Chapter 16

Transformations of functions

Contents:

OPENING PROBLEM

In our study of quadratic functions, we saw that the completed square form $y = (x-h)^2 + k$ was extremely useful in identifying the vertex (h, k).

Things to think about:

a What transformation maps the graph $y = x^2$ onto the graph $y = (x-h)^2 + k$?

b If we let $f(x) = x^2$, what function is $f(x-h) + k$?

c In general terms, what transformation maps $y = f(x)$ onto $y = f(x-h) + k$?

In this Chapter we perform **transformations** of graphs to produce the graph of a related function.

The transformations of $y = f(x)$ we consider include:

- **translations** $y = f(x) + b$ and $y = f(x-a)$
- **stretches** $y = p\,f(x)$, $p > 0$ and $y = f(qx)$, $q > 0$
- **reflections** $y = -f(x)$ and $y = f(-x)$
- transformations of the form $y = \dfrac{1}{f(x)}$
- combinations of these transformations.

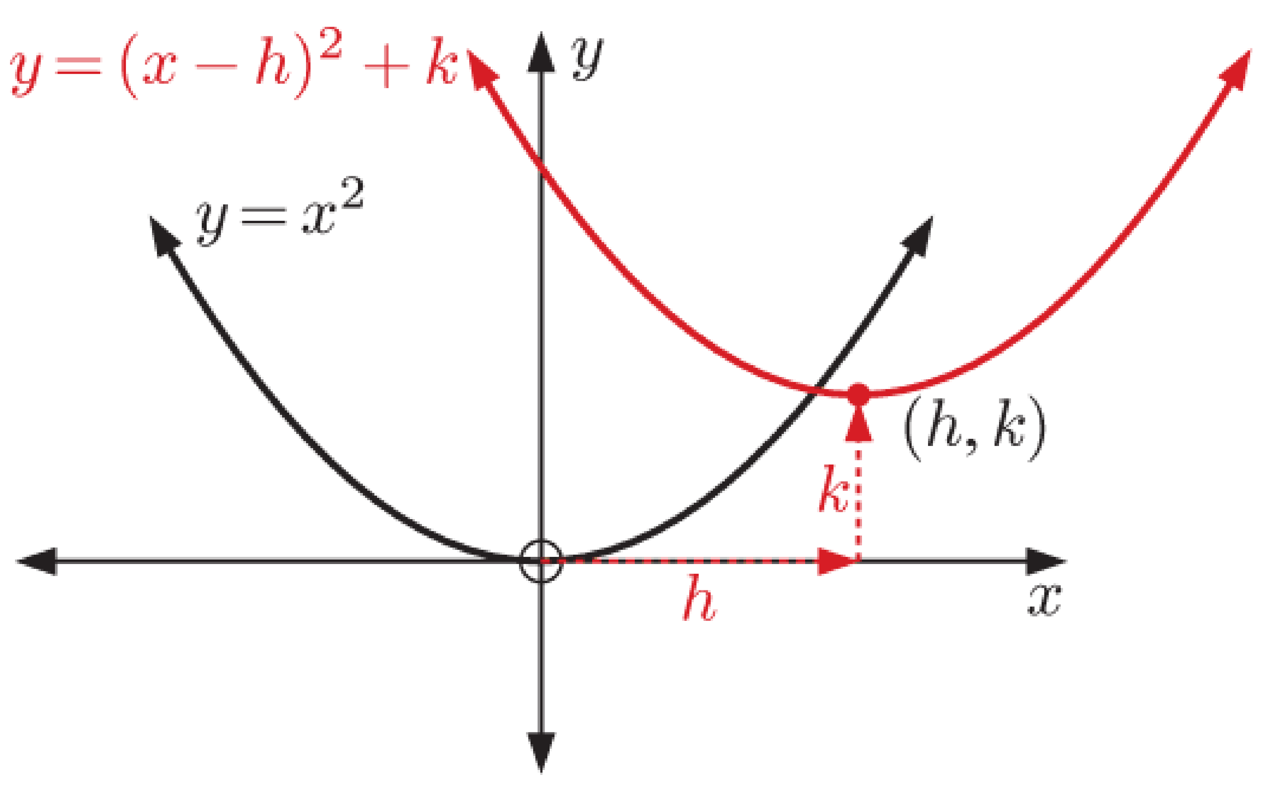
A TRANSLATIONS

If $f(x) = x^2$ then $f(x-h) + k = (x-h)^2 + k$.

The graph $y = (x-h)^2 + k$ has the same shape as $y = x^2$.

It can be produced from $y = x^2$ by a translation h units to the right and k units upwards.

This shifts the vertex of the parabola from the origin O(0, 0) to (h, k).

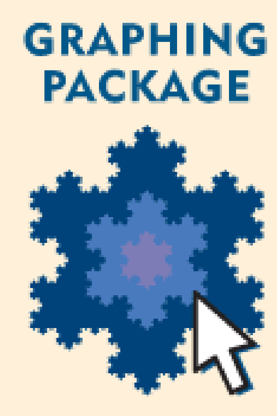


INVESTIGATION 1 TRANSLATIONS

Our observations of quadratics suggest that $y = f(x)$ can be transformed into $y = f(x-a) + b$ by a *translation*. In this Investigation we test this theory with other functions.

What to do:

1 Let $f(x) = x^3$.



a Write down:

i $f(x) + 2$ **ii** $f(x) - 3$ **iii** $f(x) + 6$

Graph $y = f(x)$ and the other three functions on the same set of axes. Record your observations.

b Write down:

i $f(x-2)$ **ii** $f(x+3)$ **iii** $f(x-6)$

Graph $y = f(x)$ and the other three functions on the same set of axes. Record your observations.

c Write down:

i $f(x-1)+3$ **ii** $f(x+2)+1$ **iii** $f(x-3)-4$

Graph $y=f(x)$ and the other three functions on the same set of axes.

2 Repeat **1** for the function $y=\dfrac{1}{x}$.

3 Describe the transformation which maps $y=f(x)$ onto:

a $y=f(x)+b$ **b** $y=f(x-a)$ **c** $y=f(x-a)+b$

4 Do any of these transformations change the *shape* of the graph?

From the **Investigation** you should have found:

- For $y=f(x)+b$, the effect of b is to **translate** the graph **vertically** through b units.
  - If $b>0$ it moves **upwards**.
  - If $b<0$ it moves **downwards**.
- For $y=f(x-a)$, the effect of a is to **translate** the graph **horizontally** through a units.
  - If $a>0$ it moves to the **right**.
  - If $a<0$ it moves to the **left**.
- For $y=f(x-a)+b$, the graph is translated horizontally a units and vertically b units.

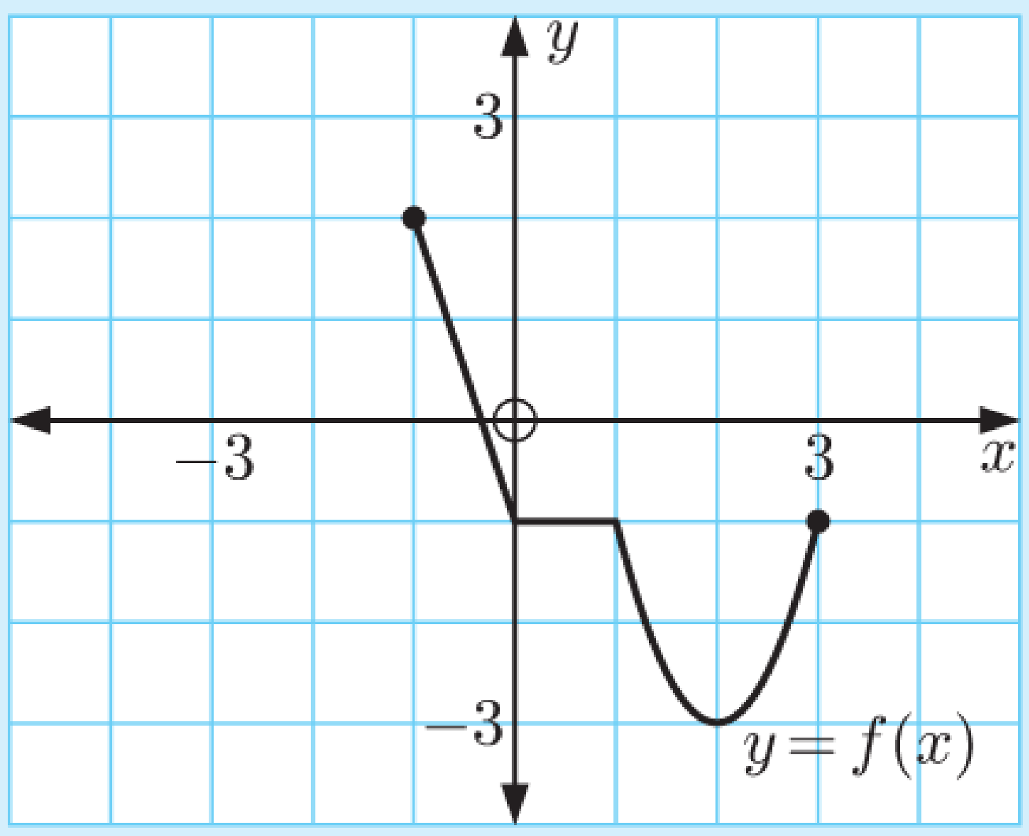
  We say it is **translated by the vector** $\begin{pmatrix} a \\ b \end{pmatrix}$.

Example 1 **Self Tutor**

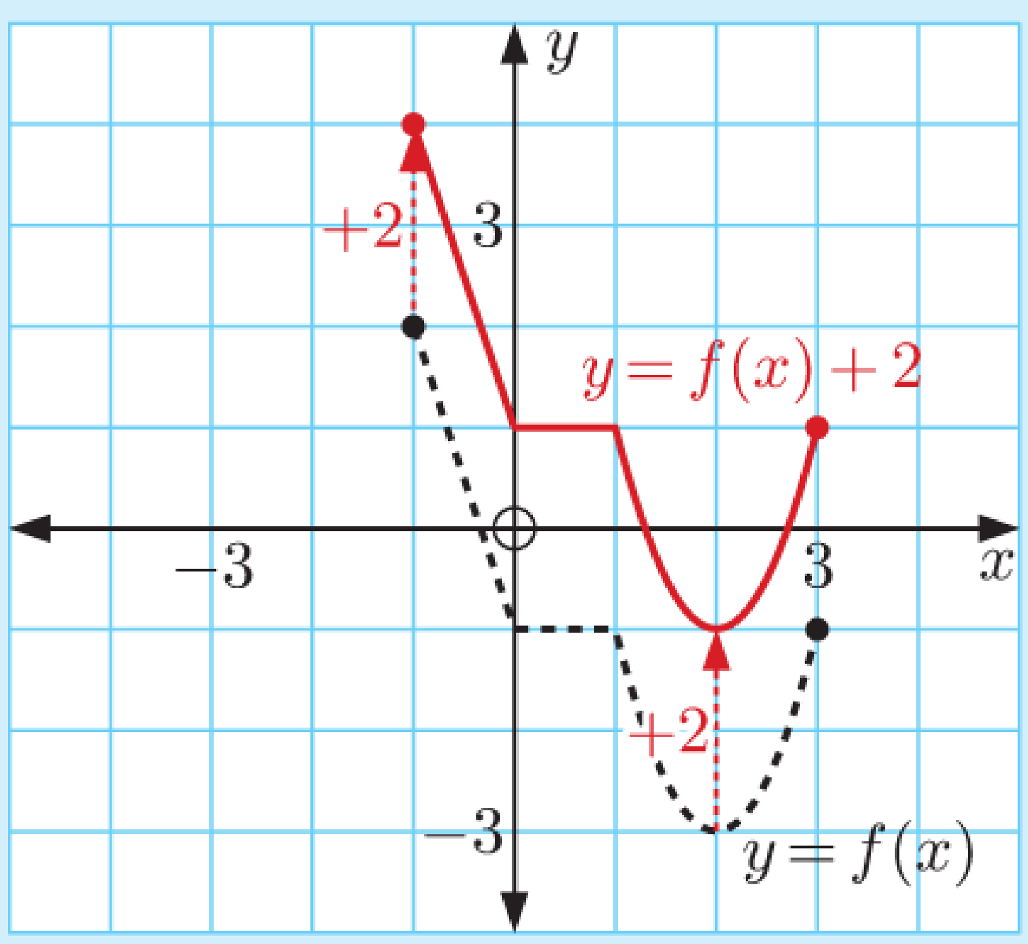
Consider the graph of $y=f(x)$ alongside.

On separate axes, draw the graphs of:

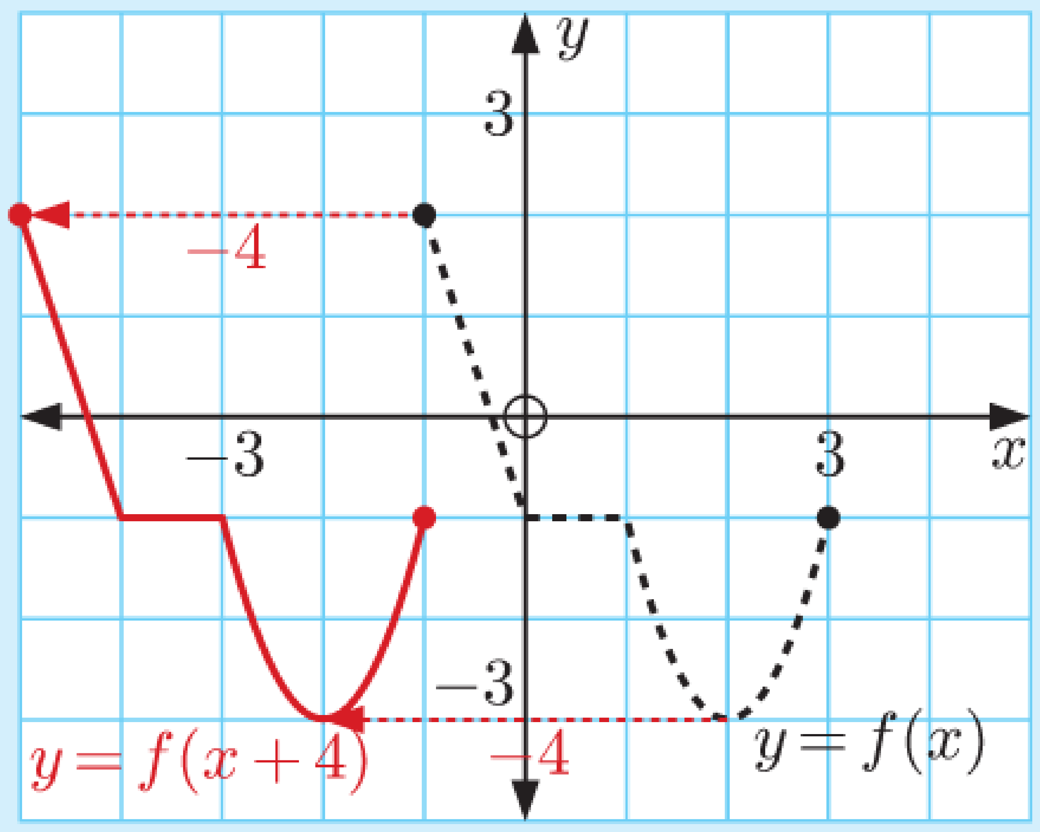
a $y=f(x)+2$ **b** $y=f(x+4)$



a The graph of $y=f(x)+2$ is found by translating $y=f(x)$ 2 units upwards.



b The graph of $y = f(x+4)$ is found by translating $y = f(x)$ 4 units to the left.



