3.4 Rational functions

Another important category of algebraic functions is rational functions, which are functions in the form $R(x) = \frac{f(x)}{g(x)}$ where f and g are polynomials and the domain of the function R is the set of all real numbers except the real zeros of polynomial g in the denominator. Some examples of rational functions are

$$p(x) = \frac{1}{x-5}, \quad q(x) = \frac{x+2}{(x+3)(x-1)}, \quad \text{and} \quad r(x) = \frac{x}{x^2+1}$$

The domain of p excludes $x = 5$, and the domain of q excludes $x = -3$ and $x = 1$. The domain of r is all real numbers because the polynomial $x^2 + 1$ has no real zeros.

Example 24

Find the domain and range of $h(x) = \frac{1}{x-2}$. Sketch the graph of h.

Solution

Because the denominator is zero when $x = 2$, the domain of h is all real numbers except $x = 2$, i.e. $x \in \mathbb{R}, x \neq 2$. Determining the range of the function is a little less straightforward. It is clear that the function could never take on a value of zero because that will only occur if the numerator is zero. And since the denominator can have any value except zero it seems that the function values of h could be any real number except zero. To confirm this and to determine the behaviour of the function (and shape of the graph), some values of the domain and range (pairs of coordinates) are displayed in the tables below.

● **Hint:** A fraction is only zero if its numerator is zero.

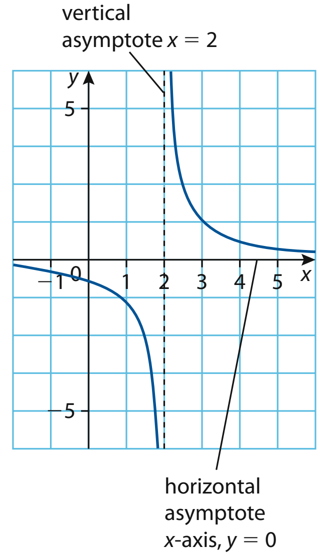
x approaches 2 from the left

x	$h(x)$
−98	−0.01
−8	−0.1
0	−0.5
1	−1
1.5	−2
1.9	−10
1.99	−100
1.999	−1000

x approaches 2 from the right

x	$h(x)$
102	0.01
12	0.1
4	0.5
3	1
2.5	2
2.1	10
2.01	100
2.001	1000

The values in the tables provide clear evidence that the range of h is all real numbers except zero, i.e. $h(x) \in \mathbb{R}, h(x) \neq 0$. The values in the tables also show that as $x \to -\infty$, $h(x) \to 0$ from below (sometimes written $h(x) \to 0^-$) and as $x \to +\infty$, $h(x) \to 0$ from above ($h(x) \to 0^+$). It follows that the line with equation $y = 0$ (the x-axis) is a horizontal asymptote for the graph of h. As $x \to 2$ from the left (sometimes written $x \to 2^-$), $h(x)$ appears to decrease without bound, whereas as $x \to 2$ from the right ($x \to 2^+$), $h(x)$ appears to increase without bound. This indicates that the graph of h will have a vertical asymptote at $x = 2$. This behaviour is confirmed by the graph at left.



Horizontal and vertical asymptotes

The line $y = c$ is a **horizontal asymptote** of the graph of the function f if at least one of the following statements is true:

- as $x \to +\infty$, then $f(x) \to c^+$
- as $x \to -\infty$, then $f(x) \to c^+$
- as $x \to +\infty$, then $f(x) \to c^-$
- as $x \to -\infty$, then $f(x) \to c^-$

The line $x = d$ is a **vertical asymptote** of the graph of the function f if at least one of the following statements is true:

- as $x \to d^+$, then $f(x) \to +\infty$
- as $x \to d^+$, then $f(x) \to -\infty$
- as $x \to d^-$, then $f(x) \to +\infty$
- as $x \to d^-$, then $f(x) \to -\infty$

Example 25

Consider the function $f(x) = \dfrac{3x^2 - 12}{x^2 + 3x - 4}$. Sketch the graph of f and identify any asymptotes and any x- or y-intercepts. Use the sketch to confirm the domain and range of the function.

Solution

Firstly, let's completely factorize both the numerator and denominator.

$$f(x) = \frac{3x^2 - 12}{x^2 + 3x - 4} = \frac{3(x + 2)(x - 2)}{(x - 1)(x + 4)}$$

Axis intercepts:

The x-intercepts will occur where the numerator is zero. Hence, the x-intercepts are $(-2, 0)$ and $(2, 0)$. A y-intercept will occur when $x = 0$.

$f(0) = \dfrac{3(2)(-2)}{(-1)(4)} = 3$, so the y-intercept is $(0, 3)$.

Vertical asymptote(s):

Any vertical asymptote will occur where the denominator is zero, that is, where the function is undefined. From the factored form of f we see that the vertical asymptotes are $x = 1$ and $x = -4$. We need to determine if the graph of f falls ($f(x) \to -\infty$) or rises ($f(x) \to \infty$) on either side of each vertical asymptote. It's easiest to do this by simply analyzing what the sign of h will be as x approaches 1 and -4 from both the left and right. For example, as $x \to 1^-$ we can use a test value close to and to the left of 1 (e.g. $x = 0.9$) to check whether $f(x)$ is positive or negative to the left of 1.

$$f(x) = \frac{3(0.9 + 2)(0.9 - 2)}{(0.9 - 1)(0.9 + 4)} \Rightarrow \frac{(+)(-)}{(-)(+)} \Rightarrow f(x) > 0 \Rightarrow \text{as } x \to 1^-,$$

then $f(x) \to +\infty$ (rises)

As $x \to 1^+$ we use a test value close to and to the right of 1 (e.g. $x = 1.1$) to check whether $f(x)$ is positive or negative to the right of 1.

● **Hint:** The farther the number n is from 0, the closer the number $\frac{1}{n}$ is to 0. Conversely, the closer the number n is to 0, the farther the number $\frac{1}{n}$ is from 0. These facts can be expressed simply as:

$$\frac{1}{\text{BIG}} = \text{little and } \frac{1}{\text{little}} = \text{BIG}$$

They can also be expressed more mathematically using the concept of a limit expressed in limit notation as: $\lim_{n\to\infty}\frac{1}{n} = 0$ and $\lim_{n\to 0}\frac{1}{n} = \infty$.
Note: Infinity is not a number, so $\lim_{n\to 0}\frac{1}{n}$ actually does not exist, but writing $\lim_{n\to 0}\frac{1}{n} = \infty$ expresses the idea that $\frac{1}{n}$ increases without bound as n approaches 0.

$$f(x) = \frac{3(1.1+2)(1.1-2)}{(1.1-1)(1.1+4)} \Rightarrow \frac{(+)(-)}{(+)(+)} \Rightarrow f(x) < 0 \Rightarrow \text{as } x \to 1^+,$$
then $f(x) \to -\infty$ (falls)

Conducting similar analysis for the vertical asymptote of $x = -4$, produces:

$$f(x) = \frac{3(-4.1+2)(-4.1-2)}{(-4.1-1)(-4.1+4)} \Rightarrow \frac{(-)(-)}{(-)(-)} \Rightarrow f(x) > 0 \Rightarrow \text{as } x \to 4^-,$$
then $f(x) \to +\infty$ (rises)

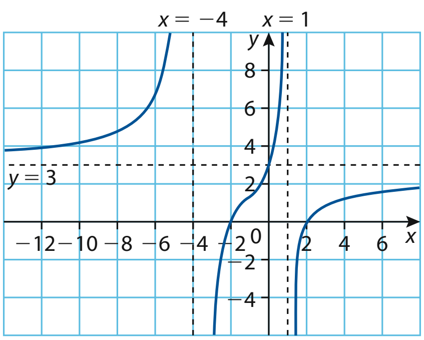
$$f(x) = \frac{3(-3.9+2)(-3.9-2)}{(-3.9-1)(-3.9+4)} \Rightarrow \frac{(-)(-)}{(-)(+)} \Rightarrow f(x) < 0 \Rightarrow \text{as } x \to 4^+,$$
then $f(x) \to -\infty$ (falls)

Horizontal asymptote(s):

A horizontal asymptote (if it exists) is the value that $f(x)$ approaches as $x \to \pm\infty$. To find this value, we divide both the numerator and denominator by the highest power of x that appears in the denominator (x^2 for function f).

$$f(x) = \frac{\frac{3x^2}{x^2} - \frac{12}{x^2}}{\frac{x^2}{x^2} + \frac{3x}{x^2} - \frac{4}{x^2}} \text{ then, as } x \to \pm\infty, f(x) = \frac{3-0}{1+0-0} = 3$$

Hence, the horizontal asymptote is $y = 3$.



Sketch of graph:

Now we know the behaviour (rising or falling) of the function on either side of each vertical asymptote and that the graph will approach the horizontal asymptote as $x \to \pm\infty$, an accurate sketch of the graph can be made as shown right.

Domain and range:

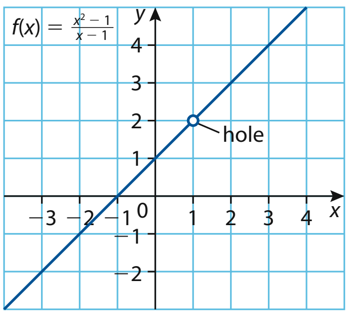
Because the zeros of the polynomial in the denominator are $x = 1$ and $x = -4$, the domain of f is all real numbers except 1 and -4.
From our analysis and from the sketch of the graph, it is clear that between $x = -4$ and $x = 1$ the function takes on all values from $-\infty$ to $+\infty$, therefore the range of f is all real numbers.

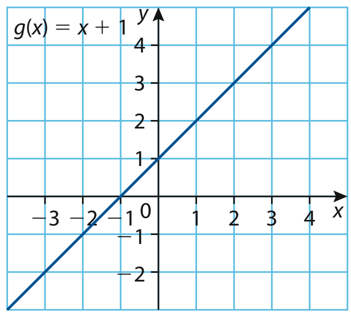
We are in the habit of cancelling factors in algebraic expressions (Section 1.5), such as

$$\frac{x^2-1}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = x+1$$

However, the function $f(x) = \frac{x^2-1}{x-1}$ and the function $g(x) = x + 1$ are **not** the same function. The difference occurs when $x = 1$.
$f(1) = \frac{1^2-1}{1-1} = \frac{0}{0}$, which is undefined, and $g(1) = 1 + 1 = 2$. So, 1 is not in the domain of f but it is in the domain of g. As we might expect the

graphs of the two functions appear identical, but upon closer inspection it is clear that there is a 'hole' in the graph of f at the point $(1, 2)$. Thus, f is a *discontinuous* function but the polynomial function g is continuous. f and g are different functions.





● **Hint:** Try graphing $\dfrac{x^2 - 1}{x - 1}$ on your GDC and zooming in closely to the region around the point $(1, 2)$. Can you see the 'hole'?

In working with rational functions, we often assume that every linear factor that appears in both the numerator and in the denominator has been cancelled. Therefore, for a rational function in the form $\dfrac{f(x)}{g(x)}$, we can usually assume that the polynomial functions f and g have no common factors.

Example 26

Find any asymptotes for the function $p(x) = \dfrac{x^2 - 9}{x - 4}$.

Solution

The denominator is zero when $x = 4$, thus the line with equation $x = 4$ is a vertical asymptote. Although the numerator $x^2 - 9$ is not divisible by $x - 4$, it does have a larger degree. Some insight into the behaviour of function p may be gained by dividing $x - 4$ into $x^2 - 9$. Since the degree of the numerator is one greater than the degree of the denominator, the quotient will be a linear polynomial. Recalling from the previous section that $\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$, where Q and R are the quotient and remainder, we can rewrite $p(x)$ as a linear polynomial plus a fraction.

Since the denominator is in the form $x - c$ we can carry out the division efficiently by means of synthetic division.

$$\begin{array}{r|rrr} 4 & 1 & 0 & -9 \\ & & 4 & 16 \\ \hline & 1 & 4 & 7 \end{array}$$

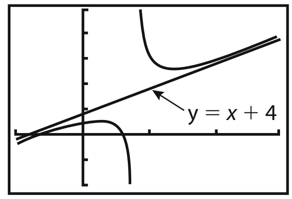
Hence, $p(x) = \dfrac{x^2 - 9}{x - 4} = x + 4 + \dfrac{7}{x - 4}$.

As $x \to \pm\infty$, the fraction $\dfrac{7}{x - 4} \to 0$. This tells us about the end behaviour of function p, namely that the graph of p will get closer and closer to the line $y = x + 4$ as the values of x get further away from the origin. Symbolically, this can be expressed as follows: as $x \to \pm\infty$, $p(x) \to x + 4$.

We can graph both the rational function $p(x)$ and the line $y = x + 4$ on our GDC to visually confirm our analysis.

```
 Plot1  Plot2  Plot3
\Y1■(X^2-9)/(X-4
)
\Y2■X+4
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=-5
 Xmax=15
 Xscl=5
 Ymin=-10
 Ymax=25
 Yscl=5
 Xres=1
```



If a line is an asymptote of a graph but it is neither horizontal nor vertical, it is called an **oblique asymptote** (sometimes called a slant asymptote).

The graph of any rational function of the form $\frac{f(x)}{g(x)}$, where the degree of function f is one more than the degree of function g will have an oblique asymptote.

Using Example 25 as a model, we can set out a general procedure for analyzing a rational function leading to a sketch of its graph and determining its domain and range.

Analyzing a rational function $R(x) = \frac{f(x)}{g(x)}$ given functions f and g have no common factors

1. **Factorize**: Completely factorize both the numerator and denominator.
2. **Intercepts**: A zero of f will be a zero of R and hence an x-intercept of the graph of R. The y-intercept is found by evaluating $R(0)$.
3. **Vertical asymptotes**: A zero of g will give the location of a vertical asymptote (if any). Then perform a sign analysis to see if $R(x) \to +\infty$ or $R(x) \to -\infty$ on either side of each vertical asymptote.
4. **Horizontal asymptote**: Find the horizontal asymptote (if any) by dividing both f and g by the highest power of x that appears in g, and then letting $x \to \pm\infty$.
5. **Oblique asymptotes**: If the degree of f is one more than the degree of g, then the graph of R will have an oblique asymptote. Divide g into f to find the quotient $Q(x)$ and remainder. The oblique asymptote will be the line with equation $y = Q(x)$.
6. **Sketch of graph**: Start by drawing dashed lines where the asymptotes are located. Use the information about the intercepts, whether $Q(x)$ falls or rises on either side of a vertical asymptote, and additional points as needed to make an accurate sketch.
7. **Domain and range**: The domain of R will be all real numbers except the zeros of g. You need to study the graph carefully in order to determine the range. Often, but not always (as in Example 25), the value of the function at the horizontal asymptote will not be included in the range.

End behaviour of a rational function

Let R be the rational function given by

$$R(x) = \frac{f(x)}{g(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$$

where functions f and g have no common factors. Then the following holds true:

1. If $n < m$, then the x-axis (line $y = 0$) is a horizontal asymptote for the graph of R.
2. If $n = m$, then the line $y = \frac{a_n}{b_m}$ is a horizontal asymptote for the graph of R.
3. If $n > m$, then the graph of R has no horizontal asymptote. However, if the degree of f is one more than the degree of g, then the graph of R will have an oblique asymptote.

Exercise 3.4

In questions 1–10, sketch the graph of the rational function without the aid of your GDC. On your sketch clearly indicate any x- or y-intercepts and any asymptotes (vertical, horizontal or oblique). Use your GDC to verify your sketch.

1 $f(x) = \dfrac{1}{x+2}$

2 $g(x) = \dfrac{3}{x-2}$

3 $h(x) = \dfrac{1-4x}{1-x}$

4 $R(x) = \dfrac{x}{x^2-9}$

5 $p(x) = \dfrac{2}{x^2+2x-3}$

6 $M(x) = \dfrac{x^2+1}{x}$

7 $f(x) = \dfrac{x}{x^2+4x+4}$

8 $h(x) = \dfrac{x^2+2x}{x-1}$

9 $g(x) = \dfrac{2x+8}{x^2-x-12}$

10 $C(x) = \dfrac{x-2}{x^2-4x}$

In questions 11–14, use your GDC to sketch a graph of the function, and state the domain and range of the function.

11 $f(x) = \dfrac{2x^2+5}{x^2-4}$

12 $g(x) = \dfrac{x+4}{x^2+3x-4}$

13 $h(x) = \dfrac{6}{x^2+6}$

14 $r(x) = \dfrac{x^2-2x+1}{x-1}$

In questions 15–18, use your GDC to sketch a graph of the function. Clearly label any x- or y-intercepts and any asymptotes.

15 $f(x) = \dfrac{2x-5}{2x^2+9x-18}$

16 $g(x) = \dfrac{x^2+x+1}{x-1}$

17 $h(x) = \dfrac{3x^2}{x^2+x+2}$

18 $g(x) = \dfrac{1}{x^3-x^2-4x+4}$

19 If a, b and c are all positive, sketch the curve $y = \dfrac{x-a}{(x-b)(x-c)}$ for each of the following conditions:

a) $a < b < c$ b) $b < a < c$ c) $b < c < a$

20 A drug is given to a patient and the concentration of the drug in the bloodstream is carefully monitored. At time $t \geqslant 0$ (in minutes after patient receiving the drug), the concentration, in milligrams per litre (mg/l) is given by the following function.

$$C(t) = \frac{25t}{t^2+4}$$

a) Sketch a graph of the drug concentration (mg/l) versus time (min).

b) When does the highest concentration of the drug occur, and what is it?

c) What eventually happens to the concentration of the drug in the bloodstream?

d) How long does it take for the concentration to drop below 0.5 mg/l?

4 Sequences and Series

Assessment statements

1.1 Arithmetic sequences and series; sum of finite arithmetic sequences; geometric sequences and series; sum of finite and infinite geometric series.
Sigma notation.

1.3 Counting principles, including permutations and combinations. The binomial theorem: expansion of $(a + b)^n$, $n \in \mathbb{N}$.

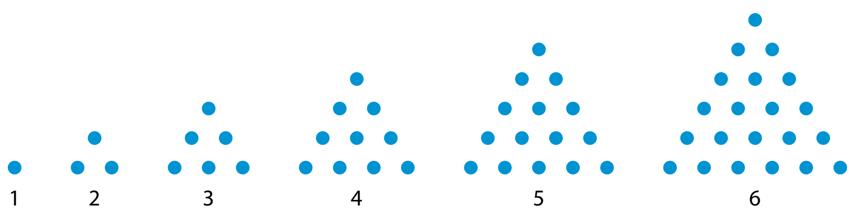
1.4 Proof by mathematical induction.

Introduction

The heights of consecutive bounds of a ball, compound interest, and Fibonacci numbers are only a few of the applications of sequences and series that you have seen in previous courses. In this chapter you will review these concepts, consolidate your understanding and take them one step further.

4.1 Sequences

Take the following pattern as an example:



The first figure represents 1 dot, the second represents 3 dots, etc. This pattern can also be described differently. For example, in function notation:

$$f(1) = 1, f(2) = 3, f(3) = 6, \text{ etc., where the domain is } \mathbb{Z}^+$$

Here are some more examples of sequences:

1 6, 12, 18, 24, 30

2 $3, 9, 27, \ldots, 3^k, \ldots$

3 $\left\{\frac{1}{i^2}; i = 1, 2, 3, \ldots, 10\right\}$

4 $\{b_1, b_2, \ldots, b_n, \ldots\}$, sometimes used with an abbreviation $\{b_n\}$

The first and third sequences are **finite** and the second and fourth are **infinite**. Notice that, in the second and third sequences, we were able to define a rule that yields the *n*th number in the sequence (called the *n*th term) as a function of n, the term's number. In this sense, a sequence is a **function** that assigns a **unique** number (a_n) to each positive integer n.

Example 1

Find the first five terms and the 50th term of the sequence $\{b_n\}$ such that $b_n = 2 - \frac{1}{n^2}$.

Solution

Since we know an *explicit* expression for the *n*th term as a *function* of its number n, we only need to find the value of that function for the required terms:

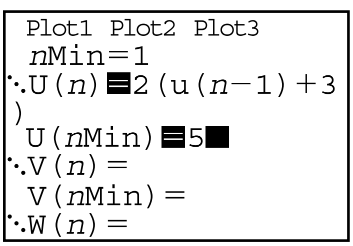
$$b_1 = 2 - \frac{1}{1^2} = 1;\ b_2 = 2 - \frac{1}{2^2} = 1\frac{3}{4};\ b_3 = 2 - \frac{1}{3^2} = 1\frac{8}{9};\ b_4 = 2 - \frac{1}{4^2} = 1\frac{15}{16};$$

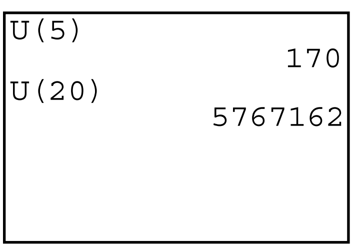
$$b_5 = 2 - \frac{1}{5^2} = 1\frac{24}{25};\ \text{and}\ \ b_{50} = 2 - \frac{1}{50^2} = 1\frac{2499}{2500}.$$

So, informally, **a sequence is an ordered set of real numbers**. That is, there is a first number, a second, and so forth. The notation used for such sets is shown above. The way we defined the function in Example 1 is called the **explicit** definition of a sequence. There are other ways to define sequences, one of which is the **recursive** definition. The following example will show you how this is used.

Example 2

Find the first five terms and the 20th term of the sequence $\{b_n\}$ such that $b_1 = 5$ and $b_n = 2(b_{n-1} + 3)$.





Solution

The defining formula for this sequence is recursive. It allows us to find the *n*th term b_n if we know the preceding term b_{n-1}. Thus, we can find the second term from the first, the third from the second, and so on. Since we know the first term, $b_1 = 5$, we can calculate the rest:

$$b_2 = 2(\mathbf{b_1} + 3) = 2(\mathbf{5} + 3) = 16$$
$$b_3 = 2(\mathbf{b_2} + 3) = 2(\mathbf{16} + 3) = 38$$
$$b_4 = 2(\mathbf{b_3} + 3) = 2(\mathbf{38} + 3) = 82$$
$$b_5 = 2(\mathbf{b_4} + 3) = 2(\mathbf{82} + 3) = 170$$

Thus, the first five terms of this sequence are 5, 16, 38, 82, 170. However, to find the 20th term, we must first find all 19 preceding terms. This is one of the drawbacks of the recursive definition, unless we can change the definition into explicit form. This can easily be done using a GDC.

