3 Algebraic Functions, Equations and Inequalities

Assessment statements

2.1 Odd and even functions (also see Chapter 7).

2.4 The rational function $x \mapsto \dfrac{ax + b}{cx + d}$ and its graph.

2.5 Polynomial functions.
The factor and remainder theorems.
The fundamental theorem of algebra.

2.6 The quadratic function $x \mapsto ax^2 + bx + c$: its graph, axis of symmetry $x = -\dfrac{b}{2a}$.
The solution of $ax^2 + bx + c = 0$, $a \neq 0$.
The quadratic formula. Use of the discriminant $\Delta = b^2 - 4ac$.
Solving equations both graphically and algebraically.
Sum and product of the roots of polynomial equations.

2.7 Solution of inequalities $g(x) \geqslant f(x)$; graphical and algebraic methods.

Introduction

A function $x \mapsto f(x)$ is called **algebraic** if, substituting for the number x in the domain, the corresponding number $f(x)$ in the range can be computed using a finite number of **elementary operations** (i.e. addition, subtraction, multiplication, division, and extracting a root). For example, $f(x) = \dfrac{x^2 + \sqrt{9 - x}}{2x - 6}$ is algebraic. For our purposes in this course, functions can be organized into three categories:

1. Algebraic functions
2. Exponential and logarithmic functions (Chapter 5)
3. Trigonometric and inverse trigonometric functions (Chapter 7)

The focus of this chapter is algebraic functions of a single variable which – given the definition above – are functions that contain polynomials, radicals (surds), rational expressions (quotients), or a combination of these. The chapter will begin by looking at polynomial functions in general and then moves onto a closer look at 2nd degree polynomial functions (quadratic functions). Solving equations containing polynomial functions is an important skill that will be covered. We will also study rational functions, which are quotients of polynomial functions and the associated topic of partial fractions (optional). The chapter will close with methods of solving inequalities and absolute value functions, and strategies for solving various equations.

The concept of a function is a fairly recent development in the history of mathematics. Its meaning started to gain some clarity about the time of René Descartes (1596–1650) when he defined a function to be any positive integral power of x (i.e. x^2, x^3, x^4, etc.). Leibniz (1646–1716) and Johann Bernoulli (1667–1748) developed the concept further. It was Euler (1707–1783) who introduced the now standard function notation $y = f(x)$.

3.1 Polynomial functions

The most common type of algebraic function is a polynomial function where, not surprisingly, the function's rule is given by a polynomial. For example,

$$f(x) = x^3, \qquad h(t) = -2t^2 + 16t - 24, \qquad g(y) = y^5 + y^4 - 11y^3 + 7y^2 + 10y - 8$$

Recalling the definition of a polynomial, we define a polynomial function.

Definition of a polynomial function in the variable x

A **polynomial function** P is a function that can be expressed as

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \qquad a_n \neq 0$$

where the non-negative integer n is the **degree** of the polynomial function. The numbers $a_0, a_1, a_2, \ldots, a_n$, are real numbers and are the **coefficients** of the polynomial. a_n is the **leading coefficient**, $a_n x^n$ is the **leading term** and a_0 is the **constant term**.

It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, the following simpler forms are often used.

Degree	Function form	Function name	Graph
Zero	$P(x) = a$	Constant function	Horizontal line
First	$P(x) = ax + b$	Linear function	Line with slope a
Second	$P(x) = ax^2 + bx + c$	Quadratic function	Parabola (∪-shape, 1 turn)
Third	$P(x) = ax^3 + bx^2 + cx + d$	Cubic function	∩∪-shape (2 or no turns)

Table 3.1 Features of polynomial functions of low degree.

To identify an individual term in a polynomial function, we use the function name correlated with the power of x contained in the term. For example, the polynomial function $f(x) = x^3 - 9x + 4$ has a *cubic* term of x^3, no *quadratic* term, a *linear* term of $-9x$, and a *constant* term of 4.

For each polynomial function $P(x)$ there is a corresponding **polynomial equation** $P(x) = 0$. When we solve polynomial equations, we often refer to solutions as **roots**.

• **Hint:** When working with a polynomial function, such as $f(x) = x^3 - 9x + 4$, it is common to refer to it in a couple of different ways – either as 'the polynomial $f(x)$', or as 'the function $x^3 - 9x + 4$.'

● **Hint:** The use of the word '**root**' here to denote the solution of a polynomial equation should not be confused with the use of the word in the context of square root, cube root, fifth root, etc.

Zeros and roots

If P is a function and c is a number such that $P(c) = 0$, then c is a **zero** of the function P (or of the polynomial P) and $x = c$ is a **root** of the equation $P(x) = 0$.

Approaches to finding zeros of various polynomial functions will be considered in the first three sections of this chapter.

Graphs of polynomial functions

As we reviewed in Section 1.6, the graph of a first-degree polynomial function (linear function), such as $P(x) = 2x - 5$, is a line (Figure 3.1a). The graph of every second-degree polynomial function (quadratic function) is a parabola (Figure 3.1b). A thorough review and discussion of quadratic functions and their graphs is in the next section.

The simplest type of polynomial function is one whose rule is given by a power of x. In Figure 3.1, the graphs of $P(x) = x^n$ for $n = 1, 2, 3, 4, 5$ and 6 are shown. As the figure suggests, the graph of $P(x) = x^n$ has the same general ∪-shape as $y = x^2$ when n is even, and the same general ∩∪ shape as $y = x^3$ when n is odd. However, as the degree n increases, the graphs of polynomial functions become flatter near the origin and steeper away from the origin.

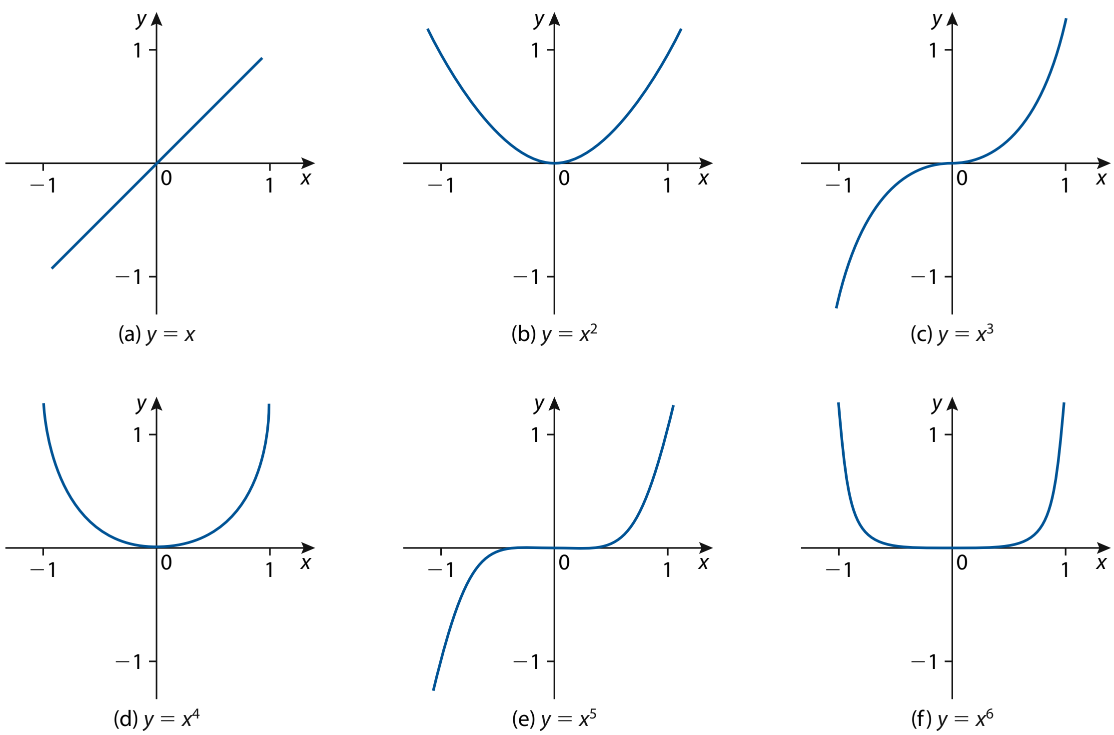
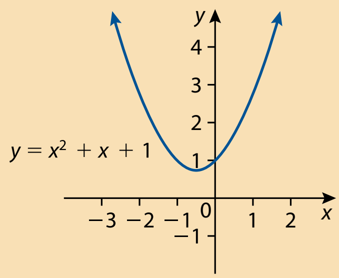


Figure 3.1 Graphs of $P(x) = x^n$ for increasing n.

Another interesting observation is that, depending on the degree of the polynomial function, its graph displays a certain type of symmetry. The graph of $P(x) = x^n$ is symmetric with respect to the origin when n is odd. Such a function is aptly called an **odd function**. The graph of $P(x) = x^n$ is symmetric with respect to the y-axis when n is even. Accordingly any such function is called an **even function**. Formal definitions for odd and even functions will be presented in Chapter 7 when we investigate the graphs of the sine and cosine functions.

Note that the graph of an **even function** may or may not intersect the x-axis (x-intercept). As we will see, where and how often the graph of a function intersects the x-axis is helpful information when trying to determine the value and nature of the roots of a polynomial equation $P(x) = 0$.

Not all polynomial functions are even or odd – that is, not all polynomial functions display rotation symmetry about the origin or reflection symmetry about the y-axis. For example, the graph of the polynomial function $y = x^2 + x + 1$ is neither even nor odd. It has line symmetry, but the line of symmetry is not the y-axis.



The graphs of polynomial functions that are not in the form $P(x) = x^n$ are more difficult to sketch. However, the graphs of all polynomial functions share these properties:

1. It is a smooth curve (i.e. it has no sharp, pointed turns – only smooth, rounded turns).
2. It is continuous (i.e. it has no breaks, gaps or holes).
3. It rises ($P(x) \to \infty$) or falls ($P(x) \to -\infty$) without bound as $x \to +\infty$ or $x \to -\infty$.
4. It extends on forever both to the left ($-\infty$) and to the right ($+\infty$); domain is $\mathbb{R}$.
5. The graph of a polynomial function of degree n has at most $n - 1$ turning points.

The property that is listed third of the five properties of the graphs of polynomial functions is referred to as the **end behaviour** of the function because it describes how the curve *behaves* at the left and right *ends* (i.e. as $x \to +\infty$ and as $x \to -\infty$). The end behaviour of a polynomial function is determined by its degree and by the sign of its leading coefficient. See Exercise 3.1, Q11.

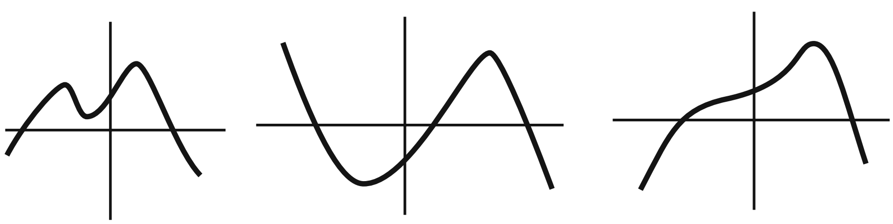


Figure 3.2 The graph of a polynomial function is a smooth, unbroken, continuous curve, such as the ones shown here.

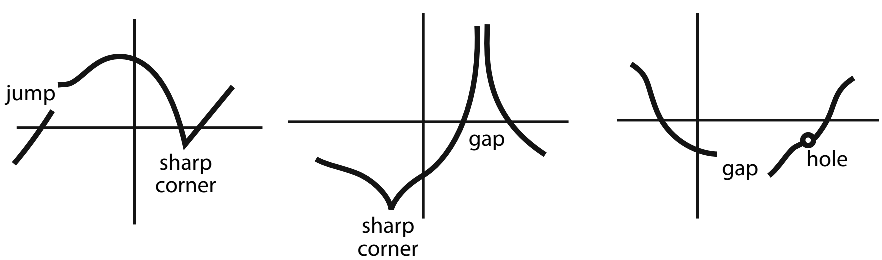


Figure 3.3 There can be no jumps, gaps, holes or sharp corners on the graph of a polynomial function. Thus none of the functions whose graphs are shown here are polynomial functions.

If we wish to sketch the graph of a polynomial function without a GDC, we need to compute some function values in order to locate a few points on the graph. This could prove to be quite tedious if the polynomial function has a high degree. We will now develop a method that provides an efficient procedure for evaluating polynomial functions. It will also be useful in the third section of this chapter for some situations when we divide polynomials. For simplicity, we give the method for a fourth-degree polynomial, but it is applicable to any nth degree polynomial.

Synthetic substitution (Optional)

Suppose we want to find the value of $P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ when $x = c$, that is, find $P(c)$. The computation of c^4 may be tricky, so rather than substituting c directly into $P(x)$ we will take a gradual approach that consists of a sequence of multiplications and additions. We define b_4, b_3, b_2, b_1, and R by the following equations.

$$b_4 = a_4 \qquad (1)$$

$$b_3 = b_4c + a_3 \qquad (2)$$

$$b_2 = b_3c + a_2 \qquad (3)$$

$$b_1 = b_2c + a_1 \qquad (4)$$

$$R = b_1c + a_0 \qquad (5)$$

Our goal is to show that the value of $P(c)$ is equivalent to the value of R. Firstly, we substitute the expression for b_3 given by equation (2) into equation (3), and also use equation (1) to replace b_4 with a_4, to produce

$$\begin{aligned} b_2 &= (a_4c + a_3)c + a_2 \\ &= a_4c^2 + a_3c + a_2 \qquad (6) \end{aligned}$$

We now substitute this expression for b_2 in (6) into (4) to give

$$\begin{aligned} b_1 &= (a_4c^2 + a_3c + a_2)c + a_1 \\ &= a_4c^3 + a_3c^2 + a_2c + a_1 \qquad (7) \end{aligned}$$

To complete our goal we substitute this expression for b_1 in (7) into (5) to give

$$\begin{aligned} R &= (a_4c^3 + a_3c^2 + a_2c + a_1)c + a_0 \\ &= a_4c^4 + a_3c^3 + a_2c^2 + a_1c + a_0 \qquad (8) \end{aligned}$$

This is the value of $P(x)$ when $x = c$. If we condense (6), (7) and (8) into one expression, we obtain

$$\begin{aligned} R &= \{[(a_4c + a_3)c + a_2]c + a_1\}c + a_0 \\ &= a_4c^4 + a_3c^3 + a_2c^2 + a_1c + a_0 = P(c) \qquad (9) \end{aligned}$$

Carrying out the computations for equation (9) can be challenging. However, a nice pattern can be found if we closely inspect the expression $\{[(a_4c + a_3)c + a_2]c + a_1\}c + a_0$. Each nested computation involves finding the product of c and one of the coefficients, a_n, (starting with the leading coefficient) and then adding the next coefficient – and repeating this process until the constant term is used. Hence, the actual computation of R is quite straightforward if we arrange the nested computations required for (9) in the following systematic manner.

$$
\begin{array}{c|ccccc l}
c & a_4 & a_3 & a_2 & a_1 & a_0 & \\
 & & c \times b_4 & c \times b_3 & c \times b_2 & c \times b_1 & \\
\hline
 & b_4 & b_3 & b_2 & b_1 & R & = P(c)
\end{array}
$$

In this procedure we place c in a small box to the upper left. The coefficients of the polynomial function $P(x)$ are placed in the first line. We start by simply rewriting the leading coefficient below the horizontal line (remember $b_4 = a_4$). The diagonal arrows indicate that we multiply the number in the row below the line by c to obtain the next number in the second row above the line. Each b_n after the leading coefficient is obtained by adding the two numbers in the first and second rows directly above b_n. At the end of the procedure, the last such sum is $R = P(c)$. This method of computing the value of $P(x)$ when $x = c$ is called **synthetic substitution**.

Example 1 – Using synthetic substitution to find function values

Given $P(x) = 2x^4 + 6x^3 - 5x^2 + 7x - 12$, find the value of $P(x)$ when $x = -4, -1$ and 2.

Solution

We use the procedure for synthetic substitution just described.

$$
\begin{array}{c|ccccc l}
-4 & 2 & 6 & -5 & 7 & -12 & \\
 & & -8 & 8 & -12 & 20 & \\
\hline
 & 2 & -2 & 3 & -5 & 8 & = P(-4)
\end{array}
$$

Therefore, $P(-4) = 8$.

Note: Contrast using synthetic substitution to evaluate $P(-4)$ with using direct substitution.

$$
\begin{aligned}
P(-4) &= 2(-4)^4 + 6(-4)^3 - 5(-4)^2 + 7(-4) - 12 \\
&= 2(256) + 6(-64) - 5(16) - 28 - 12 \\
&= 512 - 384 - 80 - 28 - 12 \\
&= 128 - 108 - 12 \\
&= 8
\end{aligned}
$$

$$
\begin{array}{c|ccccc l}
-1 & 2 & 6 & -5 & 7 & -12 & \\
 & & -2 & -4 & 9 & -16 & \\
\hline
 & 2 & 4 & -9 & 16 & -28 & = P(-1)
\end{array}
$$

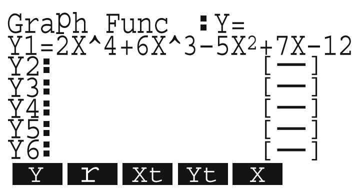
Therefore, $P(-1) = -28$.

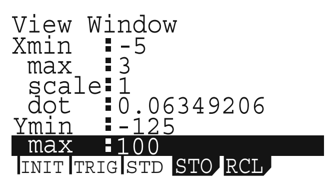
$$\begin{array}{c|ccccc}
2 & 2 & 6 & -5 & 7 & -12 \\
 & & 4 & 20 & 30 & 74 \\
\hline
 & 2 & 10 & 15 & 37 & 62 = P(2)
\end{array}$$

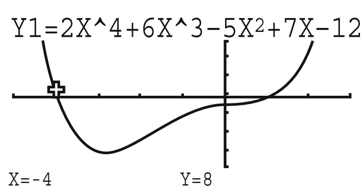
Therefore, $P(2) = 62$.

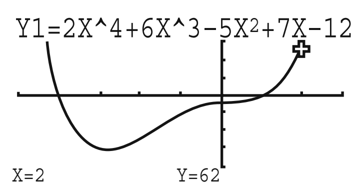
Since the graphs of all polynomial functions are continuous (no gaps or holes), then the function values we computed for the quartic polynomial function in Example 1 can give us information about the location of its zeros (i.e. x-intercepts of the graph). Since $P(-4) = 8$ and $P(-1) = -28$, then the graph of $P(x)$ must cross the x-axis ($P(x) = 0$) at least once between $x = -4$ and $x = -1$. Also, with $P(-1) = -28$ and $P(2) = 62$ there must be at least one x-intercept between $x = -1$ and $x = 2$. Hence, the polynomial equation $P(x) = 2x^4 + 6x^3 - 5x^2 + 7x - 12 = 0$ has at least one real root between -4 and -1, and at least one real root between -1 and 2. In Section 3.3 we will investigate real zeros of polynomial functions and then we will extend the investigation to include imaginary zeros, thereby extending the universal set for solving polynomial equations from the real numbers to complex numbers.

Graphing $P(x) = 2x^4 + 6x^3 - 5x^2 + 7x - 12$ on our GDC, we observe that the graph of $P(x)$ does indeed intersect the x-axis between -4 and -1 (just slightly greater than $x = -4$), and again between -1 and 2 (near $x = 1$).









● **Hint:** For some values of x, evaluating $P(x)$ by direct substitution may be quicker than using synthetic substitution. This is certainly true when $x = 0$ or $x = 1$. For example, it is easy to determine that $P(0) = -12$ for the polynomial P in Example 1; and that $P(1) = 2 + 6 - 5 + 7 - 12 = -2$.

Example 2

Use synthetic substitution to find the y-coordinates of the points on the graph of $f(x) = x^3 - 4x^2 + 24$ for $x = -3, -1, 1, 3$ and 5. Sketch the graph of f for $-4 \leqslant x \leqslant 6$.

Solution

Important: In order for the method of synthetic substitution to work properly it is necessary to insert 0 for any 'missing' terms in the polynomial. The polynomial $x^3 - 4x^2 + 24$ has no linear term so the top row in the set-up for synthetic substitution must be $1 \ -4 \ 0 \ 24$.

-3	1	-4	0	24
		-3	21	-63
	1	-7	21	-39

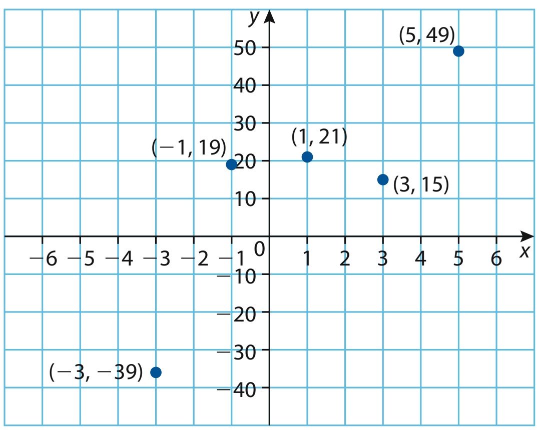
-1	1	-4	0	24
		-1	5	-5
	1	-5	5	19

1	1	-4	0	24
		1	-3	-3
	1	-3	-3	21

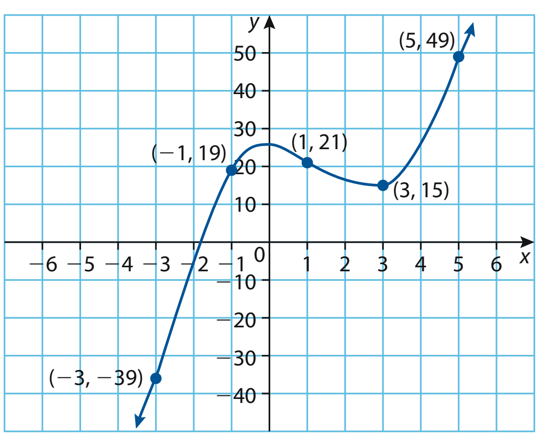
3	1	-4	0	24
		3	-3	-9
	1	-1	-3	15

5	1	-4	0	24
		5	5	25
	1	1	5	49

Therefore, the points $(-3, -39)$, $(-1, 19)$, $(1, 21)$, $(3, 15)$ and $(5, 49)$ are on the graph of f and have been plotted in the coordinate plane below.



Recall that the end behaviour of a polynomial function is determined by its degree and by the sign of its leading coefficient. Since the leading term of f is x^3 then its graph will fall $(y \to -\infty)$ as $x \to -\infty$ and will rise $(y \to \infty)$ as $x \to +\infty$. Also a polynomial function of degree n has at most $n - 1$ turning points; therefore, the graph of f has at most two turning points. Given the coordinates of the five points found with the aid of synthetic substitution, there will clearly be exactly two turning points. The graph of f can now be accurately sketched.



Exercise 3.1

In questions 1–4, use synthetic substitution to evaluate $P(x)$ for the given values of x.

1 $P(x) = x^4 + 2x^3 - 3x^2 - 4x - 20, \quad x = 2, \ x = -3$

2 $P(x) = 2x^5 - x^4 + 3x^3 - 15x - 9, \quad x = -1, \ x = 2$

3 $P(x) = x^5 + 5x^4 + 3x^3 - 6x^2 - 9x + 11, \quad x = -2, \ x = 4$

4 $P(x) = x^3 - (c + 3)x^2 + (3c + 5)x - 5c, \quad x = c, \ x = 2$

5 Given $P(x) = kx^3 + 2x^2 - 10x + 3$, for what value of k is $P(-2) = 15$?

6 Given $P(x) = 3x^4 - 2x^3 - 10x^2 + 3kx + 3$, for what value of k is $x = -\frac{1}{3}$ a zero of $P(x)$?

For questions 7 and 8, do not use your GDC.

7 a) Given $y = 2x^3 + 3x^2 - 5x - 4$, determine the y-value for each value of x such that $x \in \{-3, -2, -1, 0, 1, 2, 3\}$.

b) How many times must the graph of $y = 2x^3 + 3x^2 - 5x - 4$ cross the x-axis?

c) Sketch the graph of $y = 2x^3 + 3x^2 - 5x - 4$.

8 a) Given $y = x^4 - 4x^2 - 2x + 1$, determine the y-value for each value of x such that $x \in \{-3, -2, -1, 0, 1, 2, 3\}$.

b) How many times must the graph of $y = x^4 - 4x^2 - 2x + 1$ cross the x-axis?

c) Sketch the graph of $y = x^4 - 4x^2 - 2x + 1$.

9 Given $f(x) = x^3 + ax^2 - 5x + 7a$, find a so that $f(2) = 10$.

10 Given $f(x) = bx^3 - 5x^2 + 2bx + 10$, find b so that $f(\sqrt{3}) = -20$.

11 There are four possible end behaviours for a polynomial function $P(x)$. These are:

as $x \to \infty$, $P(x) \to \infty$ and as $x \to -\infty$, $P(x) \to \infty$ or symbolically $(\nwarrow, \nearrow)$

as $x \to \infty$, $P(x) \to -\infty$ and as $x \to -\infty$, $P(x) \to \infty$ or symbolically $(\nwarrow, \searrow)$

as $x \to \infty$, $P(x) \to -\infty$ and as $x \to -\infty$, $P(x) \to -\infty$ or symbolically $(\swarrow, \searrow)$

as $x \to \infty$, $P(x) \to \infty$ and as $x \to -\infty$, $P(x) \to -\infty$ or symbolically $(\swarrow, \nearrow)$

a) By sketching a graph on your GDC, state the type of end behaviour for each of the polynomial functions below.

(i) $P(x) = 2x^4 - 6x^3 + x^2 + 4x - 1$

(ii) $P(x) = -2x^4 - 6x^3 + x^2 + 4x - 1$

(iii) $P(x) = -6x^3 + x^2 + 4x - 1$

(iv) $P(x) = 6x^3 + x^2 - 4x - 1$

(v) $P(x) = x^2 - 4x - 1$

(vi) $P(x) = -2x^6 + x^5 + 2x^4 - 3x^3 + 4x^2 - x + 1$

(vii) $P(x) = x^5 + 2x^4 - x^3 + x^2 - x + 1$

(viii) $P(x) = -x^5 + 2x^4 - x^3 + x^2 - x + 1$

b) Use your results from a) to write a general statement about how the leading term of a polynomial function, a_nx^n, determines what type of end behaviour the graph of the function will display. Be specific about how the characteristics of the coefficient, a_n, and the power, n, of the leading term affect the function's end behaviour.