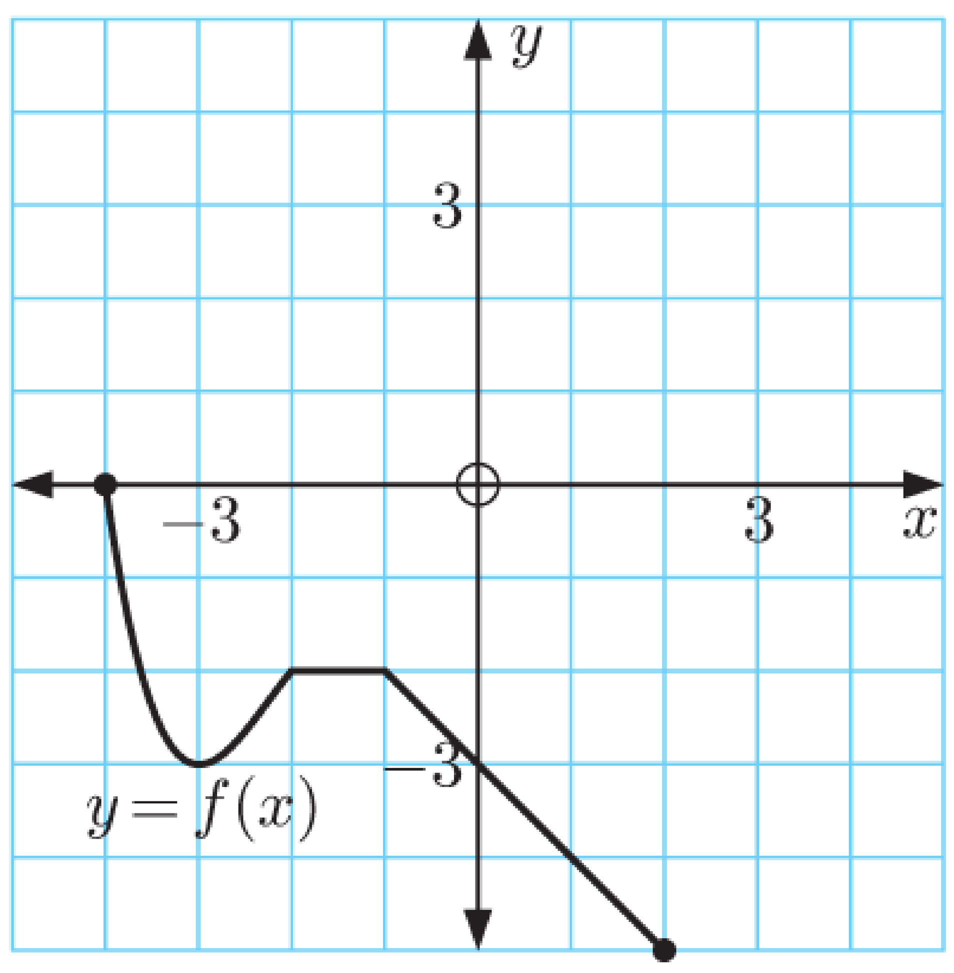
EXERCISE 16A

1 Consider the graph of $y = f(x)$ alongside. On separate axes, draw the graphs of:

a $y = f(x) + 5$ **b** $y = f(x-3)$

c $y = f(x-3) + 5$

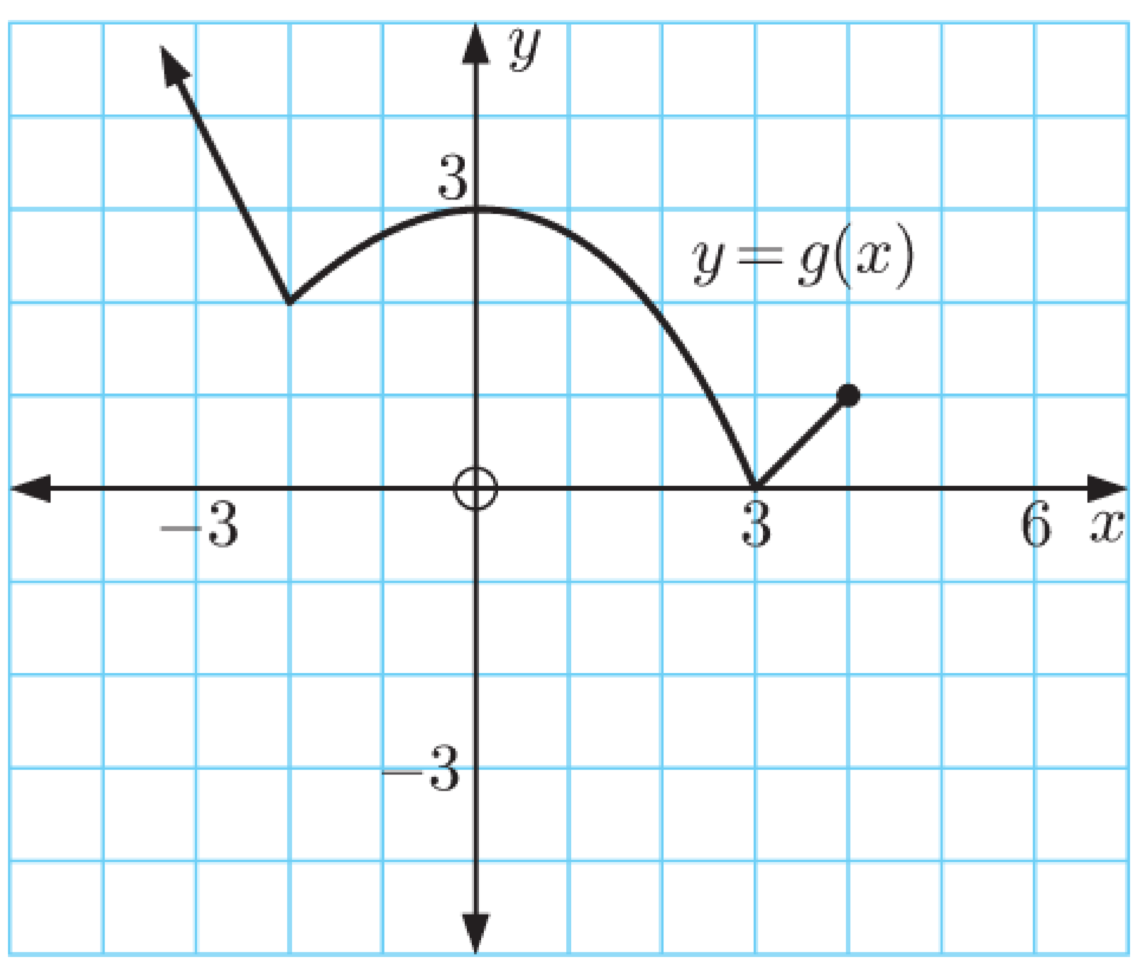
PRINTABLE GRIDS



2 Consider the graph of $y = g(x)$ alongside. On separate axes, draw the graphs of:

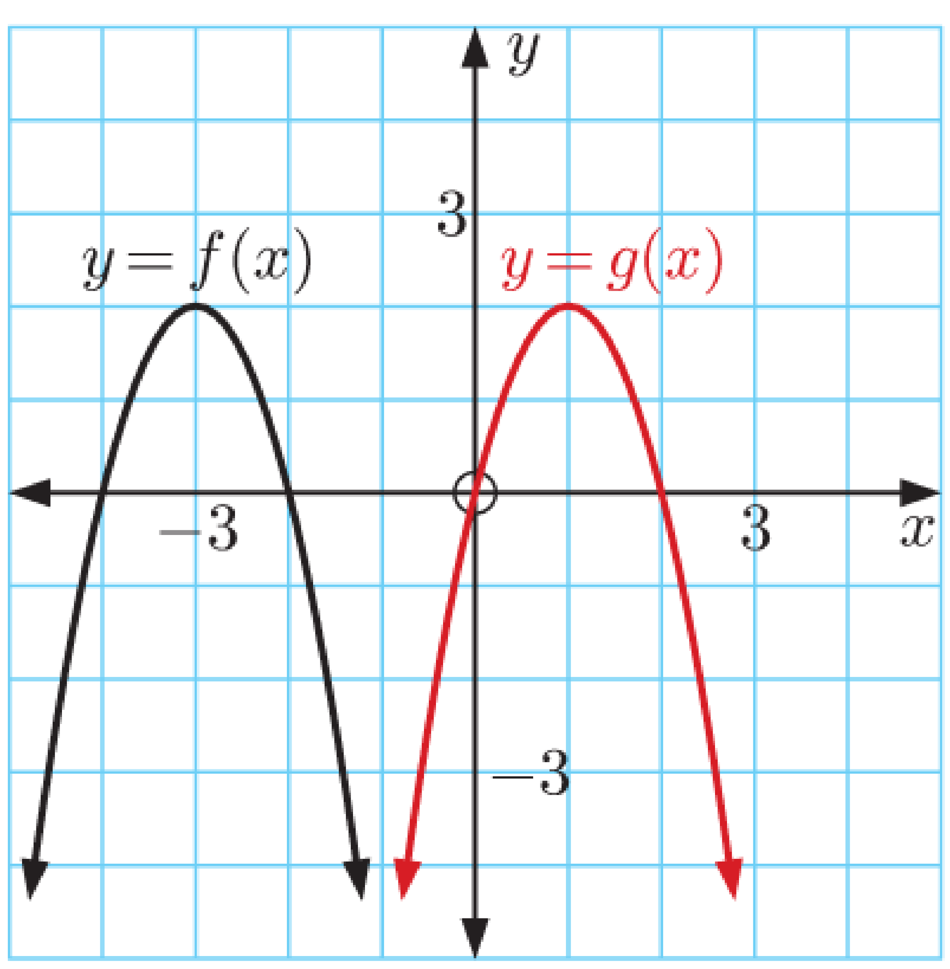
a $y = g(x) - 3$ **b** $y = g(x+1)$

c $y = g(x+1) - 3$ **d** $y = g(x-2) - 1$

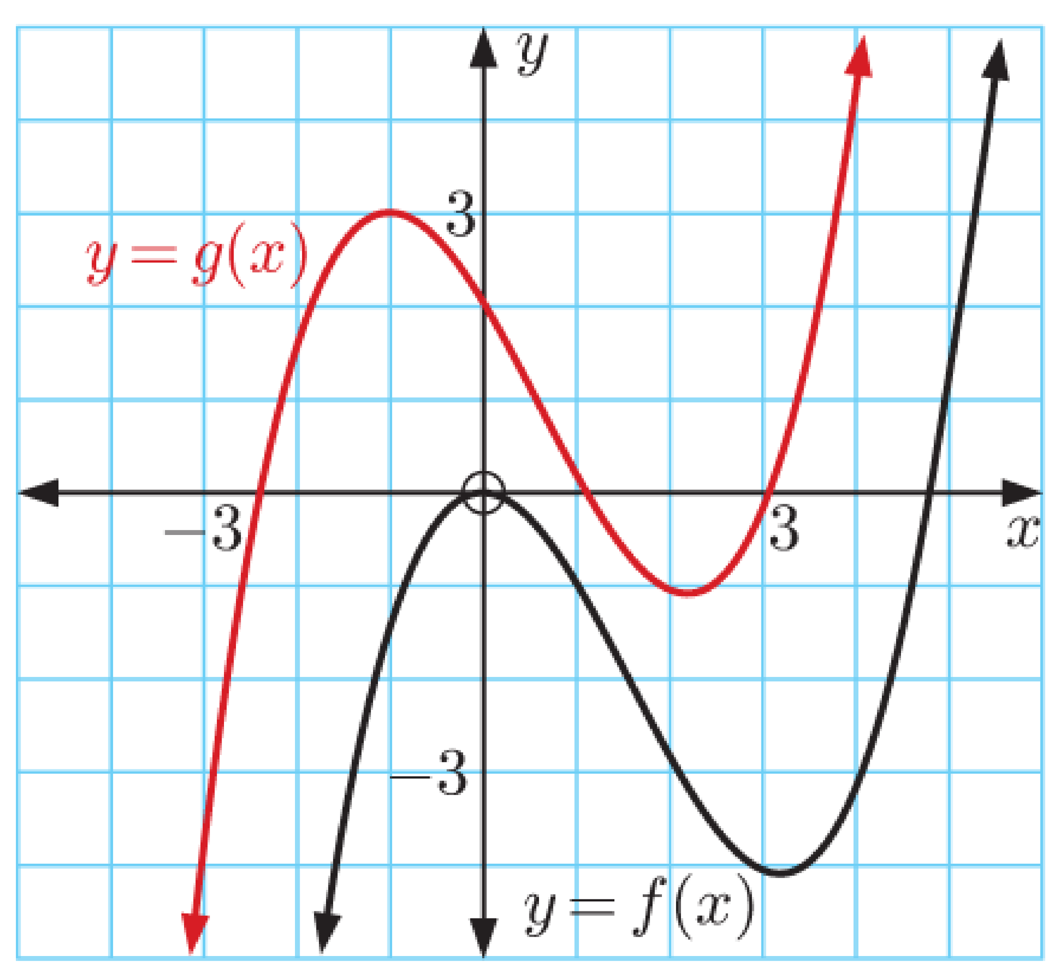


3 Write $g(x)$ in terms of $f(x)$:

a



b



4 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = 2x + 3$ is translated 4 units downwards

b $f(x) = 3x - 4$ is translated 2 units to the left

c $f(x) = -x^2 + 5x - 7$ is translated 3 units upwards

d $f(x) = x^2 + 4x - 1$ is translated 5 units to the right.

5 For each of the following functions f, sketch $y = f(x)$, $y = f(x) + 1$, and $y = f(x) - 2$ on the same set of axes.

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = \frac{1}{x}$ **d** $f(x) = (x-1)^2 + 2$

6 For each of the following functions f, sketch $y = f(x)$, $y = f(x-1)$, and $y = f(x+2)$ on the same set of axes.

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = \frac{1}{x}$ **d** $f(x) = (x-1)^2 + 2$

7 For each of the following functions f, sketch $y = f(x)$, $y = f(x-2) + 3$, and $y = f(x+1) - 4$ on the same set of axes.

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = \frac{1}{x}$ **d** $f(x) = (x-1)^2 + 2$

8 The point $(-2, -5)$ lies on the graph of $y = f(x)$. Find the coordinates of the corresponding point on the graph of $g(x) = f(x-3) - 4$.

9 Suppose the graph of $y = f(x)$ has x-intercepts -3 and 4, and y-intercept 2. What can you say about the axes intercepts of:

a $g(x) = f(x) - 3$ **b** $h(x) = f(x-1)$ **c** $j(x) = f(x+2) - 4$?

10 The graph of $f(x) = x^2 - 2x + 2$ is translated by $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ to form $g(x)$. Find $g(x)$ in the form $g(x) = ax^2 + bx + c$.

11 The graph of $f(x) = \frac{1}{x}$ is translated by $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$ to form $g(x)$. Find $g(x)$ in the form $g(x) = \dfrac{ax+b}{cx+d}$.

12 Suppose $f(x) = x^2$ is transformed to $g(x) = (x-3)^2 + 2$.

a Find the images of the following points on $f(x)$:

i $(0, 0)$ **ii** $(-3, 9)$ **iii** $(2, 4)$

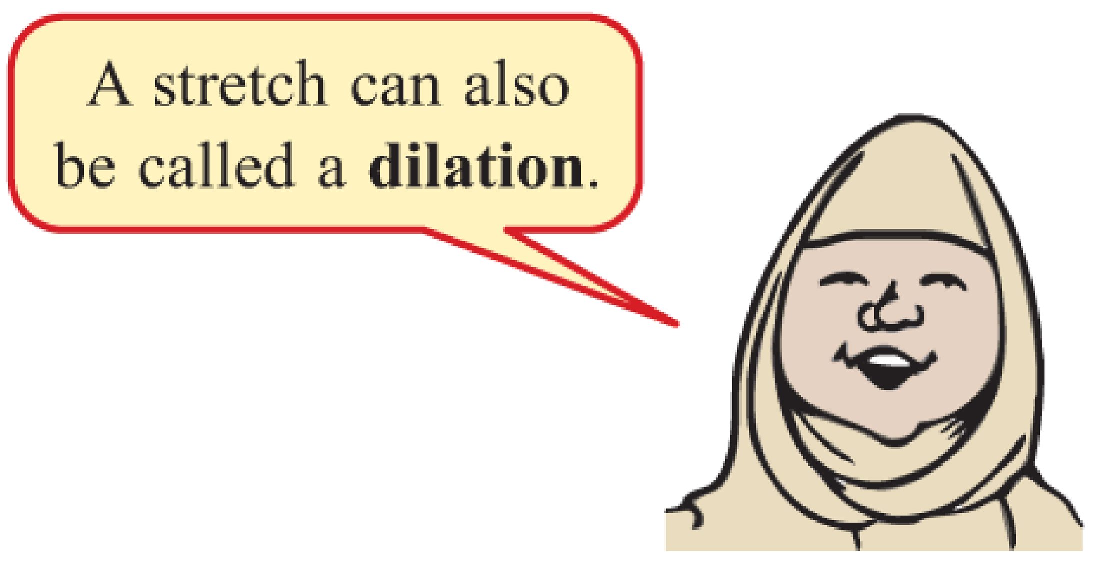
b Find the points on $f(x)$ which correspond to the following points on $g(x)$:

i $(1, 6)$ **ii** $(-2, 27)$ **iii** $(1\frac{1}{2}, 4\frac{1}{4})$

B STRETCHES

In this Section we study how a function can be manipulated to *stretch* its graph.

We will consider stretches in both the horizontal and vertical directions.

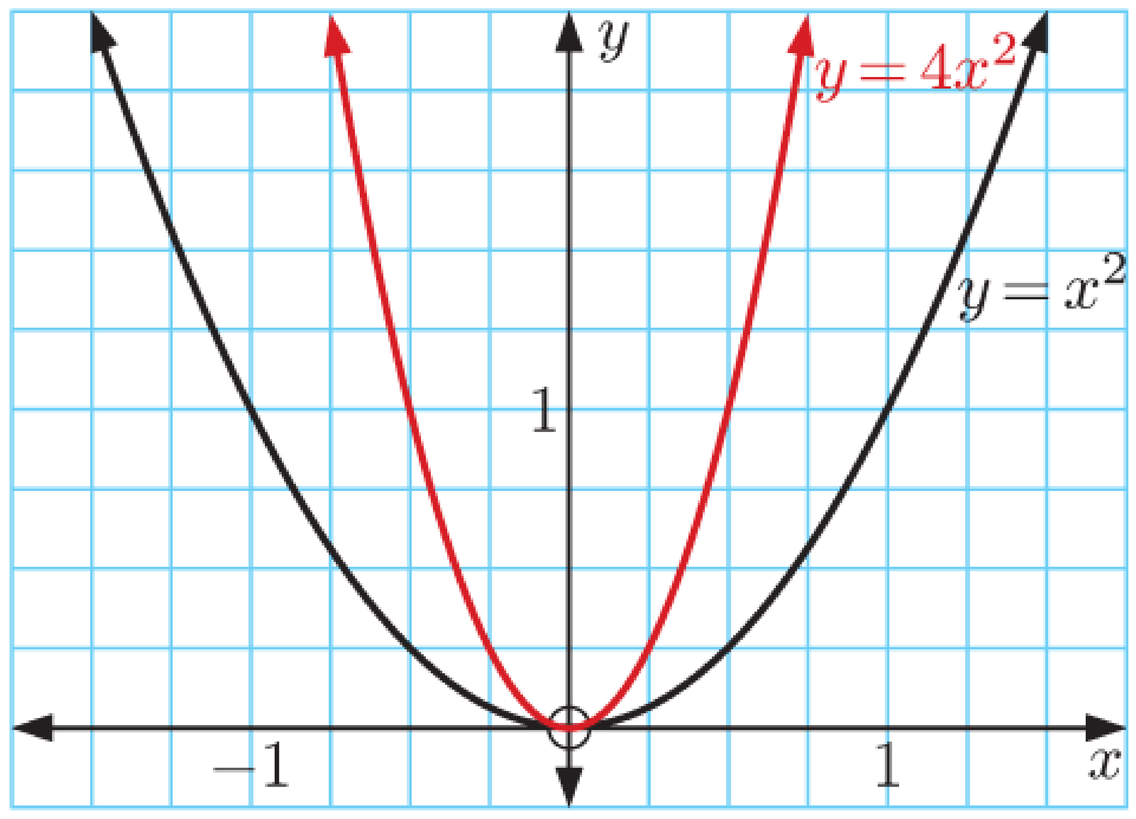


In our study of quadratic functions, we saw that the coefficient a of x^2 controls the width of the parabola.

In the case of $f(x) = x^2$,

notice that $f(2x) = (2x)^2 = 4x^2$

and $4f(x) = 4x^2$



DISCUSSION

- In what ways could $y = x^2$ be *stretched* to form $y = 4x^2$?
- Will a transformation of the form $pf(x)$, $p > 0$ always be equivalent to a transformation of the form $f(qx)$, $q > 0$?

INVESTIGATION 2 — STRETCHES

In this Investigation we consider transformations of the form $pf(x)$, $p > 0$, and $f(qx)$, $q > 0$.

What to do:

GRAPHING PACKAGE

1 Let $f(x) = x + 2$.

a Find, in simplest form:

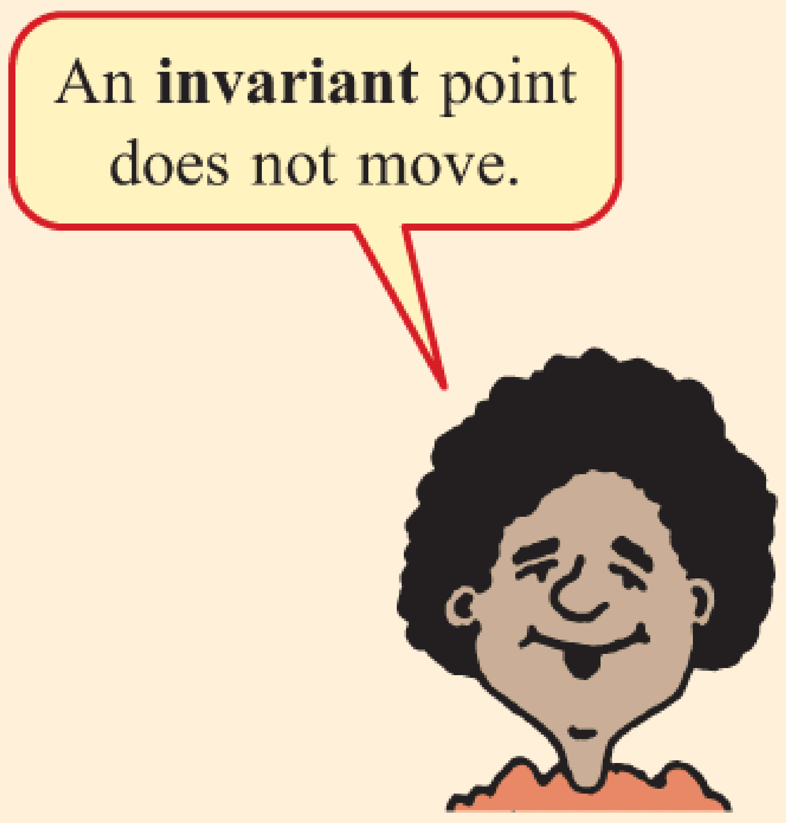
i $3f(x)$ **ii** $\frac{1}{2}f(x)$ **iii** $5f(x)$

b Graph all four functions on the same set of axes.

c Which point is *invariant* under a transformation of the form $pf(x)$, $p > 0$?

d Copy and complete:

For the transformation $y = pf(x)$, each point becomes ...... times its previous distance from the x-axis.



2 Let $f(x) = x + 2$.

a Find, in simplest form:

i $f(2x)$ **ii** $f(\frac{1}{3}x)$ **iii** $f(4x)$

b Graph all four functions on the same set of axes.

c Which point is *invariant* under a transformation of the form $f(qx)$, $q > 0$?

d Copy and complete:

For the transformation $y = f(qx)$, each point becomes ...... times its previous distance from the y-axis.

From the **Investigation** you should have found:

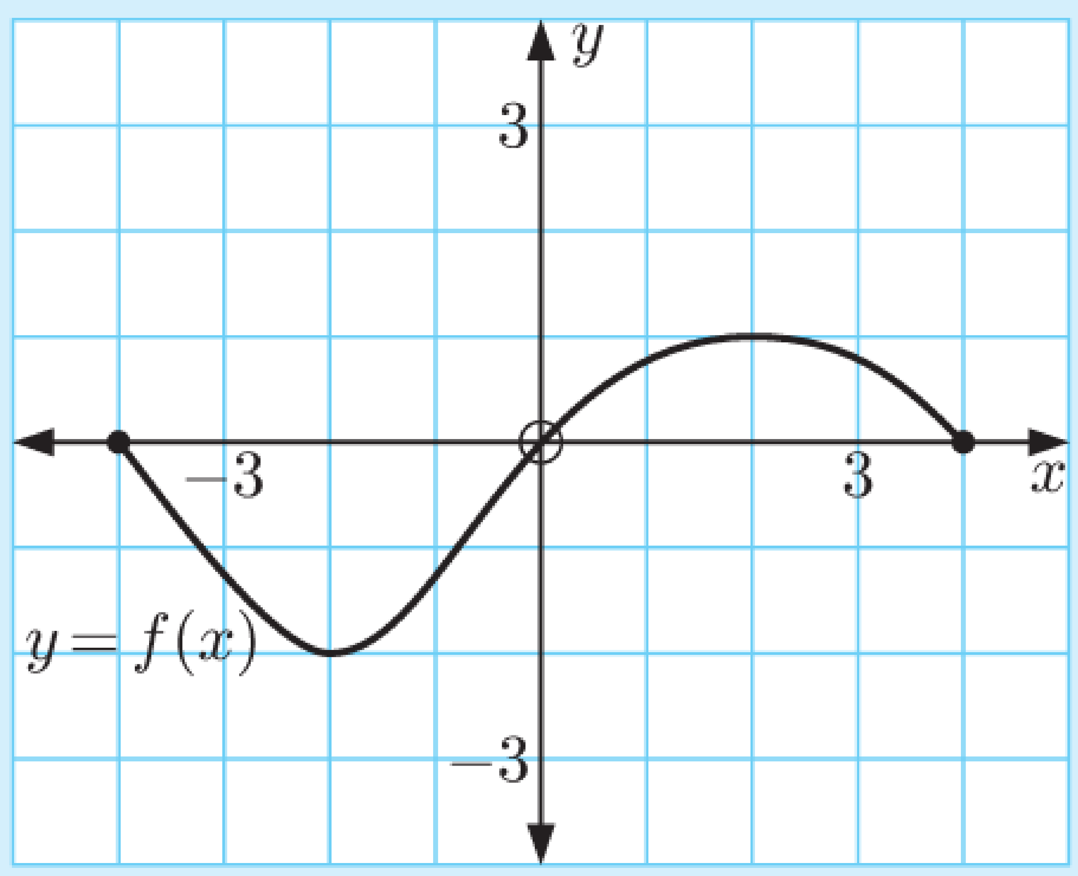
- $y = p\,f(x)$, $p > 0$ is a **vertical stretch** of $y = f(x)$ with **scale factor** p and **invariant** $\boldsymbol{x}$**-axis**.
  - Each point becomes p times its previous distance from the x-axis.
  - If $p > 1$, points move further away from the x-axis.
  - If $0 < p < 1$, points move closer to the x-axis.
- $y = f(qx)$, $q > 0$ is a **horizontal stretch** of $y = f(x)$ with **scale factor** $\frac{1}{q}$ and **invariant** $\boldsymbol{y}$**-axis**.
  - Each point becomes $\frac{1}{q}$ times its previous distance from the y-axis.
  - If $q > 1$, points move closer to the y-axis.
  - If $0 < q < 1$, points move further away from the y-axis.

