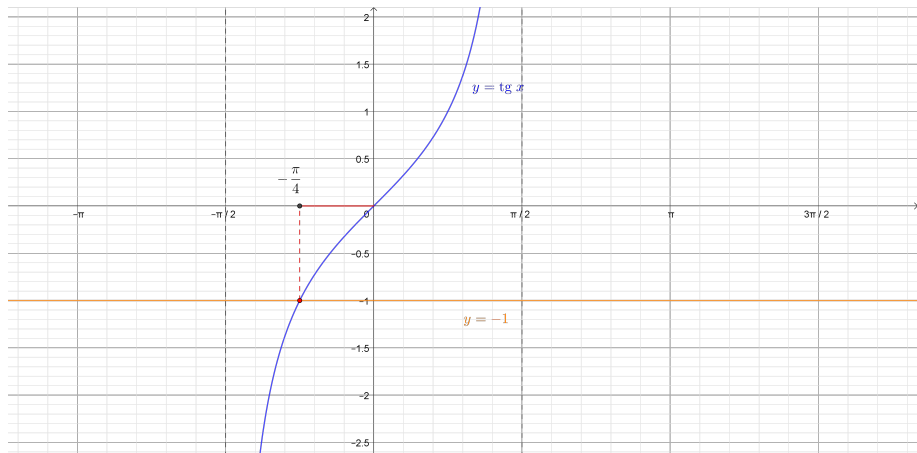


There's one solution. If we were to solve $\tan x = 1$, the solution would be $x = \frac{\pi}{4}$, so here we of course have $x = -\frac{\pi}{4}$

# Basic trigonometric equations - example 7

So the solutions to

$$\tan x = -1$$

are:

# Basic trigonometric equations - example 7

So the solutions to

$$\tan x = -1$$

are:

$$x = -\frac{\pi}{4} + k\pi$$

where $k \in \mathbb{Z}$.

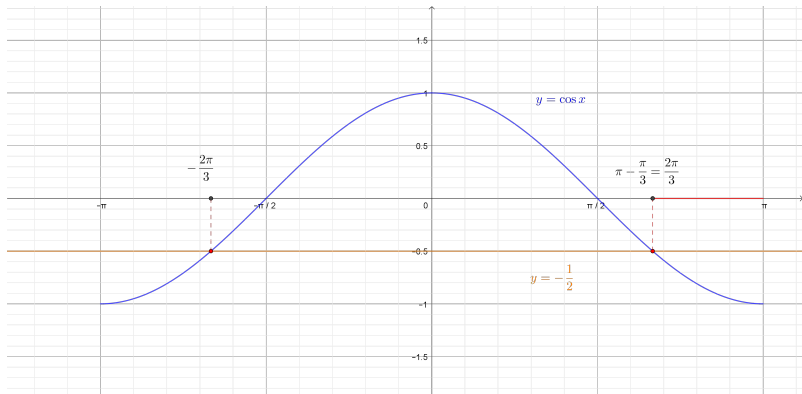# Basic trigonometric equations - example 8

Solve:

$$\cot x = -\sqrt{3}$$

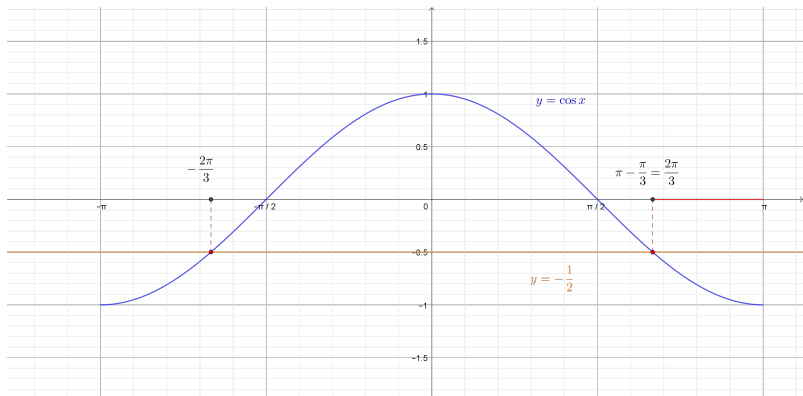# Basic trigonometric equations - example 8

Solve:

$$\cot x = -\sqrt{3}$$

We draw one period of cotangent function and the line $y = -\sqrt{3}$.

# Basic trigonometric equations - example 8

$y = \operatorname{ctg} x$

$\pi - \dfrac{\pi}{6} = \dfrac{5\pi}{6}$

$y = -\sqrt{3}$
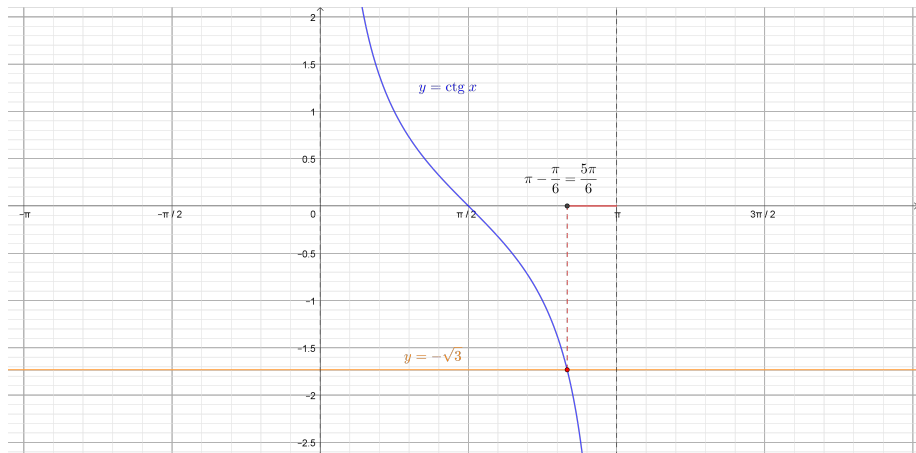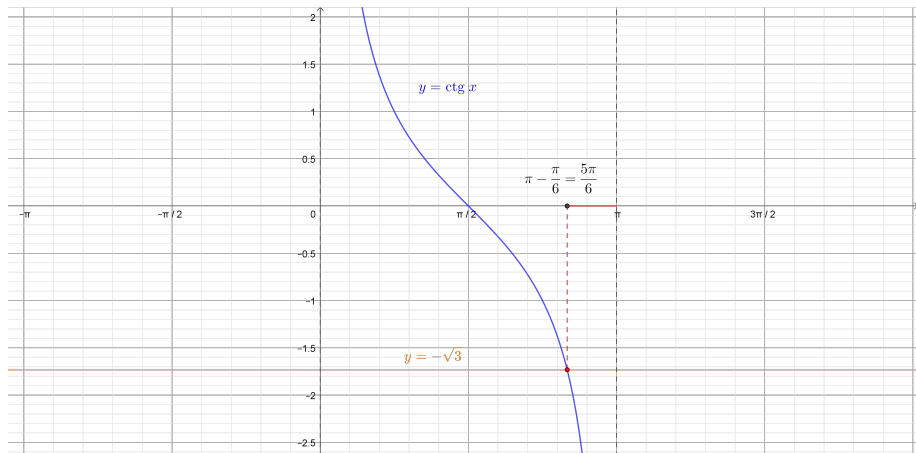
There's one solution. Solving $\cot x = \sqrt{3}$ would give us $x = \frac{\pi}{6}$, so here we have $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

So the solutions to

$$\cot x = -\sqrt{3}$$

are:

# Basic trigonometric equations - example 8

So the solutions to

$$\cot x = -\sqrt{3}$$

are:

$$x = \frac{5\pi}{6} + k\pi$$

where $k \in \mathbb{Z}$.

# Basic trigonometric equations - exercises

Solve the following equations:

# Basic trigonometric equations - exercises

Solve the following equations:

- Equation:

$$\sin x = -\frac{\sqrt{3}}{2}$$

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Solutions:

## Basic trigonometric equations - exercises

Solve the following equations:

- Equation:

$$\sin x = -\frac{\sqrt{3}}{2}$$

Solutions:

$$x = -\frac{\pi}{3} + 2k\pi \qquad \text{or} \qquad x = -\frac{2\pi}{3} + 2k\pi \qquad \text{wheree } k \in \mathbb{Z}$$

## Basic trigonometric equations - exercises

Solve the following equations:

- Equation:

$$\sin x = -\frac{\sqrt{3}}{2}$$

Solutions:

$$x = -\frac{\pi}{3} + 2k\pi \qquad \text{or} \qquad x = -\frac{2\pi}{3} + 2k\pi \qquad \text{wheree } k \in \mathbb{Z}$$

- Equation:

$$\cos x = -\frac{\sqrt{2}}{2}$$

## Basic trigonometric equations - exercises

Solve the following equations:

- Equation:

$$\sin x = -\frac{\sqrt{3}}{2}$$

  Solutions:

$$x = -\frac{\pi}{3} + 2k\pi \qquad \text{or} \qquad x = -\frac{2\pi}{3} + 2k\pi \qquad \text{wheree } k \in \mathbb{Z}$$