3.2 Quadratic functions

A **linear function** is a polynomial function of degree one that can be written in the general form $f(x) = ax + b$ where $a \neq 0$. Linear equations were briefly reviewed in Section 1.6. It is clear that any linear function will have a single solution (root) of $x = -\frac{b}{a}$. In essence, this is a formula that gives the zero of any linear polynomial.

In this section, we will focus on **quadratic functions** – functions consisting of a second-degree polynomial that can be written in the form $f(x) = ax^2 + bx + c$ such that $a \neq 0$. You are probably familiar with the quadratic formula that gives the zeros of any quadratic polynomial. We will also investigate other methods of finding zeros of quadratics and consider important characteristics of the graphs of quadratic functions.

The word *quadratic* comes from the Latin word *quadratus* that means four-sided, to make square, or simply a square. *Numerus quadratus* means a square number. Before modern algebraic notation was developed in the 17th and 18th centuries, the geometric figure of a square was used to indicate a number multiplying itself. Hence, raising a number to the power of two (in modern notation) is commonly referred to as the operation of squaring. *Quadratic* then came to be associated with a polynomial of degree two rather than being associated with the number four, as the prefix quad often indicates (e.g. quadruple).

Definition of a quadratic function

If a, b and c are real numbers, and $a \neq 0$, the function $f(x) = ax^2 + bx + c$ is a **quadratic function**. The graph of f is the graph of the equation $y = ax^2 + bx + c$ and is called a **parabola**.

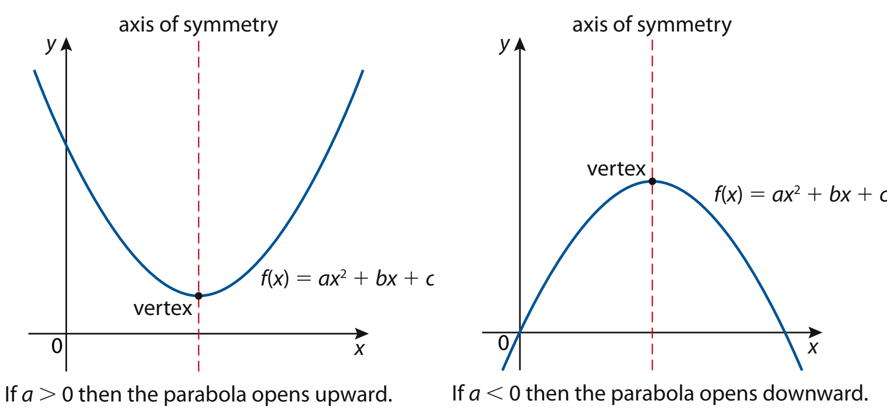


Figure 3.4 'Concave up' and 'concave down' parabolas.

Each parabola is symmetric about a vertical line called its **axis of symmetry.** The axis of symmetry passes through a point on the parabola called the **vertex** of the parabola, as shown in Figure 3.4. If the leading coefficient, a, of the quadratic function $f(x) = ax^2 + bx + c$ is positive, the parabola opens upward (concave up) – and the y-coordinate of the vertex will be a **minimum value** for the function. If the leading coefficient, a, of $f(x) = ax^2 + bx + c$ is negative, the parabola opens downward (concave down) – and the y-coordinate of the vertex will be a **maximum value** for the function.

The graph of $f(x) = a(x - h)^2 + k$

From the previous chapter, we know that the graph of the equation $y = (x + 3)^2 + 2$ can be obtained by translating $y = x^2$ three units to the left and two units up. Being familiar with the shape and position of the graph of $y = x^2$, and knowing the two translations that transform $y = x^2$ to

$y = (x + 3)^2 + 2$, we can easily visualize and/or sketch the graph of $y = (x + 3)^2 + 2$ (see Figure 3.5). We can also determine the axis of symmetry and the vertex of the graph. Figure 3.6 shows that the graph of $y = (x + 3)^2 + 2$ has an axis of symmetry of $x = -3$ and a vertex at $(-3, 2)$. The equation $y = (x + 3)^2 + 2$ can also be written as $y = x^2 + 6x + 11$. Because we can easily identify the vertex of the parabola when the equation is written as $y = (x + 3)^2 + 2$, we often refer to this as the **vertex form** of the quadratic equation, and $y = x^2 + 6x + 11$ as the **general form**.

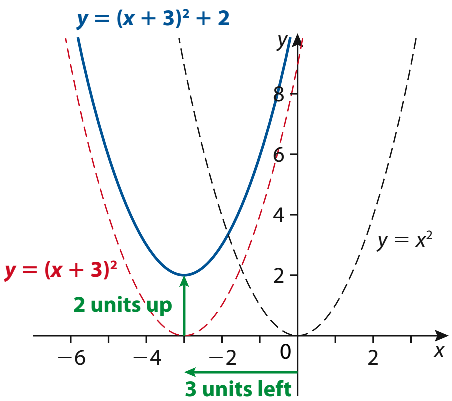


Figure 3.5 Translating $y = x^2$ to give $y = (x + 3)^2 + 2$.

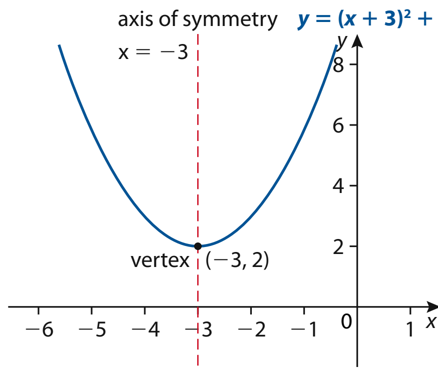


Figure 3.6 The axis of symmetry and the vertex.

• **Hint:** $f(x) = a(x - h)^2 + k$ is sometimes referred to as the **standard form** of a quadratic function.

Vertex form of a quadratic function

If a quadratic function is written in the form $f(x) = a(x - h)^2 + k$, with $a \neq 0$, the graph of f has an axis of symmetry of $x = h$ and a vertex at (h, k).

Completing the square

For visualizing and sketching purposes, it is helpful to have a quadratic function written in vertex form. How do we rewrite a quadratic function written in the form $f(x) = ax^2 + bx + c$ (general form) into the form $f(x) = a(x - h)^2 + k$ (vertex form)? We use the technique of **completing the square.**

For any real number p, the quadratic expression $x^2 + px + \left(\frac{p}{2}\right)^2$ is the square of $\left(x + \frac{p}{2}\right)$. Convince yourself of this by expanding $\left(x + \frac{p}{2}\right)^2$. The technique of *completing the square* is essentially the process of adding a constant to a quadratic expression to make it the square of a binomial. If the coefficient of the quadratic term (x^2) is positive one, the coefficient of the linear term is p, and the constant term is $\left(\frac{p}{2}\right)^2$, then $x^2 + px + \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2$ and the square is completed.

Remember that the coefficient of the quadratic term (leading coefficient) must be equal to positive one before completing the square.

Example 3

Find the equation of the axis of symmetry and the coordinates of the vertex of the graph of $f(x) = x^2 - 8x + 18$ by rewriting the function in the form $x^2 + px + \left(\frac{p}{2}\right)^2$.

Solution

To complete the square and get the quadratic expression $x^2 - 8x + 18$ in the form $x^2 + px + \left(\frac{p}{2}\right)^2$, the constant term needs to be $\left(\frac{-8}{2}\right)^2 = 16$.

We need to add 16, but also subtract 16, so that we are adding zero overall and, hence, not changing the original expression.

$f(x) = x^2 - 8x + 16 - 16 + 18$ — Actually adding zero ($-16 + 16$) to the right side.

$f(x) = x^2 - 8x + 16 + 2$ — $x^2 - 8x + 16$ fits the pattern $x^2 + px + \left(\frac{p}{2}\right)^2$ with $p = -8$.

$f(x) = (x - 4)^2 + 2$ — $x^2 - 8x + 16 = (x-4)^2$

The axis of symmetry of the graph of f is the vertical line $x = 4$ and the vertex is at $(4, 2)$. See Figure 3.7.

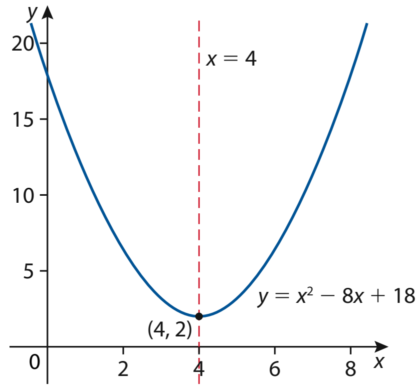


Figure 3.7

Example 4 – Properties of a parabola

For the function $g: x \mapsto -2x^2 - 12x + 7$,

a) find the axis of symmetry and the vertex of the graph
b) indicate the transformations that can be applied to $y = x^2$ to obtain the graph
c) find the minimum or maximum value.

Solution

a) $g: x \mapsto -2\left(x^2 + 6x - \frac{7}{2}\right)$ — Factorize so that the coefficient of the quadratic term is $+1$.

$g: x \mapsto -2\left(x^2 + 6x + 9 - 9 - \frac{7}{2}\right)$ — $p = 6 \Rightarrow \left(\frac{p}{2}\right)^2 = 9$; hence, add $+9 - 9$ (zero)

$g: x \mapsto -2\left[(x+3)^2 - \frac{18}{2} - \frac{7}{2}\right]$ — $x^2 + 6x + 9 = (x+3)^2$

$g: x \mapsto -2\left[(x+3)^2 - \frac{25}{2}\right]$

$g: x \mapsto -2(x+3)^2 + 25$ — Multiply through by -2 to remove outer brackets.

$g: x \mapsto -2(x - (-3))^2 + 25$ — Express in vertex form: $g: x \mapsto a(x-h)^2 + k$

The axis of symmetry of the graph of g is the vertical line $x = -3$ and the vertex is at $(-3, 25)$. See Figure 3.8.

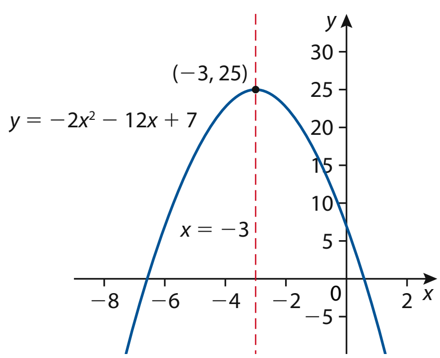


Figure 3.8

b) Since $g: x \mapsto -2x^2 - 12x + 7 = -2(x+3)^2 + 25$, the graph of g can be obtained by applying the following transformations (in the order given) on the graph of $y = x^2$: horizontal translation of 3 units left;

reflection in the x-axis (parabola opening down); vertical stretch of factor 2; and a vertical translation of 25 units up.

c) The parabola opens down because the leading coefficient is negative. Therefore, g has a maximum and no minimum value. The maximum value is 25 (y-coordinate of vertex) at $x = -3$.

The technique of completing the square can be used to derive the quadratic formula. The following example derives a general expression for the axis of symmetry and vertex of a quadratic function in the general form $f(x) = ax^2 + bx + c$ by completing the square.

Example 5 – Graphical properties of general quadratic functions

Find the axis of symmetry and the vertex for the general quadratic function $f(x) = ax^2 + bx + c$.

Solution

$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ Factorize so that the coefficient of the x^2 term is $+1$.

$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$ $p = \frac{b}{a} \Rightarrow \left(\frac{p}{2}\right)^2 = \left(\frac{b}{2a}\right)^2$

$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$ $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = x + \left(\frac{b}{2a}\right)^2$

$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$ Multiply through by a.

$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + c - \frac{b^2}{4a}$ Express in vertex form: $f(x) = a(x - h)^2 + k$

This result leads to the following generalization.

Symmetry and vertex of $f(x) = ax^2 + bx + c$

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, the axis of symmetry is the vertical line with the equation $x = -\frac{b}{2a}$ and the vertex has coordinates $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.

Check the results for Example 4 using the formulae for the axis of symmetry and vertex. For the function $g: x \mapsto -2x^2 - 12x + 7$:

$x = -\frac{b}{2a} = -\frac{-12}{2(-2)} = -3 \Rightarrow$ axis of symmetry is the vertical line $x = -3$

$c - \frac{b^2}{4a} = 7 - \frac{(-12)^2}{4(-2)} = \frac{56}{8} + \frac{144}{8} = 25 \Rightarrow$ vertex has coordinates $(-3, 25)$

These results agree with the results from Example 4.

Zeros of a quadratic function

A specific value for x is a **zero** of a quadratic function $f(x) = ax^2 + bx + c$ if it is a solution (or **root**) to the equation $ax^2 + bx + c = 0$.

As we will observe, every quadratic function will have two zeros although it is possible for the same zero to occur twice (double zero, or double root). The x-coordinate of any point(s) where f crosses the x-axis (y-coordinate is zero) is a **real zero** of the function. A quadratic function can have one, two or no real zeros as Figure 3.9 illustrates. To find non-real zeros we need to extend our search to the set of complex numbers and we will see that a quadratic function with no real zeros will have two distinct **imaginary zeros**. Finding all zeros of a quadratic function requires you to solve quadratic equations of the form $ax^2 + bx + c = 0$. Although $a \neq 0$, it is possible for b or c to be equal to zero. There are five general methods for solving quadratic equations as outlined in Table 3.2 below.

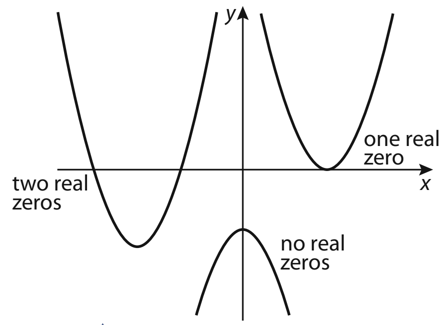
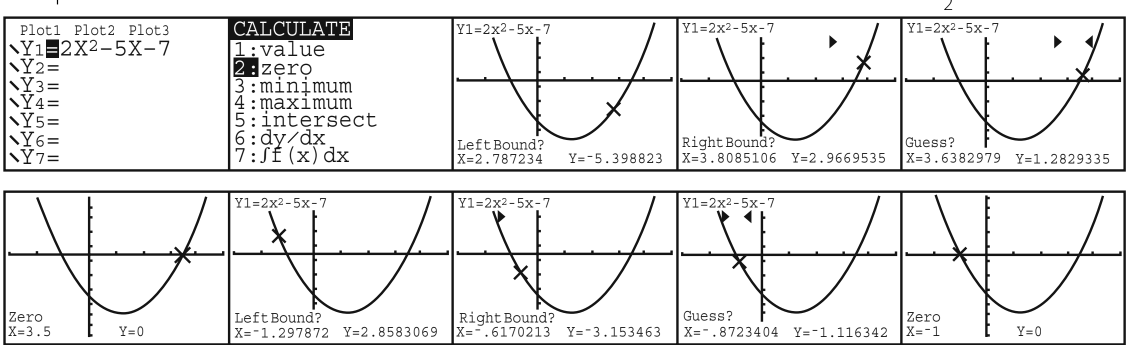


Figure 3.9

Table 3.2 Methods for solving quadratic equations.

Square root	If $a^2 = c$ and $c > 0$, then $a = \pm\sqrt{c}$.	
Examples	$x^2 - 25 = 0$ $x^2 = 25$ $x = \pm 5$	$(x + 2)^2 = 15$ $x + 2 = \pm\sqrt{15}$ $x = -2 \pm\sqrt{15}$
Factorizing	If $ab = 0$, then $a = 0$ or $b = 0$.	
Examples	$x^2 + 3x - 10 = 0$ $(x + 5)(x - 2) = 0$ $x = -5$ or $x = 2$	$x^2 - 7x = 0$ $x(x - 7) = 0$ $x = 0$ or $x = 7$
Completing the square	If $x^2 + px + q = 0$, then $x^2 + px + \left(\frac{p}{2}\right)^2 = -q + \left(\frac{p}{2}\right)^2$ which leads to $\left(x + \frac{p}{2}\right)^2 = -q + \frac{p^2}{4}$ and then the square root of both sides (as above).	
Example	$x^2 - 8x + 5 = 0$ $x^2 - 8x + 16 = -5 + 16$ $(x - 4)^2 = 11$ $x - 4 = \pm\sqrt{11}$ $x = 4 \pm \sqrt{11}$	
Quadratic formula	If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.	
Example	$2x^2 - 3x - 4 = 0$ $x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)}$ $x = \dfrac{3 \pm \sqrt{41}}{4}$	
Graphing	Graph the equation $y = ax^2 + bx + c$ on your GDC. Use the calculating features of your GDC to determine the x-coordinates of the point(s) where the parabola intersects the x-axis. *Note*: This method works for finding real solutions, but **not** imaginary solutions.	
Example	$2x^2 - 5x - 7 = 0$ GDC calculations reveal that the zeros are at $x = \frac{7}{2}$ and $x = -1$	



Sum and product of the roots of a quadratic equation

> In the next section, the Factor Theorem formally states the relationship between linear factors of the form $x - \alpha$ and the zeros for *any* polynomial.

Consider the quadratic equation $x^2 + 5x - 24 = 0$. This equation can be solved using factorization as follows.

$$x^2 + 5x - 24 = (x + 8)(x - 3) = 0 \Rightarrow x = -8 \text{ or } x = 3$$

Clearly, if $x - \alpha$ is a factor of the quadratic polynomial $ax^2 + bx + c$, then $x = \alpha$ is a root (solution) of the quadratic equation $ax^2 + bx + c = 0$.

Now let us consider the general quadratic equation $ax^2 + bx + c = 0$, whose roots are $x = \alpha$ and $x = \beta$. Given our observation from the previous paragraph, we can write the quadratic equation with roots α and β as:

$$\begin{aligned} ax^2 + bx + c = (x - \alpha)(x - \beta) &= 0 \\ x^2 - ax - \beta x + \alpha\beta &= 0 \\ x^2 - (\alpha + \beta)x + \alpha\beta &= 0 \end{aligned}$$

Since the equation $ax^2 + bx + c = 0$ can also be written as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, then:

$$x^2 - (\alpha + \beta)x + \alpha\beta = x^2 + \frac{b}{a}x + \frac{c}{a}$$

Equating coefficients of both sides, gives the following results.

$$\alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

> **Sum and product of the roots of a quadratic equation**
> For any quadratic equation in the form $ax^2 + bx + c = 0$, the **sum of the roots** of the equation is $-\frac{b}{a}$ and the **product of the roots** is $\frac{c}{a}$. (In the next section, this result is extended to polynomial equations of any degree.)

Example 6

If α and β are the roots of each equation, find the sum, $\alpha + \beta$, and product, $\alpha\beta$, of the roots.

a) $x^2 - 5x + 3 = 0$ b) $3x^2 + 4x - 7 = 0$

Solution

a) For the equation $x^2 - 5x + 3 = 0$, $a = 1$, $b = -5$ and $c = 3$.
Therefore, $\alpha + \beta = -\frac{b}{a} = -\frac{-5}{1} = 5$ and $\alpha\beta = \frac{c}{a} = \frac{3}{1} = 3$.

b) For the equation $3x^2 + 4x - 7 = 0$, $a = 3$, $b = 4$ and $c = -7$.
Therefore, $\alpha + \beta = -\frac{b}{a} = -\frac{4}{3}$ and $\alpha\beta = \frac{c}{a} = \frac{-7}{3}$.

> If the sum and product of the roots of a quadratic equation are known, then the equation can be written in the following form: $x^2 -$ (sum of roots)x + (product of roots) $= 0$

Example 7

If α and β are the roots of the equation $2x^2 + 6x - 5 = 0$, find a quadratic equation whose roots are:

a) $2\alpha, 2\beta$ b) $\frac{1}{\alpha + 1}, \frac{1}{\beta + 1}$

Solution

For the equation $2x^2 + 6x - 5 = 0$, $a = 2$, $b = 6$ and $c = -5$.

Thus, $\alpha + \beta = -\frac{b}{a} = -\frac{6}{2} = -3$ and $\alpha\beta = \frac{c}{a} = \frac{-5}{2}$.

a) Sum of the new roots $= 2\alpha + 2\beta = 2(\alpha + \beta) = 2(-3) = -6$.

Thus for the new equation, $-\frac{b}{a} = -6$.

Product of the new roots $= 2\alpha \cdot 2\beta = 4\alpha\beta = 4\left(-\frac{5}{2}\right) = -10$.

Thus for the new equation, $\frac{c}{a} = -10$.

The new equation we are looking for can be written as $ax^2 + bx + c = 0$ or $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Therefore, the quadratic equation with roots $2\alpha, 2\beta$ is $x^2 - (-6)x - 10 = 0$
$\Rightarrow x^2 + 6x - 10 = 0$

b) Sum of the new roots $\frac{1}{\alpha + 1} + \frac{1}{\beta + 1} = \frac{\beta + 1 + \alpha + 1}{(\alpha + 1)(\beta + 1)}$

$= \frac{\alpha + \beta + 2}{\alpha\beta + \alpha + \beta + 1} = \frac{-3 + 2}{-\frac{5}{2} - 3 + 1} = \frac{-1}{-\frac{9}{2}} = \frac{2}{9}$.

Thus for the new equation, $-\frac{b}{a} = \frac{2}{9}$.

Product of the new roots $\left(\frac{1}{\alpha + 1}\right)\left(\frac{1}{\beta + 1}\right) = \frac{1}{\alpha\beta + \alpha + \beta + 1}$

$= \frac{1}{-\frac{5}{2} - 3 + 1} = \frac{1}{-\frac{9}{2}} = -\frac{2}{9}$.

Thus for the new equation, $\frac{c}{a} = -\frac{2}{9}$.

The new equation we are looking for can be written as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$.

Therefore, the quadratic equation with roots
$\frac{1}{\alpha + 1}, \frac{1}{\beta + 1}$ is $x^2 - \frac{2}{9}x - \frac{2}{9} = 0$ or $9x^2 - 2x - 2 = 0$.

Example 8

Given that the roots of the equation $x^2 - 4x + 2 = 0$ are α and β, find the values of the following expressions.

a) $\alpha^2 + \beta^2$ b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

Solution

With $x^2 - 4x + 2 = 0$, $\alpha + \beta = -\frac{b}{a} = -\frac{-4}{1} = 4$ and $\alpha\beta = \frac{c}{a} = \frac{2}{1} = 2$.

Both of the expressions $\alpha^2 + \beta^2$ and $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ need to be expressed in terms of $\alpha + \beta$ and $\alpha\beta$.

a) $\alpha^2 + \beta^2 = \alpha^2 + 2\alpha\beta + \beta^2 - 2\alpha\beta = (\alpha + \beta)^2 - 2\alpha\beta$

Substituting the values for $\alpha + \beta$ and $\alpha\beta$ from above, gives
$\alpha^2 + \beta^2 = 4^2 - 2 \cdot 2 = 16 - 4 = 12$.

b) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2}{\alpha^2\beta^2} + \frac{\alpha^2}{\alpha^2\beta^2} = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2}$

From part a) we know that $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$. Substituting this into the numerator gives:

$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$ Then substituting the values for $\alpha + \beta$ and $\alpha\beta$ from above, gives:

$= \frac{4^2 - 2 \cdot 2}{2^2} = \frac{12}{4} = 3$

Therefore, $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = 3$.

The quadratic formula and the discriminant

The expression that is beneath the radical sign in the quadratic formula, $b^2 - 4ac$, determines whether the zeros of a quadratic function are real or imaginary. Because it acts to 'discriminate' between the types of zeros, $b^2 - 4ac$ is called the **discriminant**. It is often labelled with the Greek letter Δ (delta). The value of the discriminant can also indicate if the zeros are equal and if they are rational.

The discriminant and the nature of the zeros of a quadratic function

For the quadratic function $f(x) = ax^2 + bx + c$, $(a \neq 0)$ where a, b and c are real numbers:

If $\Delta = b^2 - 4ac > 0$, then f has two distinct real zeros, and the graph of f intersects the x-axis twice.

If $\Delta = b^2 - 4ac = 0$, then f has one real zero (double root), and the graph of f intersects the x-axis once (i.e. it is tangent to the x-axis).

If $\Delta = b^2 - 4ac < 0$, then f has two conjugate imaginary zeros, and the graph of f does not intersect the x-axis.

In the special case when a, b and c are integers and the discriminant is the square of an integer (a *perfect square*), the polynomial $ax^2 + bx + c$ has two distinct **rational zeros**.

When the discriminant is zero then the solution of a quadratic function is $x = \frac{-b \pm\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm\sqrt{0}}{2a} = -\frac{b}{2a}$. As mentioned, this solution of $-\frac{b}{2a}$ is called a double zero (or root) which can also be described as a **zero of multiplicity of 2**. If a and b are integers then the zero $-\frac{b}{2a}$ will be rational.

When we solve polynomial functions of higher degree later this chapter, we will encounter zeros of higher multiplicity.

● **Hint:** Remember that the **roots** of a polynomial equation are those values of x for which $P(x) = 0$. These values of x are called the **zeros** of the polynomial P.

Factorable quadratics

If the zeros of a quadratic polynomial are rational – either two distinct zeros or two equal zeros (double zero/root) – then the polynomial is factorable. That is, if $ax^2 + bx + c$ has rational zeros then $ax^2 + bx + c = (mx + n)(px + q)$ where m, n, p and q are rational numbers.

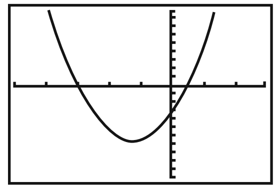
Example 9 – Using discriminant to determine the nature of the roots of a quadratic equation

Use the discriminant to determine how many real roots each equation has. Visually confirm the result by graphing the corresponding quadratic function for each equation on your GDC.

a) $2x^2 + 5x - 3 = 0$ b) $4x^2 - 12x + 9 = 0$ c) $2x^2 - 5x + 6 = 0$

Solution

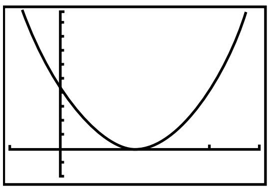
$y = 2x^2 + 5x - 3$

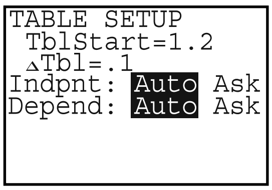


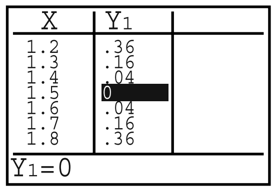
a) The discriminant is $\Delta = 5^2 - 4(2)(-3) = 49 > 0$. Therefore, the equation has two distinct real roots. This result is confirmed by the graph of the quadratic function $y = 2x^2 + 5x - 3$ that clearly shows it intersecting the x-axis twice. Also since $\Delta = 49$ is a perfect square then the two roots are also rational and the quadratic polynomial $2x^2 + 5x - 3 = 0$ is factorable: $2x^2 + 5x - 3 = (2x - 1)(x + 3) = 0$. Thus, the two rational roots are $x = \frac{1}{2}$ and $x = -3$.

b) The discriminant is $\Delta = (-12)^2 - 4(4)(9) = 0$. Therefore, the equation has one rational root (a double root). The graph on the GDC of $y = 4x^2 - 12x + 9$ appears to intersect the x-axis at only one point. We can be more confident with this conclusion by investigating further – for example, tracing or looking at a table of values on the GDC.