Example 2 **Self Tutor**

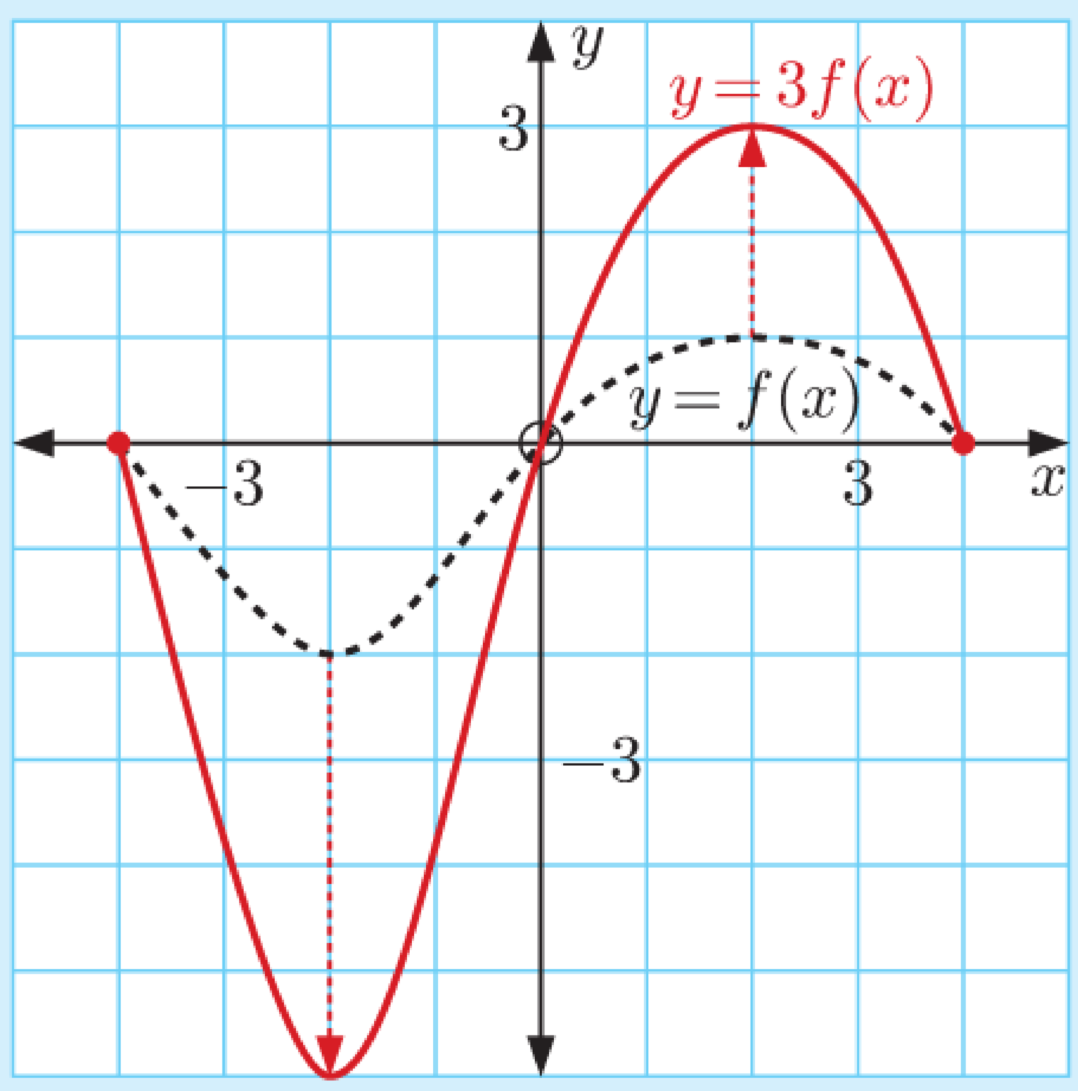
Consider the graph of $y = f(x)$ alongside.

On separate axes, draw the graphs of:

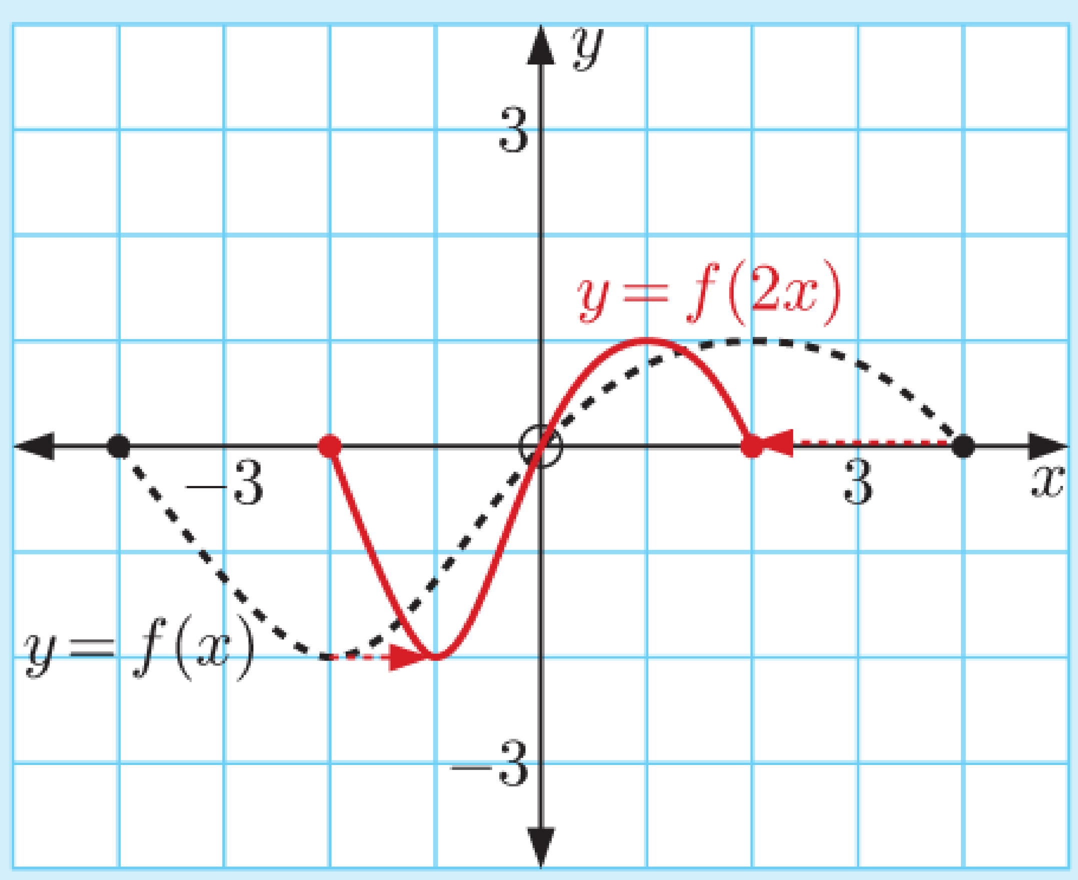
a $y = 3\,f(x)$ **b** $y = f(2x)$



a The graph of $y = 3\,f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 3.



b The graph of $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$.