- Equation:

$$\cos x = -\frac{\sqrt{2}}{2}$$

  Solutions:

## Basic trigonometric equations - exercises

Solve the following equations:

- Equation:

$$\sin x = -\frac{\sqrt{3}}{2}$$

Solutions:

$$x = -\frac{\pi}{3} + 2k\pi \qquad \text{or} \qquad x = -\frac{2\pi}{3} + 2k\pi \qquad \text{wheree } k \in \mathbb{Z}$$

- Equation:

$$\cos x = -\frac{\sqrt{2}}{2}$$

Solutions:

$$x = -\frac{3\pi}{4} + 2k\pi \qquad \text{or} \qquad x = \frac{3\pi}{4} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

# Basic trigonometric equations - exercises

- Equation:

$$\tan x = -\frac{\sqrt{3}}{3}$$

# Basic trigonometric equations - exercises

- Equation:

$$\tan x = -\frac{\sqrt{3}}{3}$$

Solutions:

# Basic trigonometric equations - exercises

- Equation:

$$\tan x = -\frac{\sqrt{3}}{3}$$

Solutions:

$$x = -\frac{\pi}{6} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

# Basic trigonometric equations - exercises

- Equation:

$$\tan x = -\frac{\sqrt{3}}{3}$$

  Solutions:

$$x = -\frac{\pi}{6} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

- Equation:

$$\cot x = -1$$

# Basic trigonometric equations - exercises

- Equation:

$$\tan x = -\frac{\sqrt{3}}{3}$$

  Solutions:

$$x = -\frac{\pi}{6} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

- Equation:

$$\cot x = -1$$

  Solutions:

# Basic trigonometric equations - exercises

- Equation:

$$\tan x = -\frac{\sqrt{3}}{3}$$

  Solutions:

$$x = -\frac{\pi}{6} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

- Equation:

$$\cot x = -1$$

  Solutions:

$$x = \frac{3\pi}{4} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

In the examples above we found **all** solutions to a given equation. However in almost all IB trig equation questions you'll be required to find solutions that are in a specific interval.

## Basic trigonometric equations - example 9

In the examples above we found **all** solutions to a given equation. However in almost all IB trig equation questions you'll be required to find solutions that are in a specific interval.
Solve

$$\sin x = \frac{\sqrt{2}}{2}$$

for $0 \le x \le 3\pi$.

In the examples above we found **all** solutions to a given equation. However in almost all IB trig equation questions you'll be required to find solutions that are in a specific interval.

Solve

$$\sin x = \frac{\sqrt{2}}{2}$$

for $0 \leq x \leq 3\pi$.

This is even simpler. We draw $y = \frac{\sqrt{2}}{2}$ and $y = \sin x$, but only for $0 \leq x \leq 3\pi$.

# Basic trigonometric equations - example 9
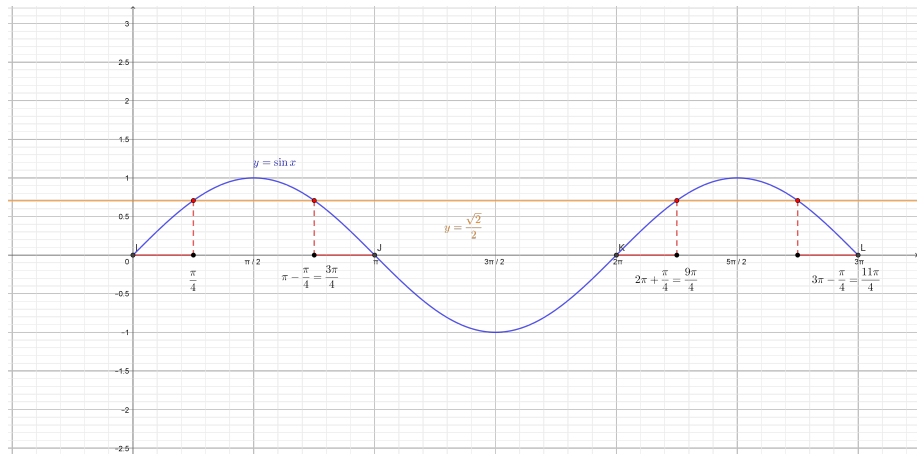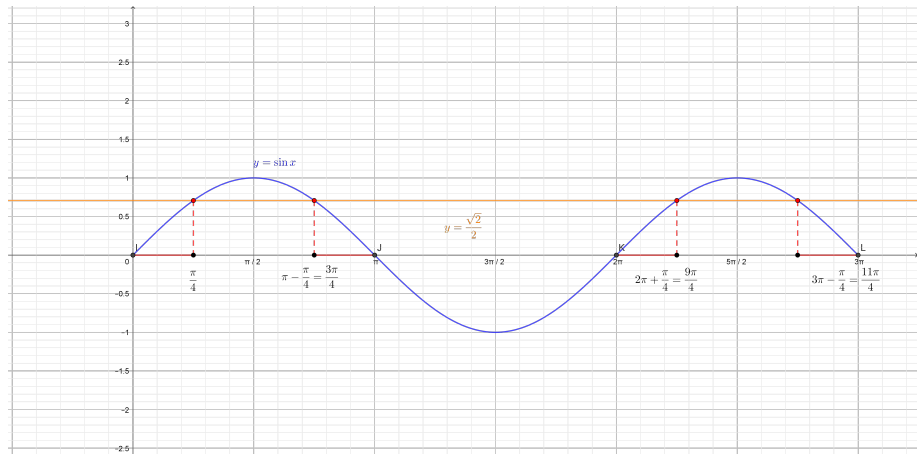
# Basic trigonometric equations - example 9



We have four solutions.
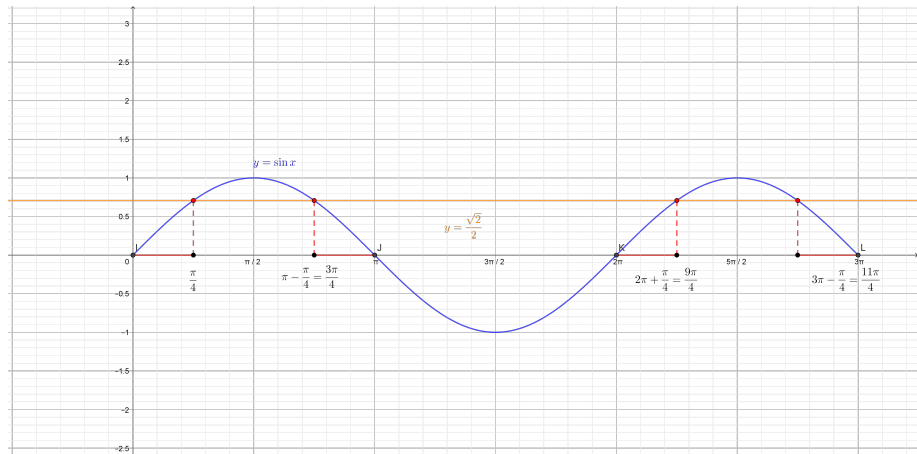
We have four solutions. We should know one from the table and find the rest using symmetries and periodicity of the graph.

# Basic trigonometric equations - example 9

The solutions to

$$\sin x = \frac{\sqrt{2}}{2}$$

for $0 \leq x \leq 3\pi$ are

# Basic trigonometric equations - example 9

The solutions to

$$\sin x = \frac{\sqrt{2}}{2}$$

for $0 \leq x \leq 3\pi$ are $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$ or $x = \frac{9\pi}{4}$ or $x = \frac{11\pi}{4}$.

# Basic trigonometric equations - example 10

Solve

$$\cos x = -\frac{\sqrt{3}}{2}$$

for $-2\pi \leq x \leq \pi$.

## Basic trigonometric equations - example 10

Solve

$$\cos x = -\frac{\sqrt{3}}{2}$$

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We draw $y = -\frac{\sqrt{3}}{2}$ and $y = \cos x$, but only for $-2\pi \leq x \leq \pi$.