$y = 4x^2 - 12x + 9$







Also, since the root is rational ($\Delta = 0$), the polynomial $4x^2 - 12x + 9$ must be factorable.

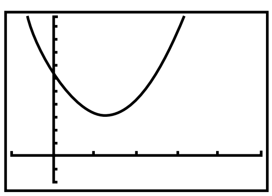
$4x^2 - 12x + 9 = (2x - 3)(2x - 3) = \left[2\left(x - \frac{3}{2}\right)2\left(x - \frac{3}{2}\right)\right] = 4\left(x - \frac{3}{2}\right)^2 = 0$

There are two equal linear factors which means there are two equal rational zeros – both equal to $\frac{3}{2}$ in this case.

c) The discriminant is $\Delta = (-5)^2 - 4(2)(6) = -23 < 0$. Therefore, the equation has no real roots. This result is confirmed by the graph of the quadratic function $y = 2x^2 - 5x + 6$ that clearly shows that the graph does not intersect the x-axis. The equation will have two imaginary roots.

• **Hint:** If a quadratic polynomial has a zero of multiplicity 2($\Delta = 0$), as in Example 6 b), then not only is the polynomial factorable but its factorization will contain two equal linear factors. In such a case then $ax^2 + bx + c = a(x - p)^2$ where $x - p$ is the linear factor and $x = p$ is the rational zero.

$y = 2x^2 - 5x + 6$



Example 10 – The discriminant and number of real zeros

For $4x^2 + 4kx + 9 = 0$, determine the value(s) of k so that the equation has: a) one real zero, b) two distinct real zeros, and c) no real zeros.

Solution

a) For one real zero $\Delta = (4k)^2 - 4(4)(9) = 0 \Rightarrow 16k^2 - 144 = 0$

$\Rightarrow 16k^2 = 144 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$

b) For two distinct real zeros $\Delta = (4k)^2 - 4(4)(9) > 0 \Rightarrow 16k^2 > 144$
$\Rightarrow k^2 > 9 \Rightarrow k < -3$ or $k > 3$

c) For no real zeros $\Delta = (4k)^2 - 4(4)(9) < 0 \Rightarrow 16k^2 < 144 \Rightarrow k^2 < 9$
$\Rightarrow k > -3$ and $k < 3 \Rightarrow -3 < k < 3$

Example 11 – Conjugate imaginary solutions

Find the zeros of the function $g: x \rightarrow 2x^2 - 4x + 7$.

Solution

Solve the equation $2x^2 - 4x + 7 = 0$ using the quadratic formula with $a = 2, b = -4, c = 7$.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(7)}}{2(2)} = \frac{4 \pm \sqrt{-40}}{4} = \frac{4 \pm \sqrt{4}\sqrt{-1}\sqrt{10}}{4}$$

$$= \frac{4 \pm 2i\sqrt{10}}{4} = 1 \pm \frac{i\sqrt{10}}{2}$$

The two zeros of g are $1 + \frac{\sqrt{10}}{2}i$ and $1 - \frac{\sqrt{10}}{2}i$.

Note that the imaginary zeros are written in the form $a + bi$ (introduced in Section 1.1) and that they clearly are a pair of conjugates, i.e. fitting the pattern $a + bi$ and $a - bi$.

Number of complex zeros of a quadratic polynomial

Every quadratic polynomial has exactly two complex zeros, provided that a zero of multiplicity 2 (two equal zeros) is counted as two zeros.

- **Hint:** Recall from Section 1.1 that the real numbers and the imaginary numbers are distinct subsets of the complex numbers. A complex number can be either real $\left(\text{e.g. } -7, \frac{\pi}{2}, 3 - \sqrt{2}\right)$ or imaginary (e.g. $4i$, $2 + i\sqrt{5}$).

The graph of $f(x) = a(x - p)(x - q)$

If a quadratic function is written in the form $f(x) = a(x - p)(x - q)$ then we can easily identify the x-intercepts of the graph of f. Consider that $f(p) = a(p - p)(p - q) = a(0)(p - q) = 0$ and that $f(q) = a(q - p)(q - q) = a(q - p)(0) = 0$. Therefore, the quadratic function $f(x) = a(x - p)(x - q)$ will intersect the x-axis at the points $(p, 0)$ and $(q, 0)$. We need to factorize in order to rewrite a quadratic function in the form $f(x) = ax^2 + bx + c$ to the form $f(x) = a(x - p)(x - q)$. Hence, $f(x) = a(x - p)(x - q)$ can be referred to as the **factorized** form of a quadratic function. Recalling the symmetric nature of a parabola, it is clear that the x-intercepts $(p, 0)$ and $(q, 0)$ will be equidistant from the axis of symmetry (see Figure 3.10). As a result, the equation of the axis of symmetry and the x-coordinate of the vertex of the parabola can be found from finding the average of p and q.

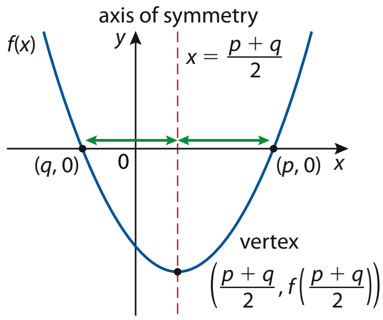


Figure 3.10

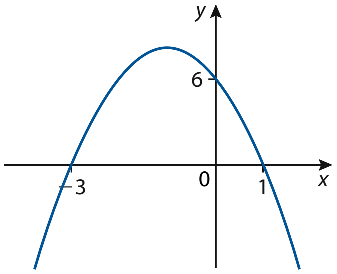
Factorized form of a quadratic function

If a quadratic function is written in the form $f(x) = a(x - p)(x - q)$, with $a \neq 0$, the graph of f has x-intercepts at $(p, 0)$ and $(q, 0)$, an axis of symmetry with equation $x = \frac{p+q}{2}$, and a vertex at $\left(\frac{p+q}{2}, f\left(\frac{p+q}{2}\right)\right)$.

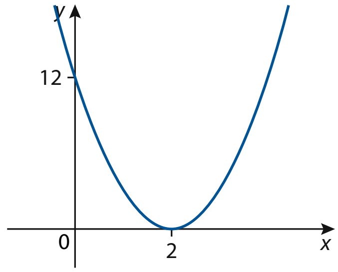
Example 12

Find the equation of each quadratic function from the graph in the form $f(x) = a(x - p)(x - q)$ and also in the form $f(x) = ax^2 + bx + c$.

a)



b)



Solution

a) Since the x-intercepts are -3 and 1 then $y = a(x + 3)(x - 1)$. The y-intercept is 6, so when $x = 0$, $y = 6$. Hence, $6 = a(0 + 3)(0 - 1) = -3a \Rightarrow a = -2$ ($a < 0$ agrees with the fact that the parabola is opening down). The function is $f(x) = -2(x + 3)(x - 1)$, and expanding to remove brackets reveals that the function can also be written as $f(x) = -2x^2 - 4x + 6$.

b) The function has one x-intercept at 2 (double root), so $p = q = 2$ and $y = a(x - 2)(x - 2) = a(x - 2)^2$. The y-intercept is 12, so when $x = 0$, $y = 12$. Hence, $12 = a(0 - 2)^2 = 4a \Rightarrow a = 3$ ($a > 0$ agrees with the parabola opening up). The function is $f(x) = 3(x - 2)^2$. Expanding reveals that the function can also be written as $f(x) = 3x^2 - 12x + 12$.

Example 13

The graph of a quadratic function intersects the x-axis at the points $(-6, 0)$ and $(-2, 0)$ and also passes through the point $(2, 16)$. a) Write the function in the form $f(x) = a(x - p)(x - q)$. b) Find the vertex of the parabola. c) Write the function in the form $f(x) = a(x - h)^2 + k$.

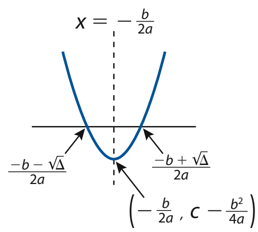
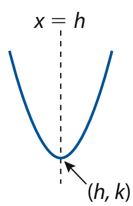
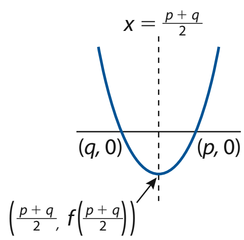
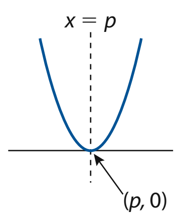
Solution

a) The x-intercepts of -6 and -2 gives $f(x) = a(x + 6)(x + 2)$. Since f passes through $(2, 16)$, then $f(2) = 16 \ \Rightarrow\ f(2) = a(2 + 6)(2 + 2) = 16$ $\Rightarrow\ 32a = 16 \ \Rightarrow\ a = \frac{1}{2}$. Therefore, $f(x) = \frac{1}{2}(x + 6)(x + 2)$.

b) The x-coordinate of the vertex is the average of the x-intercepts. $x = \dfrac{-6 - 2}{2} = -4$, so the y-coordinate of the vertex is $y = f(-4) = \frac{1}{2}(-4 + 6)(-4 + 2) = -2$. Hence, the vertex is $(-4, -2)$.

c) In vertex form, the quadratic function is $f(x) = \frac{1}{2}(x + 4)^2 - 2$.

Table 3.3 Review of properties of quadratics.

Quadratic function, $a \neq 0$	Graph of function	Results
General form $f(x) = ax^2 + bx + c$ $\Delta = b^2 - 4ac$ (discriminant)	Parabola opens up if $a > 0$ Parabola opens down if $a < 0$  If $\Delta \geqslant 0$, f has x-intercept(s) If $\Delta < 0$, f has no x-intercept(s)	Axis of symmetry is $x = -\frac{b}{2a}$ If $\Delta \geqslant 0$, f has x-intercept(s): $\left(\frac{-b \pm \sqrt{\Delta}}{2a}, 0\right)$ Vertex is: $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$
Vertex form $f(x) = a(x - h)^2 + k$		Axis of symmetry is $x = h$ Vertex is (h, k)
Factorized form (two distinct rational zeros) $f(x) = a(x - p)(x - q)$		Axis of symmetry is $x = \frac{p+q}{2}$ x-intercepts are: $(p, 0)$ and $(q, 0)$
Factorized form (one rational zero) $f(x) = a(x - p)^2$		Axis of symmetry is $x = p$ Vertex and x-intercept is $(p, 0)$

Exercise 3.2

For each of the quadratic functions f in questions 1–5, find the following:

a) the axis of symmetry and the vertex, by algebraic methods

b) the transformation(s) that can be applied to $y = x^2$ to obtain the graph of $y = f(x)$

c) the minimum or maximum value of f.

Check your results using your GDC.

1 $f: x \mapsto x^2 - 10x + 32$

2 $f: x \mapsto x^2 + 6x + 8$

3 $f: x \mapsto -2x^2 - 4x + 10$

4 $f: x \mapsto 4x^2 - 4x + 9$

5 $f: x \mapsto \frac{1}{2}x^2 + 7x + 26$

In questions 6–13, solve the quadratic equation using factorization.

6 $x^2 + 2x - 8 = 0$

7 $x^2 = 3x + 10$

8 $6x^2 - 9x = 0$

9 $6 + 5x = x^2$

10 $x^2 + 9 = 6x$

11 $3x^2 + 11x - 4 = 0$

12 $3x^2 + 18 = 15x$

13 $9x - 2 = 4x^2$

In questions 14–19, use the method of completing the square to solve the quadratic equation.

14 $x^2 + 4x - 3 = 0$

15 $x^2 - 4x - 5 = 0$

16 $x^2 - 2x + 3 = 0$

17 $2x^2 + 16x + 6 = 0$

18 $x^2 + 2x - 8 = 0$

19 $-2x^2 + 4x + 9 = 0$

20 Let $f(x) = x^2 - 4x - 1$. a) Use the quadratic formula to find the zeros of the function. b) Use the zeros to find the equation for the axis of symmetry of the parabola. c) Find the minimum or maximum value of f.

In questions 21–24, determine the number of real solutions to each equation.

21 $x^2 + 3x + 2 = 0$

22 $2x^2 - 3x + 2 = 0$

23 $x^2 - 1 = 0$

24 $2x^2 - \frac{9}{4}x + 1 = 0$

25 Find the value(s) of p for which the equation $2x^2 + px + 1 = 0$ has one real solution.

26 Find the value(s) of k for which the equation $x^2 + 4x + k = 0$ has two distinct real solutions.

27 The equation $x^2 - 4kx + 4 = 0$ has two distinct real solutions. Find the set of all possible values of k.

28 Find all possible values of m so that the graph of the function $g: x \mapsto mx^2 + 6x + m$ does not touch the x-axis.

29 Find the range of values of k such that $3x^2 - 12x + k > 0$ for all real values of x. (Hint: Consider what must be true about the zeros of the quadratic equation $y = 3x^2 - 12x + k$.)

30 Prove that the expression $x - 2 - x^2$ is negative for all real values of x.

In questions 31 and 32, find a quadratic function in the form $y = ax^2 + bx + c$ that satisfies the given conditions.

31 The function has zeros of $x = -1$ and $x = 4$ and its graph intersects the y-axis at $(0, 8)$.

32 The function has zeros of $x = \frac{1}{2}$ and $x = 3$ and its graph passes through the point $(-1, 4)$.

33 Find the range of values for k in order for the equation $2x^2 + (3 - k)x + k + 3 = 0$ to have two imaginary solutions.

34 For what values of m does the function $f(x) = 5x^2 - mx + 2$ have two distinct real zeros?

35 The graph of a quadratic function passes through the points $(-3, 10)$, $(\frac{1}{4}, -\frac{9}{16})$ and $(1, 6)$. Express the function in the form $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$.

36 The maximum value of the function $f(x) = ax^2 + bx + c$ is 10.
Given that $f(3) = f(-1) = 2$, find $f(2)$.

37 Find the values of x for which $4x + 1 < x^2 + 4$.

38 Show that there is no real value t for which the equation $2x^2 + (2 - t)x + t^2 + 3 = 0$ has real roots.

39 Show that the two roots of $ax^2 - a^2x - x + a = 0$ are reciprocals of each other.

40 Find the sum and product of the roots for each of the following quadratic equations.

a) $2x^2 + 6x - 5 = 0$ b) $x^2 = 1 - 3x$ c) $4x^2 - 6 = 0$

d) $x^2 + ax - 2a = 0$ e) $m(m - 2) = 4(m + 1)$ f) $3x - \frac{2}{x} = 1$

41 The roots of the equation $2x^2 - 3x + 6 = 0$ are α and β. Find a quadratic equation with integral coefficients whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

42 If α and β are the roots of the equation $3x^2 + 5x + 4 = 0$, find the values of the following expressions.

a) $\alpha^2 + \beta^2$ b) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

c) $\alpha^3 + \beta^3$ [Hint: factorise $\alpha^3 + \beta^3$ into a product of a binomial and a trinomial.]

43 Consider the quadratic equation $x^2 + 8x + k = 0$ where k is a constant.

a) Find both roots of the equation given that one root of the equation is three times the other.

b) Find the value of k.

44 The roots of the equation $x^2 + x + 4 = 0$ are α and β.

a) Without solving the equation, find the value of the expression $\frac{1}{\alpha} + \frac{1}{\beta}$.

b) Find a quadratic equation whose roots are $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

45 If α and β are roots of the quadratic equation $5x^2 - 3x - 1 = 0$, find a quadratic equation with integral coefficients which have the roots:

a) $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$ b) $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$

3.3 Zeros, factors and remainders

Finding the zeros of polynomial functions is a feature of many problems in algebra, calculus and other areas of mathematics. In our analysis of quadratic functions in the previous section, we saw the connection between the graphical and algebraic approaches to finding zeros. Information obtained from the graph of a function can be used to help find its zeros and, conversely, information about the zeros of a polynomial function can be used to help sketch its graph. Results and observations from the last section lead us to make some statements about real zeros of all polynomial functions. Later in this section we will extend our consideration to imaginary zeros. The following box summarizes what we have observed thus far about the zeros of polynomial functions.

Real zeros of polynomial functions

If P is a polynomial function and c is a real number, then the following statements are equivalent.

- $x = c$ is a zero of the function P.
- $x = c$ is a solution (or root) of the polynomial equation $P(x) = 0$.
- $x - c$ is a linear factor of the polynomial P.
- $(c, 0)$ is an x-intercept of the graph of the function P.

Polynomial division

As with integers, finding the factors of polynomials is closely related to dividing polynomials. An integer n is **divisible** by another integer m if m is a factor of n. If n is not divisible by m we can use the process of **long division** to find the quotient of the numbers and the remainder. For example, let's use long division to divide 485 by 34.

$$\begin{array}{r} 14 \\ 34\overline{)485} \\ \underline{34} \\ 145 \\ \underline{136} \\ 9 \end{array} \qquad\qquad \text{check:} \quad \begin{array}{rl} 14 & \text{quotient} \\ \underline{\times 34} & \text{divisor} \\ 56 & \\ \underline{420} & \\ 476 & \\ \underline{+\ \ 9} & \text{remainder} \\ 485 & \text{dividend} \end{array}$$

The number 485 is the **dividend**, 34 is the **divisor**, 14 is the **quotient** and 9 is the **remainder**. The long division process (or algorithm) stops when a remainder is less than the divisor. The procedure shown above for checking the division result may be expressed as

$$485 = 34 \times 14 + 9$$

or in words as

$$\text{dividend} = \text{divisor} \times \text{quotient} + \text{remainder}$$

The process of division for polynomials is similar to that for integers. If a polynomial $D(x)$ is a factor of polynomial $P(x)$, then $P(x)$ is divisible by $D(x)$. However, if $D(x)$ is not a factor of $P(x)$ then we can use a **long division algorithm for polynomials** to find a quotient polynomial $Q(x)$ and a remainder polynomial $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$. In the same way that the remainder must be less than the divisor when dividing integers, the remainder must be a polynomial of a lower degree than the divisor when dividing polynomials. Consequently, when the divisor is a linear polynomial (degree of 1) the remainder must be of degree 0, i.e. a constant.

● **Hint:** A common error when performing long division with polynomials is to add rather than subtract during each cycle of the process.

Example 14

Find the quotient $Q(x)$ and remainder $R(x)$ when $P(x) = 2x^3 - 5x^2 + 6x - 3$ is divided by $D(x) = x - 2$.

Solution

$$\begin{array}{rll}
 & 2x^2 - x + 4 & \\
x - 2 \,\big)\!\!\! & \overline{2x^3 - 5x^2 + 6x - 3} & \\
 & \underline{2x^3 - 4x^2} & \leftarrow 2x^2(x-2) \\
 & - x^2 + 6x & \leftarrow \text{Subtract} \\
 & \underline{- x^2 + 2x} & \leftarrow -x(x-2) \\
 & 4x - 3 & \leftarrow \text{Subtract} \\
 & \underline{4x - 8} & \leftarrow 4(x-2) \\
 & 5 & \leftarrow \text{Subtract}
\end{array}$$

Thus, the quotient $Q(x)$ is $2x^2 - x + 4$ and the remainder is 5. Therefore, we can write

$$2x^3 - 5x^2 + 6x - 3 = (x - 2)(2x^2 - x + 4) + 5$$

This equation provides a means to check the result by expanding and simplifying the right side and verifying it is equal to the left side.

$$\begin{aligned}
2x^3 - 5x^2 + 6x - 3 &= (x - 2)(2x^2 - x + 4) + 5 \\
&= (2x^3 - x^2 + 4x - 4x^2 + 2x - 8) + 5 \\
&= 2x^3 - 5x^2 + 6x - 3
\end{aligned}$$

Taking the identity $P(x) = D(x) \cdot Q(x) + R(x)$ and dividing both sides by $D(x)$ produces the equivalent identity $\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$.

Hence, the result for Example 14 could also be written as

$$\frac{2x^3 - 5x^2 + 6x - 3}{x - 2} = 2x^2 - x + 4 + \frac{5}{x - 2}.$$

Note that writing the result in this manner is the same as rewriting $17 = 5 \times 3 + 2$ as $\frac{17}{5} = 3 + \frac{2}{5}$, which we commonly write as the 'mixed number' $3\frac{2}{5}$.

● **Hint:** When performing long division with polynomials it is necessary to write all polynomials so that the powers (exponents) of the terms are in descending order. Example 12 illustrates that if there are any 'missing' terms then they have a coefficient of zero and a zero must be included in the appropriate location in the division scheme.

Example 15

Divide $f(x) = 4x^3 - 31x - 15$ by $2x + 5$, and use the result to factor $f(x)$ completely.

Solution

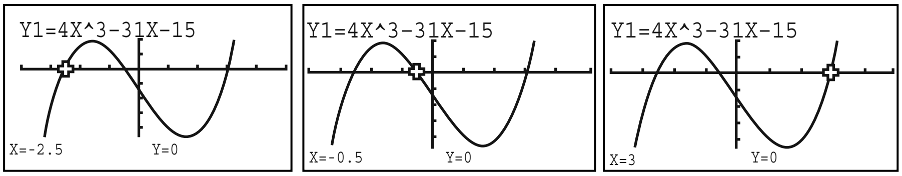
$$\begin{array}{rl}
 & 2x^2 - 5x - 3 \\
2x + 5 \,\big)\!\!\! & \overline{4x^3 + \ \ 0x^2 - 31x - 15} \\
 & \underline{4x^3 + 10x^2} \\
 & - 10x^2 - 31x \\
 & \underline{- 10x^2 - 25x} \\
 & -6x - 15 \\
 & \underline{-6x - 15} \\
 & 0
\end{array}$$

Thus $f(x) = 4x^3 - 31x - 15 = (2x + 5)(2x^2 - 5x - 3)$

… and factorizing the quadratic quotient (also a factor of $f(x)$), gives

$$\begin{aligned} f(x) = 4x^3 - 31x - 15 &= (2x + 5)(2x^2 - 5x - 3) \\ &= (2x + 5)(2x + 1)(x - 3) \end{aligned}$$

This factorization would lead us to believe that the three zeros of $f(x)$ are $x = -\frac{5}{2}$, $x = -\frac{1}{2}$ and $x = 3$. Graphing $f(x)$ on our GDC and using the 'trace' feature confirms that all three values are zeros of the cubic polynomial.



Division algorithm for polynomials

If $P(x)$ and $D(x)$ are polynomials such that $D(x) \neq 0$, and the degree of $D(x)$ is less than or equal to the degree of $P(x)$, then there exist unique polynomials $Q(x)$ and $R(x)$ such that

$$P(x) = D(x) \cdot Q(x) + R(x)$$

dividend divisor quotient remainder

and where $R(x)$ is either zero or of degree less than the degree of $D(x)$.

Remainder and factor theorems

As illustrated by Examples 14 and 15, we commonly divide polynomials of higher degree by linear polynomials. By doing so we can often uncover zeros of polynomials as occurred in Example 15. Let's look at what happens to the division algorithm when the divisor $D(x)$ is a linear polynomial of the form $x - c$. Since the degree of the remainder $R(x)$ must be less than the degree of the divisor (degree of one in this case) then the remainder will be a constant, simply written as R. Then the division algorithm for a linear divisor is the identity:

$$P(x) = (x - c) \cdot Q(x) + R$$

If we evaluate the polynomial function P at the number $x = c$, we obtain

$$P(c) = (c - c) \cdot Q(c) + R = 0 \cdot Q(c) + R = R$$

Thus the remainder R is equal to $P(c)$, the value of the polynomial P at $x = c$. Because this is true for any polynomial P and any linear divisor $x - c$, we have the following theorem.

The remainder theorem

If a polynomial function $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

Example 16

What is the remainder when $g(x) = 2x^3 + 5x^2 - 8x + 3$ is divided by $x + 4$?

Solution

The linear polynomial $x + 4$ is equivalent to $x - (-4)$. Applying the remainder theorem, the required remainder is equal to the value of $g(-4)$.

$$\begin{aligned} g(-4) &= 2(-4)^3 + 5(-4)^2 - 8(-4) + 3 = 2(-64) + 5(16) + 32 + 3 \\ &= -128 + 80 + 35 = -13 \end{aligned}$$

Therefore, when the polynomial function $g(x)$ is divided by $x + 4$ the remainder is -13.

Figure 3.11 Connection between synthetic substitution and long division.

$$\begin{array}{r|rrrr} -4 & 2 & 5 & -8 & 3 \\ & & -8 & 12 & -16 \\ \hline & 2 & -3 & 4 & -13 \end{array} = g(-4)$$

$$\begin{array}{r} 2x^2 - 3x + 4 \\ x + 4 \overline{\smash{)}\,2x^3 + 5x^2 - 8x + 3} \\ \underline{2x^3 + 8x^2} \qquad\quad \\ -3x^2 - 8x \qquad \\ \underline{-3x^2 - 12x} \qquad \\ 4x + 3 \\ \underline{4x + 16} \\ -13 \end{array}$$

We found the value of $g(-4)$ in Example 16 by directly substituting -4 into $g(x)$. Alternatively, we could have used the efficient method of synthetic substitution that we developed in Section 3.1 to evaluate $g(-4)$.

We could also have found the remainder by performing long division, which is certainly the least efficient method. However, there is a very interesting and helpful connection between the process of long division with a linear divisor and synthetic substitution.

Not only does synthetic substitution find the value of the remainder, but the numbers in the bottom row preceding the remainder (shown in red in Figure 3.11) are the same as the coefficients of the quotient (also in red) found from the long division process. Clearly, synthetic substitution is the most efficient method for finding the remainder *and* quotient when dividing a polynomial by a linear polynomial in the form $x - c$. When this method is used to find a quotient and remainder we refer to it as **synthetic division**.