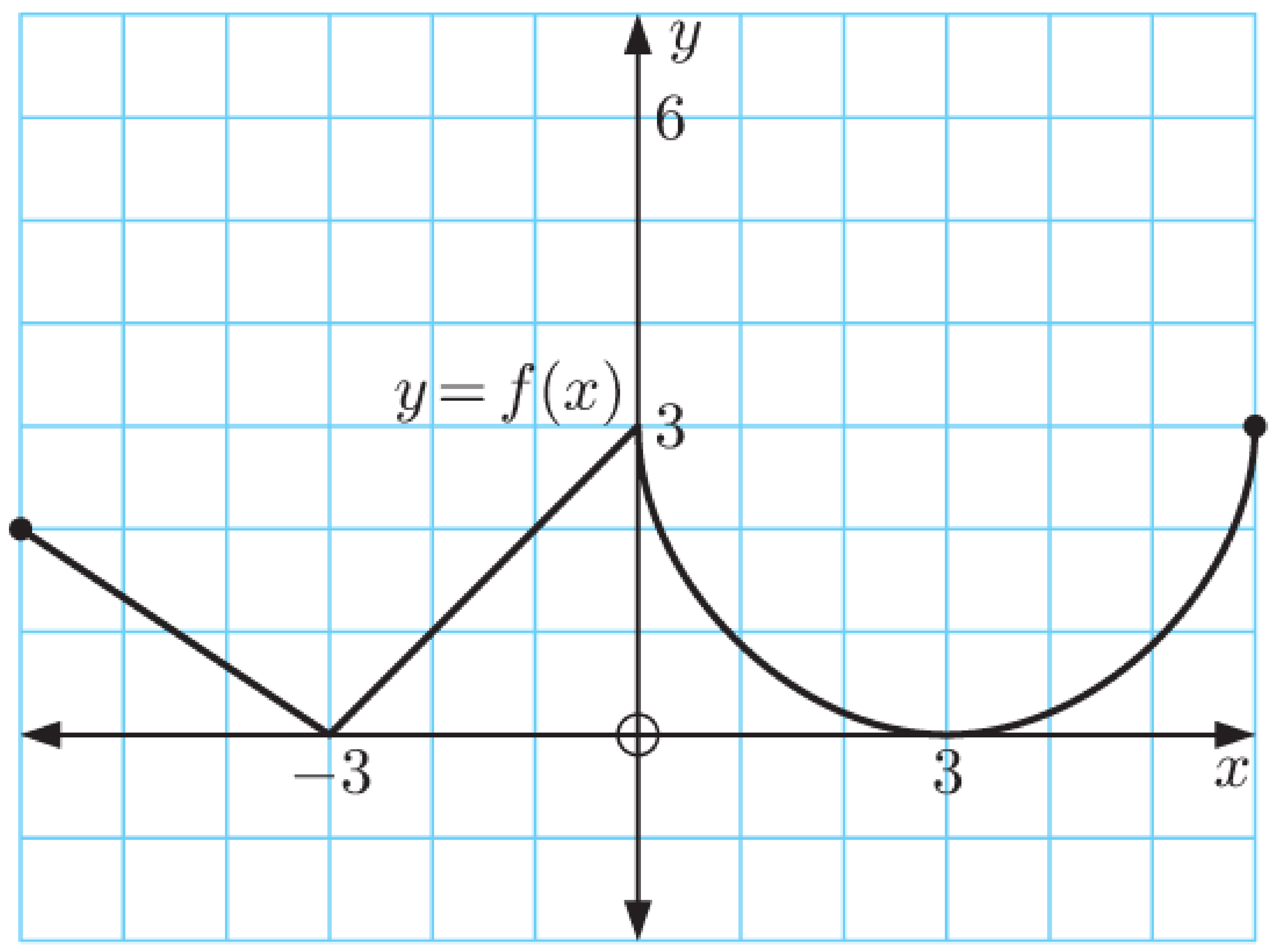


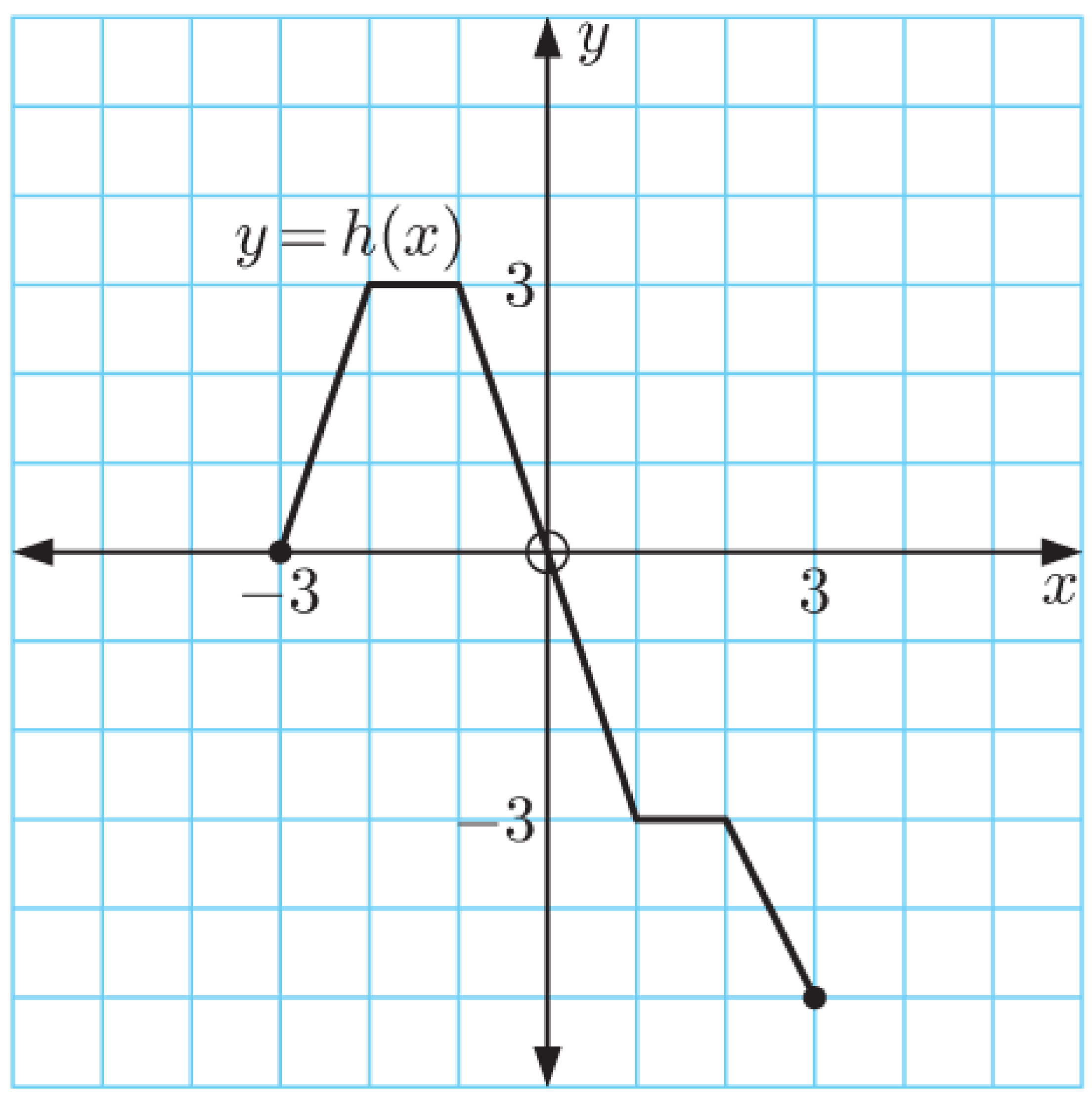
EXERCISE 16B

1 Consider the graph of $y = f(x)$ alongside.
On separate axes, draw the graphs of:

a $y = 2f(x)$ **b** $y = f(3x)$

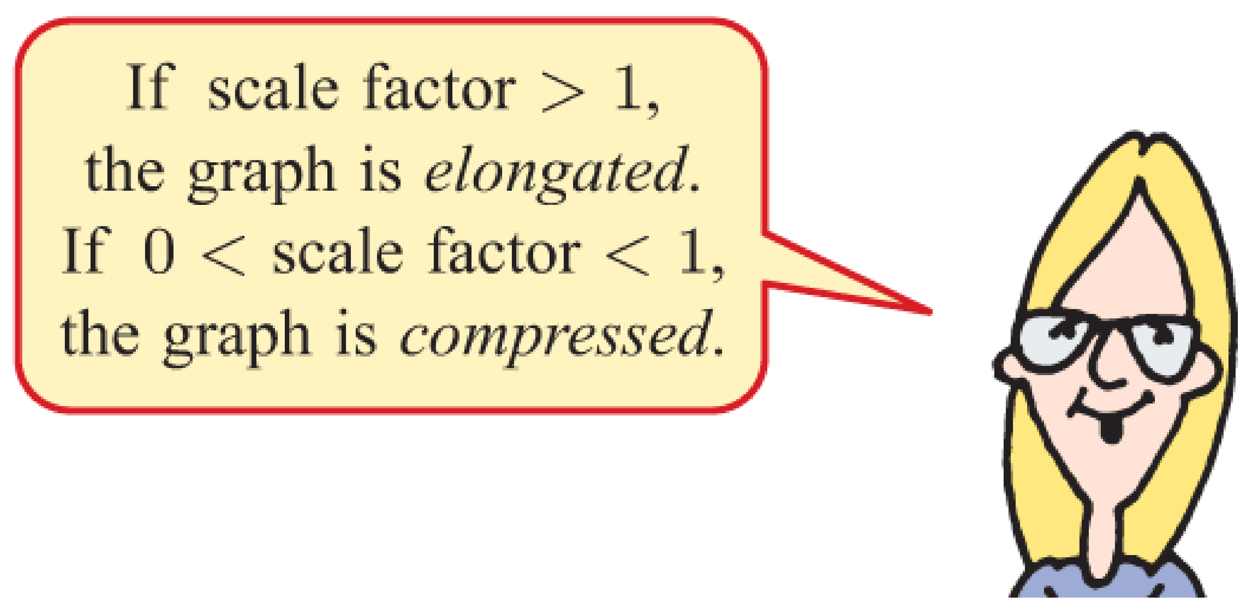
PRINTABLE GRIDS



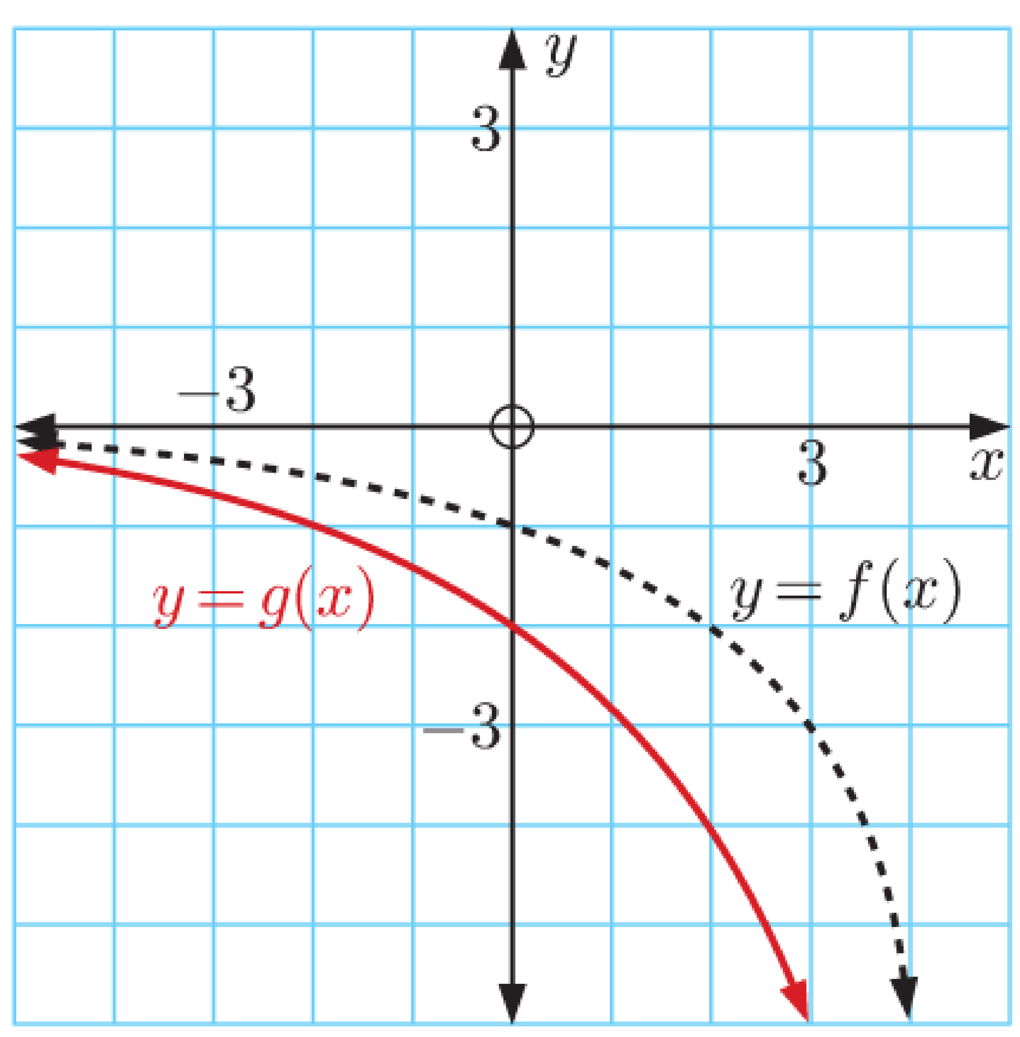
2 

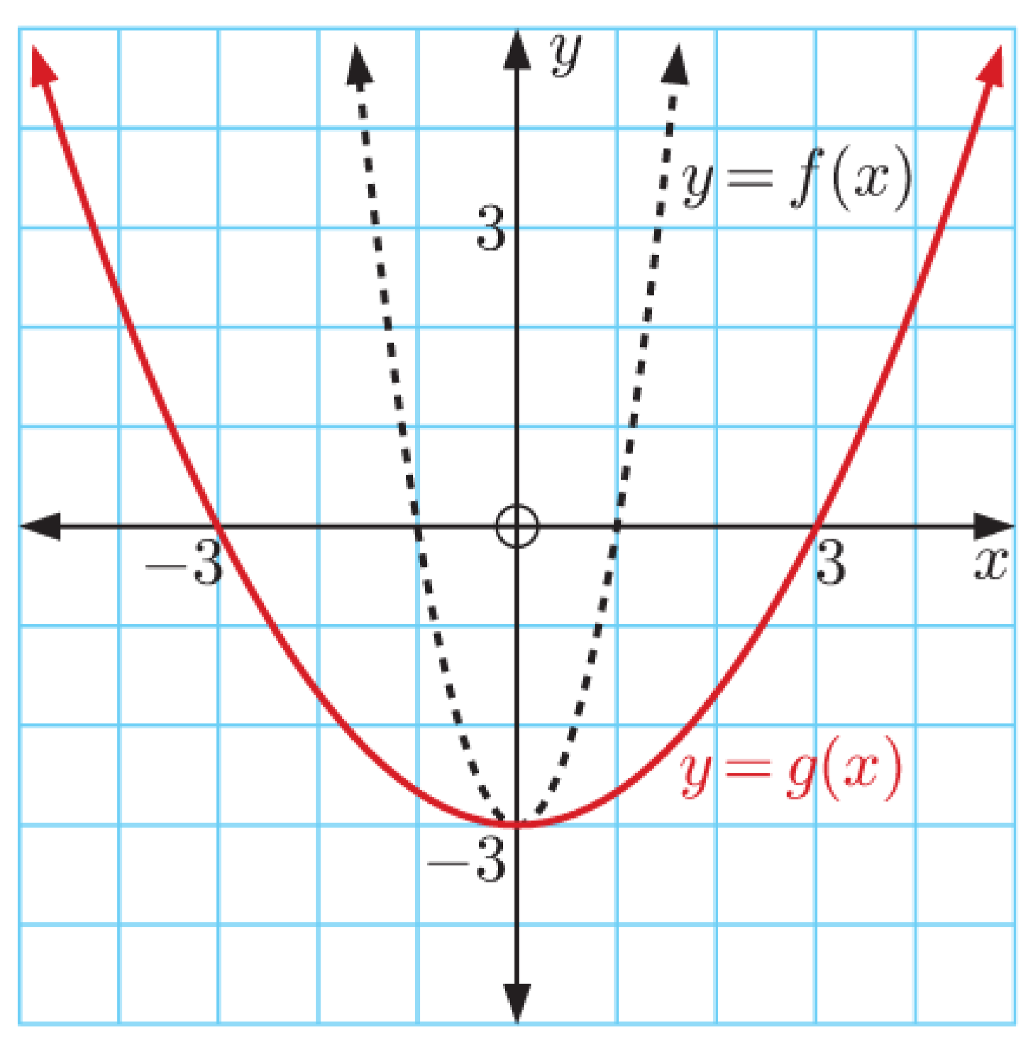
Consider the graph of $y = h(x)$ alongside.
On separate axes, draw the graphs of:

a $y = \frac{1}{3}h(x)$ **b** $y = h\left(\frac{x}{2}\right)$



3 Write $g(x)$ in terms of $f(x)$:

a 

b 

4 A linear function with gradient m is vertically stretched with scale factor c. Find the gradient of the resulting line.

5 For each of the following functions f, sketch $y = f(x)$, $y = 2f(x)$, and $y = 3f(x)$ on the same set of axes:

a $f(x) = x - 1$ **b** $f(x) = x^2$

c $f(x) = x^3$ **d** $f(x) = \frac{1}{x}$

6 For each of the following functions f, sketch $y = f(x)$, $y = \frac{1}{2}f(x)$, and $y = \frac{1}{4}f(x)$ on the same set of axes:

a $f(x) = x - 1$ **b** $f(x) = x^2$ **c** $f(x) = x^3$ **d** $f(x) = \dfrac{1}{x}$

7 Sketch, on the same set of axes, the graphs of $y = f(x)$ and $y = f(2x)$ for:

a $y = x^2$ **b** $y = (x-1)^2$ **c** $y = (x+3)^2$

8 Sketch, on the same set of axes, the graphs of $y = f(x)$ and $y = f(\frac{x}{2})$ for:

a $y = x^2$ **b** $y = 2x$ **c** $y = (x+2)^2$

9 Suppose f and g are functions such that $g(x) = f(5x)$.

a Given that $(10, 25)$ lies on $y = f(x)$, find the coordinates of the corresponding point on $y = g(x)$.

b Given that $(-5, -15)$ lies on $y = g(x)$, find the coordinates of the corresponding point on $y = f(x)$.

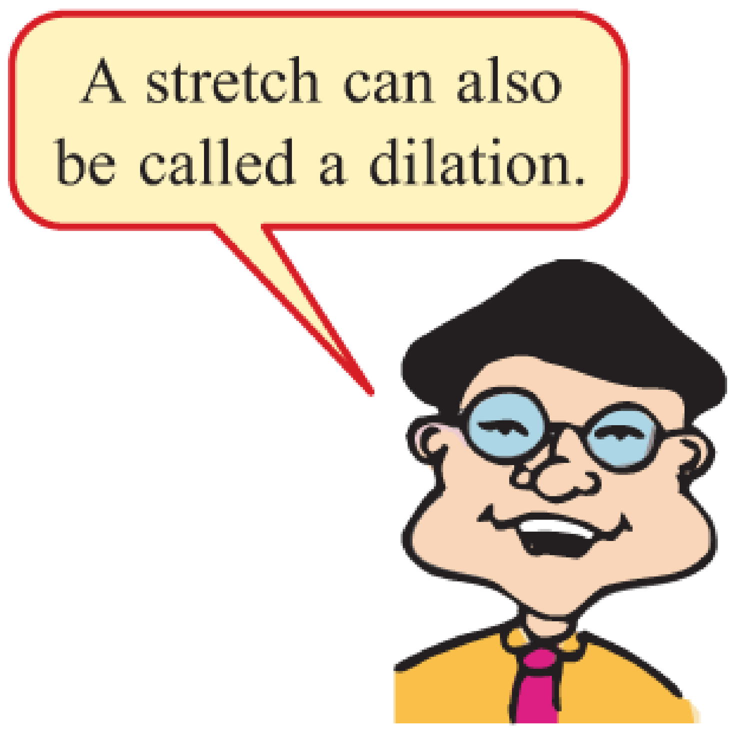
10 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = x^2 + 2$ is vertically stretched with scale factor 2

b $f(x) = 5 - 3x$ is horizontally stretched with scale factor 3

c $f(x) = x^3 + 8x^2 - 2$ is vertically dilated with scale factor $\frac{1}{4}$

d $f(x) = 2x^2 + x - 3$ is horizontally dilated with scale factor $\frac{1}{2}$.



11 Graph on the same set of axes $y = x^2$, $y = 3x^2$, and $y = 3(x+1)^2 - 2$.
Describe the combination of transformations which transform $y = x^2$ to $y = 3(x+1)^2 - 2$.

12 Graph on the same set of axes $y = x^2$, $y = \frac{1}{2}x^2$, and $y = \frac{1}{2}(x+1)^2 + 3$.
Describe the combination of transformations which transform $y = x^2$ to $y = \frac{1}{2}(x+1)^2 + 3$.

13 Graph on the same set of axes $y = x^2$, $y = 2x^2$, and $y = 2(x - \frac{3}{2})^2 + 1$.
Describe the combination of transformations which transform $y = x^2$ to $y = 2(x - \frac{3}{2})^2 + 1$.

14 Describe the combination of transformations which transform $y = x^2$ to $y = 2(x+1)^2 - 3$.
Hence sketch $y = 2(x+1)^2 - 3$.

15 Suppose f and g are functions such that $g(x) = 3f(2x)$.

a What transformations are needed to map $y = f(x)$ onto $y = g(x)$?

b Find the image of each of these points on $y = f(x)$:

i $(3, -5)$ **ii** $(1, 2)$ **iii** $(-2, 1)$

c Find the point on $y = f(x)$ which maps onto the image point:

i $(2, 1)$ **ii** $(-3, 2)$ **iii** $(-7, 3)$

16 The graph of $y = x^2$ is transformed to the graph of $y = 0.1x^2 + 5$ by a translation a units vertically followed by a horizontal stretch with scale factor b. Find a and b.

Example 3 — Self Tutor

The function $g(x)$ results when $y = \frac{1}{x}$ is transformed by a vertical stretch with scale factor 2, followed by a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

a Write an expression for $g(x)$ in the form $g(x) = \dfrac{ax+b}{cx+d}$.

b Find the asymptotes of $y = g(x)$.

c Sketch $y = g(x)$.

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a Under a vertical stretch with scale factor 2, $f(x)$ becomes $2f(x)$.

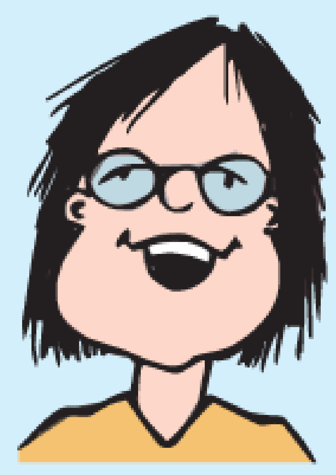
$\therefore \; \frac{1}{x}$ becomes $2\left(\frac{1}{x}\right) = \frac{2}{x}$.

Under a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $f(x)$ becomes $f(x-3)-2$.

$\therefore \; \frac{2}{x}$ becomes $\dfrac{2}{x-3} - 2$.

So, $y = \frac{1}{x}$ becomes
$$g(x) = \frac{2}{x-3} - 2 = \frac{2-2(x-3)}{x-3} = \frac{-2x+8}{x-3}$$

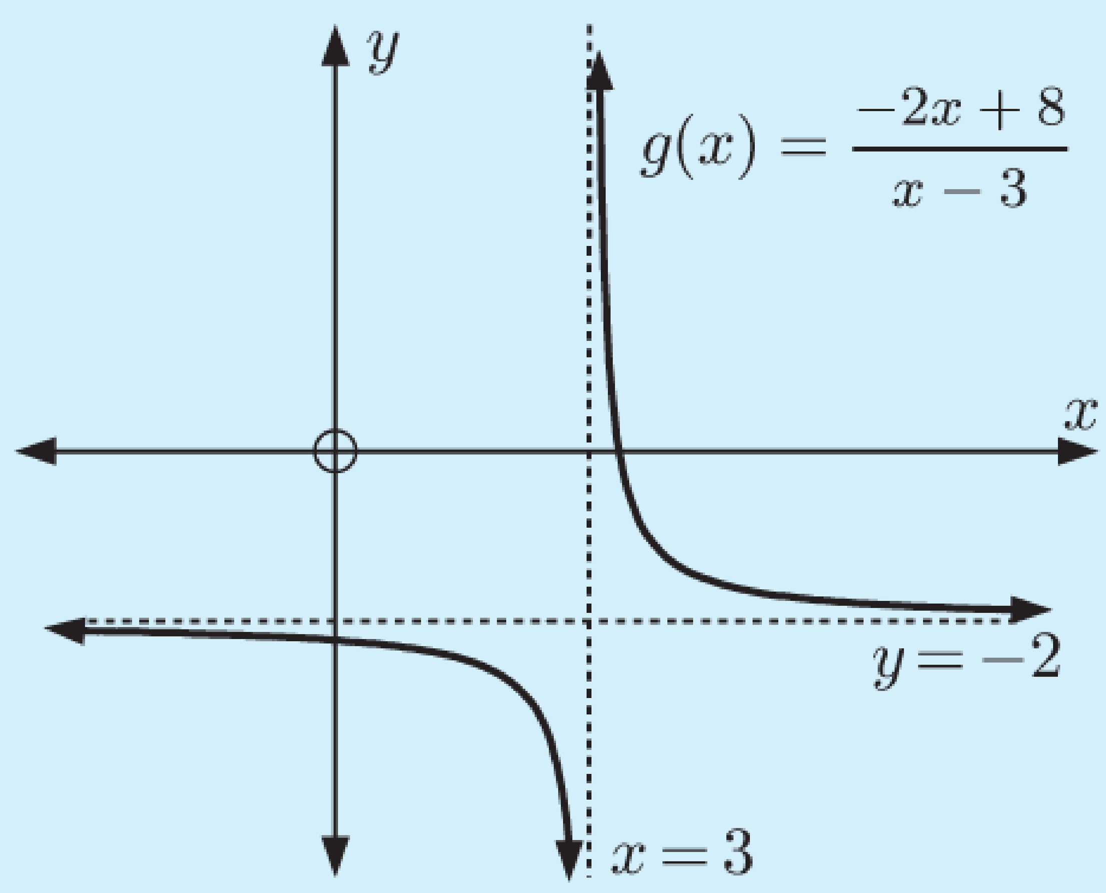
$g(x)$ is a rational function which is $\dfrac{\text{linear}}{\text{linear}}$.



b The asymptotes of $y = \frac{1}{x}$ are $x = 0$ and $y = 0$.
These are unchanged by the stretch, and shifted $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ by the translation.

$\therefore$ the vertical asymptote is $x = 3$ and the horizontal asymptote is $y = -2$.

c



17 Write, in the form $y = \dfrac{ax+b}{cx+d}$, the function that results when $y = \frac{1}{x}$ is transformed by:

a a vertical dilation with scale factor $\frac{1}{2}$

b a horizontal dilation with scale factor 3

c a horizontal translation of -3

d a vertical translation of 4.

18 The function $g(x)$ results when $y = \frac{1}{x}$ is transformed by a vertical stretch with scale factor 3, followed by a translation of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

a Write an expression for $g(x)$ in the form $g(x) = \dfrac{ax+b}{cx+d}$.

b Find the asymptotes of $y = g(x)$.

c State the domain and range of $g(x)$.

d Sketch $y = g(x)$.

19 The function $g(x)$ results when $y = \frac{1}{x}$ is transformed by a translation of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, followed by a horizontal stretch with scale factor $\frac{1}{2}$.

- **a** Write an expression for $g(x)$ in the form $g(x) = \frac{ax+b}{cx+d}$.
- **b** Find the asymptotes of $y = g(x)$.
- **c** State the domain and range of $g(x)$.
- **d** Sketch $y = g(x)$.

20 Find *two* combinations of transformations which map $f(x) = 2x^2 + 8x - 1$ onto $g(x) = 8x^2 - 16x + 5$.

DISCUSSION

For a vertical stretch with scale factor p, each point on the function is moved vertically so it is p times as far from the x-axis.

1. Using this definition of a vertical stretch, does it make sense to talk about negative values of p?
2. If a function is transformed from $f(x)$ to $-f(x)$, what transformation has actually occurred?
3. What *combinations* of transformations would transform $f(x)$ to $-2f(x)$?
4. What can we say about $y = f(qx)$ for:
   - **a** $q = -1$
   - **b** $q < 0$, $q \neq -1$?

C REFLECTIONS

INVESTIGATION 3 REFLECTIONS

In this Investigation we consider **reflections** with the forms $y = -f(x)$ and $y = f(-x)$.

What to do:

GRAPHING PACKAGE

1. Consider $f(x) = 2x + 3$.
   - **a** Find in simplest form:
     - **i** $-f(x)$
     - **ii** $f(-x)$
   - **b** Graph $y = f(x)$, $y = -f(x)$, and $y = f(-x)$ on the same set of axes.
2. Consider $f(x) = x^3 + 1$.
   - **a** Find in simplest form:
     - **i** $-f(x)$
     - **ii** $f(-x)$
   - **b** Graph $y = f(x)$, $y = -f(x)$, and $y = f(-x)$ on the same set of axes.
3. What transformation moves:
   - **a** $y = f(x)$ to $y = -f(x)$
   - **b** $y = f(x)$ to $y = f(-x)$?

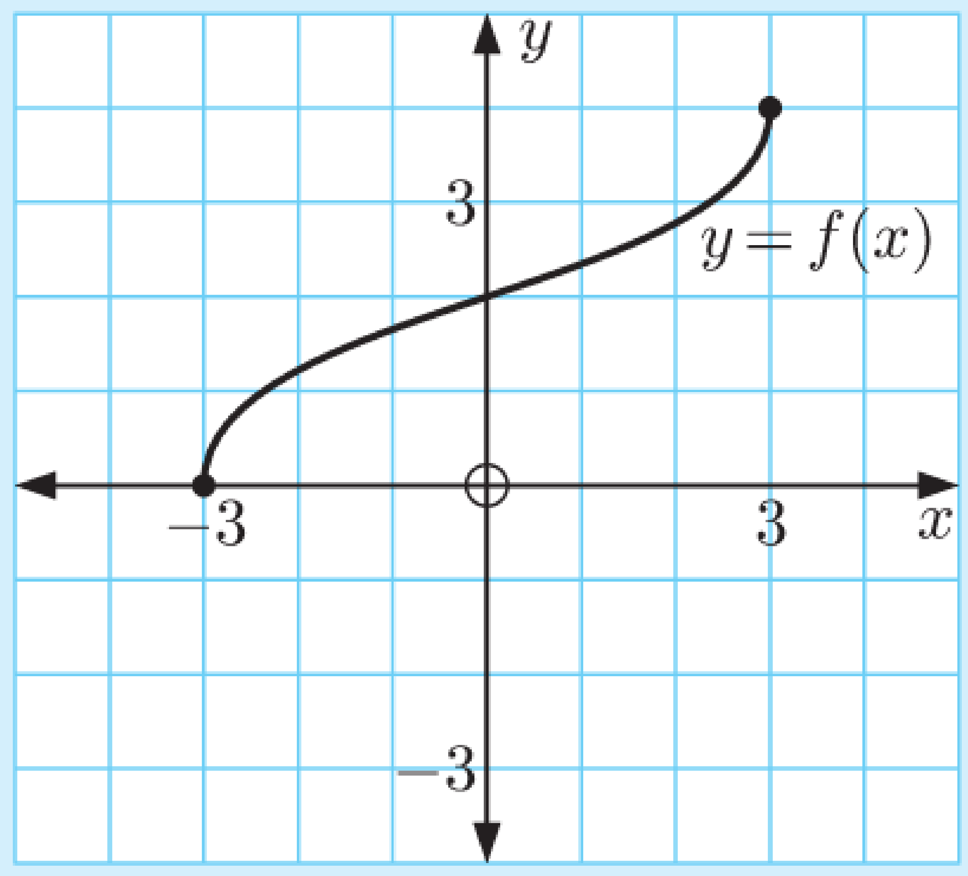
From the **Investigation** you should have discovered that:

- For $y=-f(x)$, we **reflect** $y=f(x)$ in the **x-axis**.
- For $y=f(-x)$, we **reflect** $y=f(x)$ in the **y-axis**.