Example 3

A Fibonacci sequence is defined recursively as

$$F_n = \begin{cases} 1 & n = 1 \\ 1 & n = 2 \\ F_{n-1} + F_{n-2} & n > 2 \end{cases}$$

a) Find the first 10 terms of the sequence.

b) Evaluate $S_n = \sum_{i=1}^{n} F_i$ for $n = 1, 2, 3, \ldots, 10$.

c) By observing that $F_1 = F_3 - F_2$, $F_2 = F_4 - F_3$, and so on, derive a formula for the sum of the first n Fibonacci numbers.

> **Fibonacci numbers** are a sequence of numbers named after Leonardo of Pisa, known as Fibonacci (a short form of filius Bonaccio, 'son of Bonaccio').

Solution

a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55

b) $S_1 = 1$, $S_2 = 2$, $S_3 = 4$, $S_4 = 7$, $S_5 = 12$, $S_6 = 20$, $S_7 = 33$, $S_8 = 54$, $S_9 = 88$, $S_{10} = 143$

c) Since $F_3 = F_2 + F_1$, then

$$\begin{aligned}
F_1 &= \cancel{F_3} - F_2 \\
F_2 &= \cancel{F_4} - \cancel{F_3} \\
F_3 &= \cancel{F_5} - \cancel{F_4} \\
F_4 &= \cancel{F_6} - \cancel{F_5} \\
&\vdots \quad\quad \vdots \quad\quad \vdots \\
\underline{F_n} &= \underline{F_{n+2} - \cancel{F_{n+1}}} \\
S_n &= F_{n+2} - F_2
\end{aligned}$$

Notice that $S_5 = 12 = F_7 - F_2 = 13 - 1$ and $S_8 = 54 = F_{10} - F_2 = 55 - 1$.

Note: parts a) and b) can be made easy by using a spreadsheet. Here is an example:

	A	B	C	D
1	*F*(*n*)	*S*(*n*)		
2	1	1		
3	1	2		
4	**2**	4		
5	3	7		
6	5	12		
7	8	20		
8	13	33		
9	21	54		
10	34	88		
11	55	**143**		Let this cell be A2 + A3 Then copy it down
12	89	232		
13	144	376		
14	233	609		
15	377	986		Let this cell be B10 + A11 Then copy it down
16	610	1596		
17	987	2583		

Notice that not all sequences have formulae, either recursive or explicit. Some sequences are given only by listing their terms. Among the many kinds of sequences that there are, two types are of interest to us: arithmetic and geometric sequences, which we will discuss in the next two sections.

Exercise 4.1

Find the first five terms of each infinite sequence defined in questions 1–6.

1 $s(n) = 2n - 3$

2 $g(k) = 2^k - 3$

3 $f(n) = 3 \times 2^{-n}$

4 $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + 3; \text{ for } n > 1 \end{cases}$

5 $a_n = (-1)^n(2^n) + 3$

6 $\begin{cases} b_1 = 3 \\ b_n = b_{n-1} + 2n; \text{ for } n \geqslant 2 \end{cases}$

Find the first five terms and the 50th term of each infinite sequence defined in questions 7–14.

7 $a_n = 2n - 3$

8 $b_n = 2 \times 3^{n-1}$

9 $u_n = (-1)^{n-1}\dfrac{2n}{n^2 + 2}$

10 $a_n = n^{n-1}$

11 $a_n = 2a_{n-1} + 5$ and $a_1 = 3$

12 $u_{n+1} = \dfrac{3}{2u_n + 1}$ and $u_1 = 0$

13 $b_n = 3 \cdot b_{n-1}$ and $b_1 = 2$

14 $a_n = a_{n-1} + 2$ and $a_1 = -1$

Suggest a recursive definition for each sequence in questions 15–17.

15 $\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \frac{1}{192}, \ldots$

16 $\frac{1}{2}a, \frac{2}{3}a^3, \frac{8}{9}a^5, \frac{32}{27}a^7, \ldots$

17 $a - 5k, 2a - 4k, 3a - 3k, 4a - 2k, 5a - k, \ldots$

In questions 18–21, write down a possible formula that gives the nth term of each sequence.

18 $4, 7, 12, 19, \ldots$

19 $2, 5, 8, 11, \ldots$

20 $1, \frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \ldots$

21 $\frac{1}{4}, \frac{3}{5}, \frac{5}{6}, 1, \frac{9}{8}, \ldots$

22 Define $a_n = \dfrac{F_{n+1}}{F_n}$, $n > 1$, where F_n is a member of a Fibonacci sequence.

a) Write the first 10 terms of a_n.

b) Show that $a_n = 1 + \dfrac{1}{a_{n-1}}$

23 Define the sequence

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n}\right)$$

a) Find the first 10 terms of this sequence and compare them to Fibonacci numbers.

b) Show that $3 \pm \sqrt{5} = \dfrac{(1 \pm \sqrt{5})^2}{2}$.

c) Use the result in b) to verify that F_n satisfies the recursive definition of Fibonacci sequences.

4.2 Arithmetic sequences

Examine the following sequences and the most likely recursive formula for each of them.

7, 14, 21, 28, 35, 42, … $a_1 = 7$ and $a_n = a_{n-1} + 7$, for $n > 1$

2, 11, 20, 29, 38, 47, … $a_1 = 2$ and $a_n = a_{n-1} + 9$, for $n > 1$

48, 39, 30, 21, 12, 3, −6, … $a_1 = 48$ and $a_n = a_{n-1} - 9$, for $n > 1$

Note that in each case above, every term is formed by adding a constant number to the preceding term. Sequences formed in this manner are called **arithmetic sequences**.

Definition of an arithmetic sequence

A sequence $a_1, a_2, a_3, \ldots$ is an **arithmetic sequence** if there is a constant d for which

$$a_n = a_{n-1} + d$$

for all integers $n > 1$. d is called the **common difference** of the sequence, and $d = a_n - a_{n-1}$ for all integers $n > 1$.

So, for the sequences above, 7 is the common difference for the first, 9 is the common difference for the second and −9 is the common difference for the third.

This description gives us the recursive definition of the arithmetic sequence. It is possible, however, to find the explicit definition of the sequence.

Applying the recursive definition repeatedly will enable you to see the expression we are seeking:

$$a_2 = a_1 + d;\ a_3 = a_2 + d = a_1 + d + d = a_1 + 2d;$$
$$a_4 = a_3 + d = a_1 + 2d + d = a_1 + 3d;\ \ldots$$

So, as you see, you can get to the nth term by adding d to a_1, $(n - 1)$ times, and therefore:

***n*th term of an arithmetic sequence**

The general (nth) term of an arithmetic sequence, a_n, with first term a_1 and common difference d, may be expressed explicitly as

$$a_n = a_1 + (n - 1)d$$

This result is useful in finding any term of the sequence without knowing all the previous terms.

Note: The arithmetic sequence can be looked at as a linear function as explained in the introduction to this chapter, i.e. for every increase of one unit in n, the value of the term will increase by d units. As the first term is a_1, the point $(1, a_1)$ belongs to this function. The constant increase d can be considered to be the gradient (slope) of this linear model; hence, the nth term, the dependent variable in this case, can be found by using the *point-slope* form of the equation of a line:

$$y - y_1 = m(x - x_1)$$
$$a_n - a_1 = d(n - 1) \Leftrightarrow a_n = a_1 + (n - 1)d$$

This agrees with our definition of an arithmetic sequence.

Example 4

Find the nth and the 50th terms of the sequence 2, 11, 20, 29, 38, 47, …

Solution

This is an arithmetic sequence whose first term is 2 and common difference is 9. Therefore,

$$a_n = a_1 + (n-1)d = 2 + (n-1) \times 9 = 9n - 7$$
$$\Rightarrow a_{50} = 9 \times 50 - 7 = 443$$

Example 5

Find the recursive and the explicit forms of the definition of the following sequence, then calculate the value of the 25th term.

$$13, 8, 3, -2, \ldots$$

Solution

This is clearly an arithmetic sequence, since we observe that -5 is the common difference.

Recursive definition: $a_1 = 13$

$$a_n = a_{n-1} - 5$$

Explicit definition: $a_n = 13 - 5(n-1) = 18 - 5n$, and

$$a_{25} = 18 - 5 \times 25 = -107$$

Example 6

Find a definition for the arithmetic sequence whose first term is 5 and fifth term is 11.

Solution

Since the fifth term is given, using the explicit form, we have

$$a_5 = a_1 + (5-1)d \Rightarrow 11 = 5 + 4d \Rightarrow d = \tfrac{3}{2}$$

This leads to the general term,

$a_n = 5 + \frac{3}{2}(n-1)$, or, equivalently, the recursive form

$$\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + \frac{3}{2}, n > 1 \end{cases}$$

● **Hint:** Definition: In a finite arithmetic sequence $a_1, a_2, a_3, \ldots, a_k$, the terms $a_2, a_3 \ldots, a_{k-1}$ are called **arithmetic means** between a_1 and a_k.

Example 7

Insert four arithmetic means between 3 and 7.

Solution

Since there are four means between 3 and 7, the problem can be reduced to a situation similar to Example 6 by considering the first term to be 3 and the sixth term to be 7. The rest is left as an exercise for you!

Exercise 4.2

1 Insert four arithmetic means between 3 and 7.

2 Say whether each given sequence is an arithmetic sequence. If yes, find the common difference and the 50th term; if not, say why not.

a) $a_n = 2n - 3$

b) $b_n = n + 2$

c) $c_n = c_{n-1} + 2$, and $c_1 = -1$

d) $u_n = 3u_{n-1} + 2$

e) 2, 5, 7, 12, 19, …

f) 2, −5, −12, −19, …

For each arithmetic sequence in questions 3–8, find:

a) the 8th term

b) an explicit formula for the nth term

c) a recursive formula for the nth term.

3 −2, 2, 6, 10, …

4 29, 25, 21, 17, …

5 −6, 3, 12, 21, …

6 10.07, 9.95, 9.83, 9.71, …

7 100, 97, 94, 91, …

8 $2, \frac{3}{4}, -\frac{1}{2}, -\frac{7}{4}, \ldots$

9 Find five arithmetic means between 13 and −23.

10 Find three arithmetic means between 299 and 300.

11 In an arithmetic sequence, $a_5 = 6$ and $a_{14} = 42$. Find an explicit formula for the nth term of this sequence.

12 In an arithmetic sequence, $a_3 = -40$ and $a_9 = -18$. Find an explicit formula for the nth term of this sequence.

In each of questions 13–17, the first 3 terms and the last term of an arithmetic sequence are given. Find the number of terms.

13 3, 9, 15, …, 525

14 9, 3, −3, …, −201

15 $3\frac{1}{8}, 4\frac{1}{4}, 5\frac{3}{8}, \ldots, 14\frac{3}{8}$

16 $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots, 2\frac{5}{6}$

17 $1 - k, 1 + k, 1 + 3k, \ldots, 1 + 19k$

18 Find five arithmetic means between 15 and −21.

19 Find three arithmetic means between 99 and 100.

20 In an arithmetic sequence, $a_3 = 11$ and $a_{12} = 47$. Find an explicit formula for the nth term of this sequence.

21 In an arithmetic sequence, $a_7 = -48$ and $a_{13} = -10$. Find an explicit formula for the nth term of this sequence.

22 The 30th term of an arithmetic sequence is 147 and the common difference is 4. Find a formula for the nth term.

23 The first term of an arithmetic sequence is −7 and the common difference is 3. Is 9803 a term of this sequence? If so, which one?

24 The first term of an arithmetic sequence is 9689 and the 100th term is 8996. Show that the 110th term is 8926. Is 1 a term of this sequence? If so, which one?

25 The first term of an arithmetic sequence is 2 and the 30th term is 147. Is 995 a term of this sequence? If so, which one?

4.3 Geometric sequences

Examine the following sequences and the most likely recursive formula for each of them.

7, 14, 28, 56, 112, 224, …	$a_1 = 7$ and $a_n = a_{n-1} \times 2$, for $n > 1$
2, 18, 162, 1458, 13 122, …	$a_1 = 2$ and $a_n = a_{n-1} \times 9$, for $n > 1$
48, −24, 12, −6, 3, −1.5, …	$a_1 = 48$ and $a_n = a_{n-1} \times -0.5$, for $n > 1$

Note that in each case above, every term is formed by multiplying a constant number with the preceding term. Sequences formed in this manner are called **geometric sequences**.

Definition of a geometric sequence

A sequence $a_1, a_2, a_3, \ldots$ is a **geometric sequence** if there is a constant r for which

$$a_n = a_{n-1} \times r$$

for all integers $n > 1$. r is called the **common ratio** of the sequence, and $r = a_n \div a_{n-1}$ for all integers $n > 1$.

So, for the sequences above, 2 is the common ratio for the first, 9 is the common ratio for the second and −0.5 is the common ratio for the third.

This description gives us the recursive definition of the geometric sequence. It is possible, however, to find the explicit definition of the sequence.

Applying the recursive definition repeatedly will enable you to see the expression we are seeking:

$$a_2 = a_1 \times r;\ a_3 = a_2 \times r = a_1 \times r \times r = a_1 \times r^2;$$
$$a_4 = a_3 \times r = a_1 \times r^2 \times r = a_1 \times r^3;\ \ldots$$

So, as you see, you can get to the nth term by multiplying a_1 with r, $(n - 1)$ times, and therefore:

***n*th term of geometric sequence**

The general (nth) term of a geometric sequence, a_n, with common ratio r and first term a_1, may be expressed explicitly as

$$a_n = a_1 \times r^{(n-1)}$$

This result is useful in finding any term of the sequence without knowing all the previous terms.

Example 8

a) Find the geometric sequence with $a_1 = 2$ and $r = 3$.

b) Describe the sequence 3, −12, 48, −192, 768, …

c) Describe the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

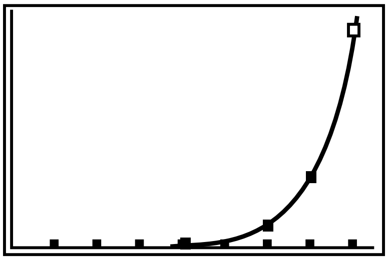
d) Graph the sequence $a_n = \frac{1}{4} \cdot 3^{n-1}$

Solution

a) The geometric sequence is 2, 6, 18, 54, …, $2 \times 3^{n-1}$. Notice that the ratio of a term to the preceding term is 3.

b) This is a geometric sequence with $a_1 = 3$ and $r = -4$. The nth term is $a_n = 3 \times (-4)^{n-1}$. Notice that, when the common ratio is negative, the terms of the sequence alternate in sign.

c) The nth term of this sequence is $a_n = 1 \cdot \left(\frac{1}{2}\right)^{n-1}$. Notice that the ratio of any two consecutive terms is $\frac{1}{2}$. Also, notice that the terms decrease in value.



d) The graph of the geometric sequence is shown on the left. Notice that the points lie on the graph of the function $y = \frac{1}{4} \cdot 3^{x-1}$.

Example 9

At 8:00 a.m., 1000 mg of medicine is administered to a patient. At the end of each hour, the concentration of medicine is 60% of the amount present at the beginning of the hour.

a) What portion of the medicine remains in the patient's body at noon if no additional medication has been given?

b) If a second dosage of 1000 mg is administered at 10:00 a.m., what is the total concentration of the medication in the patient's body at noon?

Solution

a) We use the geometric model, as there is a constant multiple by the end of each hour. Hence, the concentration at the end of any hour after administering the medicine is given by:

$a_n = a_1 \times r^{(n-1)}$, where n is the number of hours

Thus, at noon $n = 5$, and $a_5 = 1000 \times 0.6^{(5-1)} = 129.6$.

b) For the second dosage, the amount of medicine at noon corresponds to $n = 3$, and $a_3 = 1000 \times 0.6^{(3-1)} = 360$.

So, the concentration of medicine is $129.6 + 360 = 489.6$ mg.

Compound interest

Interest compounded annually

When we borrow money we pay interest, and when we invest money we receive interest. Suppose an amount of €1000 is put into a savings account that bears an annual interest of 6%. How much money will we have in the bank at the end of four years?

It is important to note that the 6% interest is given annually and is added to the savings account, so that in the following year it will also earn interest, and so on.

Time in years	Amount in the account
0	1000
1	$1000 + 1000 \times 0.06 = 1000(1 + 0.06)$
2	$1000(1 + 0.06) + (1000(1 + 0.06)) \times 0.06 = 1000(1 + 0.06)(1 + 0.06) = 1000(1 + 0.06)^2$
3	$1000(1 + 0.06)^2 + (1000(1 + 0.06)^2) \times 0.06 = 1000(1 + 0.06)^2(1 + 0.06) = 1000(1 + 0.06)^3$
4	$1000(1 + 0.06)^3 + (1000(1 + 0.06)^3) \times 0.06 = 1000(1 + 0.06)^3(1 + 0.06) = \mathbf{1000(1 + 0.06)^4}$

Table 4.1 Compound interest.

This appears to be a geometric sequence with five terms. You will notice that the number of terms is five, as both the beginning and the end of the first year are counted. (Initial value, when time = 0, is the first term.)

In general, if a **principal** of P euros is invested in an account that yields an interest rate r (expressed as a decimal) annually, and this interest is added at the end of the year, every year, to the principal, then we can use the geometric sequence formula to calculate the **future value** A, which is accumulated after t years.

If we repeat the steps above, with

$$A_0 = P = \text{initial amount}$$
$$r = \text{annual interest rate}$$
$$t = \text{number of years}$$

it becomes easier to develop the formula:

Table 4.2 Compound interest formula.

Time in years	Amount in the account
0	$A_0 = P$
1	$A_1 = P + Pr = P(1 + r)$
2	$A_2 = A_1(1 + r) = P(1 + r)^2$
⋮	
t	$A_t = P(1 + r)^t$

Notice that since we are counting from 0 to t, we have $t + 1$ terms, and hence using the geometric sequence formula,

$$a_n = a_1 \times r^{(n-1)} \Rightarrow A_t = A_0 \times (1 + r)^t$$

Interest compounded *n* times per year

Suppose that the principal P is invested as before but the interest is paid n times per year. Then $\frac{r}{n}$ is the interest paid every compounding period. Since every year we have n periods, for t years, we have nt periods. The amount A in the account after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example 10

€1000 is invested in an account paying compound interest at a rate of 6%. Calculate the amount of money in the account after 10 years if

a) the compounding is annual

b) the compounding is quarterly

c) the compounding is monthly.

Solution

a) The amount after 10 years is

$A = 1000(1 + 0.06)^{10} = €1790.85.$

b) The amount after 10 years quarterly compounding is

$A = 1000\left(1 + \frac{0.06}{4}\right)^{40} = €1814.02.$

c) The amount after 10 years monthly compounding is

$A = 1000\left(1 + \frac{0.06}{12}\right)^{120} = €1819.40.$

Example 11

You invested €1000 at 6% compounded quarterly. How long will it take this investment to increase to €2000?

Solution

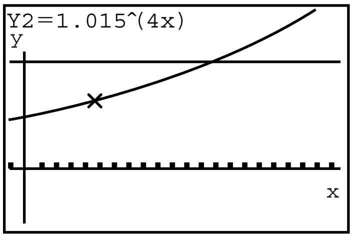
Let $P = 1000$, $r = 0.06$, $n = 4$ and $A = 2000$ in the compound interest formula:

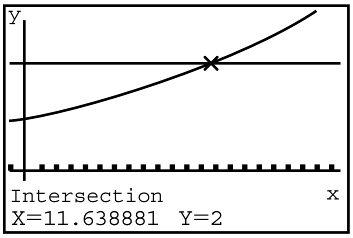
$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Then solve for t:

$$2000 = 1000\left(1 + \frac{0.06}{4}\right)^{4t} \Rightarrow 2 = 1.015^{4t}$$

Using a GDC, we can graph the functions $y = 2$ and $y = 1.015^{4t}$ and then find the intersection between their graphs.





As you can see, it will take the €1000 investment 11.64 years to double to €2000. This translates into approximately 47 quarters.

You can check your work to see that this is accurate by using the compound interest formula:

$$A = 1000\left(1 + \frac{0.06}{4}\right)^{47} = €2013.28$$

Later in the book, you will learn how to solve the problem algebraically.

Example 12

You want to invest €1000. What interest rate is required to make this investment grow to €2000 in 10 years if interest is compounded quarterly?

Solution

Let $P = 1000$, $n = 4$, $t = 10$ and $A = 2000$ in the compound interest formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Now solve for r:

$$2000 = 1000\left(1 + \frac{r}{4}\right)^{40} \Rightarrow 2 = \left(1 + \frac{r}{4}\right)^{40} \Rightarrow 1 + \frac{r}{4} = \sqrt[40]{2} \Rightarrow r = 4(\sqrt[40]{2} - 1)$$
$$= 0.0699$$

So, at a rate of 7% compounded quarterly, the €1000 investment will grow to at least €2000 in 10 years.

You can check to see whether your work is accurate by using the compound interest formula:

$$A = 1000\left(1 + \frac{0.07}{4}\right)^{40} = €2001.60$$

Population growth

The same formulae can be applied when dealing with population growth.

Example 13

The city of Baden in Lower Austria grows at an annual rate of 0.35%. The population of Baden in 1981 was 23 140. What is the estimate of the population of this city for 2013?

Solution

This situation can be modelled by a geometric sequence whose first term is 23 140 and whose common ratio is 1.0035. Since we count the population of 1981 among the terms, the number of terms is 33.

2013 is equivalent to the 33rd term in this sequence. The estimated population for Baden is, therefore,

$$\text{Population (2013)} = a_{31} = 23\,140(1.0035)^{32} = 25\,877$$

Note: Later in the book, more realistic population growth models will be explored and more efficient methods will be developed, as well as the ability to calculate interest that is continuously compounded.

Exercise 4.3

In each of questions 1–15 determine whether the sequence in each question is arithmetic, geometric, or neither. Find the common difference for the arithmetic ones and the common ratio for the geometric ones. Find the common difference or ratio and the 10th term for each arithmetic or geometric one as appropriate.

