Name:

Mathematics IB HL Test 4

December 16, 2021

1 hour 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Calculators are **not allowed** for this examination paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is [**72 marks**].
- Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to **show all working**.
- Write your solutions in the space provided.

1. [Maximum mark: 7]

The first three terms of an arithmetic sequence are $2k+1, k+2$ and $8-k$.

a) Find the value k. [2 *points*]

b) Find the sum of the first 10 terms of this sequence. [2 *points*]

c) The first three terms of this sequence have been decreased by x, $2x$ and x respectively and now form three consecutive (in the given order) terms of a geometric sequence. Find the possible values of x. [3 *points*]

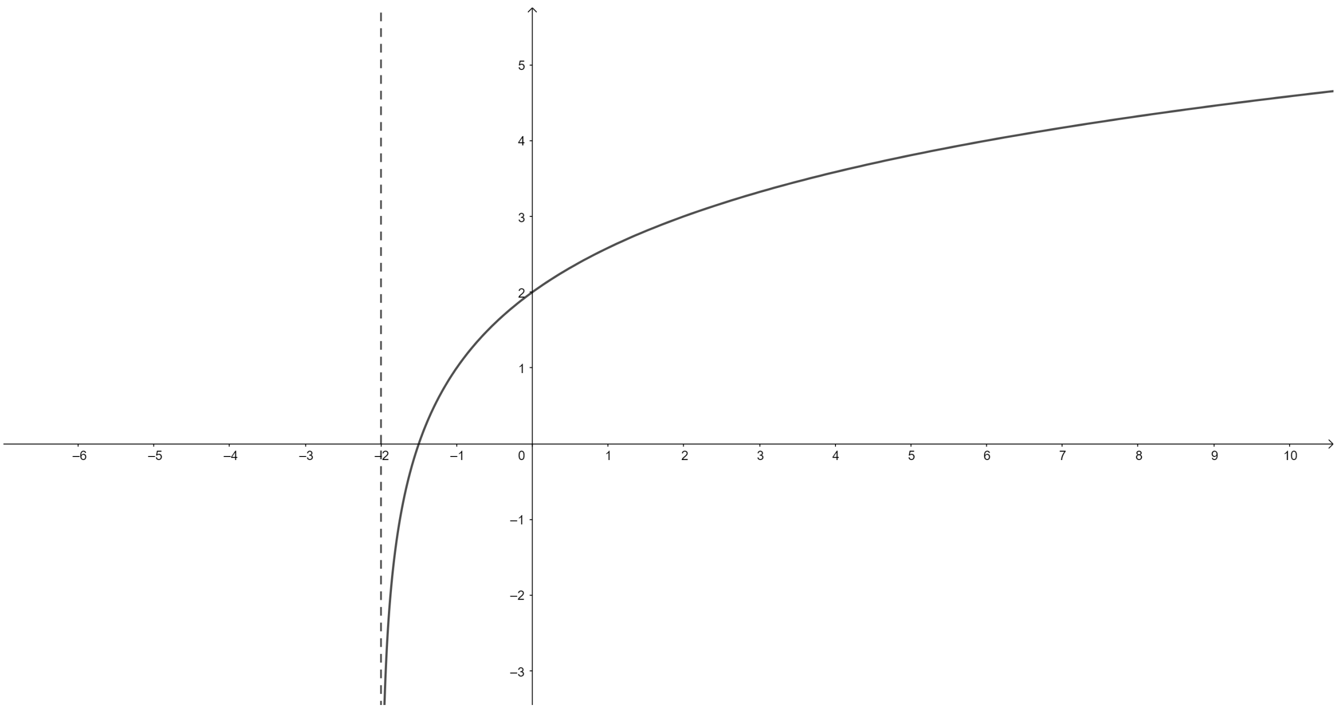
2. [Maximum mark: 5]

Solve the equation:

$$\log_2 x^2 + \log_x 32 = 11$$

3. [Maximum mark: 7]

The diagram below shows the graph of $f(x) = \log_2(ax + b)$. The graph has a vertical asymptote at $x = -2$ and crosses the y-axis at $y = 2$.



a) Find a and b. [2 *points*]

b) Find the x-intercept of the graph. [2 *points*]

c) On the same diagram sketch the graph of $y = \frac{1}{f(x)}$. Clearly indicate any axis intercepts and asymptotes. [3 *points*]

4. [Maximum mark: 7]

Let $z_1 = 8 + i$, $z_2 = 2 + pi$ and $z_3 = q - 2i$ with $p > q$.

a) Given that $z_2 z_3 = z_1^*$, find the value of p and the value of q. [4 *points*]

b) Find the value of a (with $a \in \mathbb{R}$) such that $z_1 + z_2 + az_3$ is a real number and find this number. [3 *points*]

5. [Maximum mark: 7]

Let $P(x) = x^4 + 2x^3 + 6x^2 + 8x + 8$.

a) Show that $x = 2i$ is a root of this polynomial. [2 *points*]

b) Find the remaining roots. [3 *points*]

c) Hence, or otherwise, write $P(x)$ in the form $(x^2+ax+b)(x^2+cx+d)$, where a, b, c and d are real numbers. [2 *points*]