The numbers in the last row of the synthetic substitution process give both the remainder and the coefficients of the quotient when a polynomial is divided by a linear polynomial in the form $x - c$.

A consequence of the remainder theorem is the factor theorem, which also follows intuitively from our discussion in the previous section about the zeros and factors of quadratic functions. It formalizes the relationship between zeros and linear factors of all polynomial functions with real coefficients.

It is important to understand that the factor theorem is a **biconditional** statement of the form 'A if and only if B'. Such a statement is true in either 'direction'; that is, 'If A then B', and also 'If B then A' – usually abbreviated A → B and B → A, respectively.

The factor theorem
A polynomial function $P(x)$ has a factor $x - c$ if and only if $P(c) = 0$.

To illustrate the efficiency of synthetic division, let's answer the same problem posed in Example 14 (solution reproduced in Figure 3.12) in Example 17.

Example 17

Find the quotient $Q(x)$ and remainder $R(x)$ when $P(x) = 2x^3 - 5x^2 + 6x - 3$ is divided by $D(x) = x - 2$.

Solution

Using synthetic division

$$
\begin{array}{c|cccc}
2 & 2 & -5 & 6 & -3 \\
 & & 4 & -2 & 8 \\
\hline
 & \underbrace{2 \quad -1 \quad 4}_{\text{coefficients of the quotient}} & & & 5 \leftarrow \text{remainder}
\end{array}
$$

$$
\begin{array}{rll}
 & 2x^2 - x + 4 & \\
x - 2 \,\big)\! & 2x^3 - 5x^2 + 6x - 3 & \\
 & \underline{2x^3 - 4x^2} & \leftarrow 2x^2(x-2) \\
 & -x^2 + 6x & \leftarrow \text{Subtract} \\
 & \underline{-x^2 + 2x} & \leftarrow -x(x-2) \\
 & 4x - 3 & \leftarrow \text{Subtract} \\
 & \underline{4x - 8} & \leftarrow 4(x-2) \\
 & 5 & \leftarrow \text{Subtract}
\end{array}
$$

The quotient $Q(x)$ is $2x^2 - x + 4$ and the remainder is 5.

Figure 3.12 Solution for Example 14.

Since a divisor of degree 1 is dividing a polynomial of degree 3 then the quotient must be of degree 2 and, with all polynomials written so that their terms are descending in powers (exponents), we know that the numbers in the bottom row of the synthetic division scheme are the coefficients of a quadratic polynomial. Hence, the quotient is $2x^2 - x + 4$ and the remainder is 5.

When one or more zeros of a given polynomial are known, applying the factor theorem and synthetic division is a very effective strategy to aid in finding factors and zeros of the polynomial.

Example 18

Given that $x = -\frac{1}{2}$ and $x = 8$ are zeros of the polynomial function $h(x) = x^4 - \frac{15}{2}x^3 - 30x - 16$, find the other two zeros of $h(x)$.

Solution

From the factor theorem, it follows that $x + \frac{1}{2}$ and $x - 8$ are factors of $h(x)$. Dividing the 4th degree polynomial by the two linear factors in succession will yield a quadratic factor. We can find the zeros of this quadratic factor by using known factorizing techniques or by applying the quadratic formula.

$$
\begin{array}{c|ccccc}
-\frac{1}{2} & 1 & -\frac{15}{2} & 0 & -30 & -16 \\
 & & -\frac{1}{2} & 4 & -2 & 16 \\
\hline
8 & 1 & -8 & 4 & -32 & 0 \\
 & & 8 & 0 & 32 & \\
\hline
 & 1 & 0 & 4 & 0 &
\end{array}
$$

This row shows that $x^4 - \frac{15}{2}x^3 - 30x - 16 = (x + \frac{1}{2})(x^3 - 8x^2 + 4x - 32)$.

This row shows that $x^3 - 8x^2 + 4x - 32 = (x - 8)(x^2 + 4)$.

● Hint: Example 18 indicates that if we divide the quartic polynomial $x^4 - \frac{15}{2}x^3 - 30x - 16$ by $x^2 + 4$ the remainder will be zero, since $x^2 + 4$ is a factor. Synthetic division *only* works for linear divisors of the form $x - c$ so this division could only be done by using the long division process.

Hence, $x^4 - \frac{15}{2}x^3 - 30x - 16 = (x + \frac{1}{2})(x - 8)(x^2 + 4)$.

The zeros of the quadratic factor $x^2 + 4$ must also be zeros of $h(x)$.

$$x^2 + 4 = 0 \Rightarrow x^2 = -4 \Rightarrow x = \pm\sqrt{-4} \Rightarrow x = \pm\sqrt{4}\,\sqrt{-1} \Rightarrow x = \pm 2i$$

Therefore, the other two remaining zeros of $h(x)$ are $x = 2i$ and $x = -2i$.

Note that the two imaginary zeros, $x = 2i$ and $x = -2i$, of the polynomial in Example 18 are a pair of conjugates. In the previous section we asserted that imaginary zeros of a quadratic polynomial always come in conjugate pairs. Although it is beyond the scope of this book to prove it, we will accept that this is true for imaginary zeros of any polynomial.

Conjugate zeros

If a polynomial P has real coefficients, and if the complex number $z = a + bi$ is a zero of P, then its conjugate $z^* = a - bi$ is also a zero of P.

Example 19

Given that $2 - 3i$ is a zero of the polynomial $5x^3 - 19x^2 + 61x + 13$, find all remaining zeros of the polynomial.

Solution

Firstly, we need to consider what is the maximum number of zeros that the cubic polynomial can have. In the previous section we stated that every quadratic polynomial has exactly two complex zeros. It is reasonable to conjecture that a cubic will have three complex zeros. Since $2 - 3i$ is a zero, then $2 + 3i$ must also be a zero; and the third zero must be a real number. Although not explicitly stated in the remainder and factor theorems, both theorems are true for linear polynomials $x - c$ where the number c is real *or* imaginary, i.e. it can be any complex number. Therefore, the cubic polynomial has factors $x - (2 - 3i)$ and $x - (2 + 3i)$. Rather than attempting to divide the cubic polynomial by one of these factors, let's find the product of these factors and use it as a divisor.

$$\begin{aligned}[x - (2 - 3i)][x - (2 + 3i)] &= [x - 2 + 3i][x - 2 - 3i] \\ &= [(x - 2) + 3i][(x - 2) - 3i] \\ &= (x - 2)^2 - (3i)^2 \\ &= x^2 - 4x + 4 - 9i^2 \\ &= x^2 - 4x + 4 + 9 \\ &= x^2 - 4x + 13\end{aligned}$$

We can only use synthetic division with linear divisors, so we will need to divide $5x^3 - 19x^2 + 61x + 13$ by $x^2 - 4x + 13$ using long division.

$$\begin{array}{r} 5x + 1 \\ x^2 - 4x + 13 \overline{\big) 5x^3 - 19x^2 + 61x + 13} \\ \underline{5x^3 - 20x^2 + 65x} \\ x^2 - 4x + 13 \\ \underline{x^2 - 4x + 13} \\ 0 \end{array}$$

Thus, $5x^3 - 19x^2 + 61x + 13$ also has a linear factor of $5x + 1$ and therefore has a zero of $x = -\frac{1}{5}$.

The zeros of the cubic polynomial are:

$x = 2 - 3i$, $x = 2 + 3i$ and $x = -\frac{1}{5}$.

● **Hint:** Although for this course we restrict our study to polynomials with real coefficients, it is worthwhile to note that the statement about the number of complex zeros that exist for a polynomial of degree n also holds true for a polynomial with imaginary coefficients. For example, the 2nd degree polynomial $2ix^2 + 4$ has zeros of $1 + i$ and $-1 - i$ (verify this). Note that these two imaginary zeros are not conjugates. Only if a polynomial's coefficients are real must its imaginary zeros occur in conjugate pairs.

The cubic polynomial in Example 19 had three complex zeros – one real and two imaginary. The quartic polynomial in Example 18 had four complex zeros – two real and two imaginary. In Example 15, we factored a cubic polynomial into a product of three linear polynomials, so the factor theorem says it will have three real zeros. And in the previous section we concluded that, provided we take into account the multiplicity of a zero (e.g. double root), all quadratic polynomials have two complex zeros – either two real zeros or two imaginary zeros. These examples are illustrations of the following useful fact.

Zeros of polynomials of degree *n*

A polynomial of degree $n > 0$ with complex coefficients has exactly n complex zeros, provided that each zero is counted as many times as its multiplicity.

Since imaginary zeros always exist in conjugate pairs then if a polynomial with real coefficients has any imaginary zeros there can only be an even number of them. It logically follows then that a polynomial with an odd degree has at least one real zero. One consequence of this fact is that the graph of an odd-degree polynomial function must intersect the x-axis at least once. This agrees with our claim in Section 3.1 that the end behaviour of a polynomial function is influenced by its degree. Odd-degree polynomial functions will rise as $x \to \infty$ and fall as $x \to -\infty$ (or the other way around if the leading coefficient is negative) producing the same general ∕\/ shape as $y = x^3$, and hence will cross the x-axis at least once.

Example 20

Given that $2x + 1$ is a factor of the cubic function $f(x) = 2x^3 - 15x^2 + 24x + 16$

a) completely factorize the polynomial

b) find all of the zeros and their multiplicities

c) sketch its graph for the interval $-1 \leqslant x \leqslant 6$, given that the graph of the function has a turning point at $x = 1$

Solution

a) Remember that synthetic division can only be used for linear divisors of the form $x - c$. Because $2x + 1 = 2\left(x + \frac{1}{2}\right)$, then if $2x + 1$ is a factor $x + \frac{1}{2}$ is also a factor. So we can set up synthetic division with a divisor of $x + \frac{1}{2}$, but we must take the following into account.

$$\begin{aligned} 2x^3 - 15x^2 + 24x + 16 &= (2x + 1)\cdot Q(x) \\ &= 2(x + \tfrac{1}{2})\cdot Q(x) \\ &= \left(x + \tfrac{1}{2}\right)\cdot 2Q(x) \\ \frac{2x^3 - 15x^2 + 24x + 16}{x + \frac{1}{2}} &= 2Q(x) \end{aligned}$$

When the polynomial is divided by $x + \frac{1}{2}$, the quotient will be two times the quotient from dividing by $2x + 1$. Dividing by two will give us the quotient that we want.

$$\begin{array}{r|rrrr} -\frac{1}{2} & 2 & -15 & 24 & 16 \\ & & -1 & 8 & -16 \\ \hline & 2 & -16 & 32 & 0 \end{array}$$

Hence, $2x^3 - 15x^2 + 24x + 16 = \left(x + \frac{1}{2}\right)(2x^2 - 16x + 32)$

and $2x^3 - 15x^2 + 24x + 16 = 2\left(x + \frac{1}{2}\right)\frac{1}{2}(2x^2 - 16x + 32)$

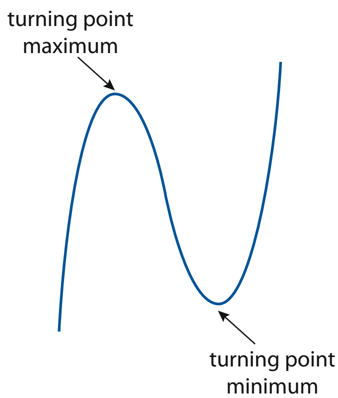
$= (2x + 1)(x^2 - 8x + 16)$ Factorize the quadratic factor.

$= (2x + 1)(x - 4)(x - 4)$ $x^2 - 8x + 16$ fits the pattern $x^2 + 2ax + a^2 = (x + a)^2$

$= (2x + 1)(x - 4)^2$

b) The zeros of $2x^3 - 15x^2 + 24x + 16$ are $x - \frac{1}{2}$ and $x = 4$ (multiplicity of two).

c) Because the polynomial is of degree 3 and its leading coefficient is positive, the end behaviour of the graph will be such that the graph rises as $x \rightarrow \infty$ and falls as $x \rightarrow -\infty$. That means the general shape of the graph will be a $\mathcal{N}$ shape with one maximum and one minimum as shown right.

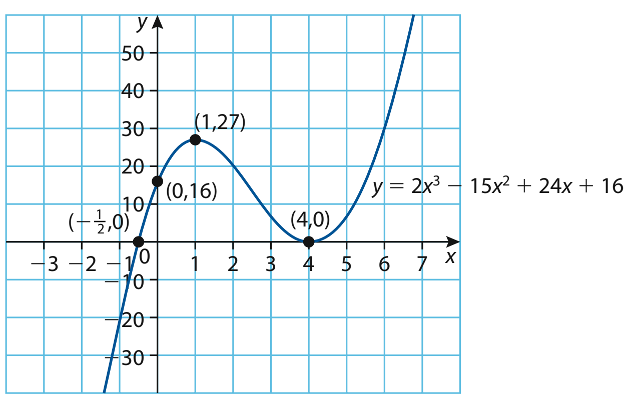


Find the coordinates of the given turning point by evaluating $f(1)$ using synthetic substitution.

$$\begin{array}{r|rrrr} 1 & 2 & -15 & 24 & 16 \\ & & 2 & -13 & 11 \\ \hline & 2 & -13 & 11 & 27 \end{array}$$

$\Rightarrow f(1) = 27$. Hence, the point $(1, 27)$ is on the graph.

Since $f(0) = 16$ then the y-intercept is $(0, 16)$, which means that $(1, 27)$ is a maximum point. Because the zero $x = 4$ has a multiplicity of two, then we know from the previous chapter on quadratic functions that the graph will be tangent to the x-axis at the point $(4, 0)$. The other x-intercept is $\left(-\frac{1}{2}, 0\right)$. We can now make a very accurate sketch of the function.



We know how to find the exact zeros of linear and quadratic functions. The quadratic formula is a general rule that gives the *exact* values of *all* complex zeros of *any* quadratic polynomial using radicals and the coefficients of the polynomial. We also know how to use our GDC to approximate real zeros. In this chapter, we have gained techniques to search for, or verify, the zeros of polynomial functions of degree 3 or higher. This leads us to an important question: Can we find exact values of all complex zeros of any polynomial function of 3rd degree and higher? This question was answered for cubic and quartic polynomials in the 16th century when the Italian mathematician Girolamo Cardano (1501–1576) presented a 'cubic formula' and a 'quartic formula'. These formulae were methods for finding all complex zeros of 3rd degree and 4th degree polynomials using only radicals and coefficients. Cardano's presentation of the formulae depended heavily on the work of other Italian mathematicians. Scipione del Ferro (1465–1526) is given credit as the first to find a general algebraic solution to cubic equations. Cardano's method of solving any cubic was obtained from Niccolo Fontana (1500–1557) known as 'Tartaglia'. Similarly, Cardano solved quartic equations using a method that he learned from his own student Lodovico Ferrari (1522–1565). The methods for solving cubic and quartic equations are quite complicated and are not part of this course. The question of finding formulae for exact zeros of polynomials of degree 5 (quintic) and higher was not resolved until the early 19th century. In 1824, a young Norwegian mathematician, Niels Henrik Abel (1802–1829), proved that it was impossible to find an algebraic formula for a general quintic equation. An even more remarkable discovery was made by the French mathematician Evariste Galois (1811–1832) who died in a pistol duel before turning 21. Galois proved that for any polynomial of degree 5 or greater, it is not possible, except in special cases, to find the exact zeros by using only radicals and the polynomial's coefficients. Mathematicians have developed sophisticated methods of approximating the zeros of polynomial equations of high degree and other types of equations for which there are no algebraic solution methods. These are studied in a branch of advanced mathematics called **numerical analysis**.

Example 21

Find a polynomial P with integer coefficients of least degree having zeros of $x = 2$, $x = -\frac{1}{3}$ and $x = 1 - i$.

Solution

Given that $1 - i$ is a zero then its conjugate $1 + i$ must also be a zero. Thus, the required polynomial has four complex zeros, and four corresponding factors. The four factors are:

$x - 2$, $x + \frac{1}{3}$, $x - (1 - i)$ and $x - (1 + i)$

$$P(x) = (x-2)\left(x+\tfrac{1}{3}\right)[x-(1-i)][x-(1+i)]$$
$$= \left(x^2 - \tfrac{5}{3}x - \tfrac{2}{3}\right)[(x-1)+i][(x-1)-i]$$

Multiplying by 3 does not change the zeros …

$$= (3x^2 - 5x - 2)[(x-1)^2 - i^2]$$

… but does guarantee integer coefficients.

$$= (3x^2 - 5x - 2)(x^2 - 2x + 1 + 1)$$
$$= (3x^2 - 5x - 2)(x^2 - 2x + 2)$$
$$= 3x^4 - 6x^3 + 6x^2 - 5x^3 + 10x^2 - 10x - 2x^2 + 4x - 4$$
$$P(x) = 3x^4 - 11x^3 + 14x^2 - 6x - 4$$

There is a theorem called the **fundamental theorem of algebra** that guarantees that *every* polynomial function of non-zero degree with complex coefficients has at least one complex zero. The theorem was first proved by the famous German mathematician Carl Friedrich Gauss (1777–1855). Many of the results in this section on the zeros of polynomials are directly connected with this important theorem.

Sum and product of the roots of any polynomial equation

In the previous section, we found a way to express the sum and product of the roots of a quadratic equation, $ax^2 + bx + c = 0$, in terms of a, b and c. It is natural to wonder whether a similar method could be found for polynomial equations of degree greater than two.

Using the same approach as in the previous section for quadratic equations, let's consider the general cubic equation $ax^3 + bx^2 + cx + d = 0$ whose roots are $x = \alpha$, $x = \beta$ and $x = \gamma$. It follows that this general cubic equation can be written in the form $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$. Applying the Factor Theorem, it can also be written in the form $(x - \alpha)(x - \beta)(x - \gamma) = 0$. Expanding the brackets gives:

$$(x - \alpha)(x - \beta)(x - \gamma) = x^3 - \alpha x^2 - \beta x^2 - \gamma x^2 + \alpha\beta x + \beta\gamma x + \alpha\gamma x - \alpha\beta\gamma = 0$$

$$x^3 - (\alpha + \beta + \gamma)\, x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$$

Equating coefficients for $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$ and $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$ gives us the following results for the sum and product of the roots for any cubic equation.

$$\alpha + \beta + \gamma = -\frac{b}{a} \text{ and } \alpha\beta\gamma = -\frac{d}{a}$$

This result for the sum and product of the roots of any cubic equation looks very similar to that for any quadratic equation. The only difference is that the product of the roots, $\alpha\beta\gamma$, is the opposite of the quotient $\frac{\text{constant term}}{\text{leading coefficient}}$.

For the general quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ with roots α, β, γ and δ, the factored form of the equation expands as follows:

$$(x - \alpha)(x - \beta)(x - \gamma)(x - \delta) =$$
$$x^4 - (\alpha + \beta + \gamma + \delta)x^2 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) + \alpha\beta\gamma\delta = 0$$

Since this is equivalent to $x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0$, then the sum and product of the roots for any quartic equation are:

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} \text{ and } \alpha\beta\gamma\delta = \frac{e}{a}.$$

These results for the sum and product of roots for polynomial equations of degree 2 (quadratic), degree 3 (cubic) and degree 4 (quartic) lead to the following result for any polynomial function of degree n that we state without a formal proof.

Sum and product of the roots (zeros) of any polynomial equation

For the **polynomial equation of degree *n*** given by $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$, $a_n \neq 0$ the **sum of the roots** is $-\frac{a_{n-1}}{a_n}$ and the **product of the roots** is $\frac{(-1)^n a_0}{a_n}$.

Example 22

Two of the roots of the equation $x^3 - 3x^2 + kx + 75 = 0$ are opposites. Find the values of all the roots and the constant k.

Solution

Let the three unknown roots be represented by α, $-\alpha$ and β.

Then $\alpha - \alpha + \beta = 3 \Rightarrow \beta = 3$ and $\alpha(-\alpha)\beta = -75 \Rightarrow \alpha(-\alpha)(3) = -75 \Rightarrow -3\alpha^2 = -75 \Rightarrow \alpha^2 = 25 \Rightarrow \alpha = \pm 5$

Therefore, the three roots are 5, -5 and 3.

To find the value of k, write the cubic in factored form and expand.

$$\begin{aligned}(x-3)(x+5)(x-5) = 0 &\Rightarrow (x-3)(x^2-25) = 0\\ &\Rightarrow x^3 - 3x^2 - 25x + 75 = 0\end{aligned}$$

Therefore, $k = -25$.

Example 23

Consider the equation $2x^4 - x^3 - 4x^2 + 10x - 4 = 0$. Given that one of the zeros of the equation is $r_1 = 1 + i$, find the other three zeros r_2, r_3 and r_4.

Solution

There are other strategies (e.g. using factors and polynomial division) but it is more efficient to apply what we know about the sum and product of the roots (zeros) of a polynomial equation.

Firstly, since $r_1 = 1 + i$ is a zero, then its conjugate must also be a zero; hence $r_2 = 1 - i$.

From the fact that the sum of the roots is $-\frac{a_{n-1}}{a_n}$, then $r_1 + r_2 + r_3 + r_4 = -\frac{a_3}{a_4}$.

Substituting in known values gives $1 + i + 1 - i + r_3 + r_4 = -\frac{-1}{2}$

$\Rightarrow 2 + r_3 + r_4 = \frac{1}{2} \Rightarrow r_3 + r_4 = -\frac{3}{2}$

Also, since the product of the roots is $\frac{(-1)^n\, a_0}{a_n}$, then $r_1r_2r_3r_4 = \frac{(-1)^n\, a_0}{a_n}$.

Substituting gives:

$$\begin{aligned}(1+i)(1-i)r_3r_4 = \frac{(-1)^4(-4)}{2} &\Rightarrow (1-i^2)r_3r_4 = -2\\ &\Rightarrow 2r_3r_4 = -2\\ &\Rightarrow r_3r_4 = -1\end{aligned}$$

To find r_3 and r_3, we need to use the pair of equations $\begin{cases} r_3 + r_4 = -\frac{3}{2} \\ r_3r_4 = -1 \end{cases}$

Solving for r_3 in the first equation gives $r_3 = -r_4 - \frac{3}{2}$.

Substituting into the other equation gives: $\left(-r_4 - \frac{3}{2}\right) r_4 = -1$

$$\begin{aligned}&\Rightarrow r_4^{\,2} + \frac{3}{2}r_4 - 1 = 0\\ &\Rightarrow 2r_4^{\,2} + 3r_4 - 2 = 0\\ &\Rightarrow (2r_4 - 1)(r_4 + 2) = 0\\ &\Rightarrow r_4 = \frac{1}{2} \text{ or } r_4 = -2\end{aligned}$$

If $r_4 = \frac{1}{2}$, then $r_3 = -\frac{1}{2} - \frac{3}{2} = -2$. $\left[\text{And if } r_4 = -2, \text{ then } r_3 = \frac{1}{2}\right]$

Therefore the other three zeros are $1 - i$, $\frac{1}{2}$ and -2.

Exercise 3.3

In questions 1–5, two polynomials P and D are given. Use either synthetic division or long division to divide $P(x)$ by $D(x)$, and express $P(x)$ in the form $P(x) = D(x) \cdot Q(x) + R(x)$.

1 $P(x) = 3x^2 + 5x - 5,\ D(x) = x + 3$

2 $P(x) = 3x^4 - 8x^3 + 9x + 5,\ D(x) = x - 2$

3 $P(x) = x^3 - 5x^2 + 3x - 7,\ D(x) = x - 4$

4 $P(x) = 9x^3 + 12x^2 - 5x + 1,\ D(x) = 3x - 1$

5 $P(x) = x^5 + x^4 - 8x^3 + x + 2,\ D(x) = x^2 + x - 7$

6 Given that $x - 1$ is a factor of the function $f(x) = 2x^3 - 17x^2 + 22x - 7$ factorize f completely.

7 Given that $2x + 1$ is a factor of the function $f(x) = 6x^3 - 5x^2 - 12x - 4$ factorize f completely.

8 Given that $x + \frac{2}{3}$ is a factor of the function $f(x) = 3x^4 + 2x^3 - 36x^2 + 24x + 32$ factorize f completely.

In questions 9–12, find the quotient and the remainder.

9 $\dfrac{x^2 - 5x + 4}{x - 3}$

10 $\dfrac{x^3 + 2x^2 + 2x + 1}{x + 2}$

11 $\dfrac{9x^2 - x + 5}{3x^2 - 7x}$

12 $\dfrac{x^5 + 3x^3 - 6}{x - 1}$

In questions 13–16, use synthetic division and the remainder theorem to evaluate $P(c)$.

13 $P(x) = 2x^3 - 3x^2 + 4x - 7,\ c = 2$

14 $P(x) = x^5 - 2x^4 + 3x^2 + 20x + 3,\ c = -1$

15 $P(x) = 5x^4 + 30x^3 - 40x^2 + 36x + 14,\ c = -7$

16 $P(x) = x^3 - x + 1,\ c = \frac{1}{4}$

17 Given that $x = -6$ is a zero of the polynomial $x^3 + 2x^2 - 19x + 30$ find all remaining zeros of the polynomial.

18 Given that $x = 2$ is a double root of the polynomial $x^4 - 5x^3 + 7x^2 - 4$ find all remaining zeros of the polynomial.

19 Find the values of k such that -3 is a zero of $f(x) = x^3 - x^2 - k^2x$.

20 Find the values of a and b such that 1 and 4 are zeros of $f(x) = 2x^4 - 5x^3 - 14x^2 + ax + b$.

In questions 21–23, find a polynomial with real coefficients satisfying the given conditions.

21 Degree of 3; and zeros of -2, 1 and 4

22 Degree of 4; and zeros of -1, 3 (multiplicity of 2) and -2

23 Degree of 3; and 2 is the only zero (multiplicity of 3)

In questions 24–26, find a polynomial of lowest degree with real coefficients and the given zeros.

24 $x = -1$ and $x = 1 - i$

25 $x = 2$, $x = -4$ and $x = -3i$

26 $x = 3 + i$ and $x = 1 - 2i$

27 Given that $x = 2 - 3i$ is a zero of $f(x) = x^3 - 7x^2 + 25x - 39$ find the other remaining zeros.

28 The polynomial $6x^3 + 7x^2 + ax + b$ has a remainder of 72 when divided by $x - 2$ and is exactly divisible (i.e. remainder is zero) by $x + 1$.

a) Calculate a and b.
b) Show that $2x - 1$ is also a factor of the polynomial and, hence, find the third factor.