# Basic trigonometric equations - example 10
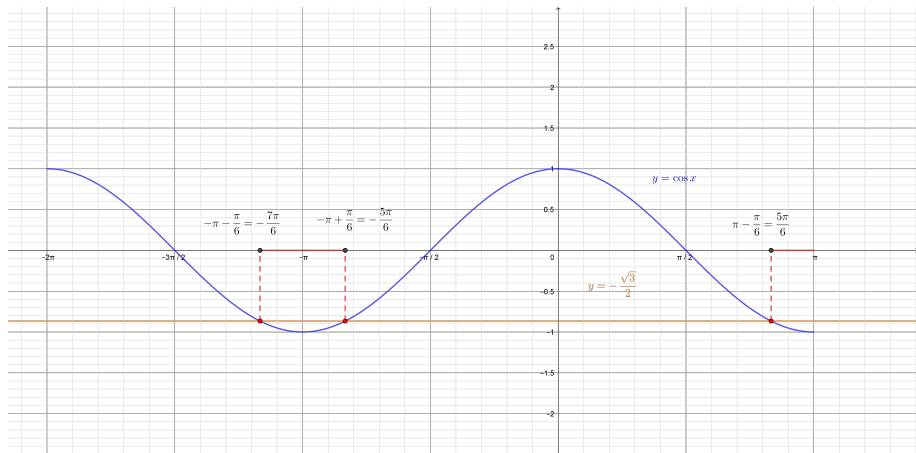
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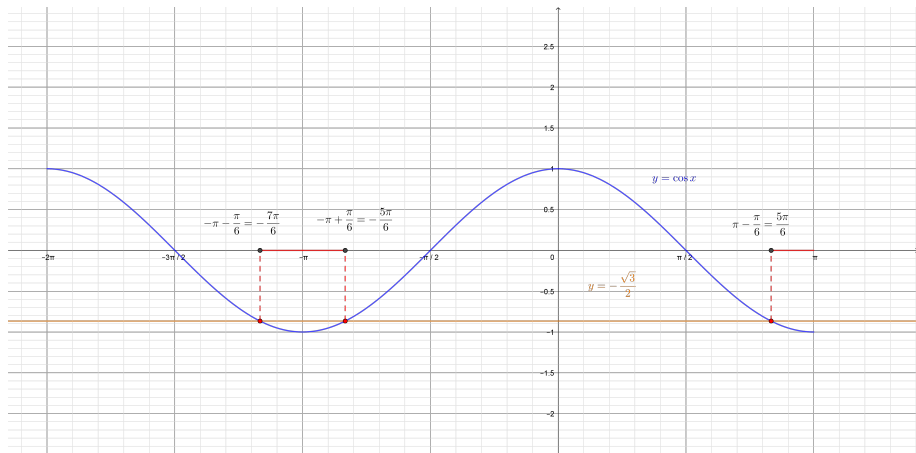


We have 3 solutions.

# Basic trigonometric equations - example 10
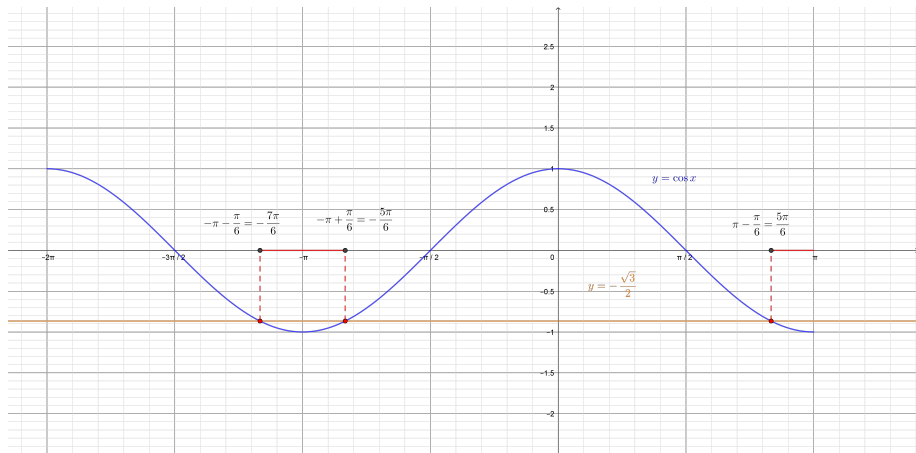


We have 3 solutions. If we were solving $\cos x = \frac{\sqrt{3}}{2}$, then we would have $x = \frac{\pi}{6}$ as a solution, based on that and symmetries we can find the actual solutions.

# Basic trigonometric equations - example 10

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# Basic trigonometric equations - example 10

The solutions to

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# Basic trigonometric equations - exercises

- Solve:

$$\tan x = -1$$

for $-\pi \leq x \leq \pi$.

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# Basic trigonometric equations - exercises

- Solve:

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$$x = -\frac{\pi}{4} \qquad \text{or} \qquad x = \frac{3\pi}{4}$$

# Basic trigonometric equations - exercises

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## Basic trigonometric equations - exercises

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On the following slides I'll skip the step with drawing graphs, but you should still do it. It is a very useful habit.

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On the following slides I'll skip the step with drawing graphs, but you should still do it. It is a very useful habit. What it means is that when you get to a basic trig equation you should solve it as above - quick sketch and read of the solutions.

We move on to equations where some algebraic manipulation is required.

# Variations of basic equations - example 1

Solve:

$$2\sin(3x) + 4 = 3$$

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Now we divide by 3, to get $x$:

$$x = -\frac{\pi}{18} + \frac{2k\pi}{3} \qquad \text{or} \qquad x = -\frac{5\pi}{18} + \frac{2k\pi}{3} \qquad \text{where } k \in \mathbb{Z}$$

and this is our solution.

Note that when solving

$$\sin(3x) = -\frac{1}{2}$$

We don't need to draw $\sin(3x)$ (sine squeezed by a factor of $\frac{1}{3}$). It's better to draw $\sin\alpha$ (so the usual graph of sine), solve for $\alpha$ and then put $3x$ instead of $\alpha$.

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We will get back to this in a few slides.

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$$2x - \frac{\pi}{3} = \pi + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

rearrange to find $x$:

$$x = \frac{2\pi}{3} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

and that's our solution.

# Variations of basic equations - example 3

Solve:

$$\tan^2(5x) - 3 = 0$$

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Solve:

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we solve two basic equations (instead of $x$ we have $5x$), we get:

$$5x = -\frac{\pi}{3} + k\pi \qquad \text{or} \qquad 5x = \frac{\pi}{3} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

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rearrange to find $x$:

$$x = -\frac{\pi}{15} + \frac{k\pi}{5} \qquad \text{or} \qquad x = \frac{\pi}{15} + \frac{k\pi}{5} \qquad \text{where } k \in \mathbb{Z}$$

and that's it.

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# Variations of basic equations - example 4

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we now solve two basic equations (instead of $x$ we have $\frac{x}{2}$), we get:

$$\frac{x}{2} = -\frac{\pi}{3} + k\pi \qquad \text{or} \qquad \frac{x}{2} = \frac{\pi}{3} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

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Rearrange and get to:

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we now solve two basic equations (instead of $x$ we have $\frac{x}{2}$), we get:

$$\frac{x}{2} = -\frac{\pi}{3} + k\pi \qquad \text{or} \qquad \frac{x}{2} = \frac{\pi}{3} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

Multiply by 2 to get $x$:

$$x = -\frac{2\pi}{3} + 2k\pi \qquad \text{or} \qquad x = \frac{2\pi}{3} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

and that's our solution.

# Variations of basic equations - example 5

Solve:

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We rearrange and solve to get:

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we solve two basic equations (instead of $x$ we have $3x$), we get:

$$3x = \frac{\pi}{2} + k\pi \qquad \text{or} \qquad 3x = 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

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divide by 3 to get $x$:

$$x = \frac{\pi}{6} + \frac{k\pi}{3} \qquad \text{or} \qquad x = \frac{2k\pi}{3} \qquad \text{where } k \in \mathbb{Z}$$

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the first equation has no solutions, we solve the second one (instead of $x$ we have $7x$), we get:

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the first equation has no solutions, we solve the second one (instead of $x$ we have $7x$), we get:

$$7x = \frac{\pi}{6} + 2k\pi \qquad \text{or} \qquad 7x = \frac{5\pi}{6} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

divide by 7 to get $x$:

$$x = \frac{\pi}{42} + \frac{2k\pi}{7} \qquad \text{or} \qquad x = \frac{5\pi}{42} + \frac{2k\pi}{7} \qquad \text{where } k \in \mathbb{Z}$$

and that's our solution.

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# Variations of basic equations - exercises

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# Variations of basic equations - exercises

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Solution:

$$x = \frac{\pi}{16} + \frac{k\pi}{4} \qquad \text{or} \qquad x = \frac{\pi}{8} + \frac{k\pi}{4} \qquad \text{where } k \in \mathbb{Z}$$

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so:

$$x = \frac{\pi}{12} + \frac{k\pi}{2} \qquad \text{or} \qquad x = -\frac{\pi}{12} + \frac{k\pi}{2} \qquad \text{where } k \in \mathbb{Z}$$

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Now we need to choose values of $k$, so that our solutions will satisfy $0 \leq x \leq \pi$. After brief deliberation we get: $x = \frac{\pi}{12}$ or $x = \frac{\pi}{12} + \frac{\pi}{2} = \frac{7\pi}{12}$ or $x = -\frac{\pi}{12} + \frac{\pi}{2} = \frac{5\pi}{12}$ or $x = -\frac{\pi}{12} + \frac{2\pi}{2} = \frac{11\pi}{12}$.

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## Variations of basic equations - example 7

$$x = \frac{\pi}{12} + \frac{k\pi}{2} \qquad \text{or} \qquad x = -\frac{\pi}{12} + \frac{k\pi}{2} \qquad \text{where } k \in \mathbb{Z}$$

Now we need to choose values of $k$, so that our solutions will satisfy $0 \leq x \leq \pi$. After brief deliberation we get: $x = \frac{\pi}{12}$ or $x = \frac{\pi}{12} + \frac{\pi}{2} = \frac{7\pi}{12}$ or $x = -\frac{\pi}{12} + \frac{\pi}{2} = \frac{5\pi}{12}$ or $x = -\frac{\pi}{12} + \frac{2\pi}{2} = \frac{11\pi}{12}$. So we have four solutions.