1 $3, 3^{a+1}, 3^{2a+1}, 3^{3a+1}, \ldots$

2 $a_n = 3n - 3$

3 $b_n = 2^{n+2}$

4 $c_n = 2c_{n-1} - 2$, and $c_1 = -1$

5 $u_n = 3u_{n-1}, u_1 = 4$

6 2, 5, 12.5, 31.25, 78.125, …

7 2, −5, 12.5, −31.25, 78.125, …

8 2, 2.75, 3.5, 4.25, 5, …

9 $18, -12, 8, -\frac{16}{3}, \frac{32}{9}, \ldots$

10 52, 55, 58, 61, …

11 −1, 3, −9, 27, −81, …

12 0.1, 0.2, 0.4, 0.8, 1.6, 3.2, …

13 3, 6, 12, 18, 21, 27, …

14 6, 14, 20, 28, 34, …

15 2.4, 3.7, 5, 6.3, 7.6, …

For each arithmetic or geometric sequence in questions 16–32 find
a) the 8th term
b) an explicit formula for the nth term
c) a recursive formula for the nth term.

16 $-3, 2, 7, 12, \ldots$

17 $19, 15, 11, 7, \ldots$

18 $-8, 3, 14, 25, \ldots$

19 $10.05, 9.95, 9.85, 9.75, \ldots$

20 $100, 99, 98, 97, \ldots$

21 $2, \frac{1}{2}, -1, -\frac{5}{2}, \ldots$

22 $3, 6, 12, 24, \ldots$

23 $4, 12, 36, 108, \ldots$

24 $5, -5, 5, -5, \ldots$

25 $3, -6, 12, -24, \ldots$

26 $972, -324, 108, -36, \ldots$

27 $-2, 3, -\frac{9}{2}, \frac{27}{4} \ldots$

28 $35, 25, \frac{125}{7}, \frac{625}{49}, \ldots$

29 $-6, -3, -\frac{3}{2}, -\frac{3}{4}, \ldots$

30 $9.5, 19, 38, 76, \ldots$

31 $100, 95, 90.25, \ldots$

32 $2, \frac{3}{4}, \frac{9}{32}, \frac{27}{256}, \ldots$

33 Insert 4 geometric means between 3 and 96.
● **Hint:** Definition: In a finite geometric sequence $a_1, a_2, a_3, \ldots, a_k$, the terms $a_2, a_3, \ldots, a_{k-1}$ are called *geometric means* between a_1 and a_k.

34 Find 3 geometric means between 7 and 4375.

35 Find a geometric mean between 16 and 81.
● **Hint:** This is also called the *mean proportional*.

36 Find 4 geometric means between 7 and 1701.

37 Find a geometric mean between 9 and 64.

38 The first term of a geometric sequence is 24 and the fourth term is 3, find the fifth term and an expression for the nth term.

39 The first term of a geometric sequence is 24 and the third term is 6, find the fourth term and an expression for the nth term.

40 The common ratio in a geometric sequence is $\frac{2}{7}$ and the fourth term is $\frac{14}{3}$. Find the third term.

41 Which term of the geometric sequence 6, 18, 54, … is 118 098?

42 The 4th term and the 7th term of a geometric sequence are 18 and $\frac{729}{8}$. Is $\frac{59049}{128}$ a term of this sequence? If so, which term is it?

43 The 3rd term and the 6th term of a geometric sequence are 18 and $\frac{243}{4}$. Is $\frac{19683}{64}$ a term of this sequence? If so, which term is it?

44 Jim put €1500 into a savings account that pays 4% interest compounded semiannually. How much will his account hold 10 years later if he does not make any additional investments in this account?

45 At her daughter Jane's birth, Charlotte set aside £500 into a savings account. The interest she earned was 4% compounded quarterly. How much money will Jane have on her 16th birthday?

46 How much money should you invest now if you wish to have an amount of €4000 in your account after 6 years if interest is compounded quarterly at an annual rate of 5%?

47 In 2007, the population of Switzerland was estimated to be 7554 (in thousands). How large would the Swiss population be in 2012 if it grows at a rate of 0.5% annually?

48 The common ratio in a geometric sequence is $\frac{3}{7}$ and the fourth term is $\frac{14}{3}$. Find the third term.

49 Which term of the geometric sequence 7, 21, 63, … is 137 781?

50 Tim put €2500 into a savings account that pays 4% interest compounded semiannually. How much will his account hold 10 years later if he does not make any additional investments in this account?

51 At her son William's birth, Jane set aside £1000 into a savings account. The interest she earned was 6% compounded quarterly. How much money will William have on his 18th birthday?

4.4 Series

The word 'series' in common language implies much the same thing as 'sequence'. But in mathematics when we talk of a series, we are referring in particular to sums of terms in a sequence, e.g. for a sequence of values a_n, the corresponding series is the sequence of S_n with

$$S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$$

If the terms are in an arithmetic sequence, we call the sum an **arithmetic series**.

Sigma notation

Most of the series we consider in mathematics are **infinite** series. This name is used to emphasize the fact that the series contain infinitely many terms. Any sum in the series S_k will be called a partial sum and is given by

$$S_k = a_1 + a_2 + \ldots + a_{k-1} + a_k$$

For convenience, this partial sum is written using the sigma notation:

$$S_k = \sum_{i=1}^{i=k} a_i = a_1 + a_2 + \ldots + a_{k-1} + a_k$$

Sigma notation is a concise and convenient way to represent long sums. Here, the symbol Σ is the Greek capital letter *sigma* that refers to the initial letter of the word 'sum'. So, the expression $\sum_{i=1}^{i=k} a_i$ means the sum of all the terms a_i, where i takes the values from 1 to k. We can also write $\sum_{i=m}^{n} a_i$ to mean the sum of the terms a_i, where i takes the values from m to n. In such a sum, m is called the lower limit and n the upper limit.

Example 14

Write out what is meant by:

a) $\sum_{i=1}^{5} i^4$ b) $\sum_{r=3}^{7} 3^r$ c) $\sum_{j=1}^{n} x_j p(x_j)$

Solution

a) $\sum_{i=1}^{5} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + 5^4$

b) $\sum_{r=3}^{7} 3^r = 3^3 + 3^4 + 3^5 + 3^6 + 3^7$

c) $\sum_{j=1}^{n} x_j p(x_j) = x_1 p(x_1) + x_2 p(x_2) + \ldots + x_n p(x_n)$

Example 15

Evaluate $\sum_{n=0}^{5} 2^n$

Solution

$$\sum_{n=0}^{5} 2^n = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$$

Example 16

Write the sum $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \ldots + \frac{99}{100}$ in sigma notation.

Solution

We notice that each term's numerator and denominator are consecutive integers, so they take on the absolute value of $\frac{k}{k+1}$ or any equivalent form. We also notice that the signs of the terms alternate and that we have 99 terms. To take care of the sign, we use some power of (-1) that will start with a positive value. If we use $(-1)^k$, the first term will be negative, so we can use $(-1)^{k+1}$ instead. We can, therefore, write the sum as

$$(-1)^{1+1}\frac{1}{2} + (-1)^{2+1}\frac{2}{3} + (-1)^{3+1}\frac{3}{4} + \ldots + (-1)^{99+1}\frac{99}{100} = \sum_{k=1}^{99}(-1)^{k+1}\frac{k}{k+1}$$

Properties of the sigma notation

There are a number of useful results that we can obtain when we use sigma notation.

1 For example, suppose we had a sum of constant terms

$$\sum_{i=1}^{5} 2$$

What does this mean? If we write this out in full, we get

$$\sum_{i=1}^{5} 2 = 2 + 2 + 2 + 2 + 2 = 5 \times 2 = 10.$$

In general, if we sum a constant n times then we can write

$$\sum_{i=1}^{n} k = k + k + \dots + k = n \times k = nk.$$

2 Suppose we have the sum of a constant times i. What does this give us? For example,

$$\sum_{i=1}^{5} 5i = 5 \times 1 + 5 \times 2 + 5 \times 3 + 5 \times 4 + 5 \times 5 = 5 \times (1 + 2 + 3 + 4 + 5) = 75.$$

However, this can also be interpreted as follows

$$\sum_{i=1}^{5} 5i = 5 \times 1 + 5 \times 2 + 5 \times 3 + 5 \times 4 + 5 \times 5 = 5 \times (1 + 2 + 3 + 4 + 5) = 5\sum_{i=1}^{5} i$$

which implies that

$$\sum_{i=1}^{5} 5i = 5\sum_{i=1}^{5} i$$

In general, we can say

$$\begin{aligned}\sum_{i=1}^{n} ki &= k \times 1 + k \times 2 + \dots + k \times n \\ &= k \times (1 + 2 + \dots + n) \\ &= k\sum_{i=1}^{n} i\end{aligned}$$

3 Suppose that we need to consider the summation of two different functions, such as

$$\begin{aligned}\sum_{k=1}^{n} (k^2 + k^3) &= (1^2 + 1^3) + (2^2 + 2^3) + \dots + n^2 + n^3 \\ &= (1^2 + 2^2 + \dots + n^2) + (1^3 + 2^3 + \dots + n^3) \\ &= \sum_{k=1}^{n} (k^2) + \sum_{k=1}^{n} (k^3)\end{aligned}$$

In general,

$$\sum_{k=1}^{n} (f(k) + g(\mathrm{k})) = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k)$$

Arithmetic series

In arithmetic series, we are concerned with adding the terms of arithmetic sequences. It is very helpful to be able to find an easy expression for the partial sums of this series.

Let us start with an example:

Find the partial sum for the first 50 terms of the series

$$3 + 8 + 13 + 18 + \dots$$

We express S_{50} in two different ways:

$$\begin{aligned} S_{50} &= 3 + 8 + 13 + \dots + 248, \quad \text{and} \\ S_{50} &= 248 + 243 + 238 + \dots + 3 \\ \hline 2S_{50} &= 251 + 251 + 251 + \dots + 251 \end{aligned}$$

There are 50 terms in this sum, and hence

$$2S_{50} = 50 \times 251 \Rightarrow S_{50} = \frac{50}{2}(251) = 6275.$$

This reasoning can be extended to any arithmetic series in order to develop a formula for the nth partial sum S_n.

Let $\{a_n\}$ be an arithmetic sequence with first term a_1 and a common difference d. We can construct the series in two ways: Forward, by adding d to a_1 repeatedly, and backwards by subtracting d from a_n repeatedly. We get the following two expressions for the sum:

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d)$$

and

$$S_n = a_n + a_{n-1} + a_{n-2} + \ldots + a_1 = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_n - (n-1)d)$$

By adding, term by term vertically, we get

$$\begin{array}{rcl}
S_n &=& a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d) \\
S_n &=& a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_n - (n-1)d) \\
\hline
2S_n &=& (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n)
\end{array}$$

Since we have n terms, we can reduce the expression above to

$2S_n = n(a_1 + a_n)$, which can be reduced to

$S_n = \frac{n}{2}(a_1 + a_n)$, which in turn can be changed to give an interesting perspective of the sum,

i.e. $S_n = n\left(\frac{a_1 + a_n}{2}\right)$ is n times the average of the first and last terms!

If we substitute $a_1 + (n-1)d$ for a_n then we arrive at an alternative formula for the sum:

$$S_n = \frac{n}{2}(a_1 + a_1 + (n-1)d) = \frac{n}{2}(2a_1 + (n-1)d)$$

Sum of an arithmetic series

The sum, S_n, of n terms of an arithmetic series with common difference d, first term a_1, and nth term a_n is:

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{or} \quad S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

Example 17

Find the partial sum for the first 50 terms of the series

$$3 + 8 + 13 + 18 + \ldots$$

Solution

Using the second formula for the sum, we get

$$S_{50} = \frac{50}{2}(2 \times 3 + (50 - 1)5) = 25 \times 251 = 6275.$$

Using the first formula requires that we know the nth term. So, $a_{50} = 3 + 49 \times 5 = 248$, which now can be used:

$$S_{50} = 25(3 + 248) = 6275.$$

Geometric series

As is the case with arithmetic series, it is often desirable to find a general expression for the nth partial sum of a geometric series.

Let us start with an example:

Find the partial sum for the first 20 terms of the series

$$3 + 6 + 12 + 24 + \dots$$

We express S_{20} in two different ways and subtract them:

$$\begin{array}{lllll}
S_{20} & = 3 + 6 + 12 + \dots + 1\,572\,864 & \\
2S_{20} & = \phantom{3 +{}} 6 + 12 + \dots + 1\,572\,864 & + 3\,145\,728 \\
\hline
-S_{20} & = 3 & - 3\,145\,728 \\
\Rightarrow S_{20} & = 3\,145\,725 &
\end{array}$$

This reasoning can be extended to any geometric series in order to develop a formula for the nth partial sum S_n.

Let $\{a_n\}$ be a geometric sequence with first term a_1 and a common ratio $r \neq 1$. We can construct the series in two ways as before and using the definition of the geometric sequence, i.e. $a_n = a_{n-1} \times r$, then

$$\begin{aligned}
S_n &= a_1 + \mathbf{a_2} + \mathbf{a_3} + \dots + \mathbf{a_{n-1}} + \mathbf{a_n}, \text{ and} \\
rS_n &= ra_1 + ra_2 + ra_3 + \dots + ra_{n-1} + ra_n \\
&= \mathbf{a_2} + \mathbf{a_3} + \dots + \mathbf{a_{n-1}} + \mathbf{a_n} + ra_n
\end{aligned}$$

Now, we subtract the first and last expressions to get

$$S_n - rS_n = a_1 - ra_n \Rightarrow S_n(1 - r) = a_1 - ra_n \Rightarrow \mathbf{S_n = \frac{a_1 - ra_n}{1 - r}};\ r \neq 1.$$

This expression, however, requires that r, a_1, as well as a_n be known in order to find the sum. However, using the nth term expression developed earlier, we can simplify this sum formula to

$$S_n = \frac{a_1 - r\mathbf{a_n}}{1 - r} = \frac{a_1 - r\mathbf{a_1 r^{n-1}}}{1 - r} = \frac{a_1(1 - r^n)}{1 - r};\ r \neq 1.$$

Sum of a geometric series

The sum, S_n, of n terms of a geometric series with common ratio r ($r \neq 1$) and first term a_1, is:

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \left[\text{equivalent to } S_n = \frac{a_1(r^n - 1)}{r - 1}\right]$$

Example 18

Find the partial sum for the first 20 terms of the series $3 + 6 + 12 + 24 + \dots$ in the opening example for this section.

Solution

$$S_{20} = \frac{3(1 - 2^{20})}{1 - 2} = \frac{3(1 - 1\,048\,576)}{-1} = 3\,145\,725$$

Infinite geometric series

Consider the series

$$\sum_{k=1}^{n} 2\left(\tfrac{1}{2}\right)^{k-1} = 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots$$

Consider also finding the partial sums for 10, 20 and 100 terms. The sums we are looking for are the partial sums of a geometric series. So,

$$\sum_{k=1}^{10} 2\left(\frac{1}{2}\right)^{k-1} = 2 \times \frac{1-\left(\frac{1}{2}\right)^{10}}{1-\frac{1}{2}} \approx 3.996$$

$$\sum_{k=1}^{20} 2\left(\frac{1}{2}\right)^{k-1} = 2 \times \frac{1-\left(\frac{1}{2}\right)^{20}}{1-\frac{1}{2}} \approx 3.999\,996$$

$$\sum_{k=1}^{100} 2\left(\frac{1}{2}\right)^{k-1} = 2 \times \frac{1-\left(\frac{1}{2}\right)^{100}}{1-\frac{1}{2}} \approx 4$$

As the number of terms increases, the partial sum appears to be approaching the number 4. This is no coincidence. In the language of limits,

$$\lim_{n\to\infty} \sum_{k=1}^{n} 2\left(\frac{1}{2}\right)^{k-1} = \lim_{n\to\infty} 2 \times \frac{1-\left(\frac{1}{2}\right)^{k}}{1-\frac{1}{2}} = 2 \times \frac{1-0}{\frac{1}{2}} = 4, \text{since } \lim_{n\to\infty}\left(\frac{1}{2}\right)^{n} = 0.$$

This type of problem allows us to extend the usual concept of a 'sum' of a **finite** number of terms to make sense of sums in which an **infinite** number of terms is involved. Such series are called **infinite series.**

One thing to be made clear about infinite series is that they are not true sums! The associative property of addition of real numbers allows us to extend the definition of the sum of two numbers, such as $a + b$, to three or four or n numbers, but not to an infinite number of numbers. For example, you can add any specific number of 5s together and get a real number, but if you add an *infinite* number of 5s together, you cannot get a real number! The remarkable thing about infinite series is that, in some cases, such as the example above, the sequence of partial sums (which are true sums) approach a finite limit L. The limit in our example is 4. This we write as

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} (a_1 + a_2 + \ldots + a_n) = L.$$

We say that the series **converges** to L, and it is convenient to define L as the **sum of the infinite series.** We use the notation

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = L.$$

We can, therefore, write the limit above as

$$\sum_{k=1}^{\infty} 2\left(\frac{1}{2}\right)^{k-1} = \lim_{n\to\infty} \sum_{k=1}^{n} 2\left(\frac{1}{2}\right)^{k-1} = 4.$$

If the series does not have a limit, it **diverges** and does not have a sum.

We are now ready to develop a general rule for **infinite geometric series.** As you know, the sum of the geometric series is given by

$$S_n = \frac{a_1 - r\boldsymbol{a_n}}{1-r} = \frac{a_1 - r\boldsymbol{a_1 r^{n-1}}}{1-r} = \frac{a_1(1-r^n)}{1-r}; r \neq 1.$$

If $|r| < 1$, then $\lim\limits_{n\to\infty} r^n = 0$ and

$$S_n = S = \lim_{n\to\infty} \frac{a_1(1-r^n)}{1-r} = \frac{a_1}{1-r}.$$

We will call this **the sum of the infinite geometric series**. In all other cases the series diverges. The proof is left as an exercise.

$$\sum_{k=1}^{\infty} 2\left(\frac{1}{2}\right)^{k-1} = \frac{2}{1-\frac{1}{2}} = 4, \text{ as already shown.}$$

Sum of an infinite geometric series

The sum, S_∞, of an infinite geometric series with first term a_1, such that the common ratio r satisfies the condition $|r| < 1$ is given by:

$$S_\infty = \frac{a_1}{1-r}$$

Example 19

A rational number is a number that can be expressed as a quotient of two integers. Show that $0.\overline{6} = 0.666\ldots$ is a rational number.

Solution

$$\begin{aligned} 0.\overline{6} &= 0.666\ldots = 0.6 + 0.06 + 0.006 + 0.0006 + \ldots \\ &= \frac{6}{10} + \frac{6}{10}\cdot\frac{1}{10} + \frac{6}{10}\cdot\left(\frac{1}{10}\right)^2 + \frac{6}{10}\cdot\left(\frac{1}{10}\right)^3 + \ldots \end{aligned}$$

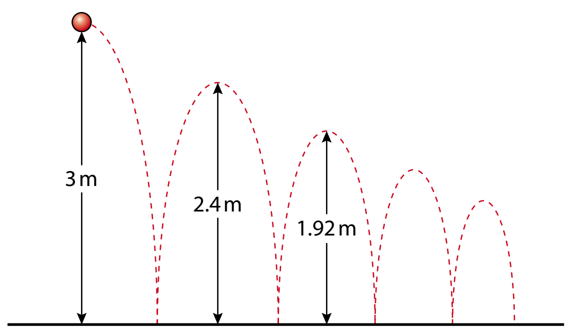
This is an infinite geometric series with $a_1 = \frac{6}{10}$ and $r = \frac{1}{10}$; therefore,

$$0.\overline{6} = \frac{\frac{6}{10}}{1-\frac{1}{10}} = \frac{6}{10}\cdot\frac{10}{9} = \frac{2}{3}$$

Example 20

If a ball has elasticity such that it bounces up 80% of its previous height, find the total vertical distances travelled down and up by this ball when it is dropped from an altitude of 3 metres. Ignore friction and air resistance.

Solution



After the ball is dropped the initial 3 m, it bounces up and down a distance of 2.4 m. Each bounce after the first bounce, the ball travels 0.8 times the previous height twice – once upwards and once downwards. So, the total vertical distance is given by

$$h = 3 + 2(2.4 + (2.4 \times 0.8) + (2.4 \times 0.8^2) + \ldots) = 3 + 2 \times l$$

The amount in parenthesis is an infinite geometric series with $a_1 = 2.4$ and $r = 0.8$. The value of that quantity is

$$l = \frac{2.4}{1-0.8} = 12.$$

Hence, the total distance required is

$$h = 3 + 2(12) = 27 \text{ m}.$$

Applications of series to compound interest calculations

Annuities

An **annuity** is a sequence of equal periodic payments. If you are saving money by depositing the same amount at the end of each compounding period, the annuity is called **ordinary annuity**. Using geometric series you can calculate the **future value (FV)** of this annuity, which is the amount of money you have after making the last payment.

You invest €1000 at the end of each year for 10 years at a fixed annual interest rate of 6%. See table below.

Year	Amount invested	Future value
10	1000	1000
9	1000	$1000(1 + 0.06)$
8	1000	$1000(1 + 0.06)^2$
⋮		
1	1000	$1000(1 + 0.06)^9$

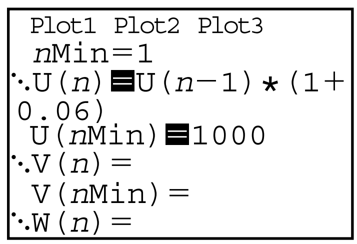
Table 4.3 Calculating the future value.

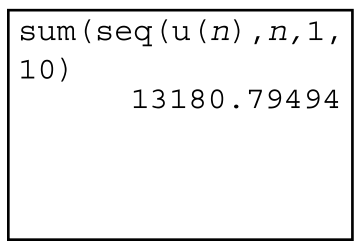
The future value of this investment is the sum of all the entries in the last column, so it is

$$FV = 1000 + 1000(1 + 0.06) + 1000(1 + 0.06)^2 + \ldots + 1000(1 + 0.06)^9$$

This sum is a partial sum of a geometric series with $n = 10$ and $r = 1 + 0.06$. Hence,

$$FV = \frac{1000(1 - (1 + 0.06)^{10})}{1 - (1 + 0.06)} = \frac{1000(1-(1 + 0.06)^{10})}{-0.06} = 13\,180.79.$$





This result can also be produced with a GDC, as shown.