29 The polynomial $p(x) = (ax + b)^3$ leaves a remainder of -1 when divided by $x + 1$, and a remainder of 27 when divided by $x - 2$. Find the values of the real numbers a and b.

30 The quadratic polynomial $x^2 - 2x - 3$ is a factor of the quartic polynomial function $f(x) = 4x^4 - 6x^3 - 15x^2 - 8x - 3$. Find all of the zeros of the function f. Express the zeros exactly and completely simplified.

31 $x - 2$ and $x + 2$ are factors of $x^3 + ax^2 + bx + c$, and it leaves a remainder of 10 when divided by $x - 3$. Find the values of a, b and c.

32 Let $P(x) = x^3 + px^2 + qx + r$. Two of the zeros of $P(x) = 0$ are 3 and $1 + 4i$. Find the value of p, q and r.

33 When divided by $(x + 2)$ the expression $5x^3 - 3x^2 + ax + 7$ leaves a remainder of R. When the expression $4x^3 + ax^2 + 7x - 4$ is divided by $(x + 2)$ there is a remainder of $2R$. Find the value of the constant a.

34 The polynomial $x^3 + mx^2 + nx - 8$ is divisible by $(x + 1 + i)$. Find the value of m and n.

35 Given that the roots of the equation $x^3 - 9x^2 + bx - 216 = 0$ are consecutive terms in a geometric sequence, find the value of b and solve the equation.

36 a) Prove that when a polynomial $P(x)$ is divided by $ax - b$ the remainder is $P\left(\frac{b}{a}\right)$.
b) Hence, find the remainder when $9x^3 - x + 5$ is divided by $3x + 2$.

37 Find the sum and product of the roots of the following equations.

a) $x^4 - \frac{2}{3}x^3 + 3x^2 - 2x + 5 = 0$
b) $(x - 2)^3 = x^4 - 1$
c) $\dfrac{3}{x^2 + 2} = \dfrac{2x^2 - x}{2x^5 + 1}$

38 If α, β and γ are the three roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, show that $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$.

39 One of the zeros of the equation $x^3 - 63x + 162 = 0$ is double another zero. Find all three zeros.

40 Find the three zeros of the equation $x^3 - 6x^2 - 24x + 64 = 0$ given that they are consecutive terms in a geometric sequence. [Hint: let the zeros be represented by $\frac{\alpha}{r}$, α, αr where r is the common ratio.]

41 Consider the equation $x^5 - 12x^4 + 62x^3 - 166x^2 + 229x - 130 = 0$. Given that two of the zeros of the equation are $x = 3 - 2i$ and $x = 2$, find the remaining three zeros.

42 Find the value of k such that the zeros of the equation $x^3 - 6x^2 + kx + 10 = 0$ are in arithmetic progression, that is, they can be represented by α, $\alpha + d$ and $\alpha + 2d$ for some constant d. [Hint: use the result from question 38.]

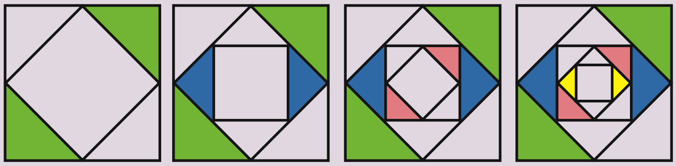
43 Find the value of k if the roots of the equation $x^3 + 3x^2 - 6x + k = 0$ are in geometric progression.

28 A ball is dropped from a height of 16 m. Every time it hits the ground it bounces 81% of its previous height.

a) Find the maximum height it reaches *after* the 10th bounce.

b) Find the total distance travelled by the ball till it rests. (Assume no friction and no loss of elasticity).

29

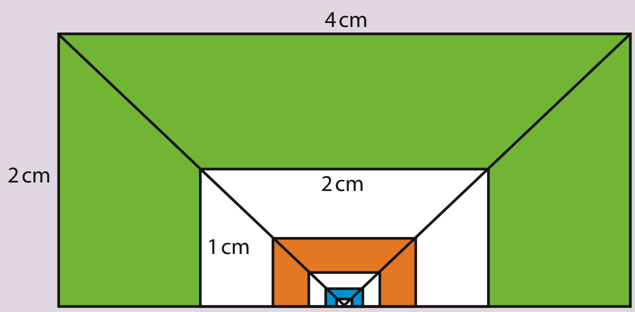


The sides of a square are 16 cm in length. A new square is formed by joining the midpoints of the adjacent sides and two of the resulting triangles are coloured as shown.

a) If the process is repeated 6 more times, determine the total area of the shaded region.

b) If the process is repeated indefinitely, find the total area of the shaded region.

30



The largest rectangle has dimensions 4 by 2, as shown; another rectangle is constructed inside it with dimensions 2 by 1. The process is repeated. The region surrounding every other inner rectangle is shaded, as shown.

a) Find the total area for the three regions shaded already.

b) If the process is repeated indefinitely, find the total area of the shaded regions.

In questions 31–34, find each sum.

31 $7 + 12 + 17 + 22 + \ldots + 337 + 342$

32 $9486 + 9479 + 9472 + 7465 + \ldots + 8919 + 8912$

33 $2 + 6 + 18 + 54 + \ldots + 3\,188\,646 + 9\,565\,938$

34 $120 + 24 + \frac{24}{5} + \frac{24}{25} + \ldots + \frac{24}{78\,125}$

4.5 Counting principles

Simple counting problems

This section will introduce you to some of the basic principles of counting. In Section 4.6 you will apply some of this in justifying the binomial theorem and in Chapter 12 you will use these principles to tackle many probability problems. We will start with two examples.

Example 21

Nine paper chips each carrying the numerals 1–9 are placed in a box. Two chips are chosen such that the first chip is chosen, the number is recorded and the chip is put back in the box, then the second chip is drawn. The numbers on the chips are added. In how many ways can you get a sum of 8?

Solution

To solve this problem, count the different number of ways that a total of 8 can be obtained:

1st chip	1	2	3	4	5	6	7
2nd chip	7	6	5	4	3	2	1

From this list, it is clear that you can have 7 different ways of receiving a sum of 8.

Example 22

Suppose now that the first chip is chosen, the number is recorded and the chip is *not* put back in the box, then the second chip is drawn. In how many ways can you get a sum of 8?

Solution

To solve this problem too, count the different number of ways that a total of 8 can be obtained:

1st chip	1	2	3	5	6	7
2nd chip	7	6	5	3	2	1

From this list, it is clear that you can have 6 different ways of receiving a sum of 8.

The difference between the two situations is described by saying that the first random selection is done **with replacement**, while the second is **without replacement**, which ruled out the use of two 4s.

Fundamental principle of counting

The above examples show you simple counting principles in which you can list each possible way that an event can happen. In many other cases, listing the ways an event can happen may not be feasible. In such cases we need to rely on counting principles. The most important of which is the **fundamental principle of counting**, also known as the multiplication principle. Consider the following situations:

Example 23

You can make a sandwich from one of three types of bread and one of four kinds of cheese, with or without pickles. How many different kinds of sandwiches can be made?

Solution

With each type of bread you can have 4 sandwiches. There are 12 possible sandwiches altogether. These are without pickles; if you want sandwiches with pickles, then you have 24 possible ones. That is, there are $3 \times 4 \times 2 = 24$ possible sandwiches.

Example 24

How many 3-digit even numbers are there?

Solution

The first digit cannot be zero, since the number has to be a 3-digit number, so there are 9 ways the hundred's digit can be. There is no condition on what the ten's digit should be, so we have 10 possibilities, and to be even, the number must end with 0, 2, 4, 6, or 8. Therefore, we have $9 \times 10 \times 5 = 450$ 3-digit even numbers.

Examples 23 and 24 are examples of the following principle:

Fundamental principle of counting

If there are m ways an event can occur followed by n ways a second event can occur, then there are a total of $(m)(n)$ ways that the two can occur.

This principle can be extended to more than two events or processes:

If there are k events than can happen in $n_1, n_2, \ldots, n_k$ ways, then the whole sequence can happen in

$$n_1 \times n_2 \times \ldots \times n_k \text{ ways.}$$

Example 25

A large school issues special coded identification cards that consist of two letters of the alphabet followed by three numerals. For example, AB 737 is such a code. How many different ID cards can be issued if the letters or numbers can be used more than once?

Solution

As the letters can be used more than once, then each letter position can be filled in 26 different ways, i.e. the letters can be filled in $26 \times 26 = 676$ ways. Each number position can be filled in 10 different ways; hence, the numerals can be filled in $10 \times 10 \times 10 = 1000$ different ways. So, the code can be formed in $676 \times 1000 = 676\,000$ different ways.

Permutations

One major application of the fundamental principle is in determining the number of ways the n objects can be arranged. Consider the following situation for example. You have 5 books you want to put on a shelf: maths (M), physics (P), English (E), biology (B), and history (H). In how many ways can you do this?

To find this out, number the positions you want to place the books in as shown

1	2	3	4	5

If we decide to put the maths book in position 1, then there are four different ways of putting a book in position 2.

M	P				M	E				M	B				M	H			
1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5

Since we can put any of the 5 books in the first position, then there will be $5 \times 4 = 20$ ways of shelving the first two books. Once you place the books in positions 1 and 2, the third book can be any one of three books left.

M	P	E			M	P	B			M	P	H		
1	2	3	4	5	1	2	3	4	5	1	2	3	4	5

Once you use three books, there are two books for the fourth position and only one way of placing the fifth book. So, the number of ways of arranging all 5 books is

$$5 \times 4 \times 3 \times 2 \times 1 = 120 = 5!$$

Factorial notation

The product of the first n positive integers is denoted by $n!$ and is called ***n* factorial:**

$$n! = 1 \times 2 \times 3 \times 4 \ldots (n-2) \times (n-1) \times n$$

We also define $0! = 1$.

Permutations

An arrangement is called a **permutation**. It is the reorganization of objects or symbols into distinguishable sequences. When we place things in order, we say we have made an arrangement. When we change the order, we say we have changed the arrangement. So each of the arrangements that can be made by taking *some* or *all* of a number of things is known as a **permutation**.

● **Hint:** A permutation of n different objects can be understood as an ordering (arrangement) of the objects such that one object is first, one is second, one is third, and so on.

Number of permutations of *n* objects

The previous set up can be applied to n objects rather than only 5. The number of ways of filling in the first position can be done in n ways.

n	$n-1$	$n-2$	$n-3$	…	1
1	2	3	4		n

Once the first position is filled, the second position can be filled by any of the $n-1$ objects left, and hence using the fundamental principle there will be $n \cdot (n-1)$ different ways for filling the first two positions. Repeating the same procedure till the nth position is filled is therefore

$$n \cdot (n-1) \cdot (n-2) \ldots 2 \cdot 1 = n!$$

Frequently, we are engaged in arranging a **subset** of the whole collection rather than the entire collection. For example, suppose we want to shelve 3 of the books rather than all 5 of them. The discussion will be analogous to the previous situation. However, we have to limit our search to the first three positions only, i.e. the number of ways we can shelve three out of the 5 books is

$$5 \times 4 \times 3 = 60$$

To change this product into factorial notation, we do the following:

$$5 \times 4 \times 3 = 5 \times 4 \times 3 \times \frac{2!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2!} = \frac{5!}{2!}$$
$$= \frac{5!}{(5-3)!}$$

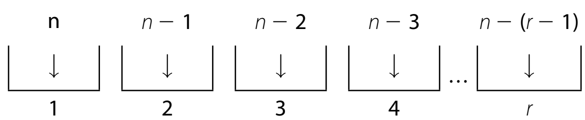
This leads us to the following general result.

Number of permutations of *n* objects taken *r* at a time

The number of permutations of n objects taken r at a time is

$$^nP_r = {}_nP_r = P_r^n = P(n, r) = \frac{n!}{(n-r)!}, \ n \geqslant r$$

To verify the formula above, you can proceed in the same manner as with the permutation of n objects.



When you arrive to the rth position, you would have used $r - 1$ objects already, and hence you are left with $n - (r - 1) = n - r + 1$ objects to fill this position. So, the number of ways of arranging n objects taken r at a time is

$$^nP_r = n \cdot (n-1) \cdot (n-2) \ldots (n-r+1)$$

Here again, to make the expression more manageable, we can write it in factorial notation:

$$\begin{aligned} ^nP_r &= n \cdot (n-1) \cdot (n-2) \ldots (n-r+1) \\ &= n \cdot (n-1) \cdot (n-2) \ldots (n-r+1)\frac{(n-r)!}{(n-r)!} \\ &= \frac{n \cdot (n-1) \cdot (n-2) \ldots (n-r+1) \cdot (n-r)!}{(n-r)!} = \frac{n!}{(n-r)!} \end{aligned}$$

Example 26

15 drivers are taking part in a Formula 1 car race. In how many different ways can the top 6 positions be filled?

Solution

Since the drivers are all different, this is a permutation of 15 'objects' taken 6 at a time.

$$^{15}P_6 = \frac{15!}{(15-6)!} = 3\,603\,600$$

This can also be easily calculated using a GDC.

```
15 nPr 6
                3603600
15!/9!
                3603600
```

Combinations

A **combination** is a selection of some or all of a number of different objects. It is an unordered collection of unique sizes. In a permutation, the order of occurrence of the objects or the arrangement is important, but in combination the order of occurrence of the objects is not important. In that sense, a combination of r objects out of n objects is a subset of the set of n objects.

For example, there are 24 permutations of three letters out of ABCD, while there are only 4 combinations! Here is why:

ABC	ABD	ACD	BCD
ACB	ADB	ADC	BDC
BAC	BAD	CAD	CBD
BCA	BDA	CDA	CDB
CAB	DAB	DAC	DBC
CBA	DBA	DCA	DCB

For one combination, ABC for example, there are $3! = 6$ permutations. This is true for all combinations. So, the number of permutations is 6 times the number of combinations, i.e.

$$^4P_3 = 3!\,{}^4C_3$$

where $^4C_3$ is the number of combinations of the 4 letters taken 3 at a time.

According to the previous result, we can write

$$^4C_3 = \frac{^4P_3}{3!} = \frac{\frac{4!}{(4-3)!}}{3!} = \frac{4!}{3!(4-3)!}$$

The last result can also be generalized to n elements combined r at a time. (The ISO notation for this quantity, which is also used by the IB is $\binom{n}{r}$. In this book, we will follow the ISO notation.)

Every subset of r objects (combination), gives rise to $r!$ permutations. So, if you have $\binom{n}{r}$ combinations, these will result in $r!\binom{n}{r}$ permutations. Therefore,

$$^nP_r = r!\binom{n}{r} \Leftrightarrow \binom{n}{r} = \frac{^nP_r}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!r!}$$

$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \binom{n}{n-r}$. This symmetry is obvious as when we pick r objects, we leave $n - r$ objects behind, and hence the number of ways of choosing r objects is the same as the number of ways of $n - r$ objects not chosen.

```
45 nCr 6
          8145060
```

Example 27

A lottery has 45 numbers. If you buy a ticket, then you choose 6 of these numbers. How many different choices does this lottery have?

Solution

Since 6 numbers will have to be chosen and order is not an issue here, this is a combination case. The number of possible choices is

$$\binom{45}{6} = 8\ 145\ 060.$$

This can also be calculated using a GDC.

Example 28

In poker, a deck of 52 cards is used, and a 'hand' is made up of 5 cards.

a) How many hands are there?

b) How many hands are there with 3 diamonds and 2 hearts?

Solution

a) Since the order is not important, as a player can reorder the cards after receiving them, this is a combination of 52 cards taken 5 at a time:
$\binom{52}{5} = 2\ 598\ 960.$

```
52 nCr 5
          2598960
13 nCr 3
              286
13 nCr 2
               78
```

b) Since there are 13 diamonds and we want 3 of them, there are $\binom{13}{3} = 286$ ways to get the 3 diamonds. Since there are 13 hearts and we want 2 of them, there are $\binom{13}{2} = 78$ ways to get the 2 hearts. Since we want them both to occur at the same time, we use the fundamental counting principle and multiply 286 and 78 together to get 22 308 possible hands.

Example 29

A code is made up of 6 different digits. How many possible codes are there?

Solution

Since there are 10 digits and we are choosing 6 of them, and since the order we use these digits makes a difference in the code, then this is a permutation case. The number of possible codes is

$${}^{10}P_6 = 151\ 200.$$

Exercise 4.5

1 Evaluate each of the following expressions.

a) ${}^{5}P_5$ b) $5!$ c) ${}^{20}P_1$ d) ${}^{8}P_3$

2 Evaluate each of the following expressions.

a) $\binom{5}{5}$ b) $\binom{5}{0}$ c) $\binom{10}{3}$ d) $\binom{10}{7}$

3 Evaluate each of the following expressions.

a) $\binom{7}{3} + \binom{7}{4}$ b) $\binom{8}{4}$ c) $\binom{10}{6} + \binom{10}{7}$ d) $\binom{11}{7}$

4 Evaluate each of the following expressions.

a) $\binom{8}{5} - \binom{8}{3}$ b) $11 \cdot 10!$ c) $\binom{10}{3} - \binom{10}{7}$ d) $\binom{10}{1}$

5 Tell whether each of the following expressions is true.

a) $\frac{10!}{5!} = 2!$ b) $(5!)^2 = 25!$ c) $\binom{101}{8} = \binom{101}{93}$

6 You are buying a computer and have the following choices: three types of HD, two types of DVD players, four types of graphic cards. How many different systems can you choose from?

7 You are going to a restaurant with a set menu. They have three starters, four main meals, two drinks, and three deserts. How many different choices are available for you to choose your meal from?

8 A school is in need of three teachers: PE, maths, and English. They have 8 applicants for the PE position, 3 applicants for the maths position and 13 applicants for English. How many different combinations of choices do they have?

9 You are given a multiple choice test where each question has four possible answers. The test is made up of 12 questions and you are guessing at random. In how many ways can you answer all the questions on the test?

10 The test in question 9 is divided into two parts, the first six are true/false questions and the last six are multiple choice as described. In how many different ways can you answer all questions on that test?

11 Passwords on a network are made up of two parts. One part consists of three letters of the alphabet, not necessarily different, and five digits, also not necessarily different. How many passwords are possible on this network?

12 How many 5-digit numbers can be made if the units digit cannot be 0?

13 Four couples are to be seated in a theatre row. In how many different ways can they be seated if

a) no restrictions are made

b) every two members of each couple like to sit together?

14 Five girls and three boys should go through a doorway in single file. In how many orders can they do that if

a) there are no constraints

b) the girls must go first?

15 Write all the permutations of the letters in JANE.

16 Write all the permutations of the letters in MAGIC taken three at a time.

17 A computer code is made up of three letters followed by four digits.

a) In how many ways is the code possible?

b) If 97 of the three-letter combinations cannot be used because they are offensive, how many codes are still possible?

18 A local bridge club has 17 members, 10 females and 7 males. They have to elect three officers: president, deputy, and treasurer. In how many ways is this possible if

a) there are no restrictions

b) the president is a male

c) the deputy must be a male, the president can be any gender, but the treasurer must be a female

d) the president and deputy are of the same gender

e) all three officers are not the same gender.

19 The research and development department for a computer manufacturer has 26 employees: 8 mathematicians, 12 computer scientists, and 6 electrical engineers. They need to select three employees to be leaders of the group. In how many ways can they do this if

a) the three officers are of the same specialization

b) at least one of them must be an engineer

c) two of them must be mathematicians?

20 A 'combination' lock has three numbers, each in the range 1 to 50.

a) How many different combinations are possible?

b) How many combinations do not have duplicates?

c) How many have the first and second numbers matching?

d) How many have exactly two of the numbers matching?

21 In how many ways can five married couples be seated around a circle so that spouses sit together?

22 a) How many subsets of $\{1, 2, 3, \dots, 9\}$ have two elements?

b) How many subsets of $\{1, 2, 3, \dots, 9\}$ have an odd number of elements?

23 Nine seniors and 12 juniors make up the maths club at a school. They need four members for an upcoming competition.

a) How many 4-member teams can they form?

b) How many of these 4-member teams have the same number of juniors and seniors?

c) How many of these 4-member teams have more juniors than seniors?

24 This problem uses the same data as question 23 above. Tim, a junior, is the strongest 'mathlete' among his group while senior Gwen is the strongest among her group. Either Tim or Gwen must be on the team, but they cannot both be on the team. Answer the same questions as above.

25 A shipment of 100 hard disks contains 4 defective disks. We choose a sample of 6 disks for inspection.

a) How many different possible samples are there?

b) How many samples could contain all 4 defective disks? What percentage of the total is that?

c) How many samples could contain at least 1 defective disk? What percentage of the total is that?

26 There are three political parties represented in a parliament: 10 conservatives, 8 liberals, and 4 independents. A committee of 6 members is needed to be set up.

a) How many different committees are possible?

b) How many committees with equal representation are possible?

27 How many ways are there for 9 boys and 6 girls to stand in a line so that no two girls stand next to each other?

4.6 The binomial theorem

A binomial is a polynomial with two terms. For example, $x + y$ is a binomial. In principle, it is easy to raise $x + y$ to any power; but raising it to high powers would be tedious. We will find a formula that gives the expansion of $(x + y)^n$ for any positive integer n. The proof of the binomial theorem is given in Section 4.7.

Let us look at some special cases of the expansion of $(x + y)^n$:

$$(x + y)^0 = 1$$
$$(x + y)^1 = x + y$$
$$(x + y)^2 = x^2 + 2xy + y^2$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$
$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

There are several things that you will have noticed after looking at the expansion:

- There are $n + 1$ terms in the expansion of $(x + y)^n$.
- The degree of each term is n.
- The powers on x begin with n and decrease to 0.
- The powers on y begin with 0 and increase to n.
- The coefficients are symmetric.

For instance, notice how the exponents of x and y behave in the expansion of $(x + y)^5$.

The exponents of x decrease:

$$(x + y)^5 = x^{\boxed{5}} + 5x^{\boxed{4}}y + 10x^{\boxed{3}}y^2 + 10x^{\boxed{2}}y^3 + 5x^{\boxed{1}}y^4 + x^{\boxed{0}}y^5$$

The exponents of y increase:

$$(x + y)^5 = x^5y^{\boxed{0}} + 5x^4y^{\boxed{1}} + 10x^3y^{\boxed{2}} + 10x^2y^{\boxed{3}} + 5xy^{\boxed{4}} + y^{\boxed{5}}$$

Using this pattern, we can now proceed to expand any binomial raised to power n: $(x + y)^n$. For example, leaving a blank for the missing coefficients, the expansion for $(x + y)^7$ can be written as

$$(x + y)^7$$
$$= \square x^7 + \square x^6y + \square x^5y^2 + \square x^4y^3 + \square x^3y^4 + \square x^2y^5 + \square xy^6 + \square y^7$$

To finish the expansion we need to determine these coefficients. In order to see the pattern, let us look at the coefficients of the expansion we started the section with.

$(x+y)^0$	1							row 0
$(x+y)^1$	1	1						row 1
$(x+y)^2$	1	2	1					row 2
$(x+y)^3$	1	3	3	1				row 3
$(x+y)^4$	1	4	6	4	1			row 4
$(x+y)^5$	1	5	10	10	5	1		row 5
$(x+y)^6$	1	6	15	20	15	6	1	row 6
	column 0	column 1	column 2	column 3	column 4	column 5	column 6	

A triangle like the one above is known as Pascal's triangle. Notice how the first and **second** terms in row **3** give you the **second** term in row **4**; the third and **fourth** terms in row **3** give you the **fourth** term of row **4**; the second and **third** terms in row **5** give you the **third** term in row **6**; and the fifth and **sixth** terms in row **5** give you the **sixth** term in row **6**, and so on. So now we can state the key property of Pascal's triangle.

Pascal's triangle was known to Persian and Chinese mathematicans in the 13th century.

Pascal's triangle

Every entry in a row is the sum of the term directly above it and the entry diagonally above and to the left of it. When there is no entry, the value is considered zero.

Take the last entry in row 5, for example; there is no entry directly above it, so its value is $0 + 1 = 1$.

From this property it is easy to find all the terms in any row of Pascal's triangle from the row above it. So, for the expansion of $(x + y)^7$, the terms are found from row 6 as follows:

$$\begin{array}{ccccccccccccccccc}
0 & \to & 1 & \to & 6 & \to & 15 & \to & 20 & \to & 15 & \to & 6 & \to & 1 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

So, $(x+y)^7 = x^7 + \boxed{7}\,x^6y + \boxed{21}\,x^5y^2 + \boxed{35}\,x^4y^3 + \boxed{35}\,x^3y^4 + \boxed{21}\,x^2y^5 + \boxed{7}\,xy^6 + y^7$.

Note: Several sources use a slightly different arrangement for Pascal's triangle. The common usage considers the triangle as isosceles and uses the principle that every two entries add up to give the entry diagonally below them, as shown in the following diagram.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Example 30

Use Pascal's triangle to expand $(2k - 3)^5$.

Solution

We can find the expansion above by replacing x by $2k$ and y by -3 in the binomial expansion of $(x + y)^5$.

Using the fifth row of Pascal's triangle for the coefficients will give us the following:

$$\mathbf{1}(2k)^5 + \mathbf{5}(2k)^4(-3) + \mathbf{10}(2k)^3(-3)^2 + \mathbf{10}(2k)^2(-3)^3 + \mathbf{5}(2k)(-3)^4 + \mathbf{1}(-3)^5 = 32k^5 - 240k^4 + 720k^3 - 1080k^2 + 810k - 243.$$

Pascal's triangle is an easy and useful tool in finding the coefficients of the binomial expansion for relatively small values of n. It is not very efficient doing that for large values of n. Imagine you want to evaluate $(x + y)^{20}$. Using Pascal's triangle, you will need the terms in the 19th row and the 18th row and so on. This makes the process tedious and not practical.

Luckily, we have a formula that can find the coefficients of any Pascal's triangle row. This formula is the binomial formula, whose proof is beyond the scope of this book. Every entry in Pascal's triangle is denoted by $\binom{n}{r}$, which is also known as the binomial coefficient.

In $\binom{n}{r}$, n is the row number and r is the column number.

The factorial notation makes many formulae involving the multiplication of consecutive positive integers shorter and easier to write. That includes the binomial coefficient.