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Let's go back to the beginning:

$$2\cos 4x - 1 = 0$$

with $0 \le x \le \pi$.

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but $0 \leq \alpha \leq 4\pi$. Remember to change the interval!

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We draw $\cos\alpha$ for $0 \leq \alpha \leq 4\pi$ and we find the solutions:

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Solve:

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$$\alpha \in \left\{ -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3} \right\}$$

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$$x \in \left\{ -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3} \right\}$$

# Variations of basic equations - exercises

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  with $-\pi \leq x \leq \frac{\pi}{2}$. Solution:

$$x \in \left\{ -\frac{5\pi}{6}, -\frac{2\pi}{3}, -\frac{\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{3} \right\}$$

Now we move on to equations which can be easily factored resulting in two or more basic equations.

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Solve:

$$2\sin^2 x + \sin x - 1 = 0$$

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We solve these basic equations to get:

$$x = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2k\pi \quad \text{or} \quad x = -\frac{\pi}{2} + 2k\pi \quad \text{where } k \in \mathbb{Z}$$

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There's no solutions to the second equation, solving the first one gives:

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Now we solve and get:

$$x = -\frac{\pi}{6} + 2k\pi \quad \text{or} \quad x = -\frac{5\pi}{6} + 2k\pi \quad \text{or} \quad x = 2k\pi \quad \text{where } k \in \mathbb{Z}$$

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Think about this solution. Make sure you understand where it came from.

# Factoring - exercises

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We increase the difficulty slightly. We add the Pythagorean identity to our arsenal.

# Pythagorean identity - introduction

The Pythagorean identity is probably the most famous trigonometric identity. For any angle $x$ we have:

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### Simple problem

Given an angle $\alpha$, such that $\cos \alpha = \frac{1}{3}$ and $\frac{3\pi}{2} < \alpha < 2\pi$, calculate $\sin \alpha$ and $\cot \alpha$.

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We have $\sin x = -\frac{2\sqrt{2}}{3}$ and $\cot x = -\frac{\sqrt{2}}{4}$.

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### Simple problem

Given an angle $\alpha$, such that $\cos \alpha = \frac{1}{3}$ and $\frac{3\pi}{2} < \alpha < 2\pi$, calculate $\sin \alpha$ and $\cot \alpha$.

We have $\sin x = -\frac{2\sqrt{2}}{3}$ and $\cot x = -\frac{\sqrt{2}}{4}$. Refer to chapter 8E in Core HL if you forgot about these types of problems.

# Pythagorean identity - introduction

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Solving trig equations using Pythagorean identity boils down to simplifying the equation so that it can be solved using previous methods.

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They of course can be derived by dividing the Pythagorean identity by $\cos^2 x$ and $\sin^2 x$ respectively.

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We now solve and get:

$$x = \frac{\pi}{6} + 2k\pi \qquad \text{or} \qquad x = \frac{5\pi}{6} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

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Factorize:

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$$x = -\frac{2\pi}{9} + \frac{2k\pi}{3} \quad \text{or} \quad x = \frac{2\pi}{9} + \frac{2k\pi}{3} \quad \text{or} \quad x = \frac{2k\pi}{3} \qquad \text{where } k \in \mathbb{Z}$$

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It may however happen that it's not so simple and then the goal would be to make sure that we have the same angle everywhere.

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In case of *cosine* we in fact have 3 formulae and we use the one which suits us.

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

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The angle on the left hand side has to be twice the angle on the right hand side.

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$$x = k\pi \quad \text{or} \quad x = -\frac{2\pi}{3} + 2k\pi \quad \text{or} \quad x = \frac{2\pi}{3} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

Solve

$$\cos(6x) - 3\cos(3x) + 1 = 0$$

## Double angle - example 2

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Divide by $-2$ and factorize:

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solve and get (the second equation has no solutions):

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$$x = \frac{\pi}{4} + \frac{k\pi}{2} \qquad \text{or} \qquad x = -\frac{\pi}{8} + \frac{k\pi}{2} \qquad \text{where } k \in \mathbb{Z}$$

# Formula for *sine/cosine* of sum/difference of angles - introduction

We have the following formulae for sine and cosine of sum and difference of angles:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
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They can be used to calculate for example $\sin(\frac{7\pi}{12})$ or $\cos 15°$:

$$\sin\left(\frac{7\pi}{12}\right) = \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) =$$
$$= \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

# Formula for *sine/cosine* of sum/difference of angles - introduction

$$\cos 15° = \cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30° =$$
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We get the same result. This, of course, is no accident, we have $\frac{7\pi}{12} = 105°$, so
$\sin 105° = \sin(180 - 75°) = \sin 75° = \cos(90° - 75°) = \cos 15°$.

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When solving equations we will in most cases use the formulae in the opposite direction.

# Formula for *sine/cosine* of sum/difference of angles - example 1

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$$\sin x + \sqrt{3}\cos x = \sqrt{2}$$

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We have a sum so it's appropriate to change it to cosine of a difference or sine of a sum. We will do the later. We want to change 1 into cosine and $\sqrt{3}$ into a sine. By drawing an appropriate triangle we can see that the hypotenuse is 2 (so we need to divide both sides by 2) and the required angle is $\alpha = \dfrac{\pi}{3}$:

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$$

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$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x = \frac{\sqrt{2}}{2}$$

So we get:

$$\cos\frac{\pi}{3}\sin x + \sin\frac{\pi}{3}\cos x = \frac{\sqrt{2}}{2}$$

## Formula for *sine/cosine* of sum/difference of angles - example 1

Now we can apply the formula for the sine of the sum of angles to get:

$$\sin\left(x + \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$$

Note that we could have tried to use the formula for cosine of a difference of the angles. In which case we would need to change 1 into sine and $\sqrt{3}$ into cosine. The hypotenuse is still 2, but the angle is $\alpha = \frac{\pi}{6}$, so we would get:

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Now we can apply the formula for the sine of the sum of angles to get:

$$\sin\left(x + \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$$

Note that we could have tried to use the formula for cosine of a difference of the angles. In which case we would need to change 1 into sine and $\sqrt{3}$ into cosine. The hypotenuse is still 2, but the angle is $\alpha = \frac{\pi}{6}$, so we would get:

$$\sin\frac{\pi}{6}\sin x + \cos\frac{\pi}{6}\cos x = \frac{\sqrt{2}}{2}$$

## Formula for *sine/cosine* of sum/difference of angles - example 1

Now we can apply the formula for the sine of the sum of angles to get:

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$$\sin\frac{\pi}{6}\sin x + \cos\frac{\pi}{6}\cos x = \frac{\sqrt{2}}{2}$$

Applying the formula for the cosine of a difference we get:

$$\cos\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$$

# Formula for *sine/cosine* of sum/difference of angles - example 1

Going back, we have:

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This is simple now, we have:

$$x + \frac{\pi}{3} = \frac{\pi}{4} + 2k\pi \qquad \text{lub} \qquad x + \frac{\pi}{3} = \frac{3\pi}{4} + 2k\pi \qquad \text{gdzie } k \in \mathbb{Z}$$

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so:

$$x = -\frac{\pi}{12} + 2k\pi \qquad \text{lub} \qquad x = \frac{5\pi}{12} + 2k\pi \qquad \text{gdzie } k \in \mathbb{Z}$$

# Formula for *sine/cosine* of sum/difference of angles - example 1

If we used the formula for cosine of a difference we would end up with:

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# Formula for *sine/cosine* of sum/difference of angles - example 1

If we used the formula for cosine of a difference we would end up with:

$$\cos\left(x - \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$$

This gives:

$$x - \frac{\pi}{6} = -\frac{\pi}{4} + 2k\pi \qquad \text{lub} \qquad x - \frac{\pi}{6} = \frac{\pi}{4} + 2k\pi \qquad \text{gdzie } k \in \mathbb{Z}$$

# Formula for *sine/cosine* of sum/difference of angles - example 1

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and the final answer is of course the same.