We can generalize the previous formula in the same manner. Let the periodic payment be R and the periodic interest rate be i, i.e. $i = \frac{r}{n}$. Let the number of periodic payments be m.

Period	Amount invested	Future value
m	R	R
$m - 1$	R	$R(1 + i)$
$m - 2$	R	$R(1 + i)^2$
⋮		
1	R	$R(1 + i)^{m-1}$

Table 4.4 Calculating the future value – formula.

The future value of this investment is the sum of all the entries in the last column, so it is

$$FV = R + R(1 + i) + R(1 + i)^2 + \ldots + R(1 + i)^{m-1}$$

This sum is a partial sum of a geometric series with m terms and $r = 1 + i$. Hence,

$$FV = \frac{R(1 - (1 + i)^m)}{1 - (1 + i)} = \frac{R(1 - (1 + i)^m)}{-i} = R\left(\frac{(1 + i)^m - 1}{i}\right)$$

Note: If the payment is made at the beginning of the period rather than at the end, the annuity is called **annuity due** and the future value after m periods will be slightly different. The table for this situation is given below.

Table 4.5 Calculating the future value (annuity due).

Period	Amount invested	Future value
m	R	$R(1 + i)$
$m - 1$	R	$R(1 + i)^2$
$m - 2$	R	$R(1 + i)^3$
$\vdots$		
1	R	$R(1 + i)^m$

The future value of this investment is the sum of all the entries in the last column, so it is

$$FV = R(1 + i) + R(1 + i)^2 + \ldots + R(1 + i)^{m-1} + R(1 + i)^m$$

This sum is a partial sum of a geometric series with m terms and $r = 1 + i$. Hence,

$$FV = \frac{R(1 + i(1 - (1 + i)^m))}{1 - (1 + i)} = \frac{R(1 + i - (1 + i)^{m+1})}{-i} = R\left(\frac{(1 + i)^{m+1} - 1}{i} - 1\right)$$

If the previous investment is made at the beginning of the year rather than at the end, then in 10 years we have

$$FV = R\left(\frac{(1 + i)^{m+1} - 1}{i} - 1\right) = 1000\left(\frac{(1 + 0.06)^{10+1} - 1}{0.006} - 1\right) = 13\,971.64.$$

Exercise 4.4

1 Find the sum of the arithmetic series $11 + 17 + \ldots + 365$.

2 Find the sum:

$2 - 3 + \frac{9}{2} - \frac{27}{4} + \ldots - \frac{177\,147}{1024}$

3 Evaluate $\sum_{k=0}^{13} (2 - 0.3k)$.

4 Evaluate $2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + \ldots$

5 Evaluate $\frac{1}{3} + \frac{\sqrt{3}}{12} + \frac{1}{16} + \frac{\sqrt{3}}{64} + \frac{3}{256} + \ldots$

6 Express each repeating decimal as a fraction:

a) $0.\overline{52}$ b) $0.4\overline{53}$ c) $3.01\overline{37}$

7 At the beginning of every month, Maggie invests £150 in an account that pays 6% annual rate. How much money will there be in the account after six years?

In questions 8–10, find the sum.

8 $9 + 13 + 17 + \ldots + 85$

9 $8 + 14 + 20 + \ldots + 278$

10 $155 + 158 + 161 + \ldots + 527$

11 The kth term of an arithmetic sequence is $2 + 3k$. Find, in terms of n, the sum of the first n terms of this sequence.

12 How many terms should we add to exceed 678 when we add $17 + 20 + 23 \ldots$?

13 How many terms should we add to exceed 2335 when we add $-18 - 11 - 4 \ldots$?

14 An arithmetic sequence has a as first term and $2d$ as common difference, i.e., a, $a + 2d$, $a + 4d$, ... The sum of the first 50 terms is T. Another sequence, with first term $a + d$, and common difference $2d$, is combined with the first one to produce a new arithmetic sequence. Let the sum of the first 100 terms of the new combined sequence be S. If $2T + 200 = S$, find d.

15 Consider the arithmetic sequence 3, 7, 11, ..., 999.

a) Find the number of terms and the sum of this sequence.

b) Create a new sequence by removing every third term, i.e., 11, 23,.... Find the sum of the terms of the remaining sequence.

16 The sum of the first 10 terms of an arithmetic sequence is 235 and the sum of the second 10 terms is 735. Find the first term and the common difference.

In questions 17–19, use your GDC or a spreadsheet to evaluate each sum.

17 $\displaystyle\sum_{k=1}^{20}(k^2 + 1)$

18 $\displaystyle\sum_{i=3}^{17}\frac{1}{i^2 + 3}$

19 $\displaystyle\sum_{n=1}^{100}(-1)^n\frac{3}{n}$

20 Find the sum of the arithmetic series

$13 + 19 + \ldots + 367$

21 Find the sum

$2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots - \frac{4096}{177\,147}$

22 Evaluate $\displaystyle\sum_{k=0}^{11}(3 + 0.2k)$.

23 Evaluate $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots$

24 Evaluate $\frac{1}{2} + \frac{\sqrt{2}}{2\sqrt{3}} + \frac{1}{3} + \frac{\sqrt{2}}{3\sqrt{3}} + \frac{2}{9} + \ldots$

In questions 25–27, find the first four partial sums and then the nth partial sum of each sequence.

25 $u_n = \frac{3}{5^n}$

26 $v_n = \frac{1}{n^2 + 3n + 2}$ Hint: Show that $v_n = \frac{1}{n+1} - \frac{1}{n+2}$

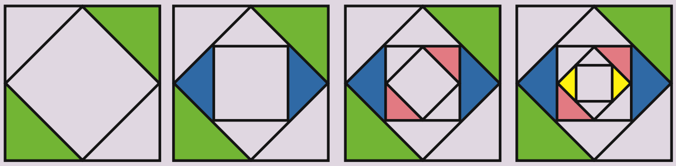
27 $u_n = \sqrt{n+1} - \sqrt{n}$

28 A ball is dropped from a height of 16 m. Every time it hits the ground it bounces 81% of its previous height.

a) Find the maximum height it reaches *after* the 10th bounce.

b) Find the total distance travelled by the ball till it rests. (Assume no friction and no loss of elasticity).

29

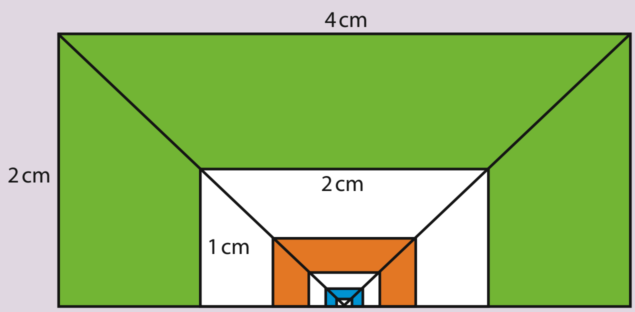


The sides of a square are 16 cm in length. A new square is formed by joining the midpoints of the adjacent sides and two of the resulting triangles are coloured as shown.

a) If the process is repeated 6 more times, determine the total area of the shaded region.

b) If the process is repeated indefinitely, find the total area of the shaded region.

30



The largest rectangle has dimensions 4 by 2, as shown; another rectangle is constructed inside it with dimensions 2 by 1. The process is repeated. The region surrounding every other inner rectangle is shaded, as shown.

a) Find the total area for the three regions shaded already.

b) If the process is repeated indefinitely, find the total area of the shaded regions.

In questions 31–34, find each sum.

31 $7 + 12 + 17 + 22 + \ldots + 337 + 342$

32 $9486 + 9479 + 9472 + 7465 + \ldots + 8919 + 8912$

33 $2 + 6 + 18 + 54 + \ldots + 3\,188\,646 + 9\,565\,938$

34 $120 + 24 + \frac{24}{5} + \frac{24}{25} + \ldots + \frac{24}{78\,125}$

4.5 Counting principles

Simple counting problems

This section will introduce you to some of the basic principles of counting. In Section 4.6 you will apply some of this in justifying the binomial theorem and in Chapter 12 you will use these principles to tackle many probability problems. We will start with two examples.

5 Exponential and Logarithmic Functions

Assessment statements

1.2 Exponents and logarithms.
Laws of exponents; laws of logarithms. Change of base.

2.4 The function $x \mapsto a^x$, $a > 0$.
The inverse function $x \mapsto \log_a x$, $x > 0$.
Graphs of $y = a^x$ and $y = \log_a x$.
The exponential function $x \mapsto e^x$.
The logarithmic function $x \mapsto \ln x$, $x > 0$.

2.6 Solutions of $a^x = b$ using logarithms.

Introduction

A variety of functions have already been considered in this text (see Figure 2.17 in Section 2.4): polynomial functions (e.g. linear, quadratic and cubic functions), functions with radicals (e.g. square root function), rational functions (e.g. inverse and inverse square functions) and the absolute value function. This chapter examines exponential and logarithmic functions.

Exponential functions help us model a wide variety of physical phenomena. The natural exponential function (or simply, *the* exponential function), $f(x) = e^x$, is one of the most important functions in calculus. Exponential functions and their applications – especially to situations involving growth and decay – will be covered at length.

Logarithms, which were originally invented as a computational tool, lead to logarithmic functions. These functions are closely related to exponential functions and play an equally important part in calculus and a range of applications. We will learn that certain exponential and logarithmic functions are inverses of each other.

5.1 Exponential functions

Characteristics of exponential functions

We begin our study of exponential functions by comparing two algebraic expressions that represent two seemingly similar but very different functions. The two expressions $y = x^2$ and $y = 2^x$ are similar in that they both contain a **base** and an **exponent** (or power). In $y = x^2$, the base is

• **Hint:** Another word for exponent is **index** (plural: **indices**).

the variable x and the exponent is the constant 2. In $y = 2^x$, the base is the constant 2 and the exponent is the variable x.

The quadratic function $y = x^2$ is in the form 'variable base$^{\text{constant power}}$', where the base is a variable and the exponent is an integer greater than or equal to zero (non-negative integer). Any function in this form is called a **power function**.

The function $y = 2^x$ is in the form 'constant base$^{\text{variable power}}$', where the base is a positive real number (not equal to one) and the exponent is a variable. Any function in this form is called an **exponential function**.

To illustrate a fundamental difference between exponential functions and power functions, consider the function values for $y = x^2$ and $y = 2^x$ when x is an integer from 0 to 10. Table 5.1 showing these results displays clearly how the values for the exponential function eventually increase at a significantly faster rate than the power function.

x	$y = x^2$	$y = 2^x$
0	0	1
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
6	36	64
7	49	128
8	64	256
9	81	512
10	100	1024

Table 5.1 Contrast between power function and exponential function.

Another important point to make is that power functions can easily be defined (and computed) for any real number. For any power function $y = x^n$, where n is any positive integer, y is found by simply taking x and repeatedly multiplying it n times. Hence, x can be any real number. For example, for the power function $y = x^3$, if $x = \pi$, then $y = \pi^3 \approx 31.006\,276\,68\ldots$. Since a power function like $y = x^3$ is defined for all real numbers, we can graph it as a continuous curve so that every real number is the x-coordinate of some point on the curve. What about the exponential function $y = 2^x$? Can we compute a value for y for any real number x? Before we try, let's first consider x being any rational number and recall the following laws of exponents (indices) that were covered in Section 1.3.

Laws of exponents

For $b > 0$ and $m, n \in \mathbb{Q}$ (rational numbers):

$b^m \cdot b^n = b^{m+n}$ $\qquad \frac{b^m}{b^n} = b^{m-n}$ $\qquad (b^m)^n = b^{mn}$ $\qquad b^0 = 1$ $\qquad b^{-m} = \frac{1}{b^m}$

Also, in Section 1.3, we covered the definition of a rational exponent.

Rational exponent

For $b > 0$ and $m, n \in \mathbb{Z}$ (integers):

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$$

From these established facts, we are able to compute b^x $(b > 0)$ when x is any rational number. For example, $b^{4.7} = b^{\frac{47}{10}}$ represents the 10th root of b raised to the 47th power, i.e. $\sqrt[10]{b^{47}}$. Now, we would like to define b^x when x is any real number such as π or $\sqrt{2}$. We know that π has a non-terminating, non-repeating decimal representation that begins $\pi = 3.141\,592\,653\,589\,793\ldots$. Consider the sequence of numbers

$$b^3, b^{3.1}, b^{3.14}, b^{3.141}, b^{3.1415}, b^{3.14159}, \ldots$$

To demonstrate just how quickly $y = 2^x$ increases, consider what would happen if you were able to repeatedly fold a piece of paper in half 50 times. A typical piece of paper is about five thousandths of a centimetre thick. Each time you fold the piece of paper the thickness of the paper doubles, so after 50 folds the thickness of the folded paper is the height of a stack of 2^{50} pieces of paper. The thickness of the paper after being folded 50 times would be $2^{50} \times 0.005$ cm – which is more than 56 million kilometres (nearly 35 million miles)! Compare that with the height of a stack of 50^2 pieces of paper that would be a meagre $12\frac{1}{2}$ cm – only 0.000 125 km.

Every term in this sequence is defined because each has a rational exponent. Although it is beyond the scope of this text, it can be proved that each number in the sequence gets closer and closer to a certain real number – defined as b^{π}. Similarly, we can define other irrational exponents in such a way that the laws of exponents hold for all real exponents. Table 5.2 shows a sequence of exponential expressions approaching the value of 2^{π}.

Table 5.2 Approaching the value of 2^{π}.

x	2^x (12 s.f.)
3	8.000 000 000 00
3.1	8.574 187 700 29
3.14	8.815 240 927 01
3.141	8.821 353 304 55
3.1415	8.824 411 082 48
3.141 59	8.824 961 595 06
3.141 592	8.824 973 829 06
3.141 5926	8.824 977 499 27
3.141 592 65	8.824 977 805 12

Your GDC will give an approximate value for 2^{π} to at least 10 significant figures, as shown below.

```
2^π
          8.824977827
```

Graphs of exponential functions

Using this definition of irrational powers, we can now construct a complete graph of any exponential function $f(x) = b^x$ such that b is a number greater than zero ($b \neq 1$) and x is any real number.

Example 1

Graph each exponential function by plotting points.

a) $f(x) = 3^x$ b) $g(x) = \left(\frac{1}{3}\right)^x$

Solution

We can easily compute values for each function for integral values of x from -3 to 3. Knowing that exponential functions are defined for all real numbers – not just integers – we can sketch a smooth curve in Figure 5.1, filling in between the ordered pairs shown in the table.

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$

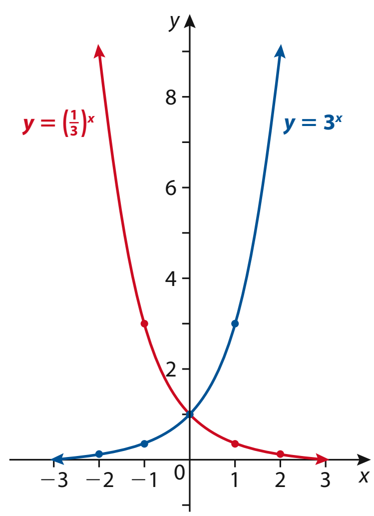


Figure 5.1

Remember that in Section 2.4 we established that the graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis. It is clear from the table and the graph in Figure 5.1 that the graph of function g is a reflection of function f about the y-axis. Let's use some laws of exponents to show that $g(x) = f(-x)$.

$$g(x) = \left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x} = f(-x)$$

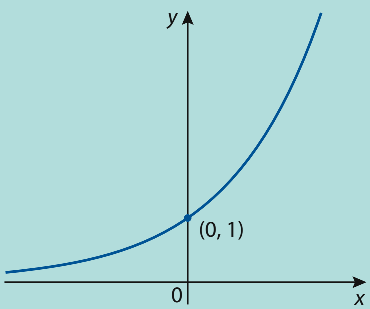
It is useful to point out that both of the graphs, $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$, pass through the point $(0, 1)$ and have a horizontal asymptote of $y = 0$ (x-axis). The same is true for the graph of all exponential functions in the form $y = b^x$ given that $b \neq 1$. If $b = 1$, then $y = 1^x = 1$ and the graph is a horizontal line rather than a constantly increasing or decreasing curve.

Exponential functions

If $b > 0$ and $b \neq 1$, the **exponential function** with base b is the function defined by

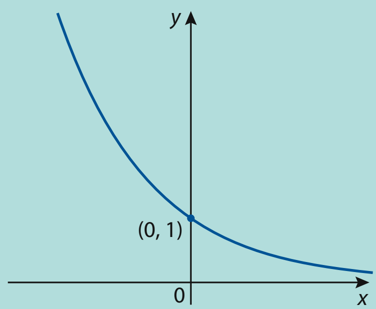
$$f(x) = b^x$$

The **domain** of f is the set of real numbers ($x \in \mathbb{R}$) and the **range** of f is the set of positive real numbers ($y > 0$). The graph of f passes through $(0, 1)$, has the x-axis as a **horizontal asymptote**, and, depending on the value of the base of the exponential function b, will either be a continually increasing **exponential growth curve** or a continually decreasing **exponential decay curve**.



$f(x) = b^x$ for $b > 1$
as $x \to \infty$, $f(x) \to \infty$

f is an increasing function
exponential growth curve



$f(x) = b^x$ for $0 < b < 1$
as $x \to \infty$, $f(x) \to 0$

f is a decreasing function
exponential decay curve

The graphs of all exponential functions will display a characteristic growth or decay curve. As we shall see, many natural phenomena exhibit exponential growth or decay. Also, the graphs of exponential functions behave **asymptotically** for either very large positive values of x (decay curve) or very large negative values of x (growth curve). This means that there will exist a horizontal line that the graph will approach, but not intersect, as either $x \to \infty$ or as $x \to -\infty$.

Transformations of exponential functions

Recalling from Section 2.4 how the graphs of functions are translated and reflected, we can efficiently sketch the graph of many exponential functions.

Example 2

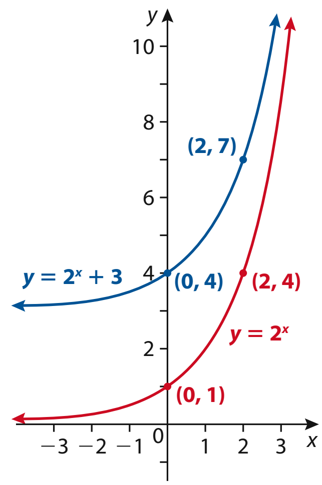
Using the graph of $f(x) = 2^x$, sketch the graph of each function. State the domain and range for each function and the equation of its horizontal asymptote.

a) $g(x) = 2^x + 3$ b) $h(x) = 2^{-x}$ c) $p(x) = -2^x$

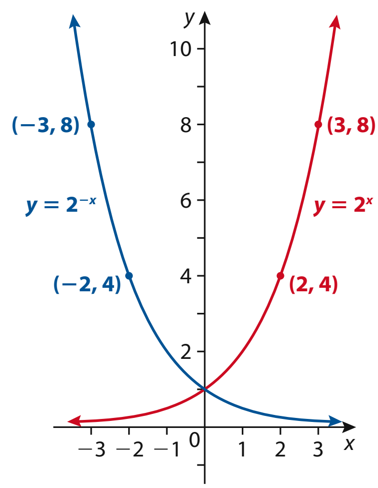
d) $r(x) = 2^{x-4}$ e) $v(x) = 3(2^x)$

Solution

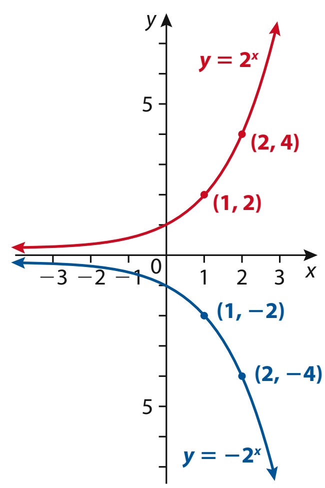
a) The graph of $g(x) = 2^x + 3$ can be obtained by translating the graph of $f(x) = 2^x$ vertically three units up. For function g, the domain is x is any real number ($x \in \mathbb{R}$) and the range is $y > 3$. The horizontal asymptote for g is $y = 3$.



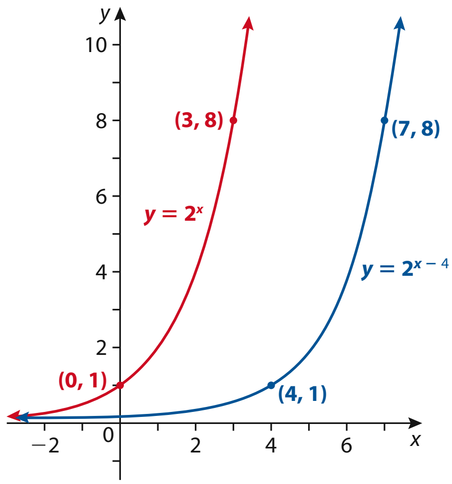
b) The graph of $h(x) = 2^{-x}$ can be obtained by reflecting the graph of $f(x) = 2^x$ across the y-axis. For function h, the domain is $x \in \mathbb{R}$ and the range is $y > 0$. The horizontal asymptote is $y = 0$ (x-axis).



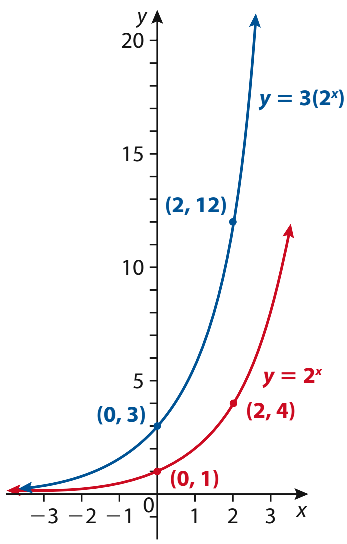
c) The graph of $p(x) = -2^x$ can be obtained by reflecting the graph of $f(x) = 2^x$ across the x-axis. For function p, the domain is $x \in \mathbb{R}$ and the range is $y < 0$. The horizontal asymptote is $y = 0$ (x-axis).



d) The graph of $r(x) = 2^{x-4}$ can be obtained by translating the graph of $f(x) = 2^x$ four units to the right. For function r, the domain is $x \in \mathbb{R}$ and the range is $y > 0$. The horizontal asymptote is $y = 0$ (x-axis).



e) The graph of $v(x) = 3(2^x)$ can be obtained by a vertical stretch of the graph of $f(x) = 2^x$ by scale factor 3. For function v, the domain is $x \in \mathbb{R}$ and the range is $y > 0$. The horizontal asymptote is $y = 0$ (x-axis).