The binomial coefficient

With n and r as non-negative integers such that $n \geqslant r$, the binomial coefficient $\binom{n}{r}$ is defined by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 31

Find the value of a) $\binom{7}{3}$ b) $\binom{7}{4}$ c) $\binom{7}{0}$ d) $\binom{7}{7}$

Solution

a) $\binom{7}{3} = \dfrac{7!}{3!(7-3)!} = \dfrac{7!}{3!4!} = \dfrac{\cancel{1\cdot2\cdot3\cdot4}\cdot5\cdot6\cdot7}{(1\cdot2\cdot3)(\cancel{1\cdot2\cdot3\cdot4})} = \dfrac{5\cdot6\cdot7}{1\cdot2\cdot3} = 35$

b) $\binom{7}{4} = \dfrac{7!}{3!(7-4)!} = \dfrac{7!}{4!3!} = \dfrac{\cancel{1\cdot2\cdot3\cdot4}\cdot5\cdot6\cdot7}{(\cancel{1\cdot2\cdot3\cdot4})(1\cdot2\cdot3)} = \dfrac{5\cdot6\cdot7}{1\cdot2\cdot3} = 35$

c) $\binom{7}{0} = \dfrac{7!}{0!(7-0)!} = \dfrac{\cancel{7!}}{0!\cancel{7!}} = \dfrac{1}{1} = 1$

d) $\binom{7}{7} = \dfrac{7!}{7!(7-7)!} = \dfrac{\cancel{7!}}{\cancel{7!}0!} = \dfrac{1}{1} = 1$

● **Hint:** Your calculator can do the tedious work of evaluating the binomial coefficient. If you have a TI, the binomial coefficient appears as ${}_nC_r$, which is another notation frequently used in mathematical literature.

```
7 nCr 3
                35
7 nCr 4
                35
7 nCr 0
                 1
█
```

Although the binomial coefficient $\binom{n}{r}$ appears as a fraction, all its results where n and r are non-negative integers are positive integers. Also, notice the **symmetry** of the coefficient in the previous examples. This is a property that you are asked to prove in the exercises:

$$\binom{n}{r} = \binom{n}{n-r}$$

Example 32

Calculate the following:

$\binom{6}{0}, \binom{6}{1}, \binom{6}{2}, \binom{6}{3}, \binom{6}{4}, \binom{6}{5}, \binom{6}{6}$

Solution

$\binom{6}{0} = \mathbf{1}, \binom{6}{1} = \mathbf{6}, \binom{6}{2} = \mathbf{15}, \binom{6}{3} = \mathbf{20}, \binom{6}{4} = \mathbf{15}, \binom{6}{5} = \mathbf{6}, \binom{6}{6} = \mathbf{1}$

The values we calculated above are precisely the entries in the sixth row of Pascal's triangle.

We can write Pascal's triangle in the following manner:

$$\begin{array}{ccccccc}
\binom{0}{0} & & & & & & \\
\binom{1}{0} & \binom{1}{1} & & & & & \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & & & \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & & & \\
\cdots & \cdots & \cdots & \cdots & & & \\
\binom{n}{0} & \binom{n}{1} & \cdots & \cdots & \cdots & \cdots & \binom{n}{n}
\end{array}$$

Example 33

Calculate $\binom{n}{r-1} + \binom{n}{r}$.

• **Hint:** You will be able to provide reasons for the steps after you do the exercises!

This is called Pascal's rule.

Solution

$$\begin{aligned}
\binom{n}{r-1} + \binom{n}{r} &= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} \\
&= \frac{n! \cdot r}{r \cdot (r-1)!(n-r+1)!} + \frac{n! \cdot (n-r+1)}{r!(n-r)! \cdot (n-r+1)} \\
&= \frac{n! \cdot r}{r!(n-r+1)!} + \frac{n! \cdot (n-r+1)}{r!(n-r+1)!} \\
&= \frac{n! \cdot r + n! \cdot (n-r+1)}{r!(n-r+1)!} = \frac{n!(r+n-r+1)}{r!(n-r+1)!} \\
&= \frac{n!(n+1)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!} = \binom{n+1}{r}
\end{aligned}$$

If we read the result above carefully, it says that the sum of the terms in the nth row $(r - 1)$th and rth columns is equal to the entry in the $(n + 1)$th row and rth column. That is, the two entries on the left are adjacent entries in the nth row of Pascal's triangle and the entry on the right is the entry in the $(n + 1)$th row directly below the rightmost entry. This is precisely the principle behind Pascal's triangle!

Using the binomial theorem

We are now prepared to state the binomial theorem. The proof of the theorem is optional and will require mathematical induction. We will develop the proof in Section 4.7.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

In a compact form, we can use sigma notation to express the theorem as follows:

$$(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$$

Example 34

Use the binomial theorem to expand $(x + y)^7$.

Solution

$$\begin{aligned}(x+y)^7 &= \binom{7}{0}x^7 + \binom{7}{1}x^{7-1}y + \binom{7}{2}x^{7-2}y^2 + \binom{7}{3}x^{7-3}y^3 + \binom{7}{4}x^{7-4}y^4 \\ &\quad + \binom{7}{5}x^{7-5}y^5 + \binom{7}{6}xy^6 + \binom{7}{7}y^7 \\ &= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7\end{aligned}$$

Example 35

Find the expansion for $(2k - 3)^5$.

Solution

$$\begin{aligned}(2k-3)^5 &= \binom{5}{0}(2k)^5 + \binom{5}{1}(2k)^4(-3) + \binom{5}{2}(2k)^3(-3)^2 + \binom{5}{3}(2k)^2(-3)^3 \\ &\quad + \binom{5}{4}(2k)(-3)^4 + \binom{5}{5}(-3)^5 \\ &= 32k^5 - 240k^4 + 720k^3 - 1080k^2 + 810k - 243\end{aligned}$$

Example 36

Find the term containing a^3 in the expansion $(2a - 3b)^9$.

Note: Why is the binomial theorem related to the number of combinations of n elements taken r at a time?

Consider evaluating $(x + y)^n$. In doing so, you have to multiply $(x + y)$ n times by itself. As you know, one term has to be x^n. How to get this term? x^n is the result of multiplying x in each of the n factors $(x + y)$ and that can only happen in one way. However, consider the term containing x^r. To have a power of r over the x, means that the x in each of r factors has to be multiplied, and the rest will be the $n - r$ y-terms. This can happen in $\binom{n}{r}$ ways. Hence, the coefficient of the term x^ry^{n-r} is $\binom{n}{r}$.

Solution

To find the term, we do not need to expand the whole expression.

Since $(x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$, the term containing a^3 is the term where $n - i = 3$, i.e. when $i = 6$. So, the required term is

$$\binom{9}{6}(2a)^{9-6}(-3b)^6 = 84 \cdot 8a^3 \cdot 729b^6 = 489\,888a^3b^6.$$

Example 37

Find the term independent of x in $\left(4x^3 - \frac{2}{x^2}\right)^5$.

Solution

The phrase 'independent of x' means the term with no x variable, i.e. the constant term. A constant is equivalent to the product of a number and x^0, since $x^0 = 1$. We are looking for the term in the expansion such that the resulting power is zero. In terms of i, each term in the expansion is given by

$$\binom{5}{i}(4x^3)^{5-i}(-2x^{-2})^i$$

Thus, for the constant term:

$$3(5 - i) - 2i = 0 \Rightarrow 15 - 5i = 0 \Rightarrow i = 3$$

Therefore, the term independent of x is:

$$\binom{5}{3}(4x^3)^2(-2x^{-2})^3 = 10 \cdot 16x^6(-8x^{-6}) = -1280$$

Example 38

Find the coefficient of b^6 in the expansion of $\left(2b^2 - \frac{1}{b}\right)^{12}$.

Solution

The general term is

$$\binom{12}{i}(2b^2)^{12-i}\left(-\frac{1}{b}\right)^i = \binom{12}{i}(2)^{12-i}(b^2)^{12-i}\left(-\frac{1}{b}\right)^i$$

$$= \binom{12}{i}(2)^{12-i}b^{24-2i}b^{-i}(-1)^i = \binom{12}{i}(2)^{12-i}b^{24-3i}(-1)^i$$

$24 - 3i = 6 \Rightarrow i = 6$. So, the coefficient in question is $\binom{12}{6}(2)^6(-1)^6 = 59\,136$.

Exercise 4.6

1 Use Pascal's triangle to expand each binomial.

a) $(x + 2y)^5$ b) $(a - b)^4$ c) $(x - 3)^6$

d) $(2 - x^3)^4$ e) $(x - 3b)^7$ f) $\left(2n + \frac{1}{n^2}\right)^6$

g) $\left(\frac{3}{x} - 2\sqrt{x}\right)^4$

2 Evaluate each expression.

a) $\binom{8}{3}$ b) $\binom{18}{5} - \binom{18}{13}$ c) $\binom{7}{4}\binom{7}{3}$

d) $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$

e) $\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6}$

3 Use the binomial theorem to expand each of the following.

a) $(x + 2y)^7$ b) $(a - b)^6$ c) $(x - 3)^5$

d) $(2 - x^3)^6$ e) $(x - 3b)^7$ f) $\left(2n + \frac{1}{n^2}\right)^6$

g) $\left(\frac{3}{x} - 2\sqrt{x}\right)^4$ h) $(1 + \sqrt{5})^4 + (1 - \sqrt{5})^4$

i) $(\sqrt{3} + 1)^8 - (\sqrt{3} - 1)^8$ j) $(1 + i)^8$, where $i^2 = -1$

k) $(\sqrt{2} - i)^6$, where $i^2 = -1$

4 Consider the expression $\left(x - \frac{2}{x}\right)^{45}$.

a) Find the first three terms of this expansion.

b) Find the constant term if it exists or justify why it does not exist.

c) Find the last three terms of the expansion.

d) Find the term containing x^3 if it exists or justify why it does not exist.

5 Prove that $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in \mathbb{N}$ and $n \geqslant k$.

6 Prove that for any positive integer n,

$\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n - 1$ • **Hint:** $2^n = (1 + 1)^n$

7 Consider all $n, k \in \mathbb{N}$ and $n \geqslant k$.

a) Verify that $k! = k(k - 1)!$

b) Verify that $(n - k + 1)! = (n - k + 1)(n - k)!$

c) Justify the steps given in the proof of $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ in the examples.

8 Find the value of the expression:

$\binom{6}{0}\left(\frac{1}{3}\right)^6 + \binom{6}{1}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \binom{6}{2}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + \ldots + \binom{6}{6}\left(\frac{2}{3}\right)^6$

9 Find the value of the expression:

$\binom{8}{0}\left(\frac{2}{5}\right)^8 + \binom{8}{1}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right) + \binom{8}{2}\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^2 + \ldots + \binom{8}{8}\left(\frac{3}{5}\right)^8$

10 Find the value of the expression:

$\binom{n}{0}\left(\frac{1}{7}\right)^n + \binom{n}{1}\left(\frac{1}{7}\right)^{n-1}\left(\frac{6}{7}\right) + \binom{n}{2}\left(\frac{1}{7}\right)^{n-2}\left(\frac{6}{7}\right)^2 + \ldots + \binom{n}{n}\left(\frac{6}{7}\right)^n$

11 Find the term independent of x in the expansion of $\left(x^2 - \frac{1}{x}\right)^6$.

12 Find the term independent of x in the expansion of $\left(3x - \frac{2}{x}\right)^8$.

13 Find the term independent of x in the expansion of $\left(2x - \frac{3}{x^3}\right)^8$.

14 Find the first three terms of the expansion of $(1 + x)^{10}$ and use them to find an approximation to

a) 1.01^{10} b) 0.99^{10}

15 Show that $\binom{n}{r-1} + 2\binom{n}{r} + \binom{n}{r+1} = \binom{n+2}{r+1}$ and interpret your result on the entries in Pascal's triangle.

16 Express each repeating decimal as a fraction:

a) $0.\overline{7}$ b) $0.3\overline{45}$ c) $3.21\overline{29}$

17 Find the coefficient of x^6 in the expansion of $(2x - 3)^9$.

18 Find the coefficient of x^3b^4 in $(ax + b)^7$.

19 Find the constant term of $\left(\frac{2}{z^2} - z\right)^{15}$.

20 Expand $(3n - 2m)^5$.

21 Find the coefficient of r^{10} in $(4 + 3r^2)^9$.

4.7 Mathematical induction

Domino effect

In addition to playing games of strategy, another familiar activity using dominoes is to place them on edge in lines, then topple the first tile, which falls on and topples the second, which topples the third, etc., resulting in all of the tiles falling. Arrangements of millions of tiles have been made that have taken many minutes to fall.

The Netherlands has hosted an annual domino toppling competition called *Domino Day* since 1986. The record, achieved in 2006, is 4 079 381 dominoes.

Similar phenomena of chains of small events each causing similar events leading to an eventual grand result, by analogy, are called *domino effects*. The phenomenon also has some theoretical bearing to familiar applications like the amplifier, digital signals, or information processing.

10 Complex Numbers

Assessment statements

1.5 Complex numbers: the number $i = \sqrt{-1}$; the term's real part, imaginary part, conjugate, modulus and argument.
Cartesian form $z = a + ib$.
Sums, products and quotients of complex numbers.

1.6 Modulus–argument (polar) form $z = r(\cos\theta + i\sin\theta) = r\text{cis}(\theta) = re^{i\theta}$.
The complex plane.

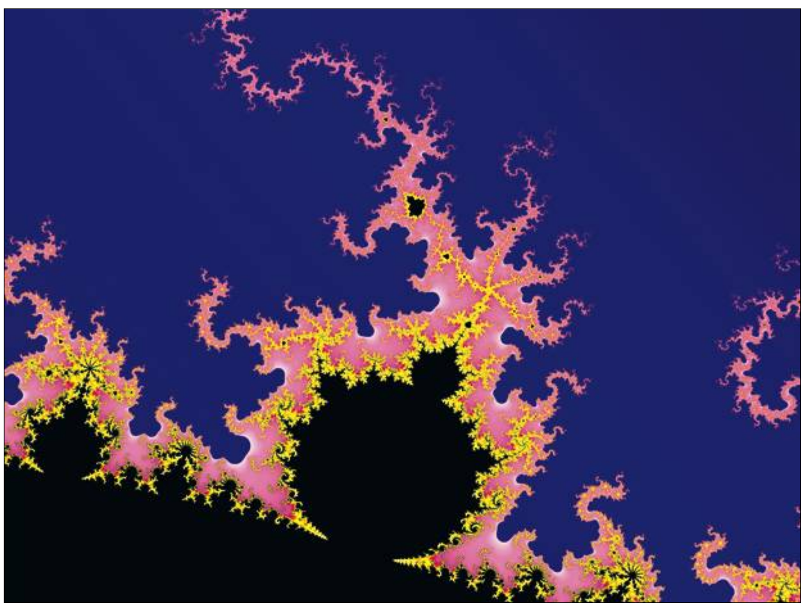
1.7 De Moivre's theorem.
Powers and roots of a complex number.

1.8 Conjugate roots of polynomial equations with real coefficients.

Introduction

You have already met complex numbers in Chapters 1 and 3. This chapter will broaden your understanding to include trigonometric representation of complex numbers and some applications.

Fractals can be generated using complex numbers.



Solving a linear equation of the form

$$ax + b = 0, \text{ with } a \neq 0$$

is a straightforward procedure if we are using the set of real numbers. The situation, as you already know, is different with quadratic equations. For example, as you have seen in Chapter 3, solving the quadratic equation

$x^2 + 1 = 0$ *over the set of real numbers* is not possible. The square of any real number has to be non-negative, i.e.

$(x^2 \geqslant 0 \Leftrightarrow x^2 + 1 \geqslant 1) \Rightarrow x^2 + 1 > 0$ for any choice of a real number x.

This means that $x^2 + 1 = 0$ is impossible for every real number x. This forces us to introduce a new set where such a solution is possible.

Numbers such as $\sqrt{-1}$ are not intuitive and many mathematicians in the past resisted their introduction, so they are called **imaginary numbers**.

The situation with finding a solution to $x^2 + 1 = 0$ is analogous to the following scenario: For a child in the first or second grade, a question such as $5 + ? = 9$ is manageable. However, a question such as $5 + ? = 2$ is impossible because the student's knowledge is *restricted* to the set of positive integers.

However, at a later stage when the same student is faced with the same question, he/she can solve it because their scope has been *extended* to include negative numbers too.

Also, at early stages an equation such as

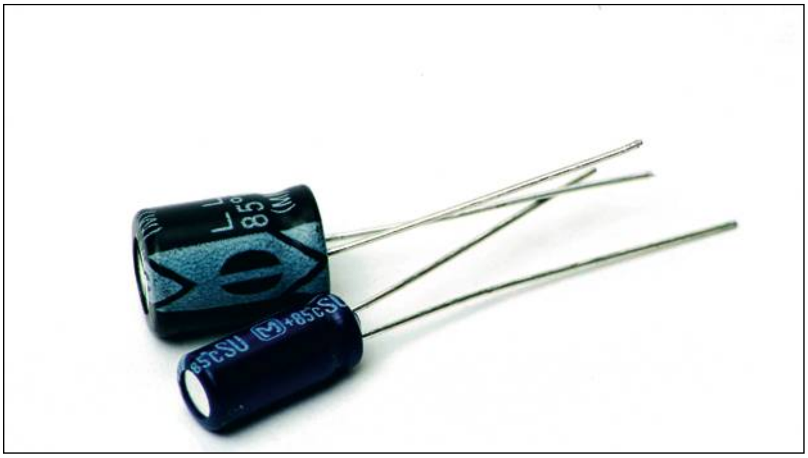
$x^2 = 5$

cannot be solved till the student's knowledge of sets is extended to include irrational numbers where he/she can recognize numbers such as $x = \pm\sqrt{5}$.

The situation is much the same for $x^2 + 1 = 0$. We *extend* our number system to include numbers such as $\sqrt{-1}$; i.e. a number whose square is -1.

Thanks to Euler's (1707–1783) seminal work on imaginary numbers, they now feature prominently in the number system. Euler skilfully employed them to obtain many interesting results. Later, Gauss (1777–1855) represented them as points in the plane and renamed them as **complex numbers**, using them to obtain various significant results in number theory.

10.1 Complex numbers, sums, products and quotients



Electronic components like capacitors are used in AC circuits. Their effects are represented using complex numbers.

As you have seen in the introduction, the development of complex numbers had its origin in the search for methods of solving polynomial equations. The quadratic formula

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

had been used earlier than the 16th century to solve quadratic equations – in more primitive notations, of course. However, mathematicians stopped short of using it for cases where $b^2 - 4ac$ was negative. The use of the formula in cases where $b^2 - 4ac$ is negative depends on two principles (in addition to the other principles inherent in the set of real numbers, such as associativity and commutativity of multiplication).

1. $\sqrt{-1} \cdot \sqrt{-1} = -1$
2. $\sqrt{-k} = \sqrt{k} \cdot \sqrt{-1}$ **for any real number $k > 0$**

Example 1

Multiply $\sqrt{-36} \cdot \sqrt{-49}$.

Solution

First we simplify each square root using rule 2.

$$\sqrt{-36} = \sqrt{36} \cdot \sqrt{-1} = 6 \cdot \sqrt{-1}$$
$$\sqrt{-49} = \sqrt{49} \cdot \sqrt{-1} = 7 \cdot \sqrt{-1}$$

And hence using rule 1 with the other obvious rules:

$$\sqrt{-36} \cdot \sqrt{-49} = 6 \cdot \sqrt{-1} \cdot 7 \cdot \sqrt{-1} = 42 \cdot \sqrt{-1} \cdot \sqrt{-1} = -42$$

To deal with the quadratic formula expressions that consist of combinations of real numbers and square roots of negative numbers, we can apply the rules of binomials to numbers of the form

$$a + b\sqrt{-1}$$

where a and b are real numbers. For example, to add $5 + 7\sqrt{-1}$ to $2 - 3\sqrt{-1}$ we combine 'like' terms as we do in polynomials:

$$\begin{aligned}(5 + 7\sqrt{-1}) + (2 - 3\sqrt{-1}) &= 5 + 2 + 7\sqrt{-1} - 3\sqrt{-1}\\ &= (5 + 2) + (7 - 3)\sqrt{-1} = 7 + 4\sqrt{-1}\end{aligned}$$

Similarly, to multiply these numbers we use the binomial multiplication procedures:

$$\begin{aligned}(5 + 7\sqrt{-1}) \cdot (2 - 3\sqrt{-1}) &= 5 \cdot 2 + (7\sqrt{-1}) \cdot (-3\sqrt{-1}) + 5 \cdot (-3\sqrt{-1})\\ &\quad + (7\sqrt{-1}) \cdot 2\\ &= 10 - 21 \cdot (\sqrt{-1})^2 - 15 \cdot \sqrt{-1} + 14 \cdot \sqrt{-1}\\ &= 10 - 21 \cdot (-1) + (-15 + 14)\sqrt{-1}\\ &= 31 - \sqrt{-1}\end{aligned}$$

Euler introduced the symbol i for $\sqrt{-1}$.

A **pure imaginary number** is a number of the form ki, where k is a real number and i, the **imaginary unit**, is defined by $i^2 = -1$.

Note: In some cases, especially in engineering sciences, the number i is sometimes denoted as j.

Note: With this definition of i, a few interesting results are immediately apparent. For example,

$$i^3 = i^2 \cdot i = -1 \cdot i = -i, \text{ and}$$
$$i^4 = i^2 \cdot i^2 = (-1) \cdot (-1) = 1, \text{ and so}$$
$$i^5 = i^4 \cdot i = 1 \cdot i = i, \text{ and also}$$
$$i^6 = i^4 \cdot i^2 = i^2 = -1;\ i^7 = -i, \text{ and finally } i^8 = 1.$$

This leads you to be able to evaluate any positive integer power of i using the following property:

$$i^{4n+k} = i^k,\ k = 0, 1, 2, 3.$$

So, for example $i^{2122} = i^{2120+2} = i^2 = -1$.

Example 2

Simplify

a) $\sqrt{-36} + \sqrt{-49}$ b) $\sqrt{-36} \cdot \sqrt{-49}$

Solution

a) $\sqrt{-36} + \sqrt{-49} = \sqrt{36}\sqrt{-1} + \sqrt{49}\sqrt{-1}$
$= 6i + 7i = 13i$

b) $\sqrt{-36} \cdot \sqrt{-49} = 6i \cdot 7i = 42i^2$
$= 42(-1) = -42$

We do not define $i = \sqrt{-1}$ for a reason. It is the convention in mathematics that when we write $\sqrt{9}$ then we mean the non-negative square root of 9, namely 3. We do not mean -3! i does not belong to this category since we cannot say that $\boldsymbol{i}$ is the positive square root of -1, i.e. $i > 0$. If we do, then $-1 = i \cdot i > 0$, which is false, and if we say $i < 0$, then $-i > 0$, and $-1 = -i \cdot -i > 0$, which is also false. Actually $-i$ is also a square root of -1 because $-i \cdot -i = i^2 = -1$.

With this in mind, we can use a 'convention' which calls i the **principal** square root of -1 and write $i = \sqrt{-1}$.

Gauss introduced the idea of complex numbers by giving them the following definition.

A **complex number** is a number that can be written in the form $a + bi$ where a and b are real numbers and $i^2 = -1$. a is called the **real part** of the number and b is the **imaginary part**.

Notation

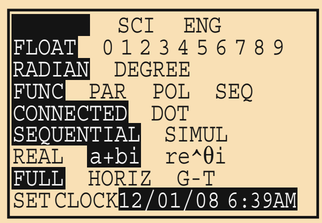
It is customary to denote complex numbers with the variable z.

$z = 5 + 7i$ is the complex number with real part 5 and imaginary part 7 and $z = 2 - 3i$ has 2 as real part and -3 as imaginary.

It is usual to write $\mathbf{Re}(z)$ for the real part of z and $\mathbf{Im}(z)$ for the imaginary part. So, $\text{Re}(2 + 3i) = 2$ and $\text{Im}(2 + 3i) = 3$.

Note that both the real and imaginary parts are real numbers!

A GDC can be set up to do basic complex number operations. For example, if you have a TI-84 Plus, the set up is as follows.



Algebraic structure of complex numbers

Gauss' definition of the complex numbers triggers the following understanding of the set of complex numbers as an extension to our number sets in algebra.

The set of *complex numbers* $\mathbb{C}$ is the set of ordered pairs of real numbers $\mathbb{C} = \{z = (x, y): x, y \in \mathbb{R}\}$, with the following additional structure:

Equality

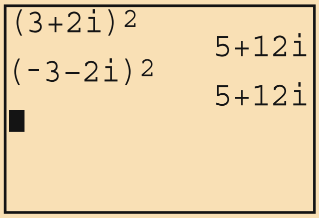
Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are equal if their corresponding components are equal: $(x_1, y_1) = (x_2, y_2)$ if $x_1 = x_2$ and $y_1 = y_2$. That is, *two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.*

This is equivalent to saying: $a + bi = c + di \Leftrightarrow a = c$ and $b = d$.

For example, if $2 - (y - 2)i = x + 3 + 5i$, then x must be -1 and y must be -3. **Explain why.**

An interesting application of the way equality works is in finding the square roots of complex numbers without a need for the trigonometric forms developed later in the chapter.

Find the square root(s) of $z = 5 + 12i$. Let the square root of z be $x + yi$, then $(x + yi)^2 = 5 + 12i \Rightarrow x^2 - y^2 + 2xyi = 5 + 12i \Rightarrow x^2 - y^2 = 5$ and $2xy = 12 \Rightarrow xy = 6 \Rightarrow y = \frac{6}{x}$, and when we substitute this value in $x^2 - y^2 = 5$, we have $x^2 - \left(\frac{6}{x}\right)^2 = 5$. This simplifies to $x^4 - 5x^2 - 36 = 0$ which yields $x^2 = -4$ or $x^2 = 9, \Rightarrow x = \pm 3$. This leads to $x = \pm 2i$, that is, the two square roots of $5 + 12i$ are $3 + 2i$ or $-3 - 2i$.



Addition and subtraction for complex numbers are defined as follows:

Addition

$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_2 + y_2)$

This is equivalent to saying: $(a + bi) + (c + di) = (a + c) + (b + d)i$.