# Formula for *sine/cosine* of sum/difference of angles - example 2

Solve:

$$\sin x - \cos x = \sqrt{2}$$

Solve:

$$\sin x - \cos x = \sqrt{2}$$

We can use sine of a difference here. We draw an appropriate triangle, the hypotenuse is $\sqrt{2}$ and the angle is $\alpha = \dfrac{\pi}{4}$. So we divide both side by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} \sin x - \frac{1}{\sqrt{2}} \cos x = 1$$

so:

# Formula for *sine/cosine* of sum/difference of angles - example 2

Solve:

$$\sin x - \cos x = \sqrt{2}$$

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$$\frac{1}{\sqrt{2}} \sin x - \frac{1}{\sqrt{2}} \cos x = 1$$

so:

$$\cos \frac{\pi}{4} \sin x - \sin \frac{\pi}{4} \cos x = 1$$

# Formula for *sine/cosine* of sum/difference of angles - example 2

Solve:

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We can use sine of a difference here. We draw an appropriate triangle, the hypotenuse is $\sqrt{2}$ and the angle is $\alpha = \dfrac{\pi}{4}$. So we divide both side by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} \sin x - \frac{1}{\sqrt{2}} \cos x = 1$$

so:

$$\cos \frac{\pi}{4} \sin x - \sin \frac{\pi}{4} \cos x = 1$$

Now apply the formula for sine of a difference:

$$\sin \left( x - \frac{\pi}{4} \right) = 1$$

# Formula for *sine/cosine* of sum/difference of angles - example 2

$$\sin\left(x - \frac{\pi}{4}\right) = 1$$

# Formula for *sine/cosine* of sum/difference of angles - example 2

$$\sin\left(x - \frac{\pi}{4}\right) = 1$$

this gives:

$$x - \frac{\pi}{4} = \frac{\pi}{2} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

# Formula for *sine/cosine* of sum/difference of angles - example 2

$$\sin\left(x - \frac{\pi}{4}\right) = 1$$

this gives:

$$x - \frac{\pi}{4} = \frac{\pi}{2} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

So finally we get:

$$x = \frac{3\pi}{4} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

# Formula for *sine/cosine* of sum/difference of angles - example 3

Solve:

$$\sqrt{3}\sin x + \cos x = 1$$

# Formula for *sine/cosine* of sum/difference of angles - example 3

Solve:

$$\sqrt{3}\sin x + \cos x = 1$$

We can apply the formula for sine of a sum. We draw a triangle with adjacent side $\sqrt{3}$ and opposite side 1. The hypotenuse is 2 and the angle is $\alpha = \dfrac{\pi}{6}$. So we divide both sides by 2:

$$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \frac{1}{2}$$

so:

# Formula for *sine/cosine* of sum/difference of angles - example 3

Solve:

$$\sqrt{3}\sin x + \cos x = 1$$

We can apply the formula for sine of a sum. We draw a triangle with adjacent side $\sqrt{3}$ and opposite side 1. The hypotenuse is 2 and the angle is $\alpha = \dfrac{\pi}{6}$. So we divide both sides by 2:

$$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \frac{1}{2}$$

so:

$$\cos\frac{\pi}{6}\sin x + \sin\frac{\pi}{6}\cos x = \frac{1}{2}$$

# Formula for *sine/cosine* of sum/difference of angles - example 3

Solve:

$$\sqrt{3}\sin x + \cos x = 1$$

We can apply the formula for sine of a sum. We draw a triangle with adjacent side $\sqrt{3}$ and opposite side 1. The hypotenuse is 2 and the angle is $\alpha = \dfrac{\pi}{6}$. So we divide both sides by 2:

$$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x = \frac{1}{2}$$

so:

$$\cos\frac{\pi}{6}\sin x + \sin\frac{\pi}{6}\cos x = \frac{1}{2}$$

Applying the formula we get:

$$\sin\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

# Formula for *sine/cosine* of sum/difference of angles - example 3

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so finally we get:

$$x = 2k\pi \qquad \text{lub} \qquad x = \frac{2\pi}{3} + 2k\pi \qquad \text{gdzie } k \in \mathbb{Z}$$

# Formula for *sine/cosine* of sum/difference of angles - example 4

Solve:

$$\sin 3x + \cos 3x = -\frac{\sqrt{6}}{2}$$

# Formula for *sine/cosine* of sum/difference of angles - example 4

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$$\sin 3x + \cos 3x = -\frac{\sqrt{6}}{2}$$

The fact that instead of $x$ we have $3x$ makes no difference.

# Formula for *sine/cosine* of sum/difference of angles - example 4

Solve:

$$\sin 3x + \cos 3x = -\frac{\sqrt{6}}{2}$$

The fact that instead of $x$ we have $3x$ makes no difference. We draw a triangle with both adjacent and opposite sides being 1. The hypotenuse is $\sqrt{2}$ and the angle is $\alpha = \frac{\pi}{6}$. We divide both side by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}}\sin 3x + \frac{1}{\sqrt{2}}\cos 3x = -\frac{\sqrt{12}}{4}$$

so:

## Formula for *sine/cosine* of sum/difference of angles - example 4

Solve:

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The fact that instead of $x$ we have $3x$ makes no difference. We draw a triangle with both adjacent and opposite sides being 1. The hypotenuse is $\sqrt{2}$ and the angle is $\alpha = \frac{\pi}{6}$. We divide both side by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}}\sin 3x + \frac{1}{\sqrt{2}}\cos 3x = -\frac{\sqrt{12}}{4}$$

so:

$$\cos\frac{\pi}{4}\sin 3x + \sin\frac{\pi}{4}\cos 3x = -\frac{\sqrt{3}}{2}$$

## Formula for *sine/cosine* of sum/difference of angles - example 4

Solve:

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$$\frac{1}{\sqrt{2}} \sin 3x + \frac{1}{\sqrt{2}} \cos 3x = -\frac{\sqrt{12}}{4}$$

so:

$$\cos \frac{\pi}{4} \sin 3x + \sin \frac{\pi}{4} \cos 3x = -\frac{\sqrt{3}}{2}$$

Using formula for sine of a sum we get:

$$\sin \left( 3x + \frac{\pi}{4} \right) = -\frac{\sqrt{3}}{2}$$

# Formula for *sine/cosine* of sum/difference of angles - example 4

$$\sin\left(3x + \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2}$$

# Formula for *sine/cosine* of sum/difference of angles - example 4

$$\sin\left(3x + \frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2}$$

we get

$$3x + \frac{\pi}{4} = -\frac{\pi}{3} + 2k\pi \qquad \text{or} \qquad 3x + \frac{\pi}{4} = -\frac{2\pi}{3} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

## Formula for *sine/cosine* of sum/difference of angles - example 4

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So in the end we get:

$$x = -\frac{7\pi}{36} + \frac{2k\pi}{3} \qquad \text{or} \qquad x = -\frac{11\pi}{36} + \frac{2k\pi}{3} \qquad \text{where } k \in \mathbb{Z}$$

# Formula for *sine/cosine* of sum/difference of angles - exercise

- Solve:

$$\sin 2x - \sqrt{3}\cos 2x = 1$$

# Formula for *sine/cosine* of sum/difference of angles - exercise

- Solve:

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Solution:

# Formula for *sine/cosine* of sum/difference of angles - exercise

- Solve:

$$\sin 2x - \sqrt{3}\cos 2x = 1$$

  Solution:

$$x = \frac{\pi}{4} + k\pi \qquad \text{or} \qquad x = \frac{7\pi}{12} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

Now we move to the final set of examples, where we apply formulae for sums and differences of sines and cosines.

# Formulae for sum and difference of *sine/cosine* - introduction

These are not required by IB, but nevertheless useful. They're **not** included in the formula booklet so you should learn them by heart (in fact it's best to learn the ones in the formula booklet by heart as well).

# Formulae for sum and difference of *sine/cosine* - introduction

These are not required by IB, but nevertheless useful. They're **not** included in the formula booklet so you should learn them by heart (in fact it's best to learn the ones in the formula booklet by heart as well).

$$\sin \alpha + \sin \beta = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin \alpha - \sin \beta = 2 \sin\left(\frac{\alpha - \beta}{2}\right) \cos\left(\frac{\alpha + \beta}{2}\right)$$

$$\cos \alpha + \cos \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos \alpha - \cos \beta = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

# Formulae for sum and difference of *sine/cosine* - example 1

Solve:

$$\sin x + \sin 2x = 0$$

## Formulae for sum and difference of *sine/cosine* - example 1

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We of course use the formula for the sum of the sine:

$$2 \sin \frac{3x}{2} \cos \frac{x}{2} = 0$$

so we get:

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$$\frac{3x}{2} = k\pi \qquad \text{or} \qquad \frac{x}{2} = \frac{\pi}{2} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

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and finally:

$$x = \frac{2k\pi}{3} \qquad \text{or} \qquad x = \pi + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

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We've solve the above equation earlier using $\sin 2x = 2 \sin x \cos x$.

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We've solve the above equation earlier using $\sin 2x = 2\sin x \cos x$. Compare the answers.

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We've solve the above equation earlier using $\sin 2x = 2 \sin x \cos x$.
Compare the answers. At first glance you may think that we got different solutions, but if you study it carefully you will see that they are indeed the same.

Solve:

$$\cos x + \cos 2x + \cos 3x = 0$$

Solve:

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We use the formula for the sum of cosine to $\cos x + \cos 3x$.

# Formulae for sum and difference of *sine/cosine* - example 2

Solve:

$$\cos x + \cos 2x + \cos 3x = 0$$

We use the formula for the sum of cosine to $\cos x + \cos 3x$. Why?