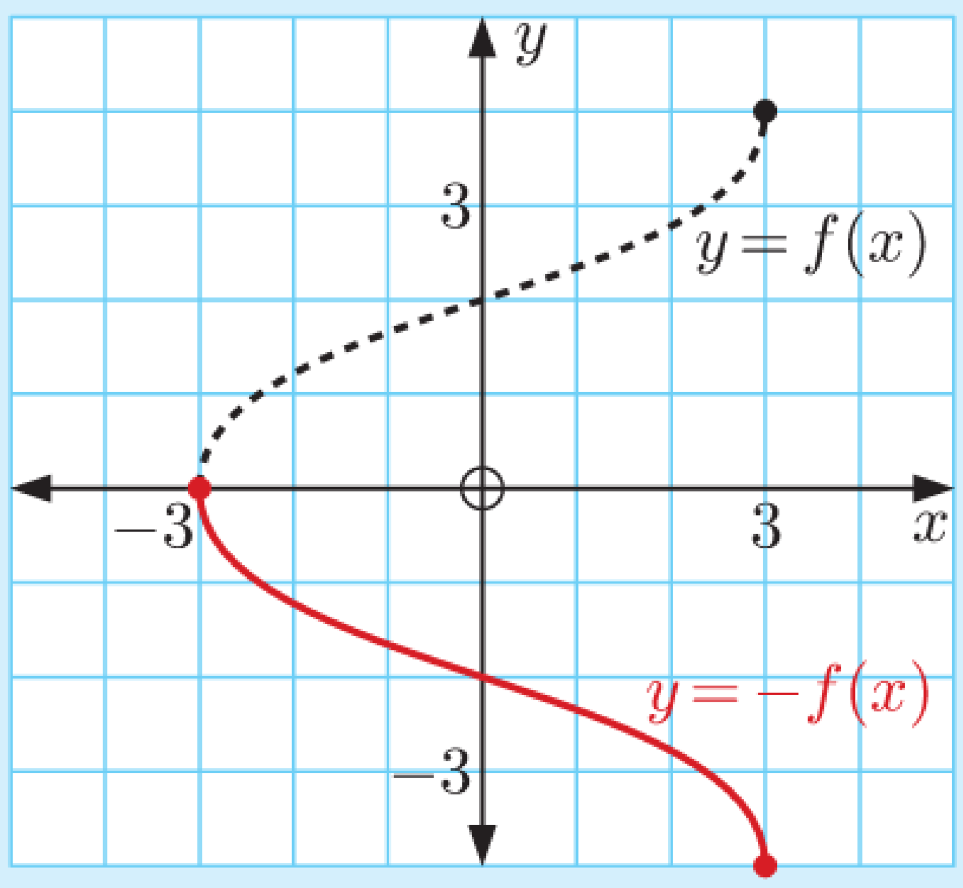
Example 4 **Self Tutor**

Consider the graph of $y=f(x)$ alongside.
On separate axes, draw the graphs of:

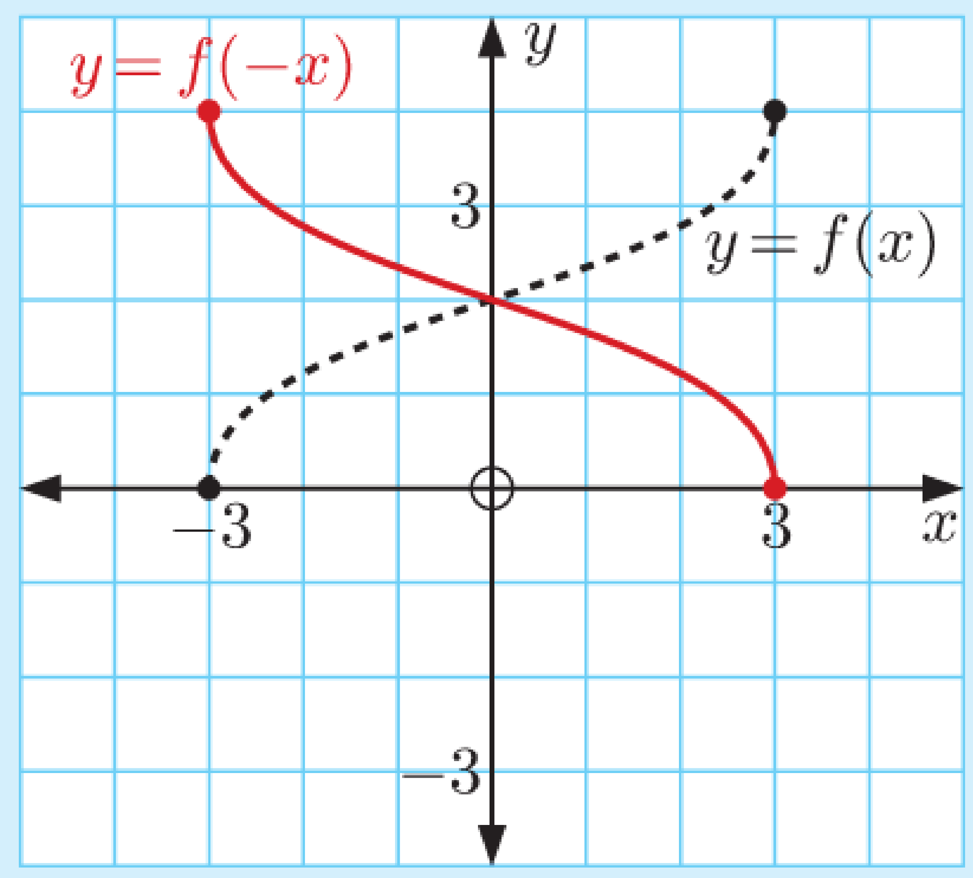
a $y=-f(x)$ **b** $y=f(-x)$



a The graph of $y=-f(x)$ is found by reflecting $y=f(x)$ in the x-axis.



b The graph of $y=f(-x)$ is found by reflecting $y=f(x)$ in the y-axis.

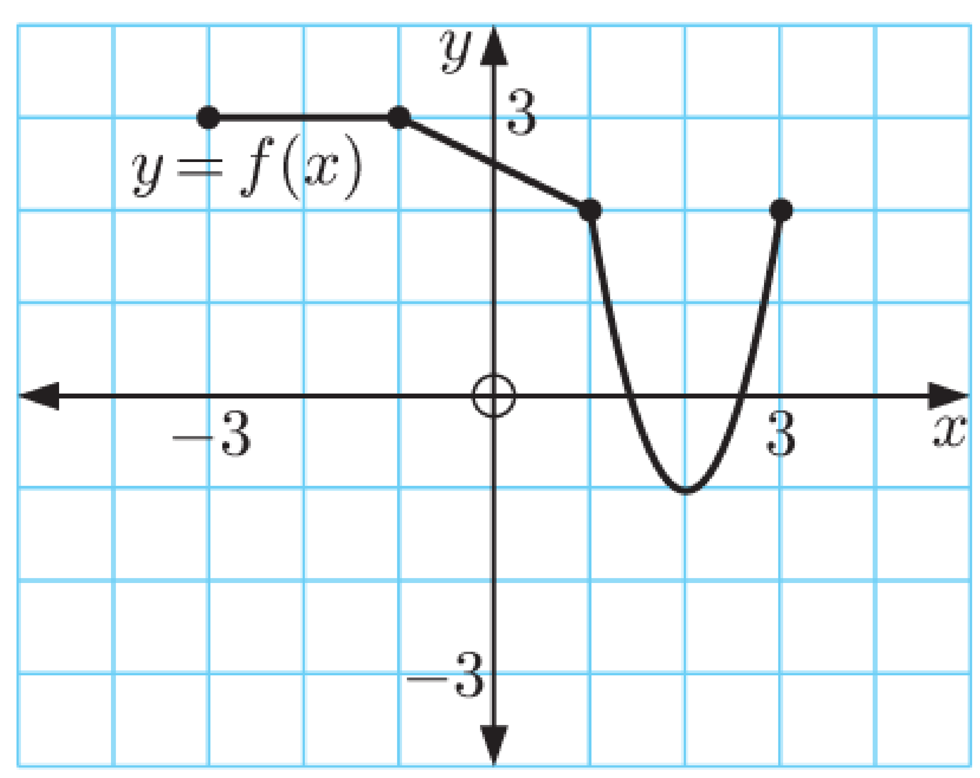


EXERCISE 16C

1 Consider the graph of $y=f(x)$ alongside.
On separate axes, draw the graphs of:

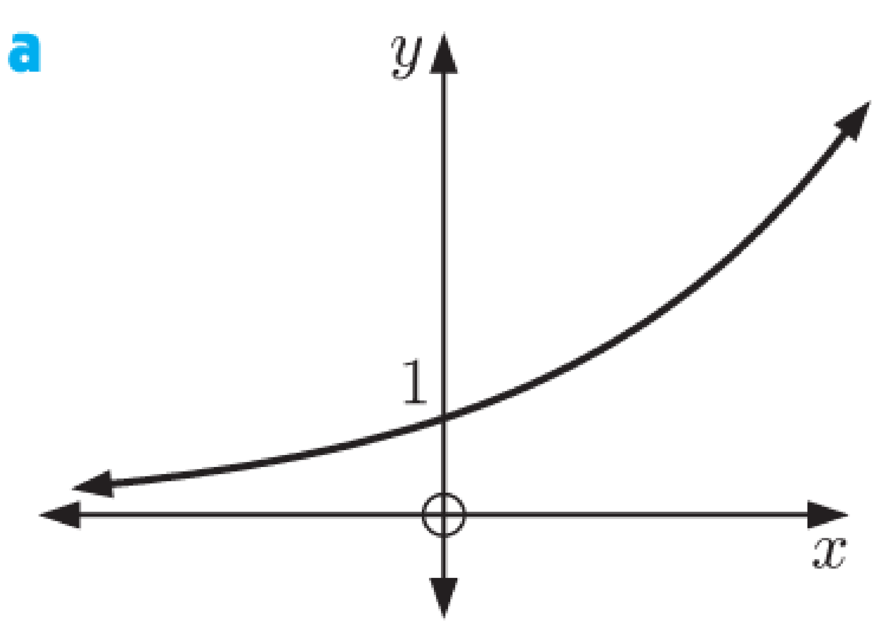
a $y=-f(x)$ **b** $y=f(-x)$

PRINTABLE GRIDS

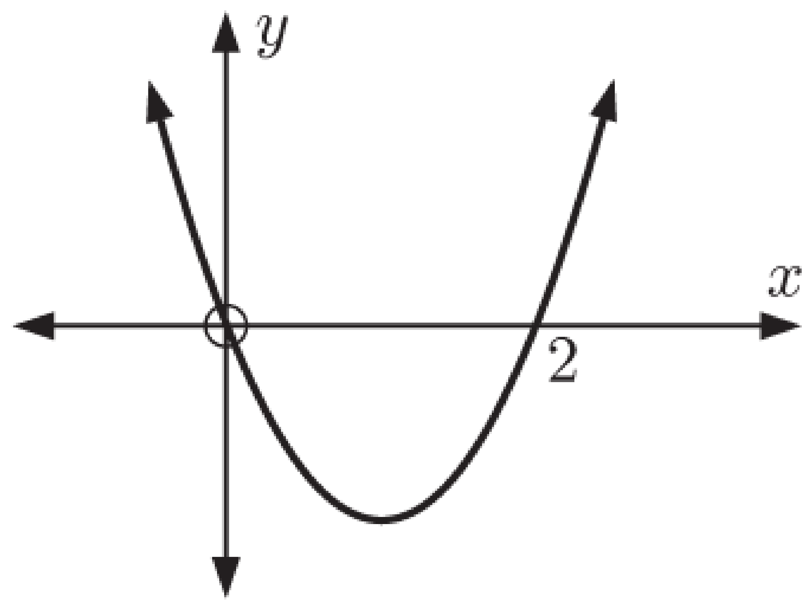


2 Copy the following graphs for $y=f(x)$ and sketch the graphs of $y=-f(x)$ on the same axes.

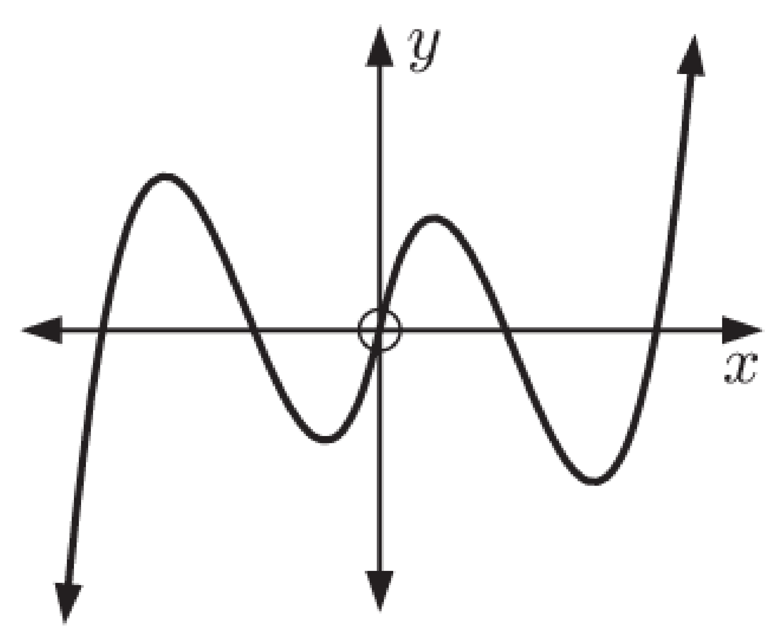
a



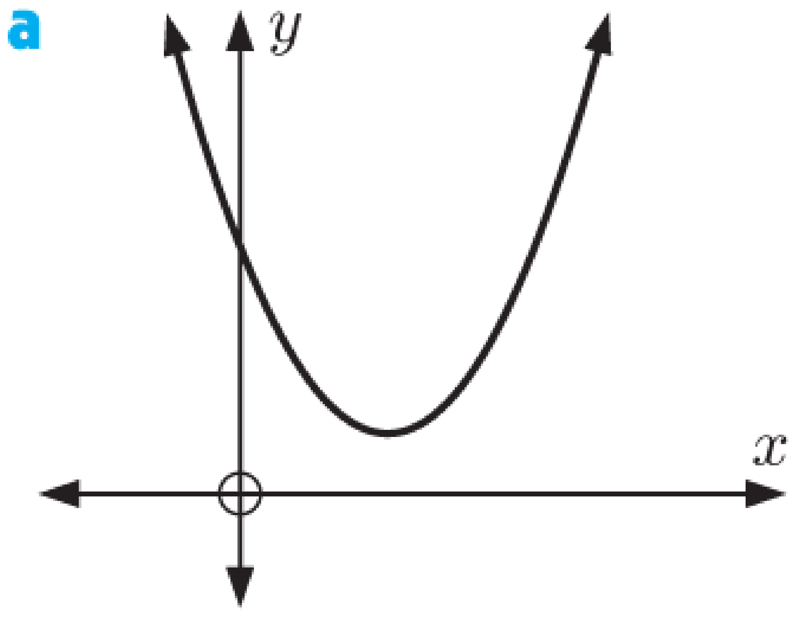
b

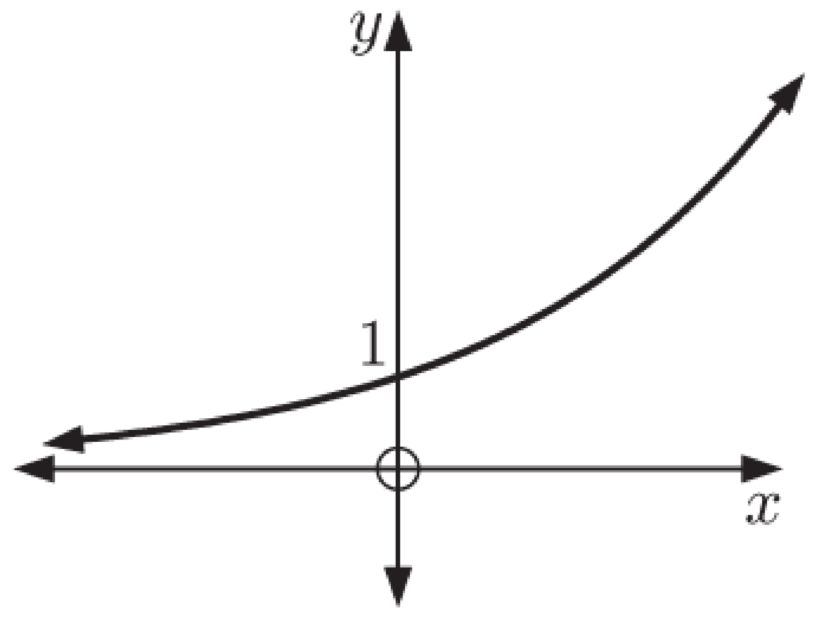


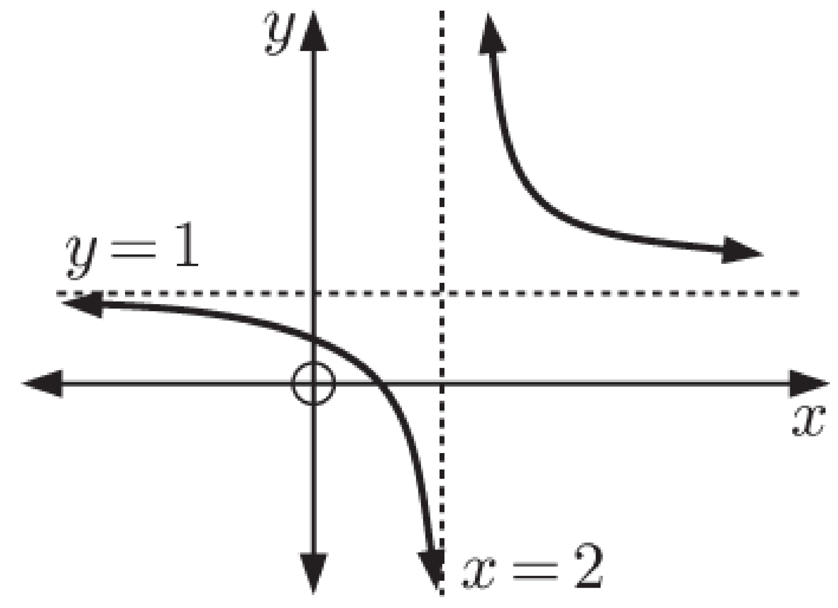
c



3 Copy the following graphs of $y = f(x)$ and sketch the graphs of $y = f(-x)$ on the same axes.

a


b


c


4 Graph $y = f(x)$ and $y = -f(x)$ for:

a $f(x) = 3x$ **b** $f(x) = x^3 - 2$ **c** $f(x) = 2(x+1)^2$

5 Graph $y = f(x)$ and $y = f(-x)$ for:

a $f(x) = 2x + 1$ **b** $f(x) = x^2 + 2x + 1$ **c** $f(x) = x^3$

6 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = 5x + 7$ is reflected in the x-axis **b** $f(x) = 2^x$ is reflected in the y-axis

c $f(x) = 2x^2 + 1$ is reflected in the x-axis

d $f(x) = x^4 - 2x^3 - 3x^2 + 5x - 7$ is reflected in the y-axis.

7 The function $y = f(x)$ is transformed to $g(x) = -f(x)$.

a Find the image points on $y = g(x)$ corresponding to the following points on $y = f(x)$:

i $(3, 0)$ **ii** $(2, -1)$ **iii** $(-3, 2)$

b Find the points on $y = f(x)$ which are transformed to the following points on $y = g(x)$:

i $(7, -1)$ **ii** $(-5, 0)$ **iii** $(-3, -2)$

8 The function $y = f(x)$ is transformed to $h(x) = f(-x)$.

a Find the image points on $y = h(x)$ for the following points on $y = f(x)$:

i $(2, -1)$ **ii** $(0, 3)$ **iii** $(-1, 2)$

b Find the points on $y = f(x)$ corresponding to the following points on $y = h(x)$:

i $(5, -4)$ **ii** $(0, 3)$ **iii** $(2, 3)$

9 A function $f(x)$ is transformed to the function $g(x) = -f(-x)$.

a What combination of transformations has taken place?

b If $(3, -7)$ lies on $y = f(x)$, find the transformed point on $y = g(x)$.

c Find the point on $f(x)$ that transforms to the point $(-5, -1)$.

10 Let $f(x) = x + 2$.

a Describe the transformation which transforms $y = f(x)$ to $y = -f(x)$.

b Describe the transformation which transforms $y = -f(x)$ to $y = -3f(x)$.

c Hence draw the graphs of $y = f(x)$, $y = -f(x)$, and $y = -3f(x)$ on the same set of axes.