Multiplication

$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

This is equivalent to using the binomial multiplication on $(a + bi)(c + di)$:

$$(a + bi) \cdot (c + di) = ac + bdi^2 + adi + bci = ac - bd + (ad + bc)i$$

Addition and multiplication of complex numbers inherit most of the properties of addition and multiplication of real numbers:

$z + w = w + z$ and $zw = wz$ (Commutativity)

$z + (u + v) = (z + u) + v$ and $z(uv) = (zu)v$ (Associativity)

$z(u + v) = zu + zv$ (Distributive property)

A number of complex numbers take up unique positions. For example, the number $(0, 0)$ has the properties of 0:

$$(x, y) + (0, 0) = (x, y) \text{ and } (x, y)(0, 0) = (0, 0).$$

It is therefore normal to identify it with 0. The symbol is exactly the same symbol used to identify the 'real' 0. So, the real and complex zeros are the same number.

Another complex number of significance is $(1, 0)$. This number plays an important role in multiplication that stems from the following property:

$$(x, y)\,(1, 0) = (x \cdot 1 - y \cdot 0, x \cdot 0 + y \cdot 1) = (x, y)$$

For complex numbers, $(1, 0)$ behaves like the identity for multiplication for real numbers. Again, it is normal to write $(1, 0) = 1$.

The third number of significance is $(0, 1)$. It has the notable characteristic of having a negative square, i.e.

$$(0, 1)(0, 1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0)$$

Using the definition above, $(0, 1) = 0 + 1i = i$. So, the last result should be no surprise to us since we know that

$$i \cdot i = -1 = (-1, 0).$$

Since (x, y) represents the complex number $x + yi$, then every real number x can be written as $x + 0i = (x, 0)$. The set of real numbers is therefore a subset of the set of complex numbers. They are the complex numbers whose imaginary part is 0. Similarly, pure imaginary numbers are of the form $0 + yi = (0, y)$. They are the complex numbers whose real part is 0.

Notation

So far, we have learned how to represent a complex number in two forms:

(x, y) and $x + yi$.

Now, from the properties above

$$(x, y) = (x, 0) + (0, y) = (x, 0) + (y, 0)(0, 1)$$

(Check the truth of this equation.)

This last equation justifies why we can write $(x, y) = x + yi$.

Example 3

Simplify each expression.

a) $(4 - 5i) + (7 + 8i)$

b) $(4 - 5i) - (7 + 8i)$

c) $(4 - 5i)(7 + 8i)$

Solution

a) $(4 - 5i) + (7 + 8i) = (4 + 7) + (-5 + 8)i = 11 + 3i$

b) $(4 - 5i) - (7 + 8i) = (4 - 7) + (-5 - 8)i = -3 - 13i$

c) $(4 - 5i)(7 + 8i) = (4 \cdot 7 - (-5) \cdot 8) + (4 \cdot 8 + (-5) \cdot 7)i = 68 - 3i$

```
(4-5i)/(8i)
         -.625-.5i
Ans▸Frac
         -5/8-1/2i
(4-5i)*(7+8i)
             68-3i
```

Division

Multiplication can be used to perform division of complex numbers.

The **division** of two complex numbers, $\frac{a + bi}{c + di}$, involves finding a complex number $(x + yi)$ satisfying $\frac{a + bi}{c + di} = x + yi$; hence, it is sufficient to find the unknowns x and y.

Example 4

Find the quotient $\frac{2 + 3i}{1 + 2i}$.

Solution

Let $\frac{2 + 3i}{1 + 2i} = x + iy$. Hence, using multiplication and the equality of complex numbers,

$2 + 3i = (1 + 2i)(x + iy) \Leftrightarrow 2 + 3i = x - 2y + i(2x + y)$

$$\Leftrightarrow \begin{cases} 2 = x - 2y \\ 3 = 2x + y \end{cases} \Rightarrow x = \frac{8}{5}, y = \frac{1}{5}$$

Thus, $\frac{2 + 3i}{1 + 2i} = \frac{8}{5} - \frac{1}{5}i$.

```
(2+3i)/(1+2i)
         1.6-.2i
Ans▸Frac
        8/5-1/5i
█
```

Now, in general, $\frac{a + bi}{c + di} = x + yi \Leftrightarrow a + bi = (x + yi)(c + di)$.

With the multiplication as described above:

$$a + bi = (cx - dy) + (dx + cy)i$$

Again by applying the equality of complex numbers property above we get a system of two equations that can be solved.

$$\begin{cases} cx - dy = a \\ dx + cy = b \end{cases} \Rightarrow x = \frac{ac + bd}{c^2 + d^2}; y = \frac{bc - ad}{c^2 + d^2}$$

The denominator $c^2 + d^2$ resulted from multiplying $c + di$ by $c - di$, which is its conjugate.

Conjugate

With every complex number $(a + bi)$ we associate another complex number $(a - bi)$ which is called its conjugate. The conjugate of number z is most often denoted with a bar over it, sometimes with an asterisk to the right of it, occasionally with an apostrophe and even less often with the plain symbol Conj as in

$$\bar{z} = z^* = z' = \text{Conj}(z).$$

> Although the conjugate notation z^* will be used in the book, in your own work you can use any notation you feel comfortable with. You just need to understand that the IB questions use this one.

In this book, we will use z^* for the conjugate.

The importance of the conjugate stems from the following property

$$(a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2$$

which is a non-negative real number. So the product of a complex number and its conjugate is always a real number.

Example 5

Find the conjugate of z and verify the property mentioned above.

a) $z = 2 + 3i$

b) $z = 5i$

c) $z = 11$

Solution

a) $z^* = 2 - 3i$, and $(2 + 3i)(2 - 3i) = 4 - 9i^2 = 4 + 9 = 13$.

b) $z^* = -5i$, and $(5i)(-5i) = -5i^2 = (-5)(-1) = 5$.

c) $z^* = 11$, and $11 \cdot 11 = 121$.

So, the method used in dividing two complex numbers can be achieved by multiplying the quotient by a fraction whose numerator and denominator are the conjugate $c - di$.

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$$

Example 6

Find each quotient and write your answer in standard form.

a) $\dfrac{4-5i}{7+8i}$

b) $\dfrac{4-5i}{8i}$

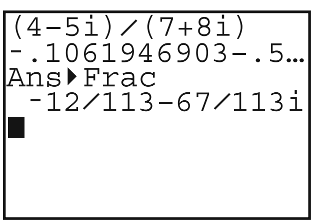
c) $\dfrac{4-5i}{7}$

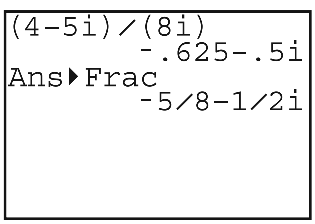
Solution

a) $\dfrac{4-5i}{7+8i} = \dfrac{4-5i}{7+8i} \cdot \dfrac{7-8i}{7-8i} = \dfrac{28-40+(-32-35)i}{49+64} = -\dfrac{12}{113} - \dfrac{67}{113}i$

b) $\dfrac{4-5i}{8i} = \dfrac{4-5i}{8i} \cdot \dfrac{-8i}{-8i} = \dfrac{-32i-40}{64} = -\dfrac{5}{8} - \dfrac{1}{2}i$

c) $\dfrac{4-5i}{7} = \dfrac{4}{7} - \dfrac{5}{7}i$





Example 7

Solve the system of equations and express your answer in Cartesian form.

$$(1+i)z_1 - iz_2 = -3$$
$$2z_1 + (1-i)z_2 = 3 - 3i$$

Solution

Multiply the first equation by 2, and the second equation by $(1 + i)$.

$$2(1 + i)z_1 - 2iz_2 = -6 \qquad (\mathbf{1})$$

$$2(1 + i)z_1 + (1 + i)(1 - i)z_2 = (1 + i)(3 - 3i)$$

$$2(1 + i)z_1 + 2z_2 = 6 \qquad (\mathbf{2})$$

By subtracting (**2**) from (**1**), we get

$$(-2 - 2i)z_2 = -12$$

And hence $\quad z_2 = \dfrac{-12}{-2 - 2i} = 3 - 3i$

$$z_1 = \frac{-3 + i(3 - 3i)}{1 + i} = \frac{3}{2} + \frac{3}{2i}$$

Properties of conjugates

Here is a theorem that lists some of the important properties of conjugates. In the next section, we will add a few more to the list.

Theorem

Let z, z_1 and z_2 be complex numbers, then

(1) $(z^*)^* = z$

(2) $z^* = z$ if and only if z is real.

(3) $(z_1 + z_2)^* = z_1^* + z_2^*$ The conjugate of the sum is the sum of conjugates.

(4) $(-z)^* = -z^*$

(5) $(z_1 \cdot z_2)^* = z_1^* \cdot z_2^*$ The conjugate of the product is the product of conjugates.

(6) $(z^{-1})^* = (z^*)^{-1}$, if $z \neq 0$.

> The product can be extended to powers of complex numbers, i.e. $(z^2)^* = (z \cdot z)^* = z^* \cdot z^* = (z^*)^2$. This result can be generalized for any non-negative integer power n, i.e. $(z^n)^* = (z^*)^n$ and can be proved by mathematical induction.
>
> The basis case, when $n = 0$, is obviously true: $(z^0)^* = 1 = (z^*)^0$.
>
> Now assume $(z^k)^* = (z^*)^k$. $(z^{k+1})^* = (z^k z)^* = (z^k)^* z^* = (z^*)^k z^*$ (using the product rule).
>
> Therefore, $(z^{k+1})^* = (z^*)^k z^* = (z^*)^{k+1}$.
>
> So, since if the statement is true for $n = k$, it is also true for $n = k + 1$, then by the principle of mathematical induction it is true for all $n \geq 0$.

Proof

(1) and (2) are obvious. For (1), $((a + bi)^*)^* = (a - bi)^* = a + bi$, and for (2), $a - bi = a + bi \Rightarrow 2bi = 0 \Rightarrow b = 0$.

(3) is proved by straightforward calculation:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned}(z_1 + z_2)^* &= ((x_1 + iy_1) + (x_2 + iy_2))^* = ((x_1 + x_2) + i(y_1 + y_2))^* \\ &= (x_1 + x_2) - i(y_1 + y_2) = (x_1 - iy_1) + (x_2 - iy_2) = z_1^* + z_2^*.\end{aligned}$$

(4) can now be proved using the above results:

$$\begin{aligned}&(z + (-z))^* = 0^* = 0 \\ \text{but, } &(z + (-z))^* = 0^* = z^* + (-z)^*, \\ \text{so } &z^* + (-z)^* = 0, \text{ and } (-z)^* = -z^*.\end{aligned}$$

Also (5) is proved by straightforward calculation:

$$\begin{aligned}(z_1 \cdot z_2)^* &= ((x_1 + iy_1) \cdot (x_2 + iy_2))^* = ((x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2))^* \\ &= (x_1x_2 - y_1y_2) - i(y_1x_2 + x_1y_2) \\ &= (x_1 - iy_1) \cdot (x_2 - iy_2) = z_1^* \cdot z_2^*\end{aligned}$$

And finally, (6):

$(z(z^{-1}))^* = 1^* = 1$

but, $(z(z^{-1}))^* = z^*(z^{-1})^*$, so $z^*(z^{-1})^* = 1$,

and $(z^{-1})^* = \dfrac{1}{z^*} = (z^*)^{-1}$.

Conjugate zeros of polynomials

In Chapter 3, you used the following result without proof.

If c is a root of a polynomial equation with real coefficients, then c^* is also a root.

Theorem: If c is a root of a polynomial equation with real coefficients, then c^* is also a root of the equation.

We give the proof for $n = 3$, but the method is general.

$$P(x) = ax^3 + bx^2 + dx + e$$

Since c is a root of $P(x) = 0$, we have

$ac^3 + bc^2 + dc + e = 0$

$\Rightarrow (ac^3 + bc^2 + dc + e)^* = 0$ Since $0^* = 0$.

$\Rightarrow (ac^3)^* + (bc^2)^* + (dc)^* + e^* = 0$ Sum of conjugates theorem.

$\Rightarrow a(c^*)^3 + b(c^*)^2 + d(c^*) + e = 0$ Result of product conjugate.

$\Rightarrow (c^*)$ is a root of $P(x) = 0$.

Example 8

$1 + 2i$ is a zero of the polynomial $P(x) = x^3 - 5x^2 + 11x - 15$. Find all other zeros.

Solution

Since the polynomial has real coefficients, then $1 - 2i$ is also a zero. Hence, using the factor theorem, $P(x) = (x - (1 + 2i))(x - (1 - 2i))(x - c)$, where c is a real number to be found.

Now, $P(x) = (x^2 - 2x + 5)(x - c)$. c can either be found by division or by factoring by trial and error. In either case, $c = 3$.

Example 9[1]

$1 + 2i$ is a zero of the polynomial
$P(x) = x^3 + (i - 2)x^2 + (2i + 5)x + 8 + i$.
Find all other zeros.

[1] Not included in present IB syllabus.

Solution

Since the polynomial does not have real coefficients, then $1 - 2i$ is not necessarily also a zero. To find the other zeros, we can perform synthetic substitution

$$\begin{array}{r|cccc} 1+2i & 1 & i-2 & 2i+5 & 8+i \\ & & 1+2i & -7+i & -8-i \\ \hline & 1 & -1+3i & -2+3i & 0 \end{array}$$

This shows that $P(x) = (x - 1 - 2i)(x^2 + (-1 + 3i)x - 2 + 3i)$. The second factor can be factored into $(x + 1)(x - 2 + 3i)$ giving us the other two zeros as -1 and $2 - 3i$.

Note: $x^2 + (-1 + 3i)x - 2 + 3i = 0$ can be solved using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 - 3i \pm \sqrt{(-1 + 3i)^2 - 4(-2 + 3i)}}{2}$$

$$= \frac{1 - 3i \pm \sqrt{-8 - 6i + 8 - 12i}}{2} = \frac{1 - 3i \pm \sqrt{-18i}}{2}$$

To find $\sqrt{-18i}$ we let $(a + bi)^2 = -18i \Rightarrow a^2 - b^2 + 2abi = -18i$, then equating the real parts and imaginary parts to each other: $a^2 - b^2 = 0$ and $2ab = -18$ will yield $\sqrt{-18i} = \pm 3 \mp 3i$, and hence

$$x = \frac{1 - 3i \pm \sqrt{-18i}}{2} = \frac{1 - 3i \pm (\pm 3 \mp 3i)}{2}$$

which will yield $x = -1$ or $x = 2 - 3i$.

Exercise 10.1

Express each of the following numbers in the form $a + bi$.

1 $5 + \sqrt{-4}$ **2** $7 - \sqrt{-7}$ **3** -6

4 $-\sqrt{49}$ **5** $\sqrt{-81}$ **6** $-\sqrt{\frac{-25}{16}}$

Perform the following operations and express your answer in the form $a + bi$.

7 $(-3 + 4i) + (2 - 5i)$ **8** $(-3 + 4i) - (2 - 5i)$

9 $(-3 + 4i)(2 - 5i)$ **10** $3i - (2 - 4i)$

11 $(2 - 7i)(3 + 4i)$ **12** $(1 + i)(2 - 3i)$

13 $\frac{3 + 2i}{2 + 5i}$ **14** $\frac{2 - i}{3 + 2i}$

15 $\left(\frac{2}{3} - \frac{1}{2}i\right) + \left(\frac{1}{3} + \frac{1}{2}i\right)$ **16** $\left(\frac{2}{3} - \frac{1}{2}i\right)\left(\frac{2}{3} + \frac{1}{2}i\right)$

17 $\left(\frac{2}{3} - \frac{1}{2}i\right) \div \left(\frac{1}{3} + \frac{1}{2}i\right)$ **18** $(2 + i)(3 - 2i)$

19 $\frac{1}{i}(3 - 7i)$ **20** $(2 + 5i) - (-2 - 5i)$

21 $\frac{13}{5 - 12i}$ **22** $\frac{12i}{3 + 4i}$

23 $3i\left(3-\frac{2}{3}i\right)$

24 $(3+5i)(6-10i)$

25 $\dfrac{39-52i}{24+10i}$

26 $(7-4i)^{-1}$

27 $(5-12i)^{-1}$

28 $\dfrac{3}{3-4i}+\dfrac{2}{6+8i}$

29 $\dfrac{(7+8i)(2-5i)}{5-12i}$

30 $\dfrac{5-\sqrt{-144}}{3+\sqrt{-16}}$

31 Let $z=a+bi$. Find a and b if $(2+3i)z=7+i$.

32 $(2+yi)(x+i)=1+3i$, where x and y are real numbers. Solve for x and y.

33 a) Evaluate $(1+i\sqrt{3})^3$.

b) Prove that $(1+i\sqrt{3})^{6n}=8^{2n}$, *where* $n\in Z^+$.

c) Hence, find $(1+i\sqrt{3})^{48}$.

34 a) Evaluate $(-\sqrt{2}+i\sqrt{2})^2$.

b) Prove that $(-\sqrt{2}+i\sqrt{2})^{4k}=(-16)^k$, *where* $k\in Z^+$.

c) Hence, find $(-\sqrt{2}+i\sqrt{2})^{46}$.

35 If z is a complex number such that $|z+4i|=2|z+i|$, find the value of $|z|$.
($|z|=\sqrt{x^2+y^2}$ where $z=x+iy$.)

36 Find the complex number z and write it in the form $a+bi$ if $z=3+\dfrac{2i}{2-i\sqrt{2}}$.

37 Find the values of the two real numbers x and y such that $(x+iy)(4-7i)=3+2i$.

38 Find the complex number z and write it in the form $a+bi$ if $i(z+1)=3z-2$.

39 Find the complex number z and write it in the form $a+bi$ if $\dfrac{2-i}{1+2i}\sqrt{z}=2-3i$.

40 Find the values of the two real numbers x and y such that $(x+iy)^2=3-4i$.

41 a) Find the values of the two real numbers x and y such that $(x+iy)^2=-8+6i$.

b) Hence, solve the following equation

$$z^2+(1-i)z+2-2i=0.$$

42 If $z\in\mathbb{C}$, find all solutions to the equation $z^3-27i=0$.

43 Given that $z=\frac{1}{2}+2i$ is a zero of the polynomial $f(x)=4x^3-16x^2+29x-51$, find the other zeros.

44 Find a polynomial function with integer coefficients and lowest possible degree that has $\frac{1}{2}$, -1 and $3+i\sqrt{2}$ as zeros.

45 Find a polynomial function with integer coefficients and lowest possible degree that has -2, -2 and $1+i\sqrt{3}$ as zeros.

46 Given that $z=5+2i$ is a zero of the polynomial $f(x)=x^3-7x^2-x+87$, find the other zeros.

47 Given that $z=1-i\sqrt{3}$ is a zero of the polynomial $f(x)=3x^3-4x^2+8x+8$, find the other zeros.

48 Let $z\in\mathbb{C}$. If $\dfrac{z}{z^*}=a+bi$, show that $|a+bi|=1$.

49 Given that $z = (k + i)^4$ where k is a real number, find all values of k such that

a) z is a real number

b) z is purely imaginary.

50 Solve the system of equations.

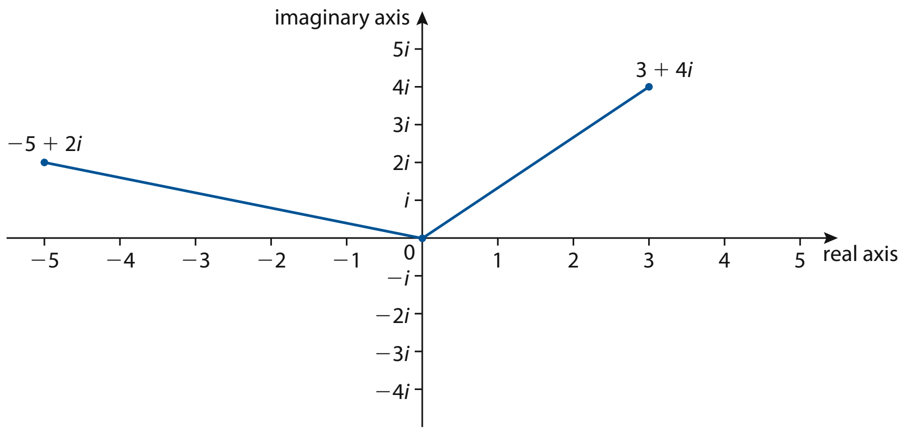
$$iz_1 + 2z_2 = 3 - i$$
$$2z_1 + (2 + i)z_2 = 7 + 2i$$

51 Solve the system of equations.

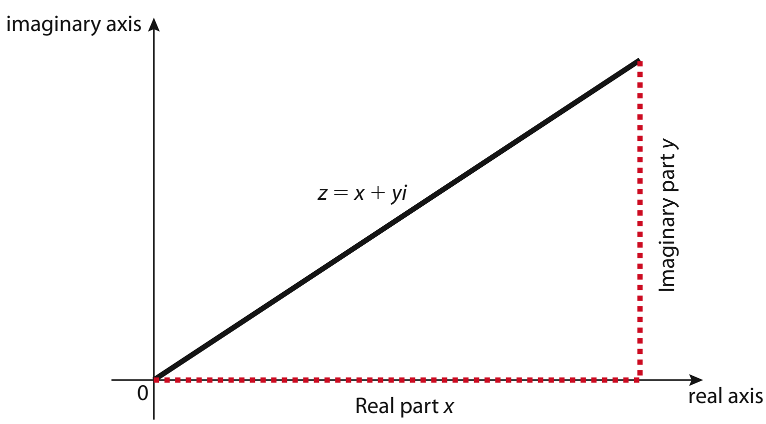
$$iz_1 - (1 + i)z_2 = 3$$
$$(2 + i)z_1 + iz_2 = 4$$

10.2 The complex plane

Our definition of complex numbers as ordered pairs of real numbers enables us to look at them from a different perspective. Every ordered pair (x, y) determines a unique complex number $x + yi$, and vice versa. This correspondence is embodied in the geometric representation of complex numbers. Looking at complex numbers as points in the plane equipped with additional structure changes the plane into what we call **complex plane**, or **Gauss plane**, or **Argand plane** (**diagram**). The complex plane has two axes, the horizontal axis is called the **real axis**, and the vertical axis is the **imaginary axis**. Every complex number $z = x + yi$ is represented by a point (x, y) in the plane. The real part is measured along the real axis and the imaginary part along the imaginary axis.



The diagram above illustrates how the two complex numbers $3 + 4i$ and $-5 + 2i$ are plotted in the complex plane.



Chapter 3

Exercise 3.1

1 $-8; -8$ **2** $0; 33$

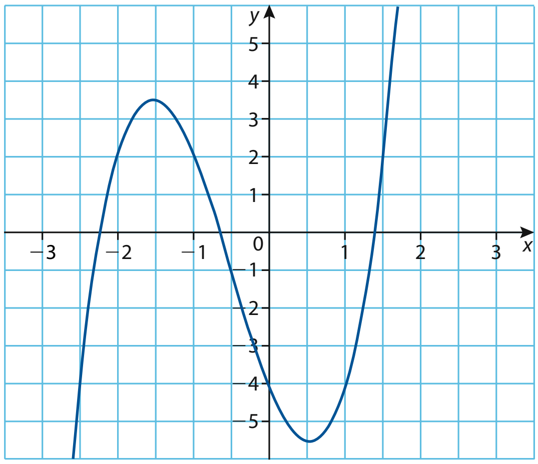
3 $29; 2375$ **4** $0; -3c+6$

5 $k=2$ **6** $k=2$

7 a) $-16, 2, 2, -4, -4, 14, 62$

b) 3

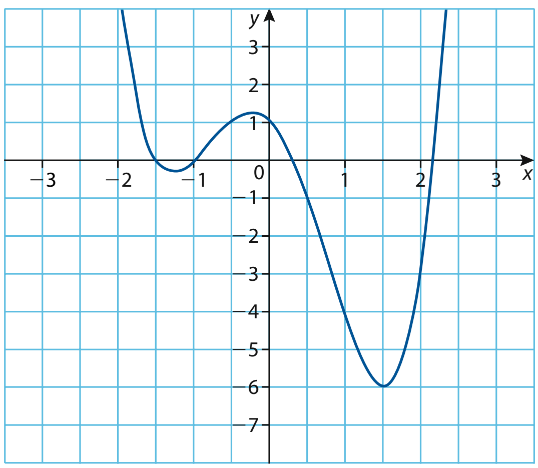
c)



8 a) $52, 5, 0, 1, -4, -3, 40$

b) 4

c)



9 $a = \frac{12}{11}$

10 $b = -\sqrt{3}$

11 a) (i) $(\nwarrow, \nearrow)$ (ii) $(\swarrow, \searrow)$

(iii) $(\nwarrow, \searrow)$ (iv) $(\swarrow, \nearrow)$

(v) $(\nwarrow, \nearrow)$ (vi) $(\swarrow, \searrow)$

(vii) $(\swarrow, \nearrow)$ (viii) $(\nwarrow, \searrow)$

b) If leading term has positive coefficient and even exponent, then $(\nwarrow, \nearrow)$.
If leading term has negative coefficient and even exponent, then $(\swarrow, \searrow)$.
If leading term has positive coefficient and odd exponent, then $(\swarrow, \nearrow)$.
If leading term has negative coefficient and odd exponent, then $(\nwarrow, \searrow)$.