Note that for function p in part c) of Example 2 the horizontal asymptote is an **upper bound** (i.e. no function value is equal to or greater than $y = 0$). Whereas, in parts a), b), d) and e) the horizontal asymptote for each function is a **lower bound** (i.e. no function value is equal to or less than the y-value of the asymptote).

5.2 Exponential growth and decay

Mathematical models of growth and decay

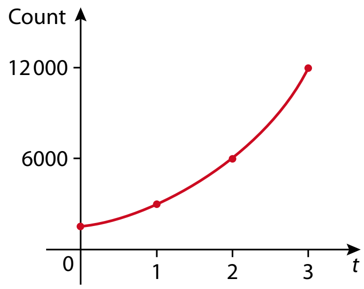
Exponential functions are well suited as a mathematical model for a wide variety of steadily increasing or decreasing phenomena of many kinds, including population growth (or decline), investment of money with compound interest and radioactive decay. Recall from the previous chapter that the formula for finding terms in a geometric sequence (repeated multiplication by common ratio r) is an exponential function. Many instances of growth or decay occur geometrically (repeated multiplication by a growth or decay factor).

Exponential models

Exponential models are equations of the form $A(t) = A_0b^t$, where $A_0 \neq 0$, $b > 0$ and $b \neq 1$. $A(t)$ is the **amount after time *t***. $A(0) = A_0b^0 = A_0(1) = A_0$, so A_0 is called the **initial amount** or value (often the value at time $(t) = 0$). If $b > 1$, then $A(t)$ is an **exponential growth model**. If $0 < b < 1$, then $A(t)$ is an **exponential decay model**. The value of b, the base of the exponential function, is often called the **growth or decay factor**.

Example 3

A sample count of bacteria in a culture indicates that the number of bacteria is doubling every hour. Given that the estimated count at 15:00 was 12 000 bacteria, find the estimated count three hours earlier at 12:00 and write an exponential growth function for the number of bacteria at any hour t.



Solution

Consider the time at 12:00 to be the starting, or initial, time and label it $t = 0$ hours. Then the time at 15:00 is $t = 3$. The amount at any time t (in hours) will double after an hour so the growth factor, b, is 2. Therefore, $A(t) = A_0(2)^t$. Knowing that $A(3) = 12\,000$, compute A_0: $12\,000 = A_0(2)^3 \Rightarrow 12\,000 = 8A_0 \Rightarrow A_0 = 1500$. Therefore, the estimated count at 12:00 was 1500, and the growth function for number of bacteria at time t is $A(t) = 1500(2)^t$.

Radioactive carbon (carbon-14 or C-14), produced when nitrogen-14 is bombarded by cosmic rays in the atmosphere, drifts down to Earth and is absorbed from the air by plants. Animals eat the plants and take C-14 into their bodies. Humans in turn take C-14 into their bodies by eating both plants and animals. When a living organism dies, it stops absorbing C-14, and the C-14 that is already in the object begins to decay at a slow but steady rate, reverting to nitrogen-14. The half-life of C-14 is 5730 years. Half of the original amount of C-14 in the organic matter will have disintegrated after 5730 years; half of the remaining C-14 will have been lost after another 5730 years, and so forth. By measuring the ratio of C-14 to N-14, archaeologists are able to date organic materials. However, after about 50 000 years, the amount of C-14 remaining will be so small that the organic material cannot be dated reliably.

Radioactive material decays at exponential rates. The **half-life** is the amount of time it takes for a given amount of material to decay to half of its original amount. An exponential function that models decay with a known value for the half-life, h, will be of the form $A(t) = A_0\left(\frac{1}{2}\right)^h$, where the decay factor is $\frac{1}{2}$ and h represents the number of half-lives that have occurred (i.e. the number of times that A_0 is multiplied by $\frac{1}{2}$). If t represents the amount of time, the number of half-lives will be $\frac{t}{h}$. For example, if the half-life of a certain material is 25 days and the amount of time that has passed since measuring the amount A_0 is 75 days, then the number of half-lives is

$k = \frac{t}{h} = \frac{75}{25} = 3$, and the amount of material remaining is equal to

$A_0\left(\frac{1}{2}\right)^3 = \frac{A_0}{8}$.

Half-life formula

If a certain initial amount, A_0, of material decays with a half-life of h, the amount of material that remains at time t is given by the exponential decay model $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$. The time units (e.g. seconds, hours, years) for h and t must be the same.

Example 4

The half-life of radioactive carbon-14 is approximately 5730 years. How much of a 10 g sample of carbon-14 remains after 15 000 years?

Solution

The exponential decay model for the carbon-14 is $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$.

What remains of 10 g after 15 000 years is given by

$A(15\,000) = 10\left(\frac{1}{2}\right)^{\frac{15\,000}{5730}} \approx 1.63$ g.

Compound interest

Recall from Chapter 4 that exponential functions occur in calculating compound interest. If an initial amount of money P, called the **principal**, is invested at an interest rate r per time period, then after one time period the amount of interest is $P \times r$ and the total amount of money is $A = P + Pr = P(1 + r)$. If the interest is added to the principal, the new principal is $P(1 + r)$, and the total amount after another time period is $A = P(1 + r)(1 + r) = P(1 + r)^2$. In the same way, after a third time period the amount is $A = P(1 + r)^3$. In general, after k periods the total amount is $A = P(1 + r)^k$, an exponential function with growth factor $1 + r$. For example, if the amount of money in a bank account is earning interest at a rate of 6.5% per time period, the growth factor is $1 + 0.065 = 1.065$. Is it possible for r to be negative? Yes, if an amount (not just money) is decreasing. For example, if the population of a town is decreasing by 12% per time period, the decay factor is $1 - 0.12 = 0.88$.

For compound interest, if the annual interest rate is r and interest is compounded (number of times added in) n times per year, then each time period the interest rate is $\frac{r}{n}$, and there are $n \times t$ time periods in t years.

Compound interest formula

The exponential function for calculating the amount of money after t years, $A(t)$, where P is the initial amount or principal, the annual interest rate is r and the number of times interest is compounded per year is n, is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example 5

An initial amount of 1000 euros is deposited into an account earning $5\frac{1}{4}\%$ interest per year. Find the amounts in the account after eight years if interest is compounded annually, semi-annually, quarterly, monthly and daily.

Solution

We use the exponential function associated with compound interest with values of $P = 1000$, $r = 0.0525$ and $t = 8$ to complete the results in Table 5.3.

Compounding	n	Amount after 8 years
Annual	1	$1000\left(1 + \frac{0.0525}{1}\right)^{8} = 1505.83$
Semi-annual	2	$1000\left(1 + \frac{0.0525}{2}\right)^{2(8)} = 1513.74$
Quarterly	4	$1000\left(1 + \frac{0.0525}{4}\right)^{4(8)} = 1517.81$
Monthly	12	$1000\left(1 + \frac{0.0525}{12}\right)^{12(8)} = 1520.57$
Daily	365	$1000\left(1 + \frac{0.0525}{365}\right)^{365(8)} = 1521.92$

Table 5.3 Compound interest calculations.

Example 6

A new car is purchased for \$22 000. If the value of the car decreases (depreciates) at a rate of approximately 15% per year, what will be the approximate value of the car to the nearest whole dollar in $4\frac{1}{2}$ years?

Solution

The decay factor for the exponential function is $1 - r = 1 - 0.15 = 0.85$. In other words, after each year the car's value is 85% of what it was one year before. We use the exponential decay model $A(t) = A_0 b^t$ with values $A_0 = 22\,000$, $b = 0.85$ and $t = 4.5$.

$$A(4.5) = 22\,000(0.85)^{4.5} \approx 10\,588$$

The value of the car will be approximately \$10 588.

Exercise 5.1 and 5.2

1 a) Write the equation for an exponential equation with base $b > 0$.

b) Given $b \neq 1$, state the domain and range of this function.

c) Sketch the general shape of the graph of this exponential function for each of two cases:
(i) $b > 1$ (ii) $0 < b < 1$

For questions 2–7, sketch a graph of the function and state its domain, range, y-intercept and the equation of its horizontal asymptote.

2 $f(x) = 3^{x+4}$

3 $g(x) = -2^x + 8$

4 $h(x) = 4^{-x} - 1$

5 $p(x) = \dfrac{1}{2^x - 1}$

6 $q(x) = 3(3^{-x}) - 3$

7 $k(x) = 2^{-|x-2|} + 1$

8 If a general exponential function is written in the form $f(x) = a(b)^{x-c} + d$, state the domain, range, y-intercept and the equation of the horizontal asymptote in terms of the parameters a, b, c and d.

9 Using your GDC and a graph-viewing window with Xmin = −2, Xmax = 2, Ymin = 0 and Ymax = 4, sketch a graph for each exponential equation on the same set of axes.

a) $y = 2^x$ b) $y = 4^x$ c) $y = 8^x$

d) $y = 2^{-x}$ e) $y = 4^{-x}$ f) $y = 8^{-x}$

10 Write equations that are equivalent to the equations in 9 d), e) and f) but have an exponent of positive x rather than negative x.

11 If $1 < a < b$, which is steeper: the graph of $y = a^x$ or $y = b^x$?

12 The population of a city triples every 25 years. At time $t = 0$, the population is 100 000. Write a function for the population $P(t)$ as a function of t. What is the population after:

a) 50 years b) 70 years c) 100 years?

13 An experiment involves a colony of bacteria in a solution. It is determined that the number of bacteria doubles approximately every 3 minutes and the initial number of bacteria at the start of the experiment is 10^4. Write a function for the number of bacteria $N(t)$ as a function of t (in minutes). Approximately how many bacteria are there after:

a) 3 minutes b) 9 minutes c) 27 minutes d) one hour?

14 A bank offers an investment account that will double your money in 10 years.

a) Express $A(t)$, the amount of money in the account after t years, in the form $A(t) = A_0(r)^t$.

b) If interest was added into the account just once at the end of each year (simple interest), then find the annual interest rate for the account (to 3 significant figures).

15 If \$10 000 is invested at an annual interest rate of 11%, compounded quarterly, find the value of the investment after the given number of years.

a) 5 years b) 10 years c) 15 years

16 A sum of \$5000 is deposited into an investment account that earns interest at a rate of 9% per year compounded monthly.

a) Write the function $A(t)$ that computes the value of the investment after t years.

b) Use your GDC to sketch a graph of $A(t)$ with values of t on the horizontal axis ranging from $t = 0$ years to $t = 25$ years.

c) Use the graph on your GDC to determine the minimum number of years (to the nearest whole year) for this investment to have a value greater than \$20 000.

17 If \$10 000 is invested at an annual interest rate of 11% for a period of five years, find the value of the investment for the following compounding periods.

a) annually b) monthly c) daily d) hourly

18 Imagine a bank account that has the fantastic annual interest rate of 100%. If you deposit \$1 into this account, how much will be in the account exactly one year later, for the following compounding periods?

a) annually b) monthly c) daily d) hourly e) every minute

19 Each year for the past eight years, the population of deer in a national park increases at a steady rate of 3.2% per year. The present population is approximately 248 000.

a) What was the approximate number of deer one year ago?

b) What was the approximate number of deer eight years ago?

20 Radioactive carbon-14 has a half-life of 5730 years. The remains of an animal are found 20 000 years after it died. About what percentage (to 3 significant figures) of the original amount of carbon-14 (when the animal was alive) would you expect to find?

21 Once a certain drug enters the bloodstream of a human patient, it has a half-life of 36 hours. An amount of the drug, A_0, is injected in the bloodstream at 12:00 on Monday. How much of the drug will be in the bloodstream of the patient five days later at 12:00 on Friday?

22 An open can is filled with 1000 ml of fluid that evaporates at a rate of 30% per week.

a) Write a function, $A(w)$, that gives the amount of fluid after w weeks.

b) Use your GDC to find how many weeks (whole number) it will take for the volume of fluid to be less than 1 ml.

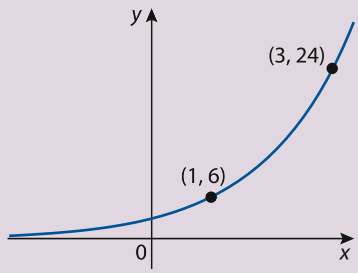
23 Why are exponential functions of the form $f(x) = b^x$ defined so that $b > 0$?

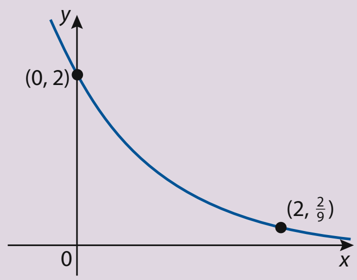
24 You are offered a highly paid job that lasts for just one month – exactly 30 days. Which of the following payment plans, I or II, would give you the largest salary? How much would you get paid?

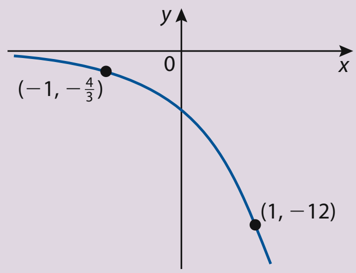
I One dollar on the first day of the month, two dollars on the second day, three dollars on the third day, and so on (getting paid one dollar more each day) until the end of the 30 days. (You would have a total of \$55 after 10 days.)

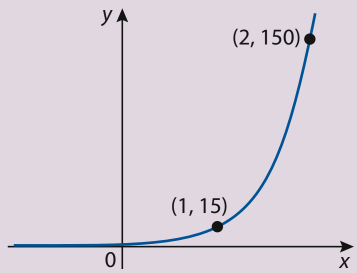
II One cent ($0.01) on the first day of the month, two cents ($0.02) on the second day, four cents on the third day, eight cents on the fourth day, and so on (each day getting paid double from the previous day) until the end of the 30 days. (You would have a total of $10.23 after 10 days.)

25 Each exponential function graphed below can be written in the form $f(x) = k(a)^x$. Find the value of a and k for each.

a) 

b) 

c) 

d) 

5.3 The number e

Recalling the definition of an exponential function, $f(x) = b^x$, we recognize that b is any positive constant and x is any real number. Graphs of $y = b^x$ for a few values where $b \geqslant 1$ are shown in Figure 5.2. As noted in the first section of this chapter, all the graphs pass through the point $(0, 1)$.

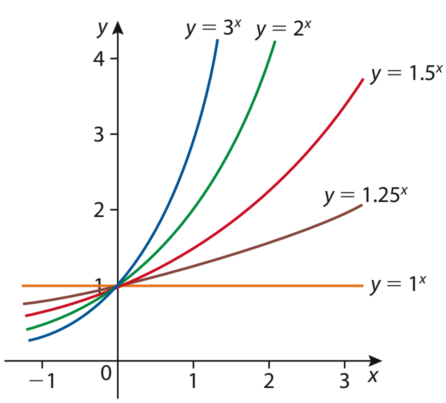


Figure 5.2 Graphs of $y = b^x$ for some values when $b \geqslant 1$.

The question arises: what is the *best* number to choose for the base b? There is a good argument for $b = 10$ since we most commonly use a base 10 number system. Your GDC will have the expression 10^x as a built-in command. The base $b = 2$ is also plausible because a binary number system (base 2) is used in many processes, especially in computer systems. However, the most important base is an irrational number that is denoted with the letter e. As we will see, the value of e approximated to six significant figures is 2.718 28. The importance of e will be clearer when we get to calculus topics. The number π – another very useful irrational number – has a natural geometric significance as the ratio of circumference to diameter for any circle. The number e also occurs in a 'natural' manner. We will illustrate this two different ways: first, by considering the **rate of change** of an exponential function, and secondly, by revisiting compound interest and considering **continuous change** rather than **incremental change**.

The 'discovery' of the constant e is attributed to Jakob Bernoulli (1654–1705). He was a member of the famous Bernoulli family of distinguished mathematicians, scientists and philosophers. This included his brother Johann (1667–1748), who made important developments in calculus, and his nephew Daniel (1700–1782), who is most well known for Bernoulli's principle in physics. The constant e is of enormous mathematical significance – and it appears 'naturally' in many mathematical processes. Jakob Bernoulli first observed e when studying sequences of numbers in connection to compound interest problems.

Rate of change (slope) of an exponential function

Since exponential functions (and associated logarithmic functions) are very important in calculus, the criteria we will use to determine the best value for b will be based on considering the slope of the curve $y = b^x$. In calculus we are interested in the rate of change (i.e. slope of the graph) of functions. Our goal to is to find a value for b such that the slope of the graph of $y = b^x$ at any value of x is equal to the function value y. We could investigate this by trial and error – and with a GDC this might prove fruitful – but it would not guarantee us an exact value and it could prove inefficient. Let's narrow our investigation to studying the slope of the curves at the point (0, 1) which is convenient because it is shared by all the curves.

To obtain a good estimate for the value of e we will use the diagram in Figure 5.3 where the scale on the x- and y-axes are equal and $P(0, 1)$ is the y-intercept of the graph of $y = e^x$. Q is a point on $y = e^x$ close to point P with coordinates (h, e^h). PR and RQ are parallel to the x- and y-axes, respectively, and they intersect at point $R(h, 1)$. The slope of the curve is always changing. It is not constant as with a straight line. As we will justify more thoroughly in our study of differential calculus in Chapter 13, the slope of a curve at a point will be equal to the slope of the line tangent to the curve at that point. PS is the tangent line to the curve at P, intersecting RQ at S. Thus, we are looking for the value e such that the slope of the tangent line PS is equal to 1. It follows that $\frac{RS}{PR} = 1$ and because $PR = h$ then $RS = h$. Since we have set Q close to P then we can assume that h is very small. Therefore, $RS \approx RQ$ and $\frac{RQ}{RS} \approx 1$. The value of $\frac{RQ}{RS}$ will get closer and closer to the value of 1 as h gets smaller (i.e. as Q gets closer to P). Since the y-coordinate of R is 1, then $RQ = e^h - 1$. Substituting h for RS and $e^h - 1$ for RQ into $\frac{RQ}{RS} \approx 1$, gives $\frac{e^h - 1}{h} \approx 1$. We wish to obtain an estimate for e so we multiply through by h to get $e^h - 1 \approx h$ leading to $e^h \approx h + 1$. To isolate e we raise both sides to the $\frac{1}{h}$ power, finally producing, $e \approx (1 + h)^{\frac{1}{h}}$.

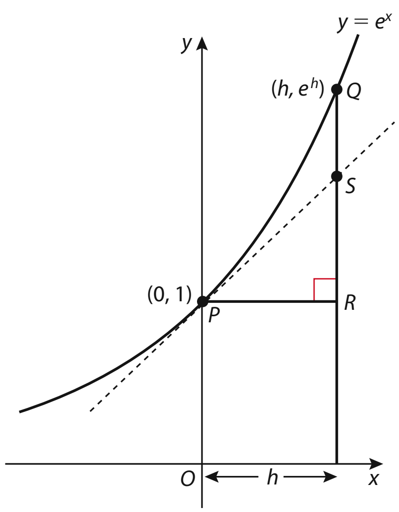


Figure 5.3 Graph of $y = e^x$; slope of the tangent line PS is equal to 1.

h	$e \approx (1+h)^{\frac{1}{h}}$
0.1	2.593742…
0.01	2.704814…
0.001	2.716924…
0.0001	2.718146…
0.00001	2.718268…
0.000001	2.718280…
0.0000001	2.718282…

Table 5.4 Values for $e \approx (1+h)^{\frac{1}{h}}$ as h approaches zero (accuracy to 7 significant figures).

Given that h is made small enough, the expression above should give a good estimation of the value of e. Using the approximation $e \approx (1+h)^{\frac{1}{h}}$, Table 5.4 shows values for e as h approaches zero.

To an accuracy of six significant figures, it appears that the value of e is approximately 2.718 28.

Definition of e (I) $\quad e = \lim_{h \to 0}(1+h)^{\frac{1}{h}}$

The definition is read 'e equals the limit of $(1+h)^{\frac{1}{h}}$ as n goes to zero.'

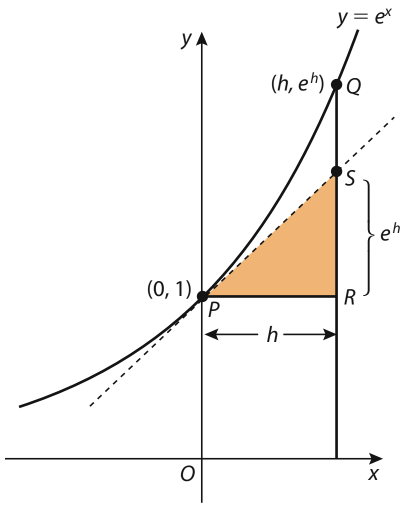


Figure 5.4 At $x = 0$, the rate of change of $y = e^x$ is equal to 1.

Geometrically speaking, as point Q gets closer to point P ($h \to 0$), and also closer to point S, we wanted the slope of the tangent line at $(0, 1)$, $\frac{RS}{PR}$, to be equal to 1. This is the same as saying that we wanted $\frac{e^h - 1}{h} \to 1$ as $h \to 0$ (see coloured triangle in Figure 5.4). The value of e approximated to increasing accuracy in Table 5.4 is the number that makes this happen. A non-geometrical way of describing this feature of the graph is to say that the **rate of change** (slope) of the function $y = e^x$ at $x = 0$ is equal to 1.

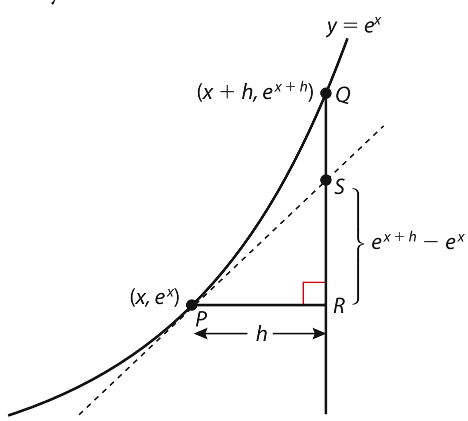
The rate of change of $y = e^x$ at a *general* value of x can be similarly obtained by fixing point P on the curve with coordinates (x, e^x) and a nearby point Q with coordinates $(x + h, e^{x+h})$. See Figure 5.5 below.