11 Let $f(x) = (x-1)^2 - 4$.

a Describe the transformation which transforms $y = f(x)$ to $y = f(-x)$.

b Describe the transformation which transforms $y = f(-x)$ to $y = f(-\frac{1}{2}x)$.

c Hence draw the graphs of $y = f(x)$, $y = f(-x)$, and $y = f(-\frac{1}{2}x)$ on the same set of axes.

12 Graph on the same set of axes $y = x^2$, $y = -x^2$, and $y = -(x+2)^2 + 3$.
Describe the combination of transformations which transform $y = x^2$ to $y = -(x+2)^2 + 3$.

13 Graph on the same set of axes $y = \frac{1}{x}$, $y = -\frac{1}{x}$, $y = -\frac{1}{x-3} + 2$.

Describe the combination of transformations which transform $y = \frac{1}{x}$ to $y = -\frac{1}{x-3} + 2$.

DISCUSSION

For which combinations of two transformations on $y = f(x)$ is the order in which the transformations are performed:

- important
- not important?

D MISCELLANEOUS TRANSFORMATIONS

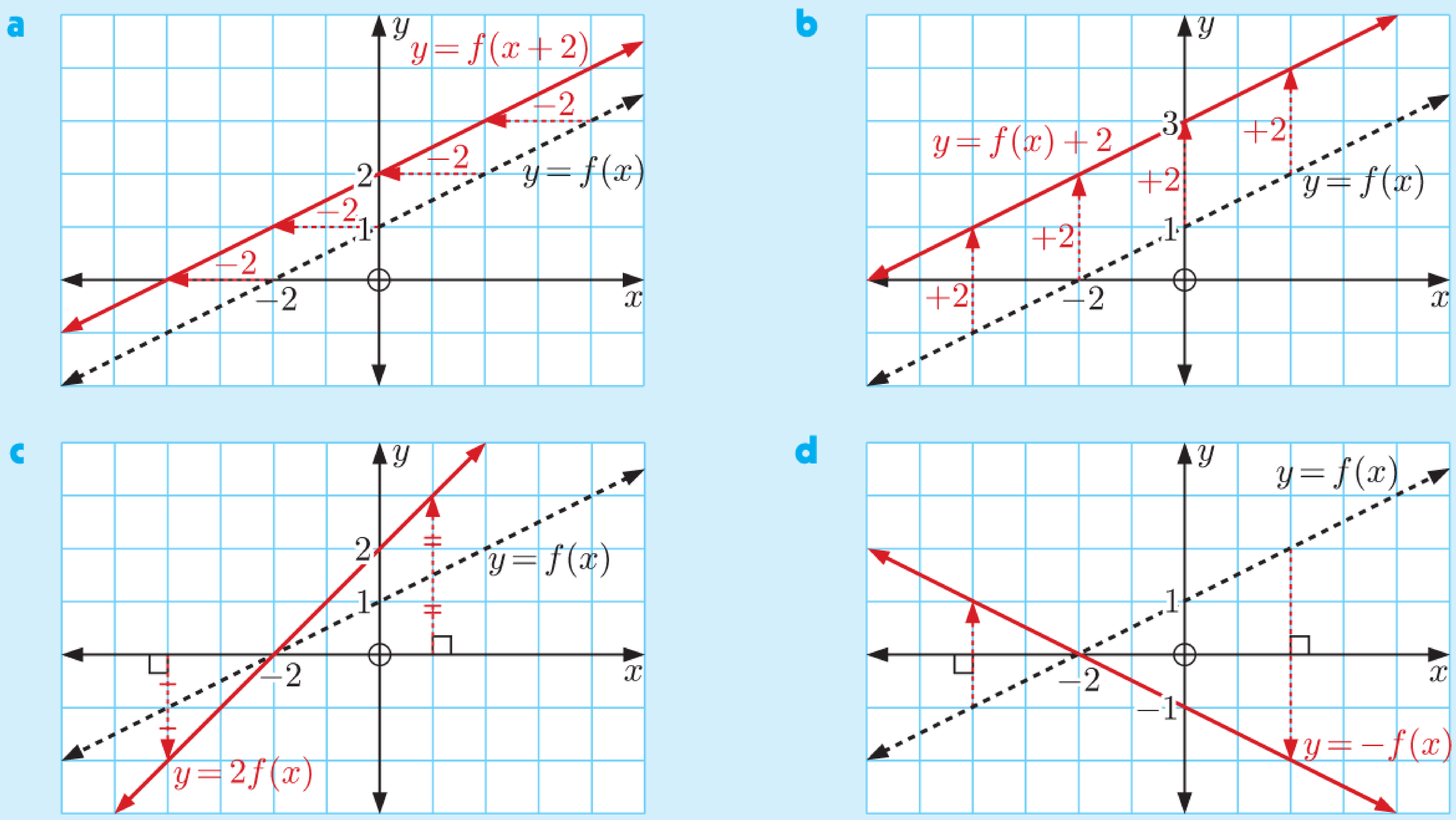
A summary of all the transformations is given in the printable concept map.

CONCEPT MAP

Example 5 — Self Tutor

Consider $f(x) = \frac{1}{2}x + 1$. On separate sets of axes graph:

a $y = f(x)$ and $y = f(x+2)$ **b** $y = f(x)$ and $y = f(x) + 2$
c $y = f(x)$ and $y = 2f(x)$ **d** $y = f(x)$ and $y = -f(x)$



EXERCISE 16D

1 Consider $f(x) = x^2 - 1$.

a Graph $y = f(x)$ and state its axes intercepts.

b Graph each function and describe the transformation which has occurred:

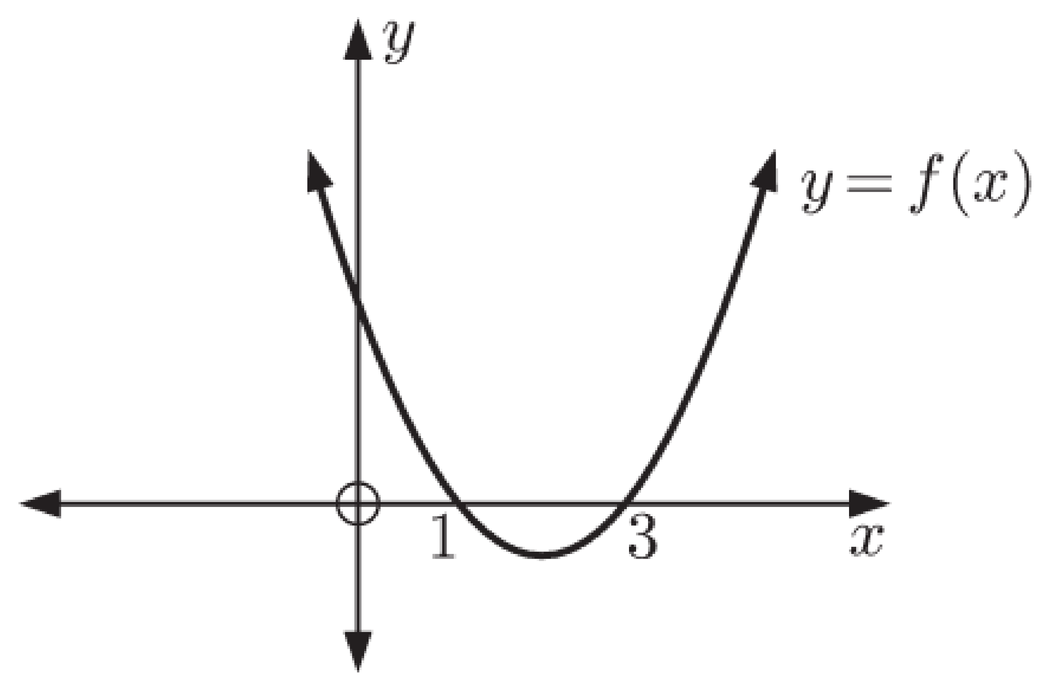
i $y = f(x) + 3$ **ii** $y = f(x - 1)$ **iii** $y = 2f(x)$ **iv** $y = -f(x)$

2 For the graph of $y = f(x)$ given, sketch the graph of:

a $y = 2f(x)$ **b** $y = \frac{1}{2}f(x)$

c $y = f(x + 2)$ **d** $y = f(2x)$

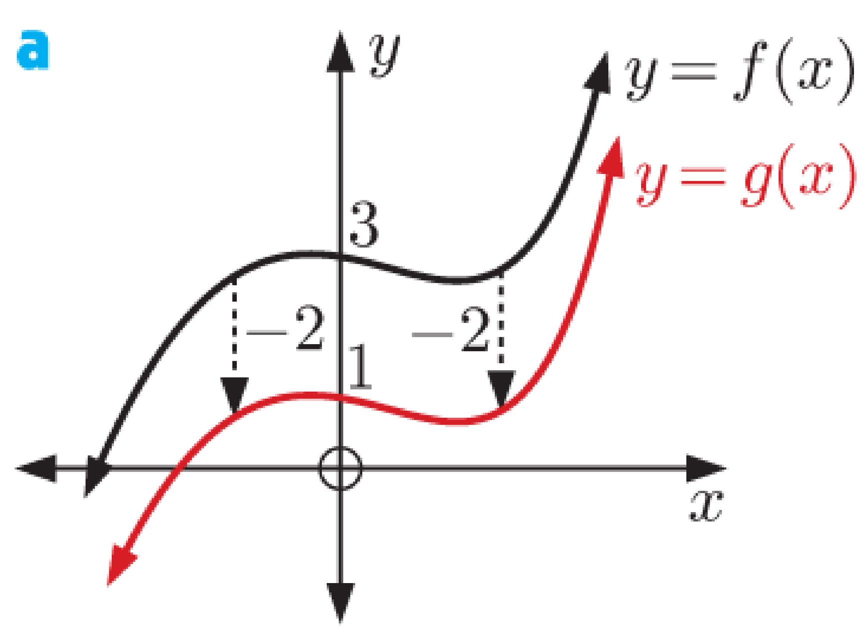
e $y = f(\frac{1}{2}x)$



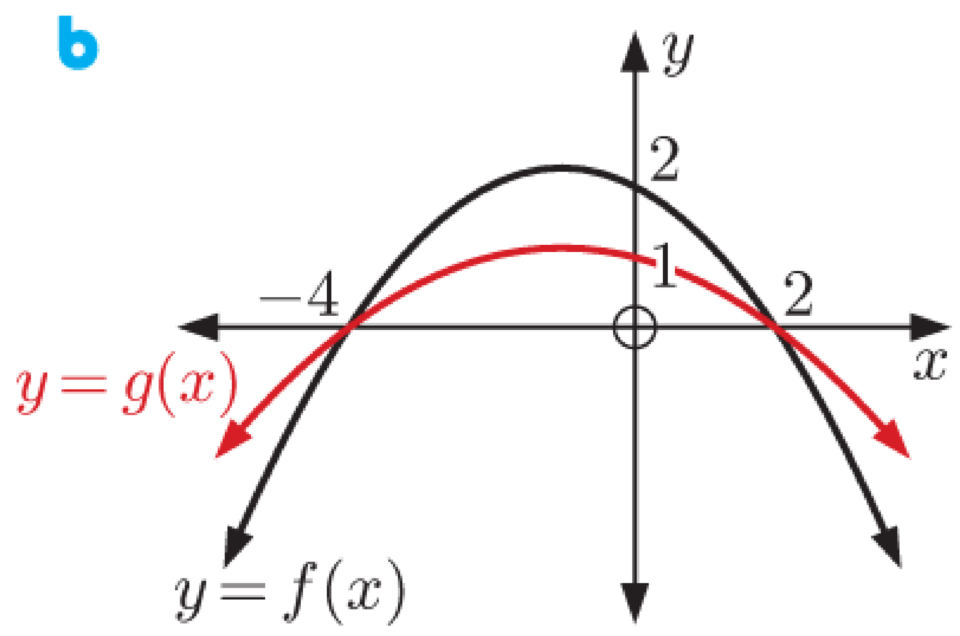
3 In each graph, $f(x)$ is transformed to $g(x)$ using a single transformation.

i Describe the transformation. **ii** Write $g(x)$ in terms of $f(x)$.

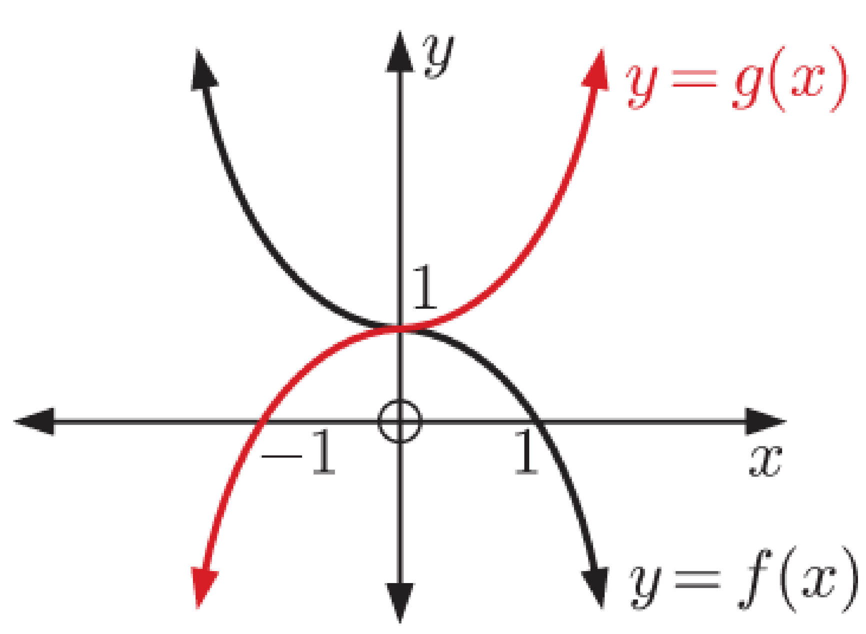
a



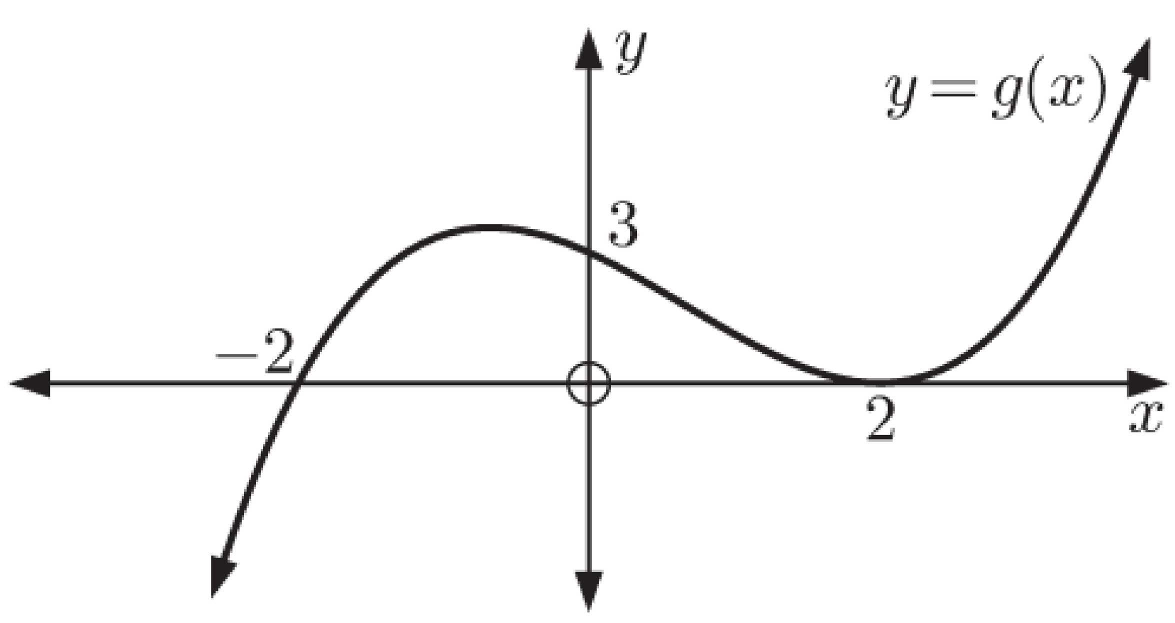
b



c



4



For the graph of $y = g(x)$ given, sketch the graph of:

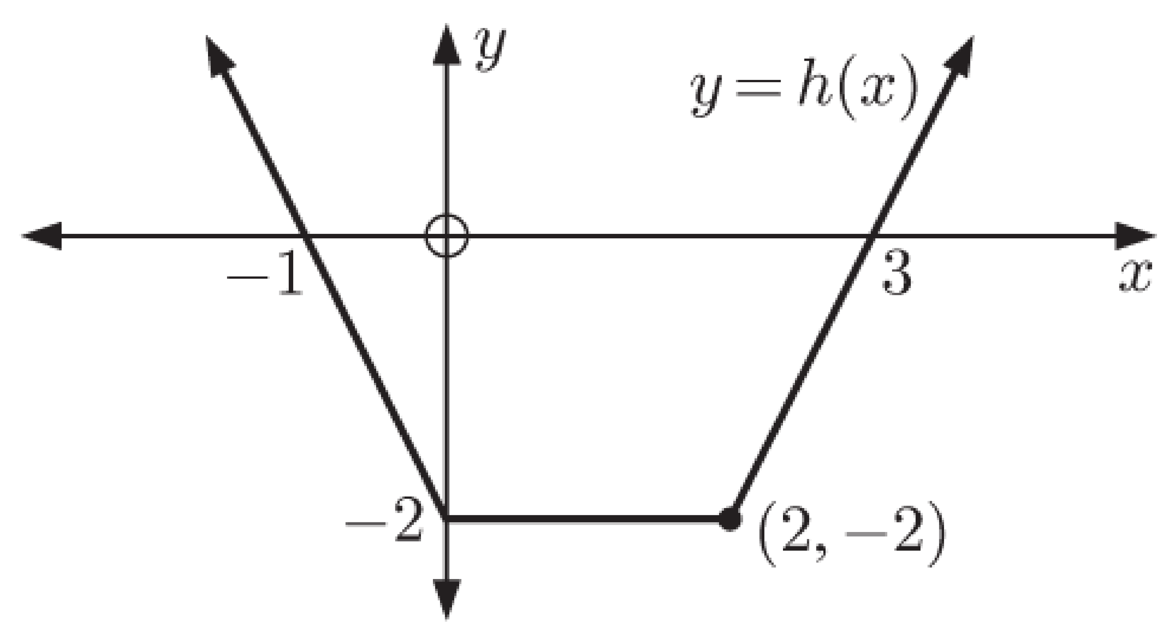
a $y = g(x) + 2$ **b** $y = -g(x)$

c $y = g(-x)$ **d** $y = g(x + 1)$

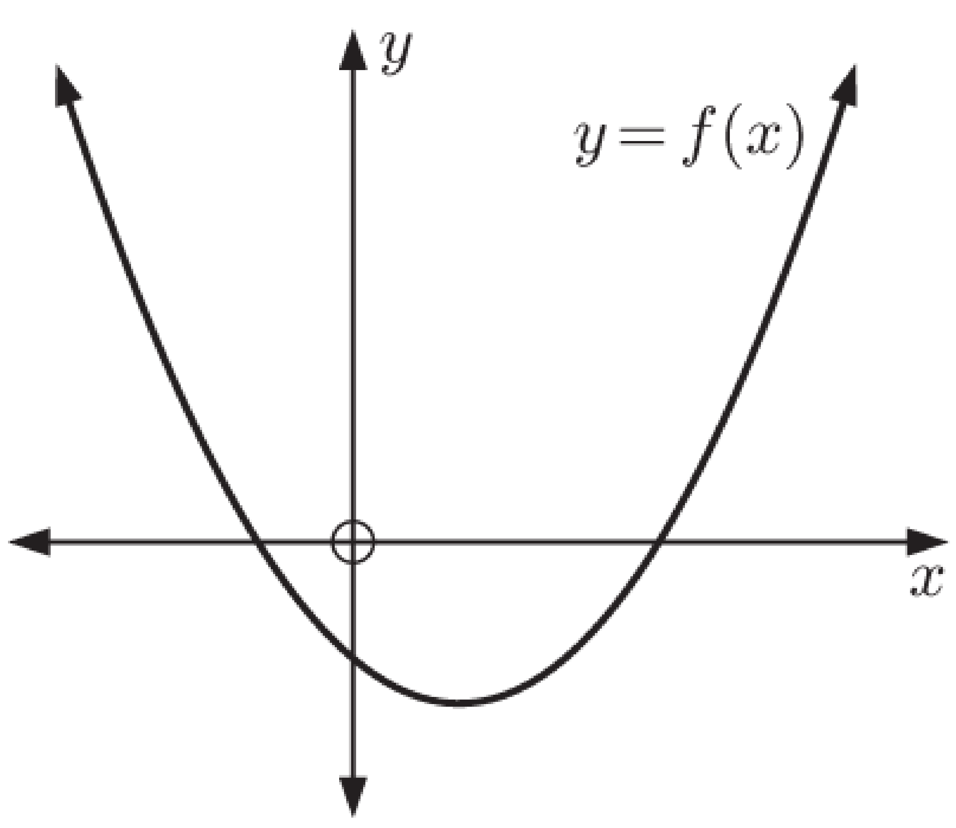
5 For the graph of $y = h(x)$ given, sketch the graph of:

a $y = h(x) + 1$ **b** $y = \frac{1}{2}h(x)$

c $y = h(-x)$ **d** $y = h\left(\frac{x}{2}\right)$



6



Consider the function $f(x) = (x + 1)(x - \beta)$ where $\beta > 0$. A sketch of the function is shown alongside.

a Determine the axes intercepts of the graph of $y = f(x)$.

b Sketch the graphs of $f(x)$ and $g(x) = -f(x - 1)$ on the same set of axes.

c Find and label the axes intercepts of $y = g(x)$.