Solve:

$$\cos x + \cos 2x + \cos 3x = 0$$

We use the formula for the sum of cosine to $\cos x + \cos 3x$. Why?
Because we get $2\cos 2x \cos(-x)$ i $\cos 2x$ and we will be able to factorize
the expression:

$$2\cos 2x \cos(-x) + \cos 2x = 0$$

factor out $\cos 2x$ (and change $\cos(-x)$ to $\cos x$):

Solve:

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factor out $\cos 2x$ (and change $\cos(-x)$ to $\cos x$):

$$\cos 2x(2\cos x + 1) = 0$$

## Formulae for sum and difference of *sine/cosine* - example 2

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This gives:

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factor out $\cos 2x$ (and change $\cos(-x)$ to $\cos x$):

$$\cos 2x(2\cos x + 1) = 0$$

This gives:

$$2x = \frac{\pi}{2} + k\pi \quad \text{or} \quad x = -\frac{2\pi}{3} + 2k\pi \quad \text{or} \quad x = \frac{2\pi}{3} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

## Formulae for sum and difference of *sine/cosine* - example 2

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## Formulae for sum and difference of *sine/cosine* - example 2

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In the end:

$$x = \frac{\pi}{4} + \frac{k\pi}{2} \quad \text{or} \quad x = -\frac{2\pi}{3} + 2k\pi \quad \text{or} \quad x = \frac{2\pi}{3} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

Solve:

$$\sin 2x - \cos 3x = 0$$

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This may seem problematic at first as there is no obvious formula that applies here, but we can simply change cosine into sine using the formula that changes a function into co-function:

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This may seem problematic at first as there is no obvious formula that applies here, but we can simply change cosine into sine using the formula that changes a function into co-function:

$$\sin 2x - \sin(\frac{\pi}{2} - 3x) = 0$$

Solve:

$$\sin 2x - \cos 3x = 0$$

This may seem problematic at first as there is no obvious formula that applies here, but we can simply change cosine into sine using the formula that changes a function into co-function:

$$\sin 2x - \sin(\frac{\pi}{2} - 3x) = 0$$

Now apply the formula for difference of sines:

# Formulae for sum and difference of *sine/cosine* - example 3

Solve:

$$\sin 2x - \cos 3x = 0$$

This may seem problematic at first as there is no obvious formula that applies here, but we can simply change cosine into sine using the formula that changes a function into co-function:

$$\sin 2x - \sin(\frac{\pi}{2} - 3x) = 0$$

Now apply the formula for difference of sines:

$$2\sin\left(\frac{2x - (\frac{\pi}{2} - 3x)}{2}\right)\cos\left(\frac{2x + (\frac{\pi}{2} - 3x)}{2}\right) = 0$$

$$2 \sin\left(\frac{2x - \left(\frac{\pi}{2} - 3x\right)}{2}\right) \cos\left(\frac{2x + \left(\frac{\pi}{2} - 3x\right)}{2}\right) = 0$$

# Formulae for sum and difference of *sine/cosine* - example 3

$$2\sin\left(\frac{2x - (\frac{\pi}{2} - 3x)}{2}\right)\cos\left(\frac{2x + (\frac{\pi}{2} - 3x)}{2}\right) = 0$$

Simplify to get:

# Formulae for sum and difference of *sine/cosine* - example 3

$$2\sin\left(\frac{2x - (\frac{\pi}{2} - 3x)}{2}\right)\cos\left(\frac{2x + (\frac{\pi}{2} - 3x)}{2}\right) = 0$$

Simplify to get:

$$2\sin\left(\frac{5x}{2} - \frac{\pi}{4}\right)\cos\left(-\frac{x}{2} + \frac{\pi}{4}\right) = 0$$

# Formulae for sum and difference of *sine/cosine* - example 3

$$2\sin\left(\frac{2x - (\frac{\pi}{2} - 3x)}{2}\right)\cos\left(\frac{2x + (\frac{\pi}{2} - 3x)}{2}\right) = 0$$

Simplify to get:

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# Formulae for sum and difference of *sine/cosine* - example 3

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Simplify to get:

$$2\sin\left(\frac{5x}{2} - \frac{\pi}{4}\right)\cos\left(-\frac{x}{2} + \frac{\pi}{4}\right) = 0$$

This gives:

$$\frac{5x}{2} - \frac{\pi}{4} = k\pi \qquad \text{or} \qquad -\frac{x}{2} + \frac{\pi}{4} = \frac{\pi}{2} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

# Formulae for sum and difference of *sine/cosine* - example 3

$$2\sin\left(\frac{2x - (\frac{\pi}{2} - 3x)}{2}\right)\cos\left(\frac{2x + (\frac{\pi}{2} - 3x)}{2}\right) = 0$$

Simplify to get:

$$2\sin\left(\frac{5x}{2} - \frac{\pi}{4}\right)\cos\left(-\frac{x}{2} + \frac{\pi}{4}\right) = 0$$

This gives:

$$\frac{5x}{2} - \frac{\pi}{4} = k\pi \qquad \text{or} \qquad -\frac{x}{2} + \frac{\pi}{4} = \frac{\pi}{2} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

And finally:

$$x = \frac{\pi}{10} + \frac{2k\pi}{5} \qquad \text{or} \qquad x = -\frac{\pi}{2} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

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# Advanced examples

On the next slides we will look at some advanced examples, where we need to make some important observations.

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Make sure you think about these example before looking at the solutions. There may be multiple ways to solve those.

Solve:

$$\sin^4 x + \cos^4 x = \cos 2x$$

## Advanced problems - example 1

Solve:
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The first observation is that the left hand side can be written as
$(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$,

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$$1 - 2\sin^2 x \cos^2 x = \cos 2x$$

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Now we have two options. We can try to make all angles equal $2x$ or make them all equal $x$. Let's analyse both options.

$$1 - 2\sin^2 x \cos^2 x = \cos 2x$$

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We try to make all angles equal to $2x$. Recall that $\sin 2x = 2\sin x \cos x$, so $\sin^2 2x = 4\sin^2 x \cos^2 x$.

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This is equivalent to:

$$(\cos 2x - 1)^2 = 0$$

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so $\sin x = 0$ or $\cos x = \pm 1$, both of these give:

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# Advanced problems - example 2

Solve:

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which gives us the following solutions:

$$x = k\pi \quad \text{or} \quad x = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

# Advanced problems - example 3

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We factor out $\sin^2 x$ and $\cos^2 x$:

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Now an important observation. $\sin^2 x \geq 0$, but $\sin x - 1 \leq 0$, because *sine* cannot be greater than 1.

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$$\sin^2 x(\sin x - 1) + \cos^2 x(\cos x - 1) = 0$$

Both terms are non-positive, but their sum is 0, so they both must be 0.

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Solving this gives::

$$x = 2k\pi \qquad \text{or} \qquad x = \frac{\pi}{2} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

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Seems obvious that we want to apply the formula for sum of sines and cosines:

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*Sine* is an odd function, so $\sin(-x) = -\sin x$, using this and moving all terms to one side:

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Factoring out $2\sin x$:

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So $\sin x = 0$ or $\sin 2x = -\cos 2x$. The second equation can be turned into $\tan 2x = -1$ (by dividing both sides by $\cos x$). We solve and get:

$$x = k\pi \qquad \text{or} \qquad 2x = -\frac{\pi}{4} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

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So finally we have:

$$x = k\pi \qquad \text{or} \qquad x = -\frac{\pi}{8} + \frac{k\pi}{2} \qquad \text{where } k \in \mathbb{Z}$$

# Advanced problems - example 5

Solve:

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Move all terms to one side:

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Move all terms to one side:

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## Advanced problems - example 5

Solve:
$$\sin^2 x + \sin^2 2x = \sin^2 3x$$

Move all terms to one side:
$$\sin^2 x - \sin^2 3x + \sin^2 2x = 0$$

We can use difference of squares (with the hope that we can get $\sin 2x$ to factor out):
$$(\sin x - \sin 3x)(\sin x + \sin 3x) + \sin^2 2x = 0$$

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We have that $\sin(-x) = -\sin x$ and we get:

$$-4\sin x \cos x \cos 2x \sin 2x + \sin^2 2x = 0$$

Now we have the expression $\sin x \cos x$ which should remind us of the formula $\sin 2x = 2\sin x \cos x$, we use it to get:

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$$-2\cos 2x \sin^2 2x + \sin^2 2x = 0$$

Now it's a breeze, we factor out $\sin^2 2x$:

$$\sin^2 2x(1 - 2\cos 2x) = 0$$

# Advanced problems - example 5

$$\sin^2 2x(1 - 2\cos 2x) = 0$$

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We get:

$$2x = k\pi \quad \text{or} \quad 2x = -\frac{\pi}{3} + 2k\pi \quad \text{or} \quad 2x = \frac{\pi}{3} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

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So the final answer is:

$$x = \frac{k\pi}{2} \quad \text{or} \quad x = -\frac{\pi}{6} + k\pi \quad \text{or} \quad x = \frac{\pi}{6} + k\pi \qquad \text{where } k \in \mathbb{Z}$$

Solve:

$$\cot 8x \cot 10x = -1$$

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$$x = \frac{\pi}{4} + \frac{k\pi}{2} \qquad \text{where } k \in \mathbb{Z}$$

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$$x \in \emptyset$$

The next slides include problems that appeared on a Polish Matura (advanced level).