Then the rate of change of the function at point P is $\frac{e^{x+h} - e^x}{h}$ as $h \to 0$. We cannot evaluate the limit of $\frac{e^{x+h} - e^x}{h}$ as $h \to 0$ directly by substituting 0 for h. By applying some algebra and knowing that $\frac{e^h - 1}{h} \to 1$ as $h \to 0$, we can evaluate the required limit.

As $h \to 0$, $\frac{e^{x+h} - e^x}{h} = \frac{e^x e^h - e^x}{h} = \frac{e^x(e^h - 1)}{h} = e^x\left[\frac{e^h - 1}{h}\right] \to e^x \cdot 1 = e^x$

Therefore, for any value of x, the rate of change of the function $y = e^x$ is e^x. In other words, the rate of change of the function at any value in the domain (x) is equal to the corresponding value of the range (y). This is the amazing feature of $y = e^x$ that makes e the most useful and 'natural' choice for the base of an exponential function, and the irrational number $e \approx 2.718\,28\ldots$ is the only base for which this is true.

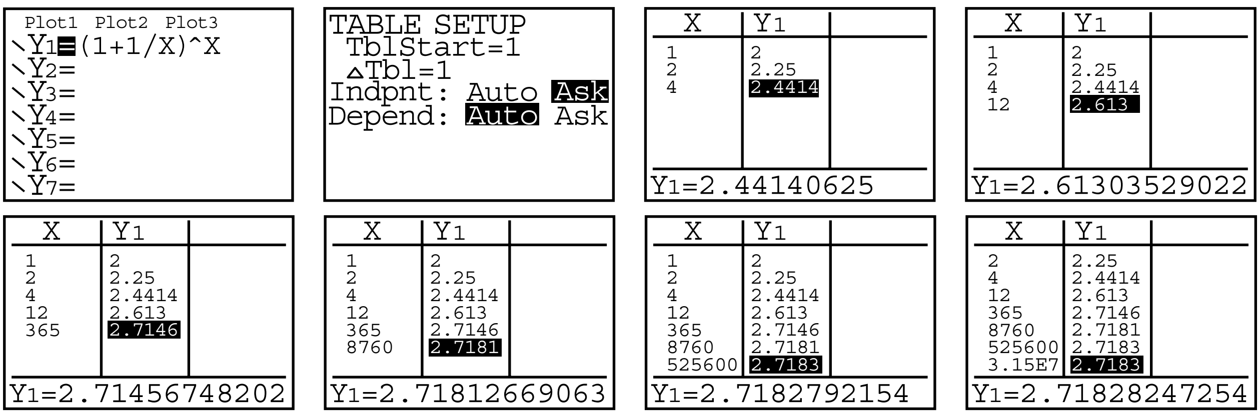
Figure 5.5 The rate of change of $y = e^x$ at a general value of x.



Continuously compounded interest

In the previous section and in Chapter 4, we computed amounts of money resulting from an initial amount (principal) with interest being compounded (added in) at discrete intervals (e.g. yearly, monthly, daily). In the formula that we used, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, n is the number of times that interest is compounded per year. Instead of adding interest only at discrete intervals, let's investigate what happens if we try to add interest continuously – that is, let the value of n increase without bound ($n \to \infty$).

Consider investing just \$1 at a very generous annual interest rate of 100%. How much will be in the account at the end of just one year? It depends on how often the interest is compounded. If it is only added at the end of the year ($n = 1$), the account will have \$2 at the end of the year. Is it possible to compound the interest more often to get a one-year balance of \$2.50 or of \$3.00? We use the compound interest formula with $P = \$1$, $r = 1.00$ (100%) and $t = 1$, and compute the amounts for increasing values of n.
$A(1) = 1\left(1 + \frac{1}{n}\right)^{n \cdot 1} = \left(1 + \frac{1}{n}\right)^{n}$. This can be done very efficiently on your GDC by entering the equation $y = \left(1 + \frac{1}{x}\right)^{x}$ to display a table showing function values of increasing values of x.



As the number of compounding periods during the year increases, the amount at the end of the year appears to approach a limiting value.

As $n \to \infty$, the quantity of $\left(1 + \frac{1}{n}\right)^{n}$ approaches the number e. To 13 decimal places, e is approximately 2.718 281 828 4590.

Table 5.5

Compounding	n	$A(1) = \left(1 + \frac{1}{n}\right)^{n}$
Annual	1	2
Semi-annual	2	2.25
Quarterly	4	2.441 406 25…
Monthly	12	2.613 035 290 22…
Daily	365	2.714 567 482 02…
Hourly	8 760	2.718 126 690 63…
Every minute	525 600	2.718 279 2154…
Every second	31 536 000	2.718 282 472 54…

Leonhard Euler (1701–1783) was the dominant mathematical figure of the 18th century and is one of the most influential and prolific mathematicians of all time. Euler's collected works fill over 70 large volumes. Nearly every branch of mathematics has significant theorems that are attributed to Euler.

Euler proved mathematically that the limit of $\left(1+\frac{1}{n}\right)^n$ as n goes to infinity is precisely equal to an irrational constant which he labelled e. His mathematical writings were influential not just because of the content and quantity but also because of Euler's insistence on clarity and efficient mathematical notation. Euler introduced many of the common algebraic notations that we use today. Along with the symbol e for the base of natural logarithms (1727), Euler introduced $f(x)$ for a function (1734), i for the square root of negative one (1777), π for pi, Σ for summation (1755), and many others. His introductory algebra text, written originally in German (Euler was Swiss), is still available in English translation. Euler spent most of his working life in Russia and Germany. Switzerland honoured Euler by placing his image on the 10 Swiss franc banknote.

Definition of *e* (II)

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

The definition is read as 'e equals the limit of $\left(1+\frac{1}{n}\right)^n$ as n goes to infinity'.

Note that the two definitions that we have provided for the number e are equivalent. Take our first limit definition for e: $e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$. Let $\frac{1}{h} = n$, it follows that $h = \frac{1}{n}$ and as $h \to 0$ then $n \to \infty$. Substituting $\frac{1}{n}$ for h, n for $\frac{1}{h}$, and evaluating the limit as $n \to \infty$ transforms $\lim_{h\to 0}(1+h)^{\frac{1}{h}}$ to $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, which is our second limit definition for e.

As the number of compoundings, n, increase without bound, we approach continuous compounding – where interest is being added continuously. In the formula for calculating amounts resulting from compound interest, letting $m = \frac{n}{r}$ produces

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt} = P\left(1+\frac{1}{m}\right)^{mrt} = P\left[\left(1+\frac{1}{m}\right)^m\right]^{rt}$$

Now if $n \to \infty$ and the interest rate r is constant, then $\frac{n}{r} = m \to \infty$. From the limit definition of e, we know that if $m \to \infty$, then $\left(1+\frac{1}{m}\right)^m \to e$.

Therefore, for continuous compounding, it follows that $A(t) = P\left[\left(1+\frac{1}{m}\right)^n\right]^{rt} = P[e]^{rt}$.

This result is part of the reason that e is the best choice for the base of an exponential function modelling change that occurs continuously (e.g. radioactive decay) rather than in discrete intervals.

Continuous compound interest formula

An exponential function for calculating the amount of money after t years, $A(t)$, for interest compounded continuously, where P is the initial amount or principal and r is the annual interest rate, is given by $A(t) = Pe^{rt}$.

Example 7

An initial investment of 1000 euros earns interest at an annual rate of $5\frac{1}{4}\%$. Find the total amount after 10 years if the interest is compounded:
a) annually (simple interest), b) quarterly, and c) continuously.

Solution

a) $A(t) = P(1 + r)^t = 1000(1 + 0.0525)^{10} = 1669.10$ euros

b) $A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = 1000\left(1 + \frac{0.0525}{4}\right)^{4(10)} = 1684.70$ euros

c) $A(t) = Pe^{rt} = 1000e^{0.0525(10)} = 1690.46$ euros

The natural exponential function and continuous change

For many applications involving continuous change, the most suitable choice for a mathematical model is an exponential function with a base having the value of e.

The natural exponential function

The natural exponential function is the function defined as

$$f(x) = e^x$$

As with other exponential functions, the domain of the natural exponential function is the set of all real numbers ($x \in \mathbb{R}$), and its range is the set of positive real numbers ($y > 0$). The natural exponential function is often referred to as *the* exponential function.

The formula developed for continuously compounded interest does not apply only to applications involving adding interest to financial accounts. It can be used to model growth or decay of a quantity that is changing *geometrically* (i.e. repeated multiplication by a constant ratio, or growth/decay factor) and the change is continuous, or approaching continuous. Another version of a formula for continuous change, which we will learn more about in calculus, is stated below:

Continuous exponential growth/decay

If an initial quantity C (when $t = 0$) increases or decreases continuously at a rate r over a certain time period, the amount $A(t)$ after t time periods is given by the function $A(t) = Ce^{rt}$. If $r > 0$, the quantity is increasing (growing). If $r < 0$, the quantity is decreasing (decaying).

Example 8

The cost of the new Boeing 787 Dreamliner airplane will be 150 million US dollars when purchased new. The airplane will lose value at a continuous rate. This is modeled by the continuous decay function $C(t) = 150e^{-0.053t}$ where $A(t)$ is the value of the airplane (in millions) after t years.

a) How much (to the nearest million dollars) would a Dreamliner jet be worth precisely five years after being purchased?

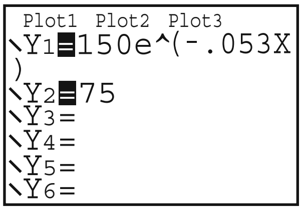
b) If a Dreamliner jet is purchased in 2010, what would be the first year that the jet is worth less than half of its original cost?

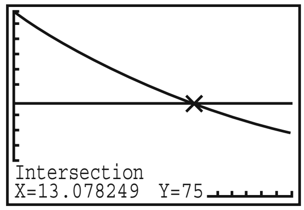
c) Find the value of b (to 4 s.f.) for a discrete decay model, $D(t) = 150b^t$, so that $D(t)$ is a suitable model to describe the same decay as $C(t)$.

Solution

a) $C(5) = 150e^{-0.053(5)} \approx 115$. The value is approximately \$115 million after five years.

b) Using a GDC, we graph the decay equation $y = 150e^{-0.053x}$ and the horizontal line $y = 75$ and determine the intersection point.





The x-coordinate of the intersection point is approximately 13.08. At the start of 2013, the jet's value is not yet half of its original value. Therefore, the first year that the jet is worth less than half of its original cost is 2014.

c) One way to find the value of b so that $D(t) = 150b^t$ serves as a reasonable substitute for $C(t) = 150e^{-0.053t}$ is to compute some function values for $C(t)$ and use them to compute the relative change from one year to the next.

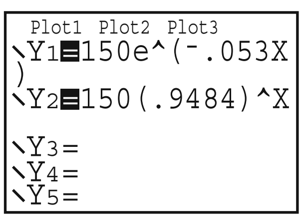
$$C(1) = 150e^{-0.053(1)} \approx 142.2570$$
$$C(2) = 150e^{-0.053(2)} \approx 134.9137$$
$$C(3) = 150e^{-0.053(3)} \approx 127.9495$$

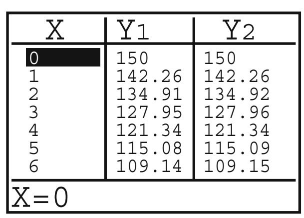
Relative change from year 1 to year 2: $\dfrac{134.9137 - 142.2570}{142.2570} \approx -0.051\,62$

Compute relative change from year 2 to year 3 to make sure it agrees with result above.

Relative change from year 2 to year 3: $\dfrac{127.9495 - 134.9137}{134.9137} \approx -0.051\,62$

The annual rate of decay, b, is the fraction of what remains after each year. Thus, $b = 1 - 0.051\,62 = 0.948\,38$; and to 4 s.f. the annual rate of decay is $b \approx 0.9484$. Therefore, the discrete decay model is $D(t) = 150(0.9484)^t$.





X	Y1	Y2
0	150	150
1	142.26	142.26
2	134.91	134.92
3	127.95	127.96
4	121.34	121.34
5	115.08	115.09
6	109.14	109.15

X=0

To check that the two decay models give similar results for each year, we can use a GDC to display a table of values for both models side by side for easy comparison.

Exercise 5.3

For questions 1–6, sketch a graph of the function and state its: a) domain and range; b) coordinates of any x-intercept(s) and y-intercept; c) and the equation of any asymptote(s).

1 $f(x) = e^{2x-1}$

2 $g(x) = e^{-x} + 1$

3 $h(x) = -2e^x$

4 $p(x) = e^{x^2} - e$

5 $h(x) = \dfrac{1}{1 - e^x}$

6 $h(x) = e^{|x+2|} - 1$

7 a) State a definition of the number e as a limit.

b) Evaluate $\left(1 - \frac{1}{n}\right)^n$ for $n = 100$, $n = 10\,000$ and $n = 1\,000\,000$.

c) To 5 significant figures, what appears to be the value of $\lim\limits_{n\to\infty}\left(1 - \frac{1}{n}\right)^n$? How does this number relate to the number e?

8 Use your GDC to graph the curve $y = \left(1 + \frac{1}{x}\right)^x$ and the horizontal line $y = 2.72$. Use a graph window so that x ranges from 0 to 20 and y ranges from 0 to 3. Describe the behaviour of the graph of $y = \left(1 + \frac{1}{x}\right)^x$. Will it ever intersect the graph of $y = 2.72$? Explain.

9 Two different banks, Bank A and Bank B, offer accounts with exactly the same annual interest rate of 6.85%. However, the account from Bank A has the interest compounded monthly whereas the account from Bank B compounds the interest continuously. To decide which bank to open an account with, you calculate the **amount of interest** you would earn after three years from an initial deposit of 500 euros in each bank's account. It is assumed that you make no further deposits and no withdrawals during the three years. How much interest would you earn from each of the accounts? Which bank's account earns more – and how much more?

10 Dina wishes to deposit \$1000 into an investment account and then withdraw the total in the account in five years. She has the choice of two different accounts. *Blue Star account*: interest is earned at an annual interest rate of 6.13% compounded weekly (52 weeks in a year). *Red Star account*: interest is earned at an annual interest rate of 5.95% compounded continuously. Which investment account – *Blue Star* **or** *Red Star* – will result in the greatest total at the end of five years? What is the total after five years for this account? How much more is it than the total for the other account?

11 Strontium-90 is a radioactive isotope of strontium. Strontium-90 decays according to the function $A(t) = Ce^{-0.0239t}$, where t is time in years and C is the initial amount of strontium-90 when $t = 0$. If you have 1 kilogram of strontium-90 to start with, how much (approximated to 3 significant figures) will you have after:

a) 1 year?

b) 10 years?

c) 100 years?

d) 250 years?

12 A radioactive substance decays in such a way that the mass (in kilograms) remaining after t days is given by the function $A(t) = 5e^{-0.0347t}$.

a) Find the mass (i.e. initial mass) at time $t = 0$.

b) What **percentage** of the initial mass remains after 10 days?

c) On your GDC and then on paper, draw a graph of the function $A(t)$ for $0 \leqslant t \leqslant 50$.

d) Use one of your graphs to approximate, to the nearest whole day, the half-life of the radioactive substance.

13 Which of the given interest rates and compounding periods would provide the better investment?

a) $8\frac{1}{2}\%$ per year, compounded semi-annually

b) $8\frac{1}{4}\%$ per year, compounded quarterly

c) 8% per year, compounded continuously

14 In certain conditions the bacterium that causes cholera, *Vibrios cholerae*, can grow rapidly in number. In a laboratory experiment a culture of *Vibrios cholerae* is started with 20 bacterium. The bacterium's growth is modeled with the following continuous growth model $A(t) = 20e^{0.068t}$ where $A(t)$ is the number of bacteria after t minutes.

a) Determine the value of r for the discrete growth model $B(t) = 20(r)^t$, so that $B(t)$ is equivalent to $A(t)$.

b) For both of these models, by what percentage does the number of bacteria grow each minute?

15 By comparing the graph of each of the following equations to the graph of $y = e^x$, determine if the slope of the tangent line at the point (0, 1) for the graph of each equation is less than or greater than 1.

a) $y = 2^x$

b) $y = \left(\frac{5}{2}\right)^x$

c) $y = \left(\frac{11}{4}\right)^x$

d) $y = 3^x$

16 Consider that £1000 is invested at 4.5% interest compounded continuously.

a) How much money is in the account after 10 years? After 20 years?

b) Use your GDC to determine how many years (to nearest tenth of a year) it takes for the initial investment to double to £2000.

c) If £5000 is invested at the same rate of interest also compounded continuously, how many years (to nearest tenth) would it take to double?

d) Are the answers to b) and c) the same or different? Why?

5.4 Logarithmic functions

In Example 7 of the previous section, we used the equation $A(t) = 1000e^{0.0525t}$ to compute the amount of money in an account after t years. Now suppose we wish to determine how much time, t, it takes for the initial investment of 1000 euros to double. To find this we need to solve the following equation for t: $2000 = 1000e^{0.0525t} \Rightarrow 2 = e^{0.0525t}$. The unknown t is in the exponent. At this point in the book, we do not have an algebraic method to solve such an equation, but developing the concept of a **logarithm** will provide us with the means to do so.

John Napier (1550–1617) was a Scottish landowner, scholar and mathematician who 'invented' logarithms – a word he coined which derives from two Greek words: *logos* – meaning ratio, and *arithmos* – meaning number. Logarithms made numerical calculations much easier in areas such as astronomy, navigation, engineering and warfare. English mathematician Henry Briggs (1561–1630) came to Scotland to work with Napier and together they perfected logarithms, which included the idea of using the base ten. After Napier died in 1617, Briggs took over the work on logarithms and published a book of tables in 1624. By the second half of the 17th century, the use of logarithms had spread around the world. They became as popular as electronic calculators in our time. The great French mathematician Pierre-Simon Laplace (1749–1827) even suggested that the logarithms of Napier and Briggs doubled the life of astronomers, because it so greatly reduced the labours of calculation. In fact, without the invention of logarithms it is difficult to imagine how Kepler and Newton could have made their great scientific advances. In 1621, an English mathematician and clergyman, William Oughtred (1574–1660) used logarithms as the basis for the invention of the slide rule. The slide rule was a very effective calculation tool that remained in common use for over three hundred years.

The inverse of an exponential function

For $b > 1$, an exponential function with base b is increasing for all x, and for $0 < b < 1$ an exponential function is decreasing for all x. It follows from this that all exponential functions must be one-to-one. Recall from Section 2.3 that a one-to-one function passes both a vertical line test and a horizontal line test. We demonstrated that an inverse function would exist for any one-to-one function. Therefore, an exponential function with base b such that $b > 0$ and $b \neq 1$ will have an inverse function, which is given in the following definition. Also recall from Section 2.3 that the domain of a function f is the range of its inverse function f^{-1}, and, similarly, the range of f is the domain of f^{-1}. The domain and range are switched around for a function and its inverse.

Definition of a logarithmic function

For $b > 0$ and $b \neq 1$, the **logarithmic function** $y = \log_b x$ (read as 'logarithm with base b of x') is the inverse of the exponential function with base b.

$$y = \log_b x \text{ if and only if } x = b^y$$

The domain of the logarithmic function $y = \log_b x$ is the set of positive real numbers ($x > 0$) and its range is all real numbers ($y \in \mathbb{R}$).

Logarithmic expressions and equations

When evaluating logarithms, note that *a logarithm is an exponent*. This means that the value of $\log_b x$ is the exponent to which b must be raised to obtain x. For example, $\log_2 8 = 3$ because 2 must be raised to the power of 3 to obtain 8 – that is, $\log_2 8 = 3$ if and only if $2^3 = 8$.

We can use the definition of a logarithmic function to translate a logarithmic equation into an exponential equation and vice versa. When doing this, it is helpful to remember, as the definition stated, that in either form – logarithmic or exponential – the base is the same.

logarithmic equation	**exponential equation**
exponent $\downarrow$	exponent $\downarrow$
$y = \log_b(x)$	$x = b^y$
$\uparrow$ base	$\uparrow$ base

Example 9

Find the value of each of the following logarithms.

a) $\log_7 49$ b) $\log_5(\frac{1}{5})$ c) $\log_6 \sqrt{6}$ d) $\log_4 64$ e) $\log_{10} 0.001$

Solution

For each logarithmic expression in a) to e), we set it equal to y and use the definition of a logarithmic function to obtain an equivalent equation in exponential form. We then solve for y by applying the logical fact that if $b > 0$, $b \neq 1$ and $b^y = b^k$ then $y = k$.

a) Let $y = \log_7 49$ which is equivalent to the exponential equation $7^y = 49$. Since $49 = 7^2$, then $7^y = 7^2$. Therefore, $y = 2 \Rightarrow \log_7 49 = 2$.

b) Let $y = \log_5(\frac{1}{5})$ which is equivalent to the exponential equation $5^y = \frac{1}{5}$. Since $\frac{1}{5} = 5^{-1}$, then $5^y = 5^{-1}$. Therefore, $y = -1 \Rightarrow \log_5(\frac{1}{5}) = -1$.

c) Let $y = \log_6 \sqrt{6}$ which is equivalent to the exponential equation $6^y = \sqrt{6}$. Since $\sqrt{6} = 6^{\frac{1}{2}}$, then $6^y = 6^{\frac{1}{2}}$. Therefore, $y = \frac{1}{2} \Rightarrow \log_6 \sqrt{6} = \frac{1}{2}$.

d) Let $y = \log_4 64$ which is equivalent to the exponential equation $4^y = 64$. Since $64 = 4^3$, then $4^y = 4^3$. Therefore, $y = 3 \Rightarrow \log_4 64 = 3$.

e) Let $y = \log_{10} 0.001$ which is equivalent to the exponential equation $10^y = 0.001$. Since $0.001 = \dfrac{1}{1000} = \dfrac{1}{10^3} = 10^{-3}$, then $10^y = 10^{-3}$. Therefore, $y = -3 \Rightarrow \log_{10} 0.001 = -3$.