Example 6 — Self Tutor

Consider a function $f(x)$.

a What function results if $y = f(x)$ is reflected in the x-axis, then translated through $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, then stretched vertically with scale factor 2?

b Fully describe the transformations which map $y = f(x)$ onto $y = 3f(2x-1) - 2$.

a $f(x) \xrightarrow{\text{reflection in } x\text{-axis}} -f(x) \xrightarrow{\text{translation } \begin{pmatrix} 3 \\ -1 \end{pmatrix}} -f(x-3)-1 \xrightarrow{\text{vertical stretch scale factor 2}} 2(-f(x-3)-1)$

The resulting function is $-2f(x-3)-2$.

b $f(x) \xrightarrow{\text{vertical stretch scale factor 3}} 3f(x) \xrightarrow{\text{translation } \begin{pmatrix} 1 \\ -2 \end{pmatrix}} 3f(x-1)-2 \xrightarrow{\text{horizontal stretch scale factor } \frac{1}{2}} 3f(2x-1)-2$

7 Consider a function $f(x)$. Find the function which results if $y = f(x)$ is:

a translated through $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ then reflected in the y-axis

b reflected in the y-axis then translated through $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

c translated through $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ then stretched vertically with scale factor $\frac{1}{2}$

d stretched vertically with scale factor $\frac{1}{2}$ then translated through $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

e translated through $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ then stretched horizontally with scale factor 4

f stretched horizontally with scale factor 4 then translated through $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$.

8 Fully describe the transformations which map $y = f(x)$ onto:

a $y = -f(x+1)+3$ **b** $y = f(\frac{1}{2}x) - 7$ **c** $y = f(3x-1)$

d $y = -1 + 2f(\frac{1}{4}x - 1)$ **e** $y = 5 + 2f(3(x-1))$ **f** $y = -4f(\frac{1}{2}(x+3)) - 1$

9 The function $f(x)$ has domain $\{x \mid x \geqslant 1\}$ and range $\{y \mid -2 \leqslant y < 5\}$.
Find the domain and range of:

a $g(x) = f(x+4) - 1$ **b** $g(x) = -2f(3x)$ **c** $g(x) = \frac{1}{3}f(2x-5) + 4$

10 Let T_A be a translation through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$,

T_B be a reflection in the y-axis, and

T_C be a vertical stretch with scale factor 5.

Find the resulting function when $f(x) = \sqrt{x}$ has the following transformations applied:

a T_A then T_B then T_C **b** T_C then T_A then T_B **c** T_C then T_B then T_A.

In each case state the domain and range of the transformed function.

11 The graph of $y = x^2$ is transformed into $y = a(x-h)^2 + k$ using three transformations:

- a vertical stretch with invariant x-axis
- a translation with vector $\begin{pmatrix} h \\ k \end{pmatrix}$
- a reflection in the x-axis.

Discuss what you know about:

a the transformations **b** the function.

12 **a** Write $\dfrac{10x+11}{2x+3}$ in the form $a + \dfrac{b}{2x+3}$, where a and b are constants.

b Hence describe the combination of transformations which map $y = \dfrac{1}{x}$ onto $y = \dfrac{10x+11}{2x+3}$.

E THE GRAPH OF $y = \dfrac{1}{f(x)}$

The **reciprocal** of a function $y = f(x)$ is the function $y = \dfrac{1}{f(x)}$.

INVESTIGATION 4 THE GRAPH OF $y = \dfrac{1}{f(x)}$

In this Investigation we will examine the reciprocals of various functions using technology.

GRAPHING PACKAGE

What to do:

1 Sketch each pair of functions on the same set of axes. Include all axes intercepts and asymptotes.

a $y = x$ and $y = \dfrac{1}{x}$

b $y = x+2$ and $y = \dfrac{1}{x+2}$

c $y = x-2$ and $y = \dfrac{1}{x-2}$

d $y = 3x+4$ and $y = \dfrac{1}{3x+4}$.

What do you notice regarding intercepts and asymptotes?

2 **a** Sketch each pair of functions on the same set of axes. Include all axes intercepts and asymptotes.

i $y = x^2$ and $y = \dfrac{1}{x^2}$

ii $y = -(x-1)^2$ and $y = -\dfrac{1}{(x-1)^2}$

iii $y = x^2+4$ and $y = \dfrac{1}{x^2+4}$

iv $y = -(x^2-4)$ and $y = -\dfrac{1}{x^2-4}$

v $y = (x-1)(x-3)$ and $y = \dfrac{1}{(x-1)(x-3)}$

b How can the vertical asymptotes of $y = \dfrac{1}{f(x)}$ be established from $f(x)$ without first viewing its graph?

c What other observations can you make about the graph of $y = \dfrac{1}{f(x)}$?

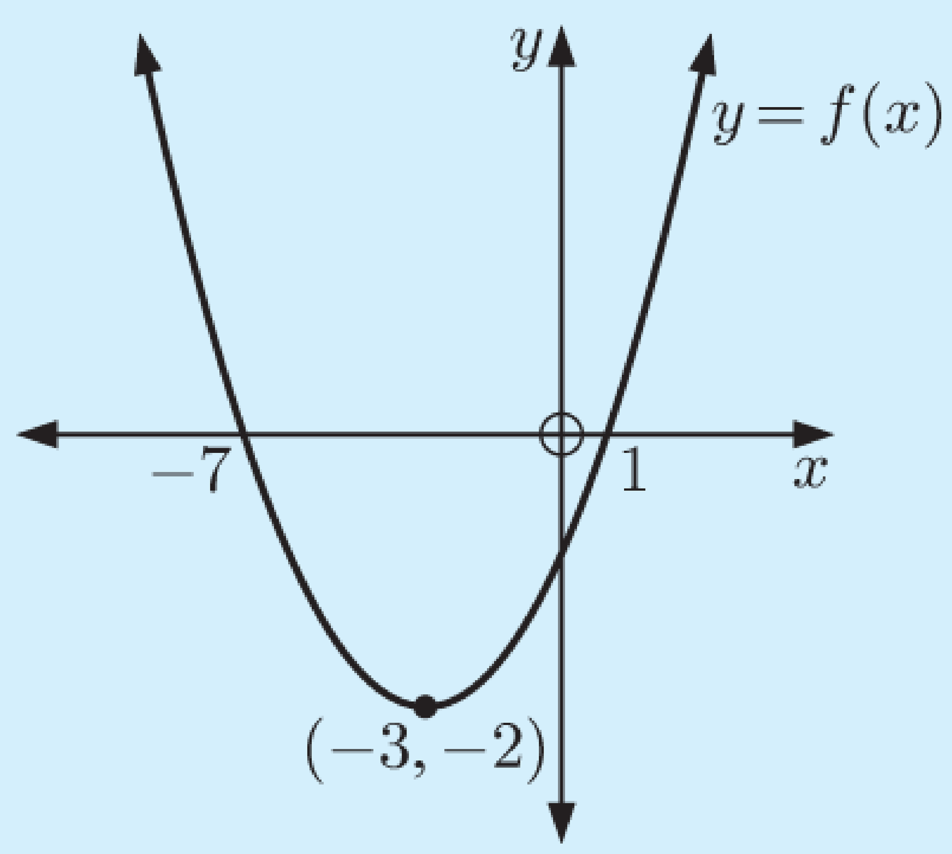
From the **Investigation**, you should have observed that:

When $y = \dfrac{1}{f(x)}$ is graphed from $y = f(x)$:

- the zeros of $y = f(x)$ become vertical asymptotes of $y = \dfrac{1}{f(x)}$
- the vertical asymptotes of $y = f(x)$ become zeros of $y = \dfrac{1}{f(x)}$
- the local maxima of $y = f(x)$ correspond to local minima of $y = \dfrac{1}{f(x)}$
- the local minima of $y = f(x)$ correspond to local maxima of $y = \dfrac{1}{f(x)}$
- when $f(x) > 0$, $\dfrac{1}{f(x)} > 0$ and when $f(x) < 0$, $\dfrac{1}{f(x)} < 0$
- when $f(x) \to 0$, $\dfrac{1}{f(x)} \to \pm\infty$ and when $f(x) \to \pm\infty$, $\dfrac{1}{f(x)} \to 0$.

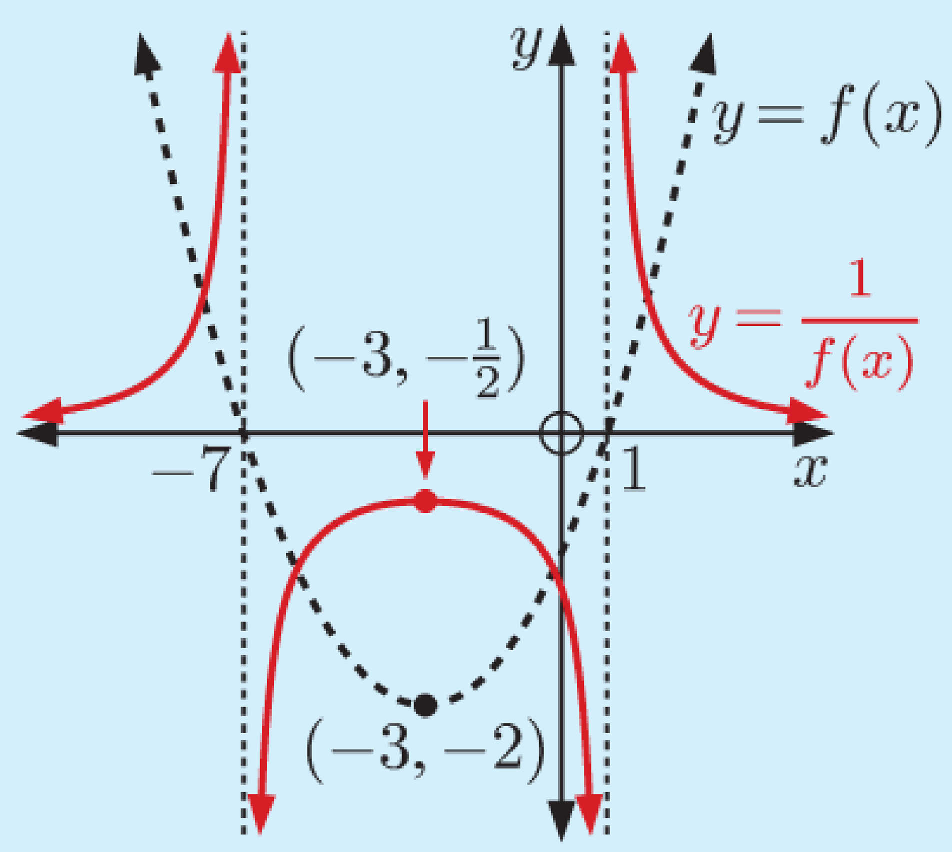
Example 7 Self Tutor

For the graph of $y = f(x)$ alongside, draw the graph of $y = \dfrac{1}{f(x)}$.



$y = f(x)$ has x-intercepts -7 and 1, so $y = \dfrac{1}{f(x)}$ has vertical asymptotes $x = -7$ and $x = 1$.

$y = f(x)$ has a local minimum at $(-3, -2)$, so $y = \dfrac{1}{f(x)}$ has a local maximum at $(-3, -\frac{1}{2})$.



EXERCISE 16E

1 Graph on the same set of axes:

a $y = x + 3$ and $y = \dfrac{1}{x+3}$

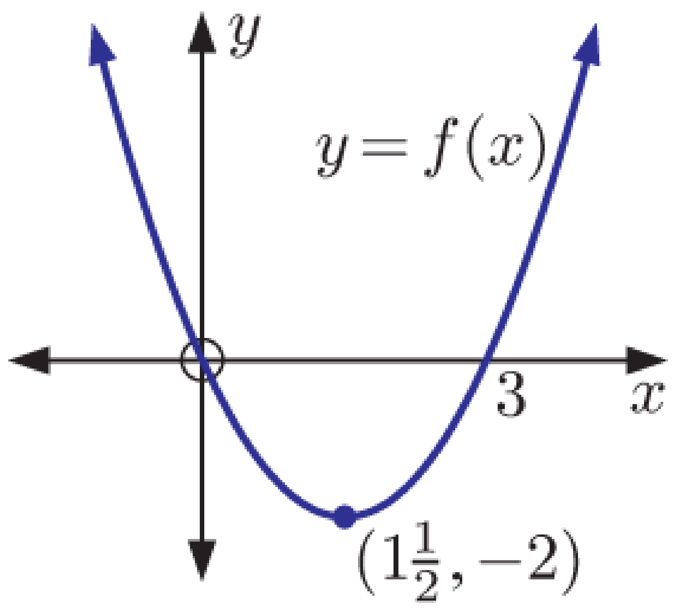
b $y = -x^2$ and $y = -\dfrac{1}{x^2}$

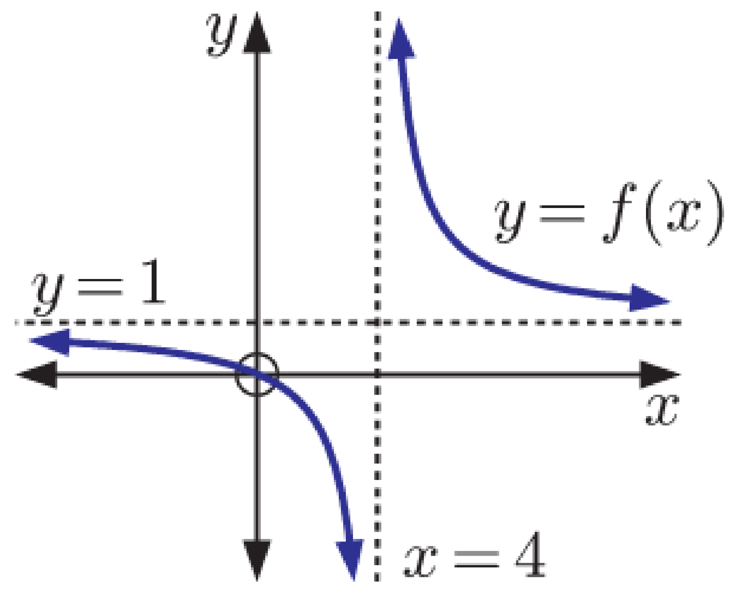
c $y = \sqrt{x}$ and $y = \dfrac{1}{\sqrt{x}}$

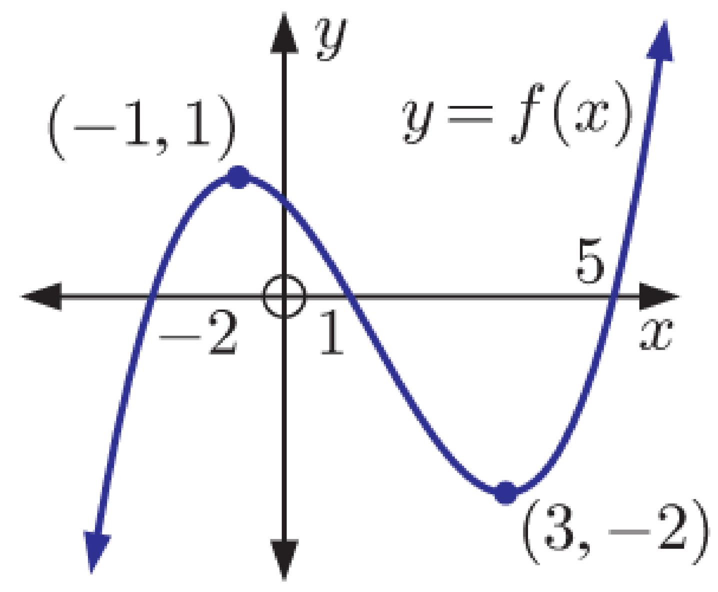
d $y = (x+1)(x-3)$ and $y = \dfrac{1}{(x+1)(x-3)}$

2 Show that if $y = f(x)$ is transformed to $y = \dfrac{1}{f(x)}$, invariant points occur at $y = \pm 1$.
Check your results in question **1** for invariant points.

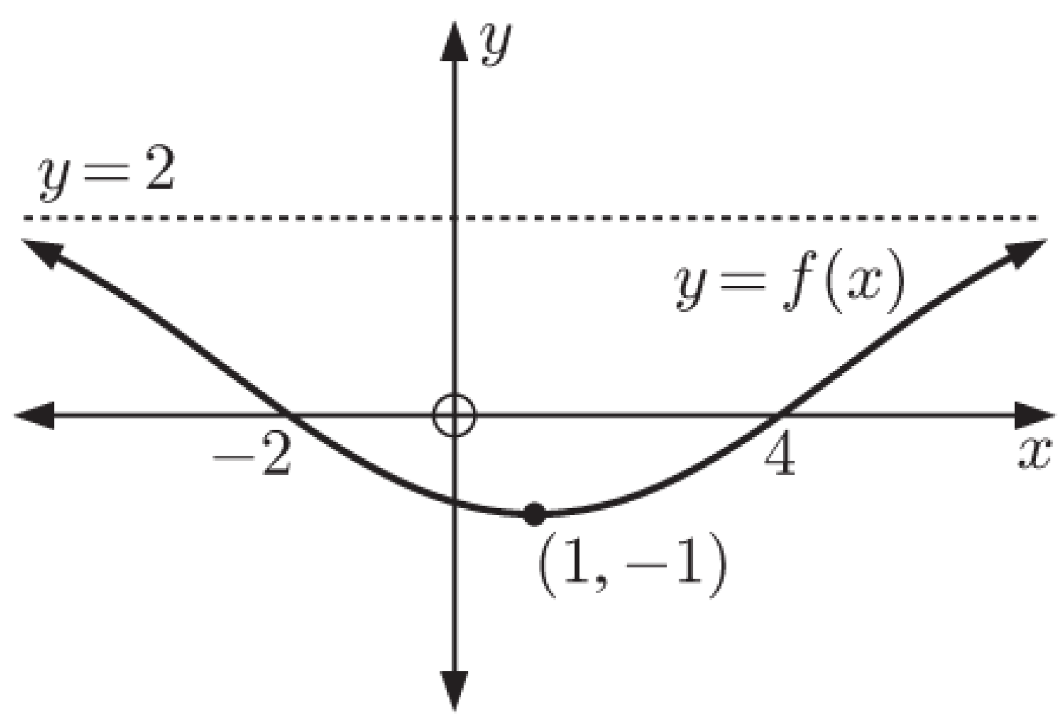
3 Copy the following graphs for $y = f(x)$ and on the same axes graph $y = \dfrac{1}{f(x)}$:

a


b


c


4 Copy the graph of $y = f(x)$ alongside, and graph $y = \dfrac{1}{f(x)} - 3$ on the same set of axes. Clearly show the asymptotes and turning points.