# Polish Matura - problem 1

May 2015,

May 2015, simple multiple choice question to begin with:

**Zadanie 4. (0–1)**

Równanie $2\sin x + 3\cos x = 6$ w przedziale $(0, 2\pi)$

**A.** nie ma rozwiązań rzeczywistych.

**B.** ma dokładnie jedno rozwiązanie rzeczywiste.

**C.** ma dokładnie dwa rozwiązania rzeczywiste.

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This is a very important question, because it shows that it's always important to think about the equation before solving it. $\sin x$ is less than or equal to 1, similarly $\cos x$, so the left hand side is certainly not greater than 5 (note that the maximum value of the left hand side is of course

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# Polish Matura - problem 2

May 2017,

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**Zadanie 10. (0–4)**

Rozwiąż równanie $\cos 2x + 3\cos x = -2$ w przedziale $\langle 0, 2\pi \rangle$.

We change $\cos 2x$ into $2\cos^2 x - 1$.

# Polish Matura - problem 2

May 2017,

**Zadanie 10. (0–4)**

Rozwiąż równanie $\cos 2x + 3\cos x = -2$ w przedziale $\langle 0, 2\pi \rangle$.

We change $\cos 2x$ into $2\cos^2 x - 1$. Move all terms to one side to get:

$$2\cos^2 x + 3\cos x + 1 = 0$$

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Factorize:

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We change $\cos 2x$ into $2 \cos^2 x - 1$. Move all terms to one side to get:

$$2 \cos^2 x + 3 \cos x + 1 = 0$$

Factorize:

$$(2 \cos x + 1)(\cos x + 1) = 0$$

# Polish Matura - problem 2

May 2017,

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Now we sketch the graph of cosine for $0 \leq x \leq 2\pi$.

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Factorize:

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Now we sketch the graph of cosine for $0 \leq x \leq 2\pi$. We get:

$$x = \frac{2\pi}{3} \qquad \text{or} \qquad x = \pi \qquad \text{or} \qquad x = \frac{4\pi}{3}$$

# Polish Matura - problem 3

May 2018,

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**Zadanie 11. (0–4)**

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Factor out $\cos 3x$ and move all terms to one side:

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Factor out $\cos 3x$ and move all terms to one side:

$$\cos 3x(2\sin 3x + 1) - (2\sin 3x + 1) = 0$$

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$$(2\sin 3x + 1)(\cos 3x - 1) = 0$$

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# Polish Matura - problem 3

$$(2\sin 3x + 1)(\cos 3x - 1) = 0$$

Now it's fairly simple, beware though that we have $3x$ and the domain is $0 \leq x \leq \pi$.

# Polish Matura - problem 3

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Now it's fairly simple, beware though that we have $3x$ and the domain is $0 \leq x \leq \pi$. We subsitute $\alpha = 3x$ and then we have $0 \leq \alpha \leq 3\pi$.

# Polish Matura - problem 3

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Now it's fairly simple, beware though that we have $3x$ and the domain is $0 \leq x \leq \pi$. We subsitute $\alpha = 3x$ and then we have $0 \leq \alpha \leq 3\pi$. We now get the following solutions:

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$$\alpha = \frac{7\pi}{6} \quad \text{or} \quad \alpha = \frac{11\pi}{6} \quad \text{or} \quad \alpha = 0 \quad \text{or} \quad \alpha = 2\pi$$

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Since $\alpha = 3x$, then $x = \frac{\alpha}{3}$, so:

## Polish Matura - problem 3

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Since $\alpha = 3x$, then $x = \frac{\alpha}{3}$, so:

$$x = \frac{7\pi}{18} \quad \text{or} \quad x = \frac{11\pi}{18} \quad \text{or} \quad x = 0 \quad \text{or} \quad x = \frac{2\pi}{3}$$

# Polish Matura - problem 4

May 2019,

# Polish Matura - problem 4

May 2019,

**Zadanie 2. (0–1)**

Liczba $\cos^2 105° - \sin^2 105°$ jest równa

**A.** $-\dfrac{\sqrt{3}}{2}$
**B.** $-\dfrac{1}{2}$
**C.** $\dfrac{1}{2}$
**D.** $\dfrac{\sqrt{3}}{2}$

# Polish Matura - problem 4

May 2019,

If we remember the formulae, then we should immediately notice $\cos 2x = \cos^2 x - \sin^2 x$, so we get:

# Polish Matura - problem 4

May 2019,

**Zadanie 2. (0–1)**

Liczba $\cos^2 105° - \sin^2 105°$ jest równa

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If we remember the formulae, then we should immediately notice
$\cos 2x = \cos^2 x - \sin^2 x$, so we get::

$$\cos^2 105° - \sin^2 105° = \cos 210° = \cos(180° + 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$$

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Answer A.

# Polish Matura - problem 5

May 2019,

# Polish Matura - problem 5

May 2019,

**Zadanie 14. (0–4)**

Rozwiąż równanie $(\cos x)\left[\sin\left(x - \dfrac{\pi}{3}\right) + \sin\left(x + \dfrac{\pi}{3}\right)\right] = \dfrac{1}{2}\sin x$.

# Polish Matura - problem 5

May 2019,

**Zadanie 14. (0–4)**

Rozwiąż równanie $(\cos x)\left[\sin\left(x - \dfrac{\pi}{3}\right) + \sin\left(x + \dfrac{\pi}{3}\right)\right] = \dfrac{1}{2}\sin x$.

Looks complicated, but the first step is obvious - we add the sines using appropriate formula. We get:

# Polish Matura - problem 5

May 2019,

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$$\cos x \cdot 2 \sin x \cos \frac{\pi}{3} = \frac{1}{2} \sin x$$

# Polish Matura - problem 5

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Looks complicated, but the first step is obvious - we add the sines using appropriate formula. We get:

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Now it becomes very simple. Of course we have $\cos\frac{\pi}{3} = \frac{1}{2}$. We move all terms to one side and we get:

$$\sin x \cos x - \frac{1}{2}\sin x = 0$$

# Polish Matura - problem 5

$$\sin x \cos x - \frac{1}{2} \sin x = 0$$

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We factor out $\sin x$:

## Polish Matura - problem 5

$$\sin x \cos x - \frac{1}{2}\sin x = 0$$

We factor out $\sin x$:

$$\sin x(\cos x - \frac{1}{2}) = 0$$

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This gives:

## Polish Matura - problem 5

$$\sin x \cos x - \frac{1}{2}\sin x = 0$$

We factor out $\sin x$:

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This gives:

$$x = k\pi \quad \text{or} \quad x = -\frac{\pi}{3} + 2k\pi \quad \text{or} \quad x = \frac{\pi}{3} + 2k\pi \qquad \text{where } k \in \mathbb{Z}$$

Now we move on to IB exam questions.

# IB exam - problem 1

[Maximum mark: 6]

Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0° \leq x < 360°$.

[Maximum mark: 6]

Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0° \leq x < 360°$.

Note that the interval is in degrees - this is unusual.

[Maximum mark: 6]

Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0° \leq x < 360°$.

Note that the interval is in degrees - this is unusual. It's quite obvious that we will use double angle formula for tangent:

# IB exam - problem 1

[Maximum mark: 6]

Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0° \leq x < 360°$.

Note that the interval is in degrees - this is unusual. It's quite obvious that we will use double angle formula for tangent:

$$\tan x + \frac{2\tan x}{1 - \tan^2 x} = 0$$

[Maximum mark: 6]

Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0° \le x < 360°$.

Note that the interval is in degrees - this is unusual. It's quite obvious that we will use double angle formula for tangent:

$$\tan x + \frac{2\tan x}{1 - \tan^2 x} = 0$$

Now it makes sense to multiply both sides by $1 - \tan^2 x$ to get:

# IB exam - problem 1

[Maximum mark: 6]

Find all solutions to the equation $\tan x + \tan 2x = 0$ where $0° \leq x < 360°$.

Note that the interval is in degrees - this is unusual. It's quite obvious that we will use double angle formula for tangent:

$$\tan x + \frac{2 \tan x}{1 - \tan^2 x} = 0$$

Now it makes sense to multiply both sides by $1 - \tan^2 x$ to get:

$$\tan x - \tan^3 x + 2 \tan x = 0$$

Now this is very easy, factoring $\tan x$ we get:

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$$\tan x(3 - \tan^2 x) = 0$$

# IB exam - problem 1

Now this is very easy, factoring $\tan x$ we get:

$$\tan x (3 - \tan^2 x) = 0$$

So $\tan x = 0$ or $\tan x = \pm\sqrt{3}$.

Now this is very easy, factoring $\tan x$ we get:

$$\tan x(3 - \tan^2 x) = 0$$

So $\tan x = 0$ or $\tan x = \pm\sqrt{3}$. We should draw the graph of $\tan x$ for $0° \leq x < 360°$, so that we don't miss any solutions.

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So $\tan x = 0$ or $\tan x = \pm\sqrt{3}$. We should draw the graph of $\tan x$ for $0° \leq x < 360°$, so that we don't miss any solutions. In the end we get:

$$x \in \{0°, 60°, 120°, 180°, 240°, 300°\}$$

[Maximum mark: 7]

Solve the equation $\sin 2x - \cos 2x = 1 + \sin x - \cos x$ for $x \in [-\pi, \pi]$.