Example 10

Find the domain of the function $f(x) = \log_2(4 - x^2)$.

Solution

From the definition of a logarithmic function the domain of $y = \log_b x$ is $x > 0$, thus for $f(x)$ it follows that $4 - x^2 > 0 \Rightarrow (2 + x)(2 - x) > 0 \Rightarrow -2 < x < 2$.

Hence, the domain is $-2 < x < 2$.

Properties of logarithms

As with all functions and their inverses, their graphs are reflections of each other over the line $y = x$. Figure 5.6 illustrates this relationship for exponential and logarithmic functions, and also confirms the domain and range for the logarithmic function stated in the previous definition.

Notice that the points (0, 1) and (1, 0) are mirror images of each other over the line $y = x$. This corresponds to the fact that since $b^0 = 1$ then $\log_b 1 = 0$. Another pair of mirror image points, $(1, b)$ and $(b, 1)$, highlight the fact that $\log_b b = 1$.

Notice also that since the x-axis is a horizontal asymptote of $y = b^x$, the y-axis is a vertical asymptote of $y = \log_b x$.

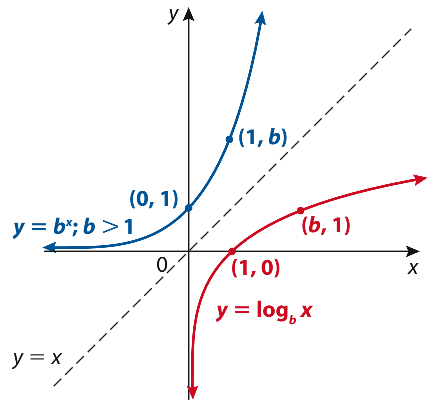


Figure 5.6 Reflection of $y = \log_b x$ over the line $y = x$.

In Section 2.3, we established that a function f and its inverse function f^{-1} satisfy the equations

$f^{-1}(f(x)) = x$ for x in the domain of f

$f(f^{-1}(x)) = x$ for x in the domain of f^{-1}

When applied to $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, these equations become

$\log_b(b^x) = x$ $x \in \mathbb{R}$

$b^{\log_b x} = x$ $x > 0$

Properties of logarithms I

For $b > 0$ and $b \neq 1$, the following statements are true:

1. $\log_b 1 = 0$ (*because* $b^0 = 1$)
2. $\log_b b = 1$ (*because* $b^1 = b$)
3. $\log_b(b^x) = x$ (*because* $b^x = b^x$)
4. $b^{\log_b x} = x$ (*because* $\log_b x$ *is the power to which b must be raised to get* x)

The logarithmic function with base 10 is called the **common logarithmic function**. On calculators and on your GDC, this function is denoted by **log**. The value of the expression $\log_{10} 1000$ is 3 because 10^3 is 1000. Generally, for common logarithms (i.e. base 10) we omit writing the base of 10. Hence, if **log** is written with no base indicated, it is assumed to have a base of 10. For example, $\log 0.01 = -2$.

Common logarithm: $\log_{10} x = \log x$

As with exponential functions, the most widely used logarithmic function – and the other logarithmic function supplied on all calculators – is the logarithmic function with the base of e. This function is known as the **natural logarithmic function** and it is the inverse of the natural exponential function $y = e^x$. The natural logarithmic function is denoted by the symbol **ln**, and the expression $\ln x$ is read as 'the natural logarithm of x'.

Natural logarithm: $\log_e x = \ln x$

Example 11

Evaluate the following expressions:

a) $\log\left(\frac{1}{10}\right)$ b) $\log(\sqrt{10})$ c) $\log 1$ d) $10^{\log 47}$ e) $\log 50$

f) $\ln e$ g) $\ln\left(\frac{1}{e^3}\right)$ h) $\ln 1$ i) $e^{\ln 5}$ j) $\ln 50$

Solution

a) $\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$

b) $\log(\sqrt{10}) = \log(10^{\frac{1}{2}}) = \frac{1}{2}$

c) $\log 1 = \log(10^0) = 0$

d) $10^{\log 47} = 47$

e) $\log 50 \approx 1.699$ (using GDC)

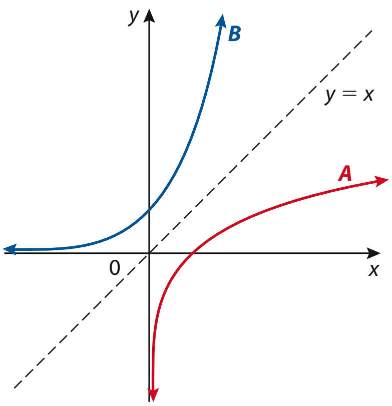
f) $\ln e = 1$

g) $\ln\left(\frac{1}{e^3}\right) = \ln(e^{-3}) = -3$

h) $\ln 1 = \ln(e^0) = 0$

i) $e^{\ln 5} = 5$

j) $\ln 50 \approx 3.912$ (using GDC)



Example 12

The diagram shows the graph of the line $y = x$ and two curves. Curve A is the graph of the equation $y = \log x$. Curve B is the reflection of curve A in the line $y = x$.

a) Write the equation for curve B.

b) Write the coordinates of the y-intercept of curve B.

Solution

a) Curve A is the graph of $y = \log x$, the common logarithm with base 10, which could also be written as $y = \log_{10} x$. Curve B is the inverse of $y = \log_{10} x$, since it is the reflection of it in the line $y = x$. Hence, the equation for curve B is the exponential equation $y = 10^x$.

b) The y-intercept occurs when $x = 0$. For curve B, $y = 10^0 = 1$. Therefore, the y-intercept for curve B is $(0, 1)$.

The logarithmic function with base b is the inverse of the exponential function with base b. Therefore, it makes sense that the laws of exponents (Section 1.3) should have corresponding properties involving logarithms. For example, the exponential property $b^0 = 1$ corresponds to the logarithmic property $\log_b 1 = 0$. We will state and prove three further important logarithmic properties that correspond to the following three exponential properties.

1. $b^m \cdot b^n = b^{m+n}$
2. $\dfrac{b^m}{b^n} = b^{m-n}$
3. $(b^m)^n = b^{mn}$

Properties of logarithms II

Given $M > 0$, $N > 0$ and k is any real number, the following properties are true for logarithms with $b > 0$ and $b \neq 1$.

Property		Description
1. $\log_b(MN)$	$= \log_b M + \log_b N$	the log of a product is the sum of the logs of its factors
2. $\log_b\left(\frac{M}{N}\right)$	$= \log_b M - \log_b N$	the log of a quotient is the log of the numerator minus the log of the denominator
3. $\log_b(M^k)$	$= k \log_b M$	the log of a number raised to an exponent is the exponent times the log of the number

Any of these properties can be applied in either direction.

Proofs

Property 1: Let $x = \log_b M$ and $y = \log_b N$.

The corresponding exponential forms of these two equations are

$$b^x = M \text{ and } b^y = N$$

Then, $\log_b(MN) = \log_b(b^x b^y) = \log_b(b^{x+y}) = x + y$.

It's given that $x = \log_b M$ and $y = \log_b N$; hence, $x + y = \log_b M + \log_b N$.

Therefore, $\log_b(MN) = \log_b M + \log_b N$.

Property 2: Again, let $x = \log_b M$ and $y = \log_b N \Rightarrow b^x = M$ and $b^y = N$.

Then, $\log_b\left(\frac{M}{N}\right) = \log_b\left(\frac{b^x}{b^y}\right) = \log_b(b^{x-y}) = x - y$.

With $x = \log_b M$ and $y = \log_b N$, then $x - y = \log_b M - \log_b N$.

Therefore, $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$.

Property 3: Let $x = \log_b M \Rightarrow b^x = M$.

Now, let's take the logarithm of M^k and substitute b^x for M:

$\log_b(M^k) = \log_b[(b^x)^k] = \log_b(b^{kx}) = kx$

It's given that $x = \log_b M$; hence, $kx = k \log_b M$.

Therefore, $\log_b(M^k) = k \log_b M$.

• **Hint:** The notation $f(x)$ uses brackets *not* to indicate multiplication but to indicate the argument of the function f. The symbol f is the name of a function, not a variable – it is not multiplying the variable x. Therefore, $f(x + y)$ is NOT equal to $f(x) + f(y)$. Likewise, the symbol **log** is also the name of a function. Therefore, $\log_b(x + y)$ is not equal to $\log_b(x) + \log_b(y)$. Other mistakes to avoid include incorrectly simplifying quotients or powers of logarithms. Specifically, $\frac{\log_b x}{\log_b y} \neq \log\left(\frac{x}{y}\right)$ and $(\log_b x)^k \neq k(\log_b x)$.

Example 13

Use the properties of logarithms to write each logarithmic expression as a sum, difference, and/or constant multiple of simple logarithms (i.e. logarithms without sums, products, quotients or exponents).

a) $\log_2(8x)$ b) $\ln\left(\frac{3}{y}\right)$ c) $\log(\sqrt{7})$

d) $\log_b\left(\frac{x^3}{y^2}\right)$ e) $\ln(5e^2)$ f) $\log\left(\frac{m+n}{n}\right)$

Solution

a) $\log_2(8x) = \log_2 8 + \log_2 x = 3 + \log_2 x$

b) $\ln\left(\frac{3}{y}\right) = \ln 3 - \ln y$

c) $\log(\sqrt{7}) = \log(7^{\frac{1}{2}}) = \frac{1}{2}\log 7$

d) $\log_b\left(\frac{x^3}{y^2}\right) = \log_b(x^3) - \log_b(y^2) = 3\log_b x - 2\log_b y$

e) $\ln(5e^2) = \ln 5 + \ln(e^2) = \ln 5 + 2\ln e = \ln 5 + 2(1) = 2 + \ln 5$
 ($2 + \ln 5 \approx 3.609$ using GDC)

f) $\log\left(\frac{m+n}{m}\right) = \log(m+n) - \log m$

(Remember: $\log(m+n) \neq \log m + \log n$)

Example 14

Write each expression as the logarithm of a single quantity.

a) $\log 6 + \log x$ b) $\log_2 5 + 2\log_2 3$

c) $\ln y - \ln 4$ d) $\log_b 12 - \frac{1}{2}\log_b 9$

e) $\log_3 M + \log_3 N - 2\log_3 P$ f) $\log_2 80 - \log_2 5$

Solution

a) $\log 6 + \log x = \log(6x)$

b) $\log_2 5 + 2\log_2 3 = \log_2 5 + \log_2(3^2) = \log_2 5 + \log_2 9 = \log_2(5 \cdot 9)$
$= \log_2 45$

c) $\ln y - \ln 4 = \ln\left(\frac{y}{4}\right)$

d) $\log_b 12 - \frac{1}{2}\log_b 9 = \log_b 12 - \log_b(9^{\frac{1}{2}}) = \log_b 12 - \log_b(\sqrt{9})$
$= \log_b 12 - \log_b 3 = \log_b\left(\frac{12}{3}\right) = \log_b 4$

e) $\log_3 M + \log_3 N - 2\log_3 P = \log_3(MN) - \log_3(P^2) = \log_3\left(\frac{MN}{P^2}\right)$

f) $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = 4$ (because $2^4 = 16$)

Change of base

The answer to part f) of Example 14 was $\log_2 16$ which we can compute to be exactly 4 because we know that $2^4 = 16$. The answer to part e) of Example 13 was $2 + \ln 5$ which we approximated to 3.609 using the natural logarithm function key (**ln**) on our GDC. But, what if we wanted to compute an approximate value for $\log_2 45$, the answer to part b) of Example 14? Our GDC can only evaluate common logarithms (base 10) and natural logarithms (base e). To evaluate logarithmic expressions and graph logarithmic functions to other bases we need to apply a **change of base formula.**

Change of base formula

Let a, b and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_b x$ can be expressed in terms of logarithms to any other base a as follows:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Proof

$y = \log_b x \Rightarrow b^y = x$ Convert from logarithmic form to exponential form.

$\log_a x = \log_a(b^y)$ If $b^y = x$, then log of each with same bases must be equal.

$\log_a x = y\log_a b$ Applying the property $\log_b(M^k) = k\log_b M$.

$y = \frac{\log_a x}{\log_a b}$ Divide both sides by $\log_a b$.

$\log_b x = \frac{\log_a x}{\log_a b}$ Substitute $\log_b x$ for y.

To apply the change of base formula, let $a = 10$ or $a = e$. Then the logarithm of any base b can be expressed in terms of either common logarithms or natural logarithms. For example:

$$\log_2 x = \frac{\log x}{\log 2} \quad \text{or} \quad \frac{\ln x}{\ln 2}$$

$$\log_5 x = \frac{\log x}{\log 5} \quad \text{or} \quad \frac{\ln x}{\ln 5}$$

$$\log_2 45 = \frac{\log 45}{\log 2} = \frac{\ln 45}{\ln 2} \approx 5.492 \quad \text{(using GDC)}$$

Example 15

Use the change of base formula and common or natural logarithms to evaluate each logarithmic expression. Start by making a rough mental estimate. Approximate your answer to 4 significant figures.

a) $\log_3 30$

b) $\log_9 6$

Solution

a) The value of $\log_3 30$ is the power to which 3 is raised to obtain 30. Because $3^3 = 27$ and $3^4 = 81$, the value of $\log_3 30$ is between 3 and 4, and will be much closer to 3 than 4 – perhaps around 3.1. Using the change of base formula and common logarithms, we obtain $\log_3 30 = \dfrac{\log 30}{\log 3} \approx 3.096$. This agrees well with the mental estimate.

After computing the answer on your GDC, use your GDC to also check it by raising 3 to the answer and confirming that it gives a result of 30.

```
log(30)/log(3)
          3.095903274
3^Ans
                   30
```

b) The value of $\log_9 6$ is the power to which 9 is raised to obtain 6. Because $9^{\frac{1}{2}} = \sqrt{9} = 3$ and $9^1 = 9$, the value of $\log_9 6$ is between $\frac{1}{2}$ and 1 – perhaps around 0.75. Using the change of base formula and natural logarithms, we obtain $\log_9 6 = \dfrac{\ln 6}{\ln 9} \approx 0.815$. This agrees well with the mental estimate.

```
ln(6)/ln(9)
          .8154648768
9^Ans
                    6
```

Exercise 5.4

In questions 1–9, express each logarithmic equation as an exponential equation.

1 $\log_2 16 = 4$ **2** $\ln 1 = 0$ **3** $\log 100 = 2$

4 $\log 0.01 = -2$ **5** $\log_7 343 = 3$ **6** $\ln\left(\frac{1}{e}\right) = -1$

7 $\log 50 = y$ **8** $\ln x = 12$ **9** $\ln(x + 2) = 3$

In questions10–18, express each exponential equation as a logarithmic equation.

10 $2^{10} = 1024$ **11** $10^{-4} = 0.0001$ **12** $4^{-\frac{1}{2}} = \frac{1}{2}$

13 $3^4 = 81$ **14** $10^0 = 1$ **15** $e^x = 5$

16 $2^{-3} = 0.125$ **17** $e^4 = y$ **18** $10^{x+1} = y$

In questions 19–38, find the exact value of the expression without using your GDC.

19 $\log_2 64$ **20** $\log_4 64$ **21** $\log_2\left(\frac{1}{8}\right)$ **22** $\log_3(3^5)$

23 $\log_{16} 8$ **24** $\log_{27} 3$ **25** $\log_{10} 0.001$ **26** $\ln e^{13}$

27 $\log_8 1$ **28** $10^{\log 6}$ **29** $\log_3\left(\frac{1}{27}\right)$ **30** $e^{\ln\sqrt{2}}$

31 $\log 1000$ **32** $\ln(\sqrt{e})$ **33** $\ln\left(\frac{1}{e^2}\right)$ **34** $\log_3(81^{22})$

35 $\log_4 2$ **36** $3^{\log_3 18}$ **37** $\log_5(\sqrt[3]{5})$ **38** $10^{\log\pi}$

In questions 39–46, use a GDC to evaluate the expression, correct to 4 significant figures.

39 $\log 50$ **40** $\log\sqrt{3}$ **41** $\ln 50$ **42** $\ln\sqrt{3}$

43 $\log 25$ **44** $\log\left(\frac{1+\sqrt{5}}{2}\right)$ **45** $\ln 100$ **46** $\ln(100^3)$

In questions 47–52, find the domain of each function.

47 $f(x) = \log(x - 2)$ **48** $g(x) = \ln(x^2)$ **49** $h(x) = \log(x) - 2$

50 $y = \log_7(8 - 5x)$ **51** $y = \sqrt{x+2} - \log_3(9 - 3x)$

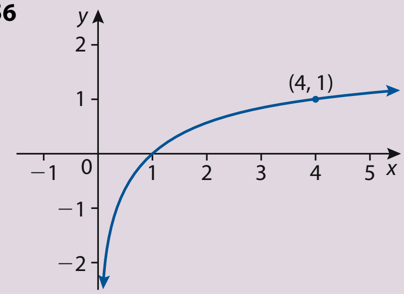
52 $y = \sqrt{\ln(1 - x)}$

In 53–55, find the domain *and* range of each function.

53 $y = \frac{1}{\ln x}$ **54** $y = |\ln(x - 1)|$ **55** $y = \frac{x}{\log x}$

For questions 56–59, find the equation of the function that is graphed in the form $f(x) = \log_b x$.

56



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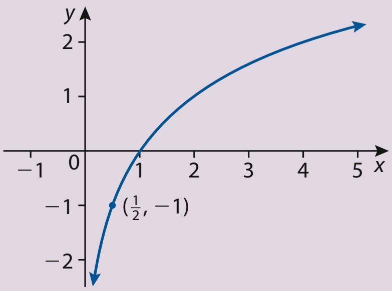

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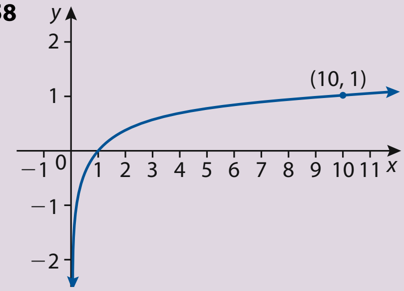

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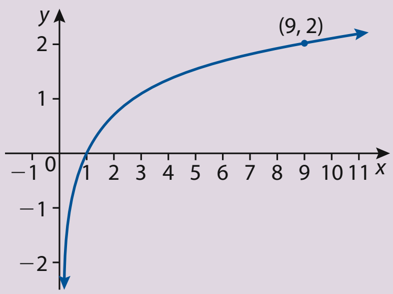


In questions 60–65, use properties of logarithms to write each logarithmic expression as a sum, difference and/or constant multiple of simple logarithms (i.e. logarithms without sums, products, quotients or exponents).

60 $\log_2(2m)$

61 $\log\left(\frac{9}{x}\right)$

62 $\ln\sqrt[5]{x}$

63 $\log_3(ab^3)$

64 $\log[10x(1 + r)^t]$

65 $\ln\left(\frac{m^3}{n}\right)$

In 66–71, write each expression in terms of $\log_b p$, $\log_b q$ and $\log_b r$.

66 $\log_b pqr$

67 $\log_b\left(\frac{p^2q^3}{r}\right)$

68 $\log_b\sqrt[4]{pq}$

69 $\log_b\sqrt{\frac{qr}{p}}$

70 $\log_b\frac{p\sqrt{q}}{r}$

71 $\log_b\frac{(pq)^3}{\sqrt{r}}$

In 72–77, write each expression as the logarithm of a single quantity.

72 $\log(x^2) + \log\left(\frac{1}{x}\right)$

73 $\log_3 9 + 3\log_3 2$

74 $4\ln y - \ln 4$

75 $\log_b 12 - \frac{1}{2}\log_b 9$

76 $\log x - \log y - \log z$

77 $2\ln 6 - 1$ • **Hint:** $\ln(?) = 1$

In questions 78–81, use the change of base formula and common or natural logarithms to evaluate each logarithmic expression. Approximate your answer to 3 significant figures.

78 $\log_2 1000$

79 $\log_{\frac{1}{2}} 40$

80 $\log_6 40$

81 $\log_5(0.75)$

In questions 82 and 83, use the change of base formula to evaluate $f(20)$.

82 $f(x) = \log_2 x$

83 $f(x) = \log_5 x$

84 Use the change of base formula to prove the following statement.

$$\log_b a = \frac{1}{\log_a b}$$

85 Show that $\log e = \frac{1}{\ln 10}$.

86 The relationship between the number of decibels dB (one variable) and the intensity I of a sound (in watts per square metre) is given by the formula $dB = 10\log\left(\frac{I}{10^{-16}}\right)$. Use properties of logarithms to write the formula in simpler form. Then find the number of decibels of a sound with an intensity of 10^{-4} watts per square metre.

87 a) Given the exponential function $f(x) = 5(2)^x$, show that $f(x)$ varies linearly with x; that is, find the linear equation in terms of x that is equal to $f(x)$.

b) Prove that for any exponential function in the form $f(x) = ab^x$, the function $\log(f(x))$ is linear and can be written in the form $\log(f(x)) = mx + c$. Find the constants m (slope) and c (y-intercept) in terms of $\log a$ and $\log b$.

5.5 Exponential and logarithmic equations

Solving exponential equations

At the start of the previous section, we wanted to find a way to determine how much time t (in years) it would take for an investment of 1000 euros to double, if the investment earns interest at an annual rate of $5\frac{1}{4}\%$. Since the interest is compounded continuously, we need to solve this equation: $2000 = 1000e^{0.0525t} \Rightarrow 2 = e^{0.0525t}$. The equation has the variable t in the exponent. With the properties of logarithms established in the previous section, we now have a way to algebraically solve such equations. Along with these properties, we need to apply the logic that if two expressions are equal then their logarithms must also be equal. That is, if $m = n$, then $\log_b m = \log_b n$.

Example 16

Solve the equation for the variable t. Give your answer accurate to 3 significant figures.

$$2 = e^{0.0525t}$$

Solution

$2 = e^{0.0525t}$

$\ln 2 = \ln(e^{0.0525t})$ Take natural logarithm of both sides.