PRINTABLE GRAPHS

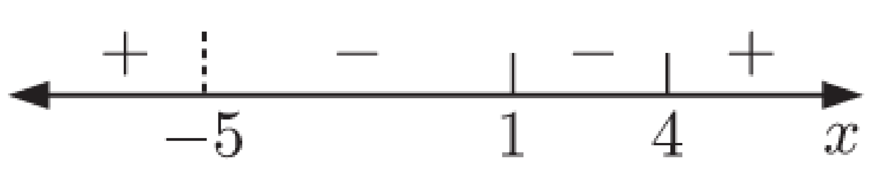
5 Let $f(x) = x^2 + 4x + 3$.

a Find the axes intercepts and vertex of $f(x)$.

b Sketch $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

c Solve for x: $\dfrac{1}{f(x)} = \dfrac{4}{21}$

6 The sign diagram of $f(x)$ is shown alongside.
Draw the sign diagram of $\dfrac{1}{f(x)}$.



7 If possible, find the domain and range of $\dfrac{1}{f(x)}$ given that:

a $f(x)$ has domain $-1 \leqslant x \leqslant 6$ and range $2 \leqslant y < 5$

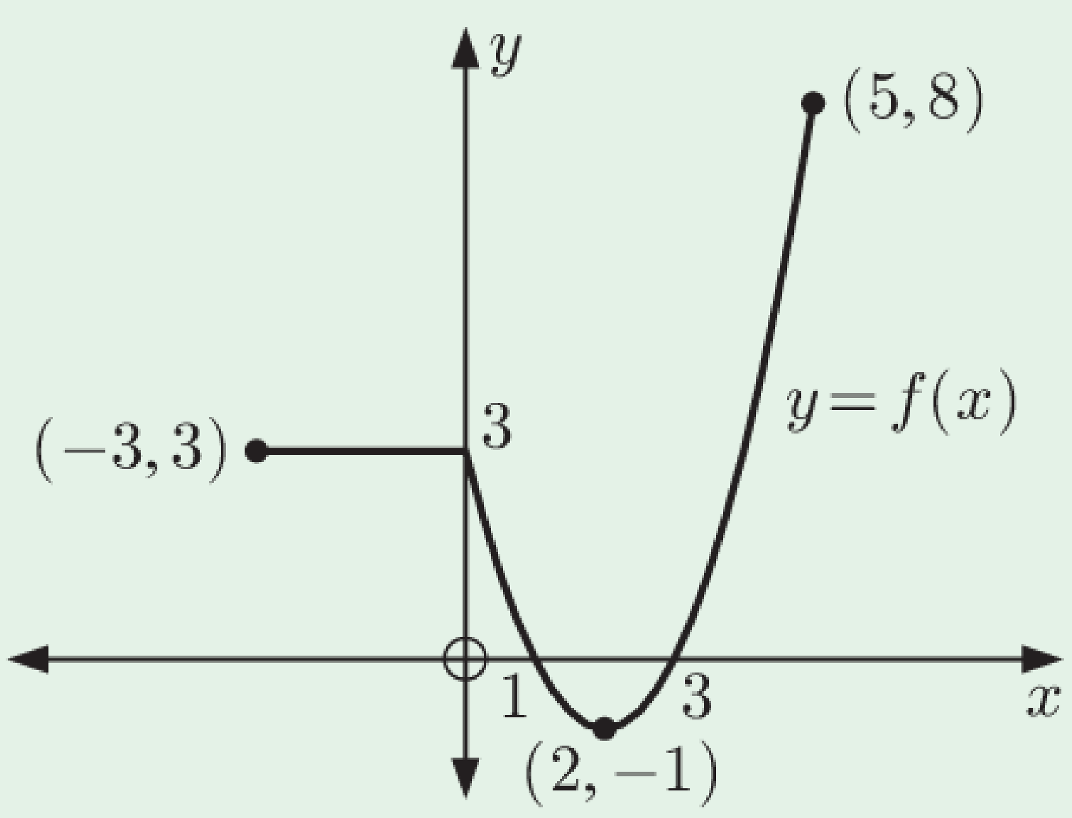
b $f(x)$ has domain $2 \leqslant x \leqslant 8$ and range $-3 \leqslant y \leqslant 3$.

REVIEW SET 16A

1 For the graph of $y = f(x)$, sketch graphs of:

a $y = f(-x)$ **b** $y = -f(x)$

c $y = f(x + 2)$ **d** $y = f(x) + 2$



2 Consider the function $f : x \mapsto x^2$.
On the same set of axes, graph $y = f(x)$, $y = 3f(x)$, and $y = 3f(x-1)+2$.

3 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = 4x - 7$ is translated 3 units downwards

b $f(x) = x^2 + 6$ is vertically stretched with scale factor 5

c $f(x) = 7 - 3x$ is translated 4 units to the left

d $f(x) = 2x^2 - x + 4$ is horizontally stretched with scale factor 3

e $f(x) = x^3$ is reflected in the y-axis.

4 Sketch the graph of $f(x) = x^2 + 1$, and on the same set of axes sketch the graph of:

a $y = -f(x)$ **b** $y = f(2x)$ **c** $y = f(x) + 3$

5 The function $f(x)$ has domain $\{x \mid -2 \leqslant x \leqslant 3\}$ and range $\{y \mid -1 \leqslant y \leqslant 7\}$.
Find the domain and range of $g(x) = f(x+3) - 4$. Explain your answers.

6 The graph of the function $f(x) = (x+1)^2 + 4$ is translated 2 units to the right and 4 units up.

a Find the function $g(x)$ corresponding to the translated graph.

b State the range of: **i** $f(x)$ **ii** $g(x)$

7 Show that the discriminant of a quadratic function is unchanged when the graph of the function is:

a reflected in the x-axis **b** reflected in the y-axis

c translated h units to the right.

8 The graph of $f(x) = 3x^2 - x + 4$ is translated by the vector $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Write the equation of the image in the form $g(x) = ax^2 + bx + c$.

9 Consider a function $f(x)$. Find the function which results if $y = f(x)$ is:

a reflected in the x-axis then translated through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

b translated through $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ then vertically stretched with scale factor 2.

10 The point A$(-2, 3)$ lies on the graph of $y = f(x)$. Find the image of A under the transformation:

a $y = f(x-2) + 1$ **b** $y = 2f(x-2)$ **c** $y = f(2x-3)$

11 Suppose the graph of $y = f(x)$ has x-intercepts -5 and 1, and y-intercept -3. What can you say about the axes intercepts of:

a $y = f(x+4)$ **b** $y = 3f(x)$ **c** $y = f\left(\frac{x}{2}\right)$ **d** $y = -f(x)$?

12 The function $g(x)$ results when $y = \frac{1}{x}$ is transformed by a translation through $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ followed by a reflection in the y-axis.

a Write an expression for $g(x)$ in the form $g(x) = \frac{ax+b}{cx+d}$.

b Find the asymptotes of $y = g(x)$.

c State the domain and range of $g(x)$.

d Sketch $y = g(x)$.

13 Graph on the same set of axes $y = x^2$, $y = \frac{1}{4}x^2$, and $y = \frac{1}{4}(x-2)^2 - 1$.
Describe the combination of transformations which transform $y = x^2$ to $y = \frac{1}{4}(x-2)^2 - 1$.

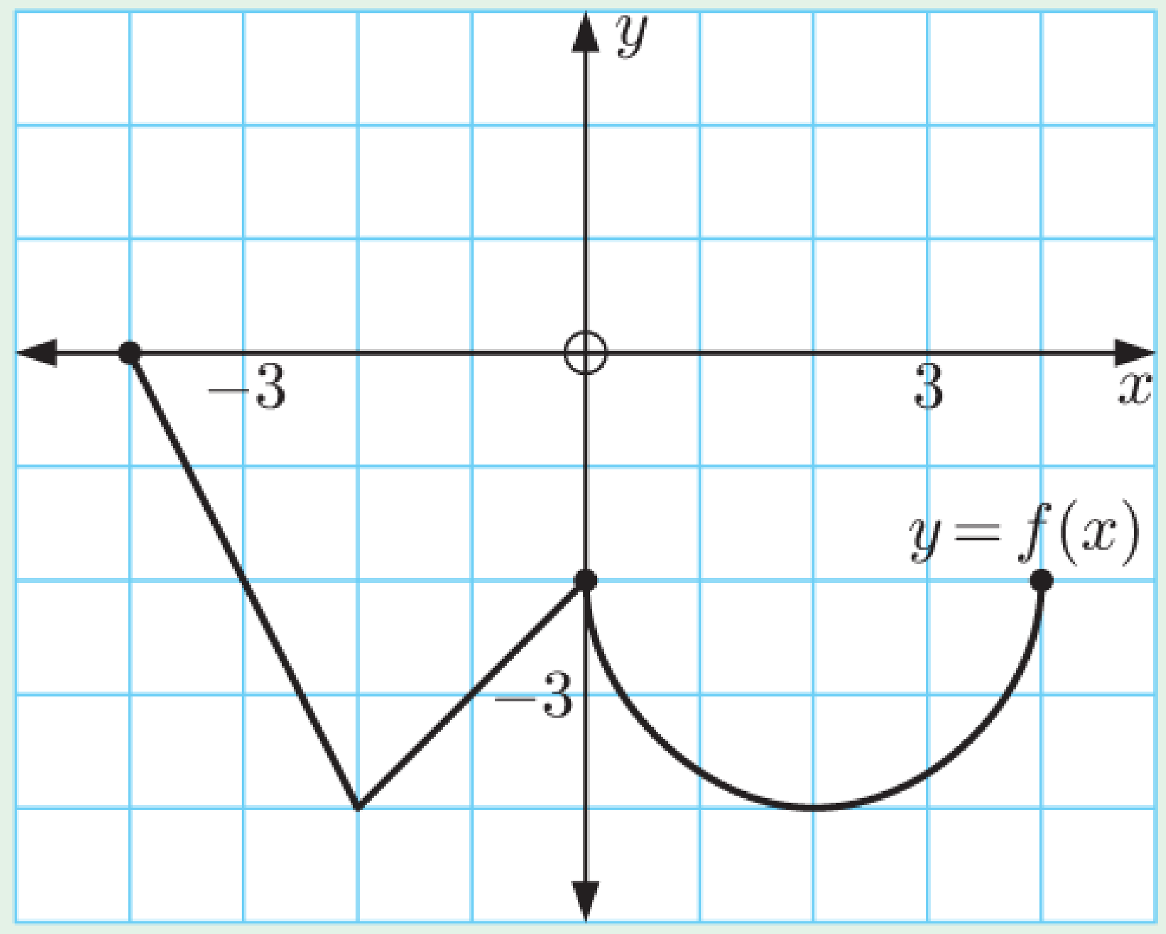
14 Sketch $y = (x-2)(x+3)$ and $y = \dfrac{1}{(x-2)(x+3)}$ on the same set of axes. Clearly label all axes intercepts and asymptotes.

REVIEW SET 16B

1 Consider the graph of $y = f(x)$ alongside.
On separate axes, draw the graphs of:

a $y = f(x-1)$ **b** $y = f(2x)$
c $y = f(x) + 3$ **d** $y = 2f(x)$
e $y = f(-x)$ **f** $y = -f(x)$

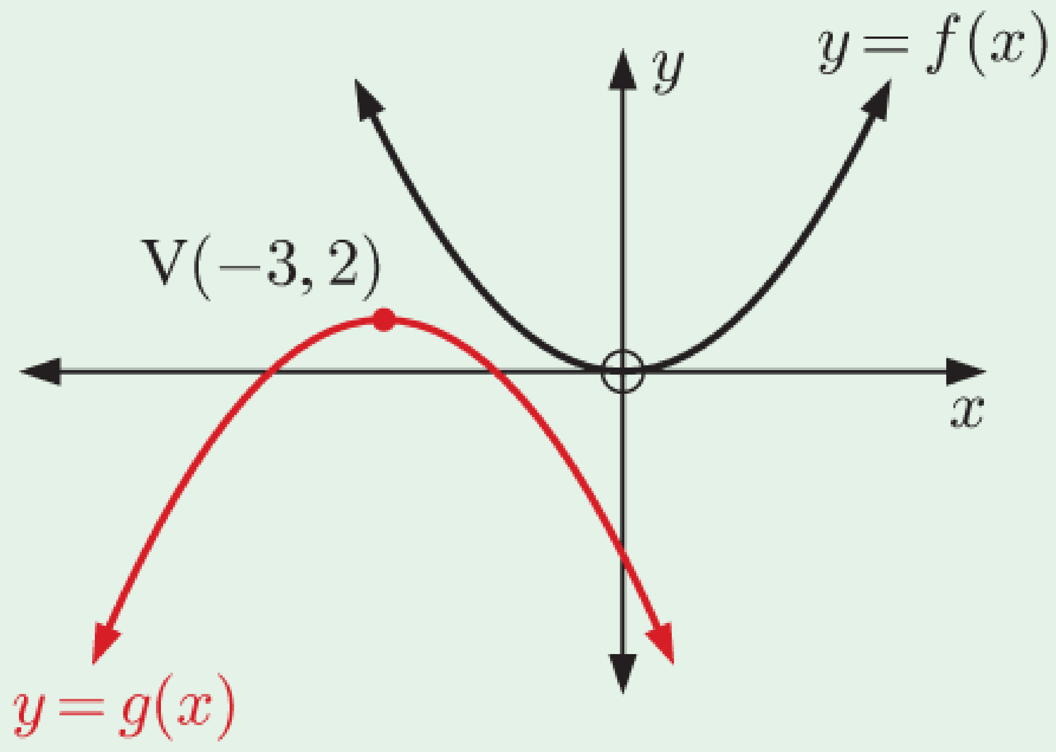
PRINTABLE GRIDS



2 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = x^2 - 3x$ is reflected in the x-axis
b $f(x) = 14 - x$ is translated 2 units upwards
c $f(x) = \frac{1}{3}x + 2$ is horizontally stretched with scale factor 4.

3 The graph of $f(x) = x^2$ is transformed to the graph of $g(x)$ by a reflection and a translation as illustrated. Find the formula for $g(x)$ in the form $g(x) = ax^2 + bx + c$.

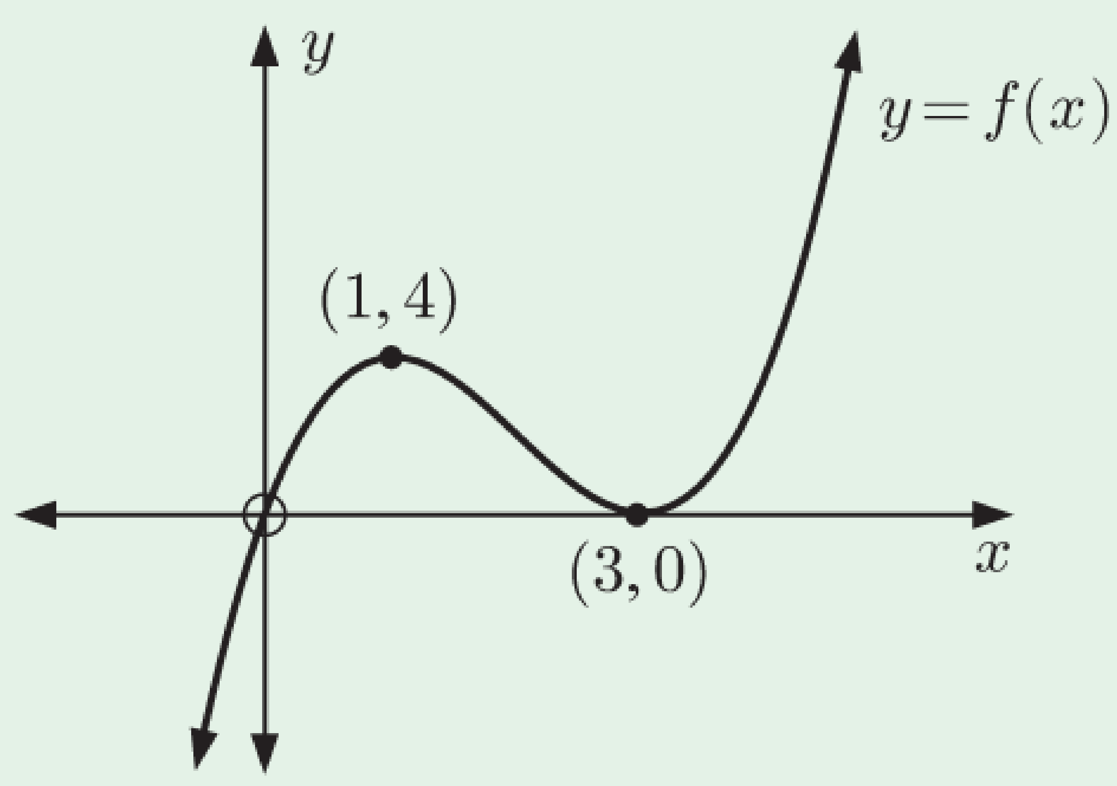


4 Sketch the graph of $f(x) = -x^2$, and on the same set of axes sketch the graph of:

a $y = f(-x)$ **b** $y = -f(x)$ **c** $y = f(2x)$ **d** $y = f(x-2)$

5 The graph of a cubic function $y = f(x)$ is shown alongside.

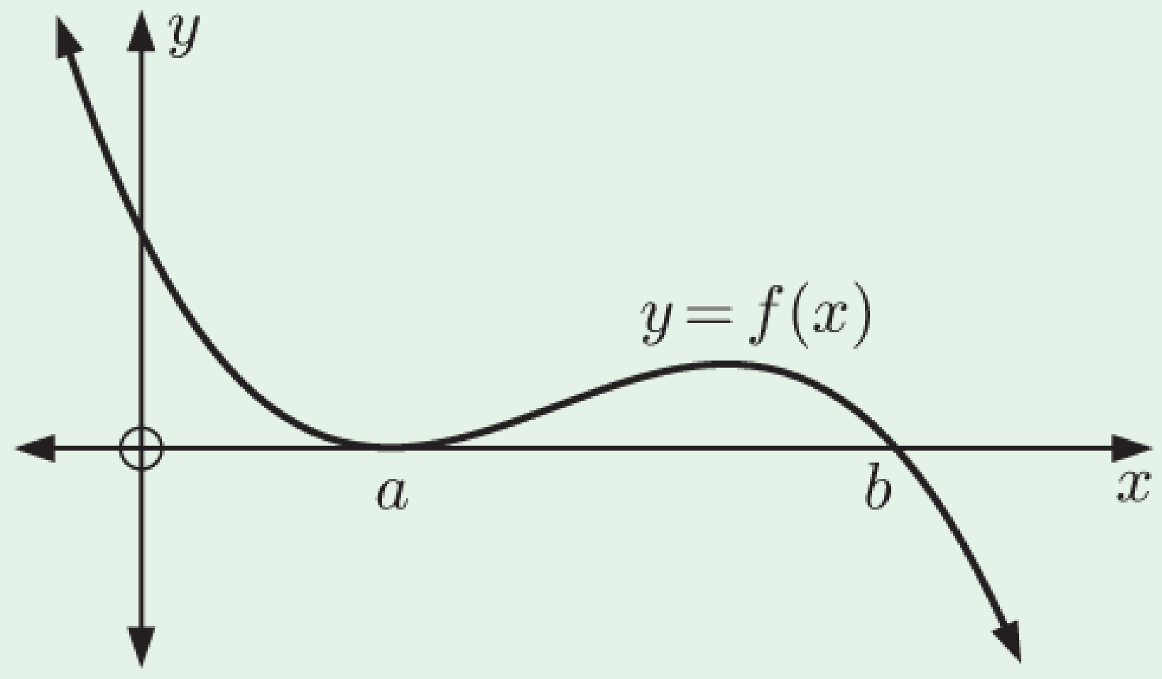
a Sketch the graph of $g(x) = -f(x-1)$.
b State the coordinates of the turning points of $y = g(x)$.



6 The graph of $f(x) = -2x^2 + x + 2$ is translated by the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Write the equation of the image in the form $y = ax^2 + bx + c$.

7 The graph of $y = f(x)$ is shown alongside.
The x-axis is a tangent to $f(x)$ at $x = a$ and $f(x)$ cuts the x-axis at $x = b$.
On the same diagram, sketch the graph of $y = f(x - c)$ where $0 < c < b - a$.
Indicate the x-intercepts of $y = f(x - c)$.



8 Find the combination of transformations which maps $f(x) = 2x^2 + 8x - 3$ onto $g(x) = -2x^2 + 2x + 7$.

9 The point $(-1, 6)$ lies on the graph of $y = f(x)$. Find the corresponding point on the graph of $y = \frac{1}{2}f(x - 2) + 3$.

10 Fully describe the transformations which map $y = f(x)$ onto:

a $y = 2f(x + 1) + 3$ **b** $y = -f(\frac{2}{3}x) - 6$ **c** $y = \frac{1}{3}f(-x + 2)$

11 The quadratic function $f(x) = x^2 + bx + c$ is reflected in the y-axis, stretched horizontally with scale factor $\frac{3}{2}$, then translated through $\begin{pmatrix} -10 \\ 20 \end{pmatrix}$. The resulting quadratic function has the same x-intercepts as $f(x)$. Find b and c.

12 **a** Graph on the same set of axes $y = \dfrac{1}{x}$, $y = -\dfrac{1}{x}$, $y = -\dfrac{1}{2x}$, and $y = -\dfrac{1}{2(x+1)} - 2$.

b Describe the combination of transformations which transform $y = \dfrac{1}{x}$ into $y = -\dfrac{1}{2(x+1)} - 2$.

c Write the resulting function in the form $y = \dfrac{ax + b}{cx + d}$, and state its domain and range.

13 **a** Sketch the graph of $f(x) = -2x + 3$, clearly showing the axes intercepts.

b Find the invariant points for the graph of $y = \dfrac{1}{f(x)}$.

c State the y-intercept and vertical asymptote of $y = \dfrac{1}{f(x)}$.

d Sketch the graph of $y = \dfrac{1}{f(x)}$ on the same axes as in part **a**, showing clearly the information you have found.

14 Let $f(x) = \dfrac{c}{x + c}$, $x \neq -c$, $c > 0$.
On a set of axes like those shown, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$. Clearly label any points of intersection with the axes and any asymptotes.