6. [Maximum mark: 6]

Given that $\tan\theta = 2$ and that $\pi < \theta < \frac{3\pi}{2}$, calculate:

a) $\sin\theta$ [2 *points*]

b) $\sin 2\theta$ [2 *points*]

c) $\sin(2\theta + \frac{\pi}{3})$ [2 *points*]

7. [Maximum mark: 5]

Consider the function $f(x) = e^{\arctan x}$.

a) Write down the domain of f. [1 *point*]

b) Find the range of f. [2 *points*]

c) Find f^{-1}, the inverse of f, and state its domain and range. [2 *points*]

8. [Maximum mark: 6]

a) By writing $5\theta = 3\theta + 2\theta$ and $\theta = 3\theta - 2\theta$ show that: [3 *points*]

$$\sin 5\theta - \sin\theta \equiv 2\cos 3\theta \sin 2\theta$$

b) Hence, or otherwise, solve the equation: [3 *points*]

$$\sin 5\theta - \sin\theta = \cos 3\theta$$

for $0 \leqslant \theta \leqslant \pi$.

9. [Maximum mark: 6]

Let $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

a) Sketch the graph of $y = f(x)$. Clearly indicate axes intercepts and asymptotes. [4 *points*]

b) By sketching the graph of $y = |f(x)|$, or otherwise, find the set of all values of k, such that the equation:

$$|f(x)| = k$$

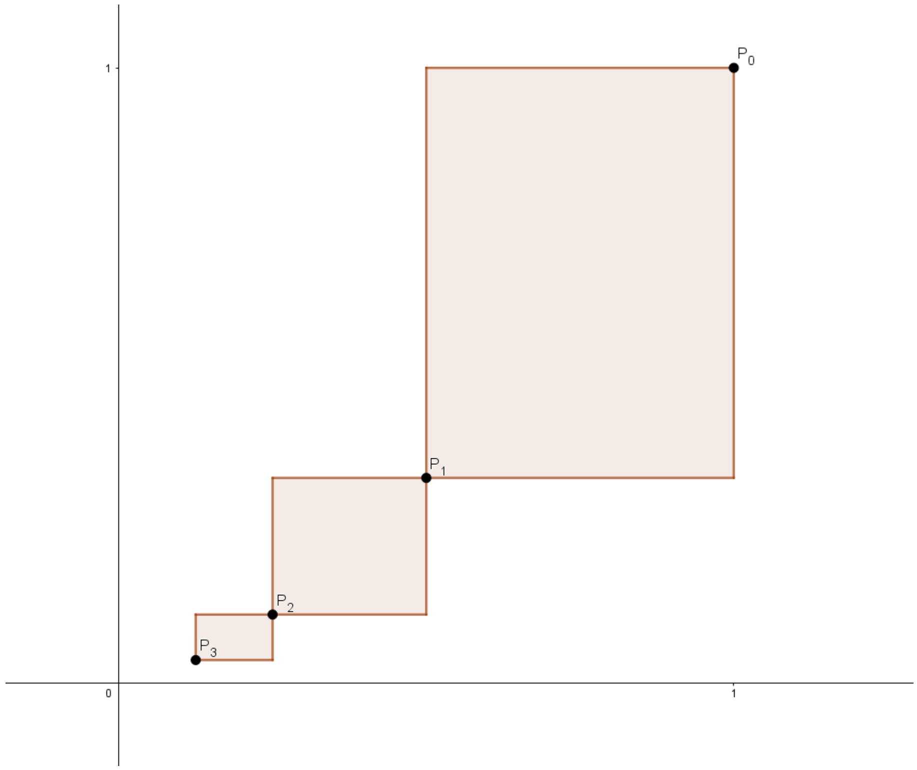
has more positive than negative solutions. [2 *points*]

10. [Maximum mark: 16]

An infinite sequence of points is defined as follows:

$$P_0 = (1, 1),\ P_1 = (\tfrac{1}{2}, \tfrac{1}{3}),\ P_2 = (\tfrac{1}{4}, \tfrac{1}{9}), ...,\ P_n = (\tfrac{1}{2^n}, \tfrac{1}{3^n}), ...$$

An infinite sequence of rectangles is created so that rectangle R_n has sides parallel to the axes and two of its vertices at points P_{n-1} and P_n (see diagram).



a) Find the area of rectangle R_1. [2 *points*]

b) Show that the area of rectangle R_n can be expressed in the form $\dfrac{a}{6^n}$, where a is a constant to be found and $a \in \mathbb{N}$. [4 *points*]

c) Show that the areas of the rectangles form a geometric sequence and state its ratio. [2 *points*]

d) Find the total area of the infinite sequence of rectangles. [2 *points*]

e) Let Q_n be a circle circumscribed on rectangle R_n. Let $A(Q_n)$ denote the area of circle Q_n.
Find $\displaystyle\sum_{j=1}^{\infty} A(Q_j)$. [6 *points*]