Exercise 3.2

1 a) $f(x) = (x-5)^2 + 7$

b) Horizontal translation 5 units right; vertical translation 7 units up.

c) Minimum: $(5, 7)$

2 a) $f(x) = (x+3)^2 - 1$

b) Horizontal translation 3 units left; vertical translation 1 unit down.

c) Minimum: $(-3, -1)$

3 a) $f(x) = -2(x+1)^2 + 12$

b) Horizontal translation 1 unit left; reflection over x-axis; vertical stretch by factor 2; vertical translation 12 units up.

c) Maximum: $(-1, 12)$

4 a) $f(x) = 4\left(x - \frac{1}{2}\right)^2 + 8$

b) Horizontal translation $\frac{1}{2}$ unit right; vertical stretch by factor 4; vertical translation 8 units up.

c) Maximum: $\left(\frac{1}{2}, 8\right)$

5 a) $f(x) = \frac{1}{2}(x+7)^2 + \frac{3}{2}$

b) Horizontal translation 7 units left; vertical shrink by factor $\frac{1}{2}$; vertical translation $\frac{3}{2}$ units up.

c) Minimum: $\left(-7, \frac{3}{2}\right)$

6 $x = 2, x = -4$ **7** $x = 5, x = -2$

8 $x = \frac{3}{2}, x = 0$ **9** $x = 6, x = -1$

10 $x = 3$ **11** $x = \frac{1}{3}, x = -4$

12 $x = 3, x = 2$ **13** $x = 2, x = \frac{1}{4}$

14 $x = -2 \pm \sqrt{7}$ **15** $x = 5, x = -1$

16 No real solution **17** $x = -4 \pm \sqrt{13}$

18 $x = 2, x = -4$ **19** $x = \frac{2 \pm \sqrt{22}}{2}$

20 a) $x = 2 \pm \sqrt{5}$

b) Axis of symmetry: $x = 2$

c) Minimum value of f is -5

21 Two real solutions **22** No real solutions

23 Two real solutions **24** No real solutions

25 $p = \pm 2\sqrt{2}$ **26** $k < 4$

27 $k < -1,\ k > 1$ **28** $m < -3,\ m > 3$

29 $k > 12$

30 $x - 2 - x^2 \Rightarrow -(x^2 - x + 2) \Rightarrow -\left(x^2 - x + \frac{1}{4}\right) - \frac{7}{4}$
$\Rightarrow -\left(x - \frac{1}{2}\right)^2 - \frac{7}{4} \leqslant -\frac{7}{4}$ for all x

31 $y = -2x^2 + 6x + 8$ **32** $y = x^2 - \frac{7}{2}x - \frac{1}{2}$

33 $-1 < k < 15$ **34** $m < -2\sqrt{10}$ or $m > 2\sqrt{10}$

35 $f(x) = 3x^2 + 5x - 2$ **36** $f(2) = 8$

37 $x < 1$ or $x > 3$

38 $\Delta = (2-t)^2 - 4(2)(t^2+3) > 0 \Rightarrow -7t^2 - 4t - 20 > 0$; because $\Delta_1 = -544$ for $-7t^2 - 4t - 20$ and leading coefficient is negative, then graph of $y = -7t^2 - 4t - 20$ is a parabola opening down and always below x-axis; hence, Δ for original equation is always negative; thus, no real roots

39 $x = \frac{-(-a^2-1) \pm \sqrt{(-a^2-1)^2 - 4a(a)}}{2a} = \frac{a^2+1 \pm \sqrt{a^4 - 2a^2 + 1}}{2a}$
$= \frac{a^2+1 \pm \sqrt{(a^2-1)^2}}{2a} = \frac{a^2+1 \pm (a^2-1)}{2a} \Rightarrow x = \frac{2a^2}{2a}$
$= a$ or $x = \frac{2}{2a} = \frac{1}{a}$

40 a) sum $= -3$, product $= -\frac{5}{2}$

b) sum $= -3$, product $= -1$

c) sum $= 0$, product $= -\frac{3}{2}$

d) sum $= a$, product $= -2a$

e) sum $= 6$, product $= -4$

f) sum $= \frac{1}{3}$, product $= -\frac{2}{3}$

41 $4x^2 + 5x + 4 = 0$

42 a) $\frac{1}{9}$ b) $\frac{1}{12}$ c) $\frac{55}{27}$

43 a) -2 and -6 b) $k = 12$

44 a) $-\frac{1}{4}$ b) $4x^2 + x + 1 = 0$

45 a) $x^2 - 19x + 25 = 0$ b) $25x^2 + 72x - 5 = 0$

Exercise 3.3

1 $3x^2 + 5x - 5 = (x+3)(3x-4) + 7$

2 $3x^4 - 8x^3 + 9x + 5 = (x-2)(3x^3 - 2x^2 - 4x + 1) + 7$

3 $x^3 - 5x^2 + 3x - 7 = (x-4)(x^2 - x - 1) - 11$

4 $9x^3 + 12x^2 - 5x + 1 = (3x-1)(3x^2 + 5x) + 1$

5 $x^5 + x^4 - 8x^3 + x + 2 = (x^2 + x - 7)(x^3 - x + 1) + (-7x + 9)$

6 $(x-7)(x-1)(2x-1)$ **7** $(x-2)(2x+1)(3x+2)$

8 $(x-2)^2(x+4)(3x+2)$ **9** $Q(x) = x - 2,\ R = -2$

10 $Q(x) = x^2 + 2,\ R = -3$ **11** $Q(x) = 3,\ R(x) = 20x + 5$

12 $Q(x) = x^4 + x^3 + 4x^2 + 4x + 4,\ R = -2$

13 $P(2) = 5$ **14** $P(-1) = -17$

15 $P(-7) = -483$ **16** $P\left(\frac{1}{4}\right) = \frac{49}{64}$

17 $x = 2 + i$ or $x = 2 - i$ **18** $x = \frac{1+\sqrt{5}}{2}$ or $x = \frac{1-\sqrt{5}}{2}$

19 $k = \sqrt{1 - x\sqrt{3}}$ or $k = -\sqrt{1 - x\sqrt{3}}$

20 $a = 5,\ b = 12$

21 $x^3 - 3x^2 - 6x + 8$ **22** $x^4 - 3x^3 - 7x^2 + 15x + 18$

23 $x^3 - 6x^2 + 12x - 8$ **24** $x^3 - x^2 + 2$

25 $x^4 + 2x^3 + x^2 + 18x - 72$ **26** $x^4 - 8x^3 + 27x^2 - 50x + 50$

27 $x = 2 + 3i,\ x = 3$

28 a) $a = -1,\ b = -2$ b) $3x + 2$

29 $a = \frac{4}{3},\ b = \frac{1}{3}$

30 $x = 3,\ x = -1,\ x = -\frac{1}{4} + \frac{\sqrt{3}}{4}i,\ x = -\frac{1}{4} - \frac{\sqrt{3}}{4}i$

31 $a = -1,\ b = -4,\ c = 4$ **32** $p = -5,\ q = 23,\ r = -51$

33 $a = -5$ **34** $m = -2,\ n = -6$

35 $b = 18$ **36** b) $R = 3$

37 a) sum $= \frac{2}{3}$, product $= 5$ b) sum $= 1$, product $= 7$

c) sum $= \frac{1}{3}$, product $= -\frac{1}{2}$

39 $-9, 3, 6$

40 $2, -4, 8$

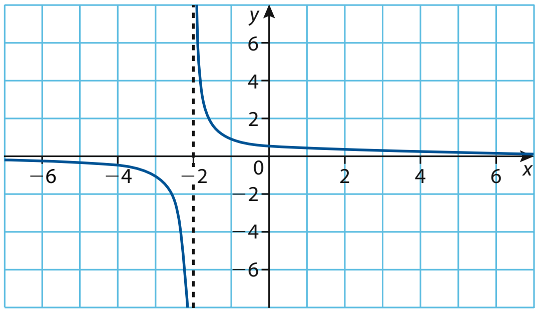
41 $3 + 2i,\ 2 + i,\ 2 - i$

42 $k = 3$

43 $k = -8$

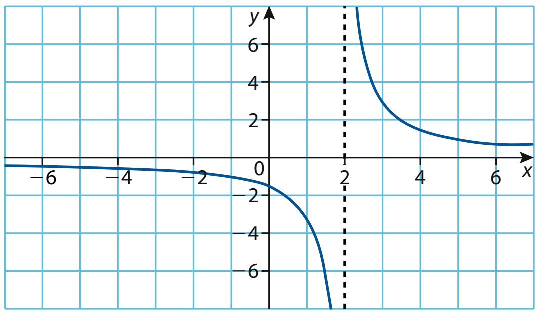
Exercise 3.4

1



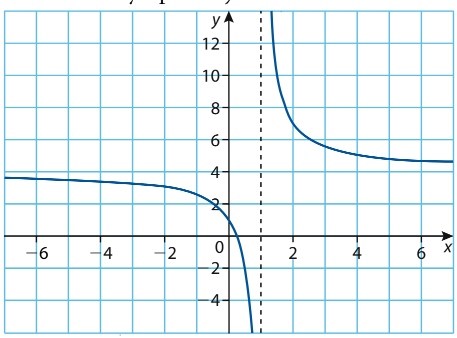
vertical asymptote: $x = -2$
horizontal asymptote: $y = 0$

2



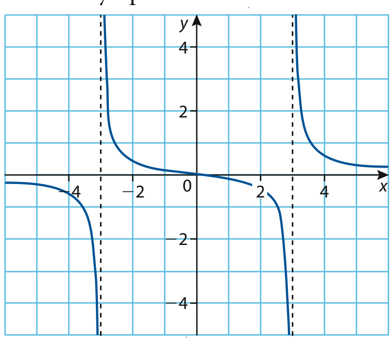
vertical asymptote: $x = 2$
horizontal asymptote: $y = 0$

3



x-intercept: $\left(\frac{1}{4}, 0\right)$, y-intercept: $(0, 1)$
vertical asymptote: $x = 1$ horizontal asymptote: $y = 4$

4



x- and y-intercept: $(0, 0)$
vertical asymptotes: $x = -3,\ x = 3$
horizontal asymptote: $y = 0$

49 10th term 50 €3714.87

51 £2921.16

Exercise 4.4

1 11 280 2 $-\dfrac{10\,5469}{1024}$ 3 0.7

4 $\dfrac{10}{7}$ 5 $\dfrac{16+4\sqrt{3}}{39}$

6 a) $\dfrac{52}{99}$ b) $\dfrac{449}{990}$ c) $\dfrac{7459}{2475}$

7 13 026.135 (£13 026.14)

8 940 9 6578

10 42 625 11 $\dfrac{n(7+3n)}{2}$

12 17 terms 13 85 terms

14 $d = 4$ 15 a) 250, 125 250, b) 83 501

16 $a = 1, d = 5$ 17 2890

18 0.290 19 −2.065

20 11 400 21 1.191

22 49.2 23 $\dfrac{6}{5}$

24 $\dfrac{3+\sqrt{6}}{2}$ 25 $3, \dfrac{18}{5}, \dfrac{93}{25}, \dfrac{468}{125}, \dfrac{15}{4}\left(1-\dfrac{1}{5^n}\right)$

26 $\dfrac{1}{6}, \dfrac{1}{4}, \dfrac{3}{10}, \dfrac{1}{3}; \dfrac{n}{2n+4}$

27 $\sqrt{2}-1, \sqrt{3}-1, 1, \sqrt{5}-1; \sqrt{n+1}-1$

28 1.945, 152.42 29 127, 128

30 $\dfrac{819}{128}, \dfrac{32}{5}$ 31 11 866

32 763 517 33 14 348 906

34 ≈150

Exercise 4.5

1 a) 120 b) 120 c) 20 d) 336

2 a) 1 b) 1 c) 120 d) 120

3 a) 70 b) 70 c) 330 d) 330

4 a) 0 b) 39 916 800 c) 0 d) 10

5 a) F b) F c) T

6 24

7 72 8 312

9 16 777 216 10 262 144

11 1 757 600 000 12 81 000

13 a) 40 320 b) 384

14 a) 40 320 b) 720

15 JANE, JAEN, JNAE, JNEA, JEAN, JENA, AJNE, AJEN, ANJE, ANEJ, AEJN, AENJ, NJAE, NJEA, NEJA, NEAJ, NAJE, NAEJ, EJAN, EJNA, EAJN, EANJ, ENJA, ENAJ

16 Mag, Mga, Mai, …(60 of them)

17 a) 175 760 000 b) 174 790 000

18 a) 4080 b) 1680 c) 1050 d) 1980 e) 3150

19 a) 296 b) 1460 c) 504

20 a) 125 000 b) 117 600 c) 61 250 d) 176 400

21 768

22 a) 36 b) 256

23 a) 5985 b) 2376 c) 2475

24 a) 2280 b) 748 c) 770

25 a) 1 192 052 400 b) 4560, 0.000 38% c) 265 004 096, 22.2%

26 a) 74 613 b) 7560

27 54 867 456 000

Exercise 4.6

1 a) $x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

b) $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

c) $x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729$

d) $16 - 32x^3 + 24x^6 - 8x^9 + x^{12}$

e) $x^7 - 21bx^6 + 189b^2x^5 - 945b^3x^4 + 2835b^4x^3 - 5103b^5x^2 + 5103b^6x - 2187b^7$

f) $64n^6 + 192n^3 + 240 + \dfrac{160}{n^3} + \dfrac{60}{n^6} + \dfrac{12}{n^9} + \dfrac{1}{n^{12}}$

g) $\dfrac{81}{x^4} - \dfrac{216}{x^2\sqrt{x}} + \dfrac{216}{x} - 96\sqrt{x} + 16x^2$

2 a) 56 b) 0 c) 1225 d) 32 e) 0

3 a) $x^7 + 14x^6y + 84x^5y^2 + 280x^4y^3 + 560x^3y^4 + 672x^2y^5 + 448xy^6 + 128y^7$

b) $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$

c) $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

d) $x^{18} - 12x^{15} + 60x^{12} - 160x^9 + 240x^6 - 192x^3 + 64$

e) $x^7 - 21bx^6 + 189b^2x^5 - 945b^3x^4 + 2835b^4x^3 - 5103b^5x^2 + 5103b^6x - 2187b^7$

f) $64n^6 + 192n^3 + 240 + \dfrac{160}{n^3} + \dfrac{60}{n^6} + \dfrac{12}{n^9} + \dfrac{1}{n^{12}}$

g) $\dfrac{81}{x^4} - \dfrac{216}{x^2\sqrt{x}} + \dfrac{216}{x} - 96\sqrt{x} + 16x^2$

h) 112 i) $1792\sqrt{3}$

j) 16 k) $-23 + 10i\sqrt{2}$

4 a) $x^{45} - 90x^{43} + 3960x^{41}$

b) Does not exist as the powers of x decrease by 2's starting at 45. There is no chance for any expression to have zero exponent.

c) $\dbinom{45}{43}x^2\left(\dfrac{-2}{x}\right)^{43} + \dbinom{45}{44}x\left(\dfrac{-2}{x}\right)^{44} + \left(\dfrac{-2}{x}\right)^{45} = -\dbinom{45}{43}\dfrac{2^{43}}{x^{41}} + \dbinom{45}{44}\dfrac{2^{44}}{x^{43}} - \dfrac{2^{45}}{x^{45}}$

d) $\dbinom{45}{21}x^{24}\left(\dfrac{-2}{x}\right)^{21} = -\dbinom{45}{21}\cdot 2^{21}x^3$

5 $\dbinom{n}{k} = \dfrac{n!}{k!(n-k)!} = \dfrac{n!}{(n-k)!k!} = \dfrac{n!}{(n-k)!(n-(n-k))!} = \dbinom{n}{n-k}$

6 $(1+1)^n = \dbinom{n}{0} + \dbinom{n}{1} + \dbinom{n}{2} + \ldots + \dbinom{n}{n}$

$2^n = 1 + \dbinom{n}{1} + \dbinom{n}{2} + \ldots + \dbinom{n}{n} \Rightarrow 2^n - 1 = \dbinom{n}{1} + \dbinom{n}{2} + \ldots \dbinom{n}{n}$

7 Answers vary 8 $\left(\frac{1}{3}+\frac{2}{3}\right)^6 = 1$

9 $\left(\frac{2}{5}+\frac{3}{5}\right)^8 = 1$ 10 $\left(\frac{1}{7}+\frac{6}{7}\right)^n = 1$

11 15 12 90 720

13 16 128 14 1.1045, 0.9045

15 Proof

16 a) $\dfrac{7}{9}$ b) $\dfrac{38}{110}$ c) $\dfrac{31\,808}{9900}$

17 −145 152 18 $35a^3$ 19 96 096

20 $243n^5 - 810n^4m + 1080n^3m^2 - 720n^2m^3 + 240nm^4 - 32m^5$

21 7 838 208

Exercise 4.7

1 $2 + 4 + 6 + \ldots + 2n = n(n+1)$

2–20 All proofs

Practice questions

1 $D = 5, n = 20$

2 €2098.63

b) $\overrightarrow{IR} = \begin{pmatrix} 5 \\ -\frac{25}{6} \end{pmatrix}$

8 a) $\begin{pmatrix} 745 \\ 1000 \end{pmatrix}$ b) 600 km/h c) at 1.5 hrs

d) $\begin{pmatrix} 325 \\ 940 \end{pmatrix}$ e) 451 km

9 $2n^2 - n + 12 = 0$ does not have real solutions, so it is not possible.

10 $\alpha = \frac{\pi}{2} - 2\theta$ **11** 0

Chapter 10

Exercise 10.1

1 $5 + 2i$ **2** $7 - \sqrt{7}i$ **3** $-6 + 0i$

4 $-7 + 0i$ **5** $0 + 9i$ **6** $0 - \frac{5}{4}i$

7 $-1 - i$ **8** $-5 + 9i$ **9** $14 + 23i$

10 $-2 + 7i$ **11** $34 - 13i$ **12** $5 - i$

13 $\frac{16}{29} - \frac{11}{29}i$ **14** $\frac{4}{13} - \frac{7}{13}i$ **15** 1

16 $\frac{25}{36}$ **17** $-\frac{1}{13} - \frac{18}{13}i$ **18** $8 - i$

19 $-7 - 3i$ **20** $4 + 10i$ **21** $\frac{5}{13} + \frac{12}{13}i$

22 $\frac{48}{25} + \frac{36}{25}i$ **23** $2 + 9i$ **24** 68

25 $\frac{8}{13} - \frac{63}{26}i$ **26** $\frac{7}{65} + \frac{4}{65}i$ **27** $\frac{5}{169} + \frac{12}{169}i$

28 $\frac{12}{25} + \frac{8}{25}i$ **29** $\frac{498}{169} + \frac{553}{169}i$ **30** $-\frac{33}{25} - \frac{56}{25}i$

31 $\frac{17}{13} - \frac{19}{13}i$ **32** $x = -\frac{1}{2}, y = -2$; and $x = 1, y = 1$

33 a) -8 c) 2^{48}

34 a) $-4i$ c) 2^{46}

35 $x^2 + y^2 = 4$ **36** $\frac{9 - \sqrt{2}}{3} + \frac{2}{3}i$

37 $x = -\frac{2}{65}, y = \frac{29}{65}$ **38** $\frac{1}{2}(1 + i)$

39 $5 + 12i$ **40** $(x, y) = (2, -1)$ or $(x, y) = (-2, 1)$

41 a) $(x, y) = (1, 3)$ or $(x, y) = (-1, -3)$

b) $2i, -1 - i$

42 $\left\{-3i, \frac{3(\sqrt{3} + i)}{2}, \frac{3(-\sqrt{3} + i)}{2}\right\}$

43 $\frac{1}{2} - 2i, 3$ **44** $f(x) = 2x^4 - 11x^3 + 15x^2 + 17x - 11$

45 $f(x) = x^4 + 2x^3 + 8x + 16$

46 $5 - 2i, -3$ **47** $1 + i\sqrt{3}, -\frac{2}{3}$ **48** Verify

49 a) $k = 0 \pm 1$ b) $k = \pm\sqrt{3 \pm 2\sqrt{2}}$

50 $z_1 = 1 + i, z_2 = 2 - i$ **51** $z_1 = \frac{7 - 4i}{3}, z_2 = \frac{1 + 6i}{3}$

Exercise 10.2

1 $2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$ **2** $2\operatorname{cis}\left(\frac{\pi}{6}\right)$ **3** $2\sqrt{2}\operatorname{cis}\left(\frac{7\pi}{4}\right)$

4 $2\sqrt{2}\operatorname{cis}\left(\frac{11\pi}{6}\right)$ **5** $4\operatorname{cis}\frac{5\pi}{3}$ **6** $3\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

7 $4\operatorname{cis}\left(\frac{\pi}{2}\right)$ **8** $6\operatorname{cis}\left(\frac{7\pi}{6}\right)$ **9** $\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$

10 $15\operatorname{cis}\pi$ **11** $\frac{1}{5}\operatorname{cis}(5.64)$ **12** $3\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

13 $\pi\operatorname{cis}(0)$ **14** $e\operatorname{cis}\left(\frac{\pi}{2}\right)$ **15** $\frac{-\sqrt{3}}{2} + \frac{i}{2}, \frac{\sqrt{3}}{2} + \frac{i}{2}$

16 $1, \frac{1}{2} - \frac{\sqrt{3}\,i}{2}$ **17** $\frac{-\sqrt{3}}{2} + \frac{i}{2}, -i$ **18** $-i, \frac{-1}{2} + \frac{\sqrt{3}\,i}{2}$

19 $\frac{\sqrt{6} + \sqrt{2}}{2} + i\frac{\sqrt{6} - \sqrt{2}}{2}, \frac{9(-\sqrt{6} + \sqrt{2})}{8} - i\frac{9(\sqrt{6} + \sqrt{2})}{8}$

20 $-3\sqrt{3} - 3 + i(3\sqrt{3} - 3), \frac{3\sqrt{3} - 3}{4} - \frac{i(3\sqrt{3} + 3)}{4}$

21 $\frac{-\sqrt{2}}{2}(1 + i), \frac{\sqrt{2}}{2}(1 + i)$

22 $6, \frac{-3}{4} - \frac{3\sqrt{3}i}{4}$

23 $\frac{5\sqrt{6} - 15\sqrt{2}}{48} - i\frac{5\sqrt{6} + 15\sqrt{2}}{48}, \frac{-5\sqrt{6} - 15\sqrt{2}}{64} + i\frac{5\sqrt{6} - 15\sqrt{2}}{64}$

24 $-3\sqrt{3} + 3 + i(3\sqrt{3} + 3), \frac{3\sqrt{3} + 3}{4} + \frac{i(3\sqrt{3} - 3)}{4}$

25 $z_1 = 2\operatorname{cis}\frac{\pi}{6}, z_2 = 4\operatorname{cis}\frac{-\pi}{3}, \frac{1}{z_1} = \frac{1}{2}\operatorname{cis} -\frac{\pi}{6}, \frac{1}{z_2} = \frac{1}{4}\operatorname{cis}\frac{\pi}{3},$

$z_1z_2 = 8\operatorname{cis}\frac{-\pi}{6}, \frac{z_1}{z_2} = \frac{1}{2}\operatorname{cis}\frac{\pi}{2}$

26 $z_1 = 2\sqrt{2}\operatorname{cis}\frac{\pi}{6}, z_2 = 4\sqrt{3}\operatorname{cis}\frac{-\pi}{3}, \frac{1}{z_1} = \frac{\sqrt{2}}{4}\operatorname{cis}\frac{\pi}{6}, \frac{1}{z_2} = \frac{\sqrt{3}}{12}\operatorname{cis}\frac{\pi}{3}$

,

$z_1z_2 = 8\sqrt{6}\operatorname{cis}\frac{-\pi}{6}, \frac{z_1}{z_2} = \frac{\sqrt{6}}{6}\operatorname{cis}\frac{\pi}{2}$

27 $z_1 = 8\operatorname{cis}\frac{\pi}{6}, z_2 = 3\sqrt{2}\operatorname{cis}\frac{-3\pi}{4}, \frac{1}{z_1} = \frac{1}{8}\operatorname{cis}\frac{-\pi}{6}, \frac{1}{z_2} = \frac{\sqrt{2}}{6}\operatorname{cis}\frac{3\pi}{4},$

$z_1z_2 = 24\sqrt{2}\operatorname{cis}\frac{-7\pi}{12}, \frac{z_1}{z_2} = \frac{4\sqrt{2}}{3}\operatorname{cis}\frac{11\pi}{12}$

28 $z_1 = \sqrt{3}\operatorname{cis}\frac{\pi}{2}, z_2 = 2\sqrt{2}\operatorname{cis}\frac{-2\pi}{3}, \frac{1}{z_1} = \frac{\sqrt{3}}{3}\operatorname{cis}\frac{-\pi}{2},$

$\frac{1}{z_2} = \frac{\sqrt{2}}{8}\operatorname{cis}\frac{2\pi}{3}, z_1z_2 = 2\sqrt{6}\operatorname{cis}\frac{-\pi}{6}, \frac{z_1}{z_2} = \frac{\sqrt{6}}{4}\operatorname{cis}\frac{-5\pi}{6}$

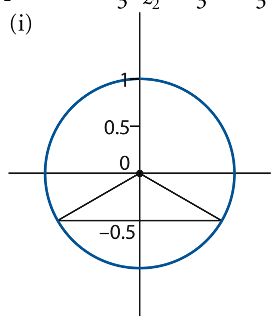
29 $z_1 = \sqrt{10}\operatorname{cis}\frac{\pi}{4}, z_2 = 2\sqrt{2}\operatorname{cis}\frac{\pi}{2}, \frac{1}{z_1} = \frac{\sqrt{10}}{10}\operatorname{cis}\frac{\pi}{4}, \frac{1}{z_2} = \frac{\sqrt{2}}{8}\operatorname{cis}\frac{\pi}{2}$

$, z_1z_2 = 4\sqrt{5}\operatorname{cis}\frac{3\pi}{4}, \frac{z_1}{z_2} = \frac{\sqrt{5}}{2}\operatorname{cis}\frac{-\pi}{4}$

30 $z_1 = 2\operatorname{cis}\frac{\pi}{3}, z_2 = 2\sqrt{3}\operatorname{cis}0, \frac{1}{z_1} = \frac{1}{2}\operatorname{cis}\frac{-\pi}{3}, \frac{1}{z_2} = \frac{\sqrt{3}}{6}\operatorname{cis}0,$

$z_1z_2 = 4\sqrt{3}\operatorname{cis}\frac{\pi}{3}, \frac{z_1}{z_2} = \frac{\sqrt{3}}{3}\operatorname{cis}\frac{\pi}{3}$

31 b) (i)



(ii) $\arg(z_1) = \frac{-\pi}{6}, \arg(z_2) = \frac{-5\pi}{6}$

32 Verify

33 a) $\frac{\sqrt{3}}{2} - \frac{3i}{2}$ b) $\frac{-2\sqrt{3}}{3}$ c) $\sqrt{3}i$

34 $|z_1| = 4, \arg(z_1) = \frac{-\pi}{6}, |z_2| = 2\sqrt{2}, \arg(z_2) = \frac{\pi}{4}, |z_3| = 8\sqrt{2},$

$\arg(z_3) = \frac{\pi}{12}$

35 $22 - 2\sqrt{3} \approx 18.5$

36 a) $\{(x, y): x^2 + y^2 = 9\}$, the circle centre $(0, 0)$ radius 3

b) $\{(x, y): x = 0\}$, the y-axis

c) $\{(x, y): x = 4\}$, the line $x = 4$

d) $\{(x, y): (x - 3)^2 + y^2 = 4\}$, the circle centre $(3, 0)$ radius 2

e) $\{(x, y): 1 - x + 3$ and $y = 0\}$, the line segment between $(1, 0)$ and $(3, 0)$