# IB exam - problem 2

[Maximum mark: 7]

Solve the equation $\sin 2x - \cos 2x = 1 + \sin x - \cos x$ for $x \in [-\pi, \pi]$.

We will start by using double angle formulae for sine and cosine. For cosine it makes sense to use $\cos 2x = 2\cos^2 x - 1$, because this will allow us to cancel 1 on both sides.

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Moving all terms to one side we get:

$$2\sin x \cos x - 2\cos^2 x - \sin x + \cos x = 0$$

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$$x \in \left\{ -\frac{3\pi}{4}, -\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{3} \right\}$$

# IB exam - problem 3

[Maximum mark: 8]

Consider the equation $\dfrac{\sqrt{3}-1}{\sin x} + \dfrac{\sqrt{3}+1}{\cos x} = 4\sqrt{2}$, $0 < x < \dfrac{\pi}{2}$. Given that $\sin\left(\dfrac{\pi}{12}\right) = \dfrac{\sqrt{6}-\sqrt{2}}{4}$ and $\cos\left(\dfrac{\pi}{12}\right) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

(a)   verify that $x = \dfrac{\pi}{12}$ is a solution to the equation;                    [3]

(b)   hence find the other solution to the equation for $0 < x < \dfrac{\pi}{2}$ .          [5]

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(a) verify that $x = \dfrac{\pi}{12}$ is a solution to the equation; [3]

(b) hence find the other solution to the equation for $0 < x < \dfrac{\pi}{2}$. [5]

The first part is easy, since we're given all the information.

We will start with the left hand side:

$$LHS = \frac{\sqrt{3} - 1}{\sin \frac{\pi}{12}} + \frac{\sqrt{3} + 1}{\cos \frac{\pi}{12}} =$$

$$= \frac{4(\sqrt{3} - 1)}{\sqrt{6} - \sqrt{2}} + \frac{4(\sqrt{3} + 1)}{\sqrt{6} + \sqrt{2}} =$$

$$= \frac{4(\sqrt{3} - 1)}{\sqrt{2}(\sqrt{3} - 1)} + \frac{4(\sqrt{3} + 1)}{\sqrt{2}(\sqrt{3} + 1)} =$$

$$= \frac{4}{\sqrt{2}} + \frac{4}{\sqrt{2}} =$$

$$= \frac{8}{\sqrt{2}} = \frac{8\sqrt{2}}{2} = 4\sqrt{2} = RHS$$

so $x = \frac{\pi}{12}$ is a solution.

We can multiply both sides by $\sin x \cos x$ to get rid of denominators. We get

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Now we can try to use the formula for sine of sums on the left hand side. The opposite side is $\sqrt{3} - 1$, the adjacent side is $\sqrt{3} + 1$. The hypotenuse is $2\sqrt{2}$.

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# IB exam - problem 3 (b)

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$$\frac{\sqrt{3} - 1}{2\sqrt{2}}\cos x + \frac{\sqrt{3} + 1}{2\sqrt{2}}\sin x = 2\sin x \cos x$$

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$$\frac{\sqrt{3} - 1}{2\sqrt{2}}\cos x + \frac{\sqrt{3} + 1}{2\sqrt{2}}\sin x = 2\sin x \cos x$$

This is happens to be perfect for the right hand side as well.

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# IB exam - problem 3 (b)

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Now there are two ways to proceed.

## IB exam - problem 3 (b)

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$$\sin \left( \frac{\pi}{12} + x \right) = \sin 2x$$

Now there are two ways to proceed. We can move all terms to one side and use the formula for difference of sines:

$$\sin \left( \frac{\pi}{12} + x \right) - \sin 2x = 0$$

## IB exam - problem 3 (b)

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Now there are two ways to proceed. We can move all terms to one side and use the formula for difference of sines:

$$\sin\left(\frac{\pi}{12} + x\right) - \sin 2x = 0$$

So:

$$2\sin\left(\frac{\frac{\pi}{12} - x}{2}\right)\cos\left(\frac{\frac{\pi}{12} + 3x}{2}\right) = 0$$

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Solving the first part (sine) in the required interval gives:

$$\frac{\frac{\pi}{12} - x}{2} = 0$$

which gives $x = \frac{\pi}{12}$ an answer we already had.

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$$\frac{\frac{\pi}{12} + 3x}{2} = \frac{\pi}{2}$$

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$$\frac{\frac{\pi}{12} + 3x}{2} = \frac{\pi}{2}$$

So $x = \frac{11\pi}{36}$ and this is our second solution.

Let's go back to:

$$\sin\left(\frac{\pi}{12} + x\right) = \sin 2x$$

and let's discuss another approach.

## IB exam - problem 3 (b)

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We have sine function on both sides. Of course if the arguments are the same then the values will also be the same, so we can have:

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But two sines are also equal if one argument is $\pi$ minus the other ($\sin\alpha = \sin(\pi - \alpha)$), so we could also have:

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But two sines are also equal if one argument is $\pi$ minus the other ($\sin \alpha = \sin(\pi - \alpha)$), so we could also have:

$$\frac{\pi}{12} + x = \pi - 2x$$

and this gives the solution $x = \frac{11\pi}{36}$.

# IB exam - problem 3 (b)

Note that in general if we have:

$$\sin \alpha = \sin \beta$$

Then:

$$\alpha = \beta \quad \text{or} \quad \alpha = \pi - \beta \quad \text{or} \quad \alpha = 2\pi + \beta \quad \text{or} \quad \alpha = 3\pi - \beta \quad \text{or} \quad \dots$$

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If we have:

$$\cos \alpha = \cos \beta$$

Then

$$\alpha = \beta \quad \text{or} \quad \alpha = 2\pi - \beta \quad \text{or} \quad \alpha = 2\pi + \beta \quad \text{or} \quad \alpha = 4\pi - \beta \quad \text{or} \quad ...$$

# IB exam - problem 4

[Maximum mark: 22]

(a) Solve $2 \sin(x + 60°) = \cos(x + 30°)$, $0° \leq x \leq 180°$. [5]

(b) Show that $\sin 105° + \cos 105° = \dfrac{1}{\sqrt{2}}$. [3]

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Note that this is part of a longer question which involved topics we haven't covered yet.

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Note that this is part of a longer question which involved topics we haven't covered yet.

For part (a) it makes sense to apply the formula for the sine and cosine of sums.

# IB exam - problem 4 (a)

We get:

$$2(\sin x \cos 60^\circ + \sin 60^\circ \cos x) = \cos x \cos 30^\circ - \sin x \sin 30^\circ$$

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We get:

$$2(\sin x \cos 60^\circ + \sin 60^\circ \cos x) = \cos x \cos 30^\circ - \sin x \sin 30^\circ$$

Now this becomes:

$$\sin x + \sqrt{3} \cos x = \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x$$

We get:

$$2(\sin x \cos 60^\circ + \sin 60^\circ \cos x) = \cos x \cos 30^\circ - \sin x \sin 30^\circ$$

Now this becomes:

$$\sin x + \sqrt{3} \cos x = \frac{\sqrt{3}}{2} \cos x - \frac{1}{2} \sin x$$

This gives:

$$3 \sin x = -\sqrt{3} \cos x$$

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which is equivalent to:

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Now this becomes:

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This gives:

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which is equivalent to:

$$\tan x = -\frac{\sqrt{3}}{3}$$

In the given interval we have only one solution, namely $x = 150^\circ$.

# IB exam - problem 4 (b)

For part (b) we can calculate both sine and cosine separately using angles $60°$ and $45°$, so that we have:

For part (b) we can calculate both sine and cosine separately using angles $60°$ and $45°$, so that we have:

$LHS = \sin 105° + \cos 105° =$

$\quad = \sin(60° + 45°) + \cos(60° + 45°) =$

$\quad = \sin 60° \cos 45° + \sin 45° \cos 60° + \cos 60° \cos 45° - \sin 60° \sin 45° =$

$\quad = \dfrac{\sqrt{6}}{4} + \dfrac{\sqrt{2}}{4} + \dfrac{\sqrt{2}}{4} - \dfrac{\sqrt{6}}{4} = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}} = RHS$

$\square$

Alternatively we can change $\cos 105°$ into $-\sin 15°$ and apply the formula for difference of sines:

Alternatively we can change $\cos 105^\circ$ into $-\sin 15^\circ$ and apply the formula for difference of sines:

$$
\begin{aligned}
LHS &= \sin 105^\circ + \cos 105^\circ = \\
&= \sin 105^\circ - \sin 15^\circ = \\
&= 2\sin 45^\circ \cos 60^\circ = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = RHS
\end{aligned}
$$

□

That's it. That's all the basics. Note that all IB questions will require you to solve trigonometric equations in a specific interval, but it's still useful to be aware of the general solution. Make sure you understand all examples discussed in the presentation. If you have any questions you can email me at t.j.lechowski@gmail.com.