$\ln 2 = 0.0525t$ Apply the property $\log_b(b^x) = x$ and $\ln e = 1$.

$$t = \frac{\ln 2}{0.0525} \approx 13.2$$

With interest compounding continuously at an annual interest rate of $5\frac{1}{4}\%$, it takes about 13.2 years for the investment to double.

This example serves to illustrate a general strategy for solving exponential equations. To solve an exponential equation, first isolate the exponential expression and take the logarithm of both sides. Then apply a property of logarithms so that the variable is no longer in the exponent and it can be isolated on one side of the equation. By taking the logarithm of both sides of an exponential equation, we are making use of the inverse relationship between exponential and logarithmic functions. Symbolically, this method can be represented as follows – solving for x:

(i) If $b = 10$ or e: $y = b^x \Rightarrow \log_b y = \log_b b^x \Rightarrow \log_b y = x$

(ii) If $b \neq 10$ or e:

$$y = b^x \Rightarrow \log_a y = \log_a b^x \Rightarrow \log_a y = x \log_a b \Rightarrow x = \frac{\log_a y}{\log_a b}$$

Example 17

Solve for x in the equation $3^{x-4} = 24$. Approximate the answer to 3 significant figures.

Solution

$3^{x-4} = 24$

$\log(3^{x-4}) = \log 24$ — Take common logarithm of both sides.

$(x-4)\log 3 = \log 24$ — Apply the property $\log_b(M^k) = k \log_b M$.

$x - 4 = \dfrac{\log 24}{\log 3}$ — Divide both sides by log 3. $\left[\text{Note: } \dfrac{\log 24}{\log 3} \neq \log 8\right]$

$x = \dfrac{\log 24}{\log 3} + 4$

$x \approx 6.89$ — Using GDC .

- **Hint:** We could have used natural logarithms instead of common logarithms to solve the equation in Example 17. Using the same method but with natural logarithms, we get

$$x = \frac{\ln 24}{\ln 3} + 4 \approx 6.89.$$

Recall Example 11 in Section 4.3 in which we solved an exponential equation graphically, because we did not yet have the tools to solve it algebraically. Let's solve it now using logarithms.

Example 18

You invested €1000 at 6% compounded quarterly. How long will it take this investment to increase to €2000?

Solution

Using the compound interest formula from Section 4.3, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, with $P = €1000$, $r = 0.06$ and $n = 4$, we need to solve for t when $A(t) = 2P$.

$2P = P\left(1 + \dfrac{0.06}{4}\right)^{4t}$ — Substitute $2P$ for $A(t)$.

$2 = 1.015^{4t}$ — Divide both sides by P.

$\ln 2 = \ln(1.015^{4t})$ — Take natural logarithm of both sides.

$\ln 2 = 4t \ln 1.015$ — Apply the property $\log_b(M^k) = k \log_b M$.

$t = \dfrac{\ln 2}{4 \ln 1.015}$

$t \approx 11.6389$ — Evaluated on GDC.

The investment will double in 11.64 years – about 11 years and 8 months.

```
ln(2)/(4ln(1.015
))
         11.63888141
```

- **Hint:** Be sure to use brackets appropriately when entering the expression $\dfrac{\ln 2}{4 \ln 1.015}$ on your GDC. Following the rules for order of operations, your GDC will give an incorrect result if entered as shown here.

```
ln(2)/ 4ln(1.015)
          .0025799999
```

missing brackets

Example 19

The bacteria that cause 'strep throat' will grow in number at a rate of about 2.3% per minute. To the nearest whole minute, how long will it take for these bacteria to double in number?

Solution

Let t represent time in minutes and let A_0 represent the number of bacteria at $t = 0$.

Using the exponential growth model from Section 5.2, $A(t) = A_0 b^t$, the growth factor, b, is $1 + 0.023 = 1.023$ giving $A(t) = A_0(1.023)^t$. The same equation would apply to money earning 2.3% annual interest with the money being added (compounded) once per year rather than once per minute. So, our mathematical model assumes that the number of bacteria increase incrementally, with the number increasing by 2.3% at the end of each minute. To find the doubling time, find the value of t so that $A(t) = 2A_0$.

$2A_0 = A_0(1.023)^t$ Substitute $2A_0$ for $A(t)$.

$2 = 1.023^t$ Divide both sides by A_0.

$\ln 2 = \ln(1.023^t)$ Take natural logarithm of both sides.

$\ln 2 = t \ln 1.023$ Apply the property $\log_b(M^k) = k \log_b M$.

$$t = \frac{\ln 2}{\ln 1.023} \approx 30.482$$

The number of bacteria will double in about 30 minutes.

Alternative solution

What if we assumed continuous growth instead of incremental growth? We apply the continuous exponential growth model from Section 5.3: $A(t) = Ce^{rt}$ with initial amount C and $r = 0.023$.

$2C = Ce^{0.023t}$ Substitute $2C$ for $A(t)$.

$2 = e^{0.023t}$ Divide both sides by C.

$\ln 2 = \ln(e^{0.023t})$ Take natural logarithm of both sides.

$\ln 2 = 0.023t$ Apply the property $\log_b(b^x) = x$.

$$t = \frac{\ln 2}{0.023} \approx 30.137$$

Continuous growth has a slightly shorter doubling time, but rounded to the nearest minute it also gives an answer of 30 minutes.

Example 20

\$1000 is invested in an investment account that earns interest at an annual rate of 10% compounded monthly. Calculate the minimum number of years needed for the amount in the account to exceed \$4000.

Solution

We use the exponential function associated with compound interest,

$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ with $P = 1000$, $r = 0.1$ and $n = 12$.

$$4000 = 1000\left(1 + \frac{0.1}{12}\right)^{12t} \Rightarrow 4 = (1.008\overline{3})^{12t} \Rightarrow \log 4 = \log[(1.008\overline{3})^{12t}] \Rightarrow$$

$$\log 4 = 12t\log(1.008\overline{3}) \Rightarrow t = \frac{\log 4}{12\log(1.008\overline{3})} \approx 13.92 \text{ years}$$

The minimum number of years needed for the account to exceed \$4000 is 14 years.

Example 21

A 20 g sample of radioactive iodine decays so that the mass remaining after t days is given by the equation $A(t) = 20e^{-0.087t}$, where $A(t)$ is measured in grams. After how many days (to the nearest whole day) is there only 5 g remaining?

Solution

$$5 = 20e^{-0.087t} \Rightarrow \frac{5}{20} = e^{-0.087t} \Rightarrow \ln 0.25 = \ln(e^{-0.087t}) \Rightarrow$$

$$\ln 0.25 = -0.087t \Rightarrow t = \frac{\ln 0.25}{-0.087} \approx 15.93$$

After about 16 days there is only 5 g remaining.

Example 22 – An equation in quadratic form

Solve for x in the equation $3^{2x} - 18 = 3^{x+1}$. Express any answers *exactly*.

Solution

The key to solving this equation is recognizing that it can be written in *quadratic form*. In Section 3.5, we solved equations of the form $at^2 + bt + c = 0$, where t is an algebraic expression. This is not immediately clear for this equation. We need to apply some laws of exponents to show that the equation is quadratic for the expression 3^x.

$3^{2x} - 18 = 3^{x+1}$

$(3^x)^2 - 3^1 \cdot 3^x - 18 = 0$ Applying rules $b^{mn} = (b^m)^n$ and $b^{m+n} = b^m b^n$.

Substituting a single variable, for example y, for the expression 3^x clearly makes the equation quadratic in terms of 3^x. We solve first for y and then solve for x after substituting 3^x back for y.

$$y^2 - 3y - 18 = 0$$
$$(y + 3)(y - 6) = 0$$
$$y = -3 \text{ or } y = 6$$
$$3^x = -3 \text{ or } 3^x = 6$$

$3^x = -3$ has no solution. Raising a positive number to a power cannot produce a negative number.

$$3^x = 6$$
$$\ln(3^x) = \ln 6 \qquad \text{Take logarithm of both sides.}$$
$$x \ln 3 = \ln 6$$

Therefore, the one solution to the equation is exactly $x = \frac{\ln 6}{\ln 3}$.

- **Hint:** There are a couple of common algebra errors to avoid in the working for Example 22.
- If $3^x = -3$, then it does not follow that $x = -1$. An exponent of -1 indicates reciprocal.
- If $x = \frac{\ln 6}{\ln 3}$, it does ***not*** follow that $x = \ln 2$. The rule $\log m - \log n = \log\left(\frac{m}{n}\right)$ does not apply to the expression $\frac{\ln 6}{\ln 3}$.

Solving logarithmic equations

A logarithmic equation is an equation where the variable appears within the argument of a logarithm. For example, $\log x = \frac{1}{2}$ or $\ln x = 4$. We can solve both of these logarithmic equations directly by applying the definition of a logarithmic function (Section 5.4):

$$y = \log_b x \text{ if and only if } x = b^y$$

The logarithmic equation $\log x = \frac{1}{2}$ is equivalent to the exponential equation $x = 10^{\frac{1}{2}} = \sqrt{10}$, which leads directly to the solution. Likewise, the equation $\ln x = 4$ is equivalent to $x = e^4 \approx 54.598$. Both of these equations could have been solved by means of another method that makes use of the following two facts:

$$\text{(i) if } a = b \text{ then } n^a = n^b; \quad \text{and (ii) } b^{\log_b x} = x$$

To understand (ii) above, remember that a **logarithm is an exponent**. The value of $\log_b x$ is the exponent to which b is raised to give x. And b is being raised to this value; hence, the expression $b^{\log_b x}$ is equivalent to x. Therefore, another method for solving the logarithmic equation $\ln x = 4$ is to **exponentiate** both sides, i.e. use the expressions on either side of the equal sign as exponents for exponential expressions with equal bases. The base needs to be the base of the logarithm. For example,

$$\ln x = 4 \Rightarrow e^{\ln x} = e^4 \Rightarrow x = e^4$$

Example 23

Solve for x: $\log_3(2x - 5) = 2$

Solution

$$\log_3(2x-5) = 2 \Rightarrow 3^{\log_3(2x-5)} = 3^2 \qquad \text{Exponentiate both side with base} = 3.$$
$$2x - 5 = 9 \qquad \text{Applying property } b^{\log_b x} = x.$$
$$2x = 14$$
$$x = 7$$

Example 24

Solve for x in terms of k: $\log_2(5x) = 3 + k$

Solution

$\log_2(5x) = 3 + k \Rightarrow 2^{\log_2(5x)} = 2^{3+k}$ Exponentiate both sides with base = 2.

$5x = 2^3 \cdot 2^k$ Law of exponents $b^m \cdot b^n = b^{m+n}$ used 'in reverse'.

$x = \frac{8}{5}(2^k)$

For some logarithmic equations, it is necessary to first apply a property, or properties, of logarithms to simplify combinations of logarithmic expressions before solving.

Example 25

Solve for x: $\log_2 x + \log_2(10 - x) = 4$

Solution

$\log_2 x + \log_2(10 - x) = 4$

$\log_2[x(10 - x)] = 4$ Property of logarithms: $\log_b M + \log_b N = \log_b(MN)$.

$10x - x^2 = 2^4$ Changing from logarithmic form to exponential form.

$x^2 - 10x + 16 = 0$

$(x - 2)(x - 8) = 0$

$x = 2$ or $x = 8$

When solving logarithmic equations, you should be careful to always check if the *original* equation is a true statement when any solutions are substituted in for the variable. For Example 25, both of the solutions $x = 2$ and $x = 8$ produce true statements when substituted into the original equations. Sometimes 'extra' (extraneous) invalid solutions (met in Chapter 3) are produced, as illustrated in the next example.

Example 26

Solve for x: $\ln(x - 2) + \ln(2x - 3) = 2\ln x$

Solution

$\ln(x - 2) + \ln(2x - 3) = 2\ln x$

$\ln[(x - 2)(2x - 3)] = \ln x^2$ Properties of logarithms.

$\ln(2x^2 - 7x + 6) = \ln x^2$

$e^{\ln(2x^2 - 7x + 6)} = e^{\ln x^2}$ Exponentiate both sides.

$2x^2 - 7x + 6 = x^2$

$x^2 - 7x + 6 = 0$

$(x - 6)(x - 1) = 0$ Factorize.

$x = 6$ or $x = 1$

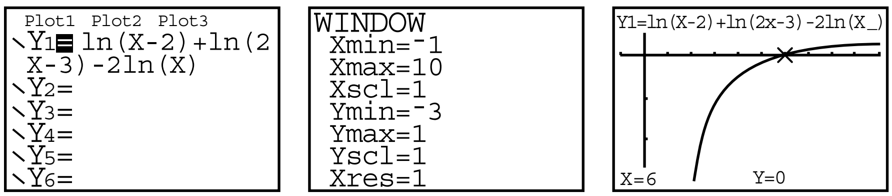
Substituting these two *possible* solutions indicates that $x = 1$ is not a valid solution. The reason is that if you try to substitute 1 for x into the original equation, we are not able to evaluate the expression $\ln(2x - 3)$ because we can only take the logarithm of a positive number. Therefore, $x = 6$ is the only solution. $x = 1$ is an extraneous solution that is not valid.

Solving, or checking the solutions to, a logarithmic equation on your GDC will help you avoid, or determine, extraneous solutions. To solve Example 26 on your GDC, a useful approach is to first set the equation equal to zero. Then graph the expression (after setting it equal to y) and observe where the graph intersects the x-axis (i.e. $y = 0$).

Graphical solution for Example 26:

$\ln(x - 2) + \ln(2x - 3) = 2\ln x \Rightarrow \ln(x - 2) + \ln(2x - 3) - 2\ln x = 0$

Graph the equation $y = \ln(x - 2) + \ln(2x - 3) - 2\ln x$ on your GDC and find x-intercepts.



The graph only intersects the x-axis at $x = 6$ and not at $x = 1$. Hence, $x = 6$ is the only valid solution and $x = 1$ is an extraneous solution.

Exponential and logarithmic inequalities

In Section 3.5, we covered methods of solving a variety of inequalities. These methods can also be applied to solving inequalities involving exponential and logarithmic functions. It is important to consider the domain of any functions in the inequality, and to check any solutions in the original inequality in case any extraneous solutions occur.

Example 27

Find the solution set to the inequality: $2\log_3 x - 1 < 0$.

Solution

Due to the domain of the logarithmic function, all solutions must be positive.

Method 1 (algebraic solution)

Solve the equation $2\log_3 x - 1 = 0$ and find the exact solution.

$$2\log_3 x = 1 \Rightarrow \log_3 x = \tfrac{1}{2} \Rightarrow x = 3^{\frac{1}{2}} = \sqrt{3}$$

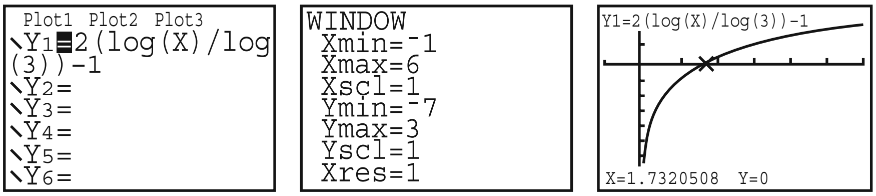
Substitute 'test' values, x_1 and x_2, into the original inequality such that $0 < x_1 < \sqrt{3}$ and $x_2 > \sqrt{3}$.

Let $x_1 = 1$: $\log_3 1 - 1 = 0 - 1 = -1 < 0$ (true)
Let $x_2 = 9$: $\log_3 9 - 1 = 2 - 1 = 1 \times 0$ (false)

Therefore, the solution set is $0 < x < \sqrt{3}$.

Method 2 (graphical solution)

Graph the equation $y = 2\log_3 x - 1$ on your GDC and use it to determine the portion of the graph that is less than zero (i.e. below the x-axis). But, how do we input the expression $\log_3 x$ on the GDC? We can use the change of base formula to write $\log_3 x = \dfrac{\log x}{\log 3}$.



The y-axis is a vertical asymptote. The graph indicates that the solution set is $0 < x < 1.732\,0508$. Although the graphical method is efficient and effective it does not give an exact result.

Example 28

Solve: $(e^x - 2)(e^x + 6) \leqslant 3e^x$

Solution

The fact that the left side is factorized is not helpful because the other side of the inequality is not zero. So we need to expand the left side and rearrange terms to get zero on the right side.

$$(e^x - 2)(e^x + 6) \leqslant 3e^x$$
$$(e^x)^2 + 4e^x - 12 \leqslant 3e^x$$
$$e^{2x} + e^x - 12 \leqslant 0$$ Now factorize this expression.
$$(e^x - 3)(e^x + 4) \leqslant 0$$ Find where each factor is zero and construct a sign chart.

$e^x - 3 = 0 \Rightarrow e^x = 3 \Rightarrow x = \ln 3$
and $\quad e^x + 4 = 0 \Rightarrow e^x = -4 \Rightarrow$ no solution

Since $x = \ln 3$ is the only zero of the expression $(e^x - 3)(e^x + 4)$ we only need to test x-values on either side of $x = \ln 3$. The factor $e^x + 4$ will always be positive.

		$\ln 3$	x
$e^x - 3$	$-$	0	$+$
$e^x + 4$	$+$		$+$
$(e^x - 3)(e^x + 4)$	$-$	0	$+$

Therefore, the solution set is $x \leqslant \ln 3$.

Exercise 5.5

In questions 1–12, solve for x. Give x accurate to 3 significant figures.

1 $10^x = 5$ **2** $4^x = 32$ **3** $8^{x-6} = 60$

4 $2^{x+3} = 100$ **5** $\left(\frac{1}{5}\right)^x = 22$ **6** $e^x = 15$

7 $10^x = e$ **8** $3^{2x-1} = 35$ **9** $2^{x+1} = 3^{x-1}$

10 $2e^{10x} = 19$ **11** $6^{\frac{x}{2}} = 5^{1-x}$ **12** $\left(1 + \frac{0.05}{12}\right)^{12x} = 3$

In questions 13–16, solve for x. Give answers **exactly**.

13 $4^x - 2^{x+1} = 48$ • **Hint:** write 4 as 2^2

14 $2^{2x+1} - 2^{x+1} + 1 = 2^x$

15 $6^{2x+1} - 17(6^x) + 12 = 0$

16 $3^{2x+1} + 3 = 10(3^x)$

17 \$5000 is invested in an account that pays 7.5% interest per year, compounded quarterly.

a) Find the amount in the account after three years.

b) How long will it take for the money in the account to double? Give the answer to the nearest quarter of a year.

18 How long will it take for an investment of €500 to triple in value if the interest is 8.5% per year, compounded continuously. Give the answer in number of years accurate to 3 significant figures.

19 A single bacterium begins a colony in a laboratory dish. If the colony doubles every hour, after how many hours does the colony first have more than one million bacteria?

20 Find the least number of years for an investment to double if interest is compounded annually with the following interest rates.

a) 3% b) 6% c) 9%

21 A new car purchased in 2005 decreases in value by 11% per year. When is the first year that the car is worth less than one-half of its original value?

22 Uranium-235 is a radioactive substance that has a half-life of 2.7×10^5 years.

a) Find the amount remaining from a 1 g sample after a thousand years.

b) How long will it take a 1 g sample to decompose until its mass is 700 milligrams (i.e. 0.7 g)? Give the answer in years accurate to 3 significant figures.

23 The stray dog population in a town is growing exponentially with about 18% more stray dogs each year. In 2008, there are 16 stray dogs.

a) Find the projected population of stray dogs after five years.

b) When is the first year that the number of stray dogs is greater than 70?

24 Initially a water tank contains one thousand litres of water. At the time $t = 0$ minutes, a tap is opened and water flows out of the tank. The volume, V litres, which remains in the tank after t minutes is given by the following exponential function: $V(t) = 1000(0.925)^t$.

a) Find the value of V after 10 minutes.

b) Find how long, to the nearest second, it takes for half of the initial amount of water to flow out of the tank.

c) The tank is considered 'empty' when only 5% of the water remains. From when the tap is first opened, how many whole minutes have passed before the tank can first be considered empty?

25 The mass m kilograms of a radioactive substance at time t days is given by $m = 5e^{-0.13t}$.

a) What is the initial mass?

b) How long does it take for the substance to decay to 0.5 kg? Give the answer in days accurate to 3 significant figures.

In questions 26–36, solve for x in the logarithmic equation. Give exact answers and be sure to check for extraneous solutions.

26 $\log_2(3x - 4) = 4$

27 $\log(x - 4) = 2$

28 $\ln x = -3$

29 $\log_{16} x = \frac{1}{2}$

30 $\log \sqrt{x + 2} = 1$

31 $\ln(x^2) = 16$

32 $\log_2(x^2 + 8) = \log_2 x + \log_2 6$

33 $\log_3(x - 8) + \log_3 x = 2$

34 $\log 7 - \log(4x + 5) + \log(2x - 3) = 0$

35 $\log_3 x + \log_3(x - 2) = 1$

36 $\log x^8 = (\log x)^4$

In questions 37–40, solve each inequality.

37 $5 \log x + 2 > 0$

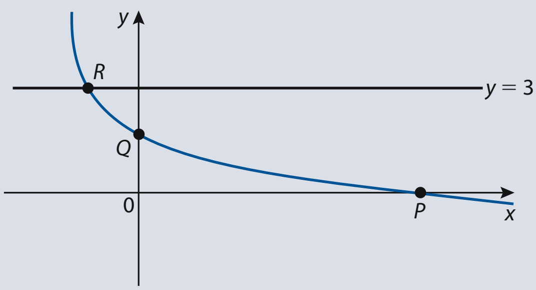
38 $2\log x^2 - 3\log x < \log 8x - \log 4x$

39 $(e^x - 2)(e^x - 3) < 2e^x$

40 $3 + \ln x > e^x$

Practice questions

1 A portion of the graph $y = 2 - \log_3(x + 1)$ is shown. It intersects the x-axis at point P, the y-axis at point Q and the line $y = 3$ at point R. Find the following:



a) The x-coordinate of point P.
b) The y-coordinate of point Q.
c) The coordinates of point R.

2 The amount $A(t)$, in grams, of a certain radioactive substance remaining after t years decays by the formula $A(t) = A_0 e^{-0.0045t}$, where A_0 is the initial amount.
a) If 5 grams are left after 800 years, how many grams were present initially?
b) What is the half-life of the substance?

3 **a)** Find expressions for the nth term and the sum to n terms of the following arithmetic series, $\ln y + \ln y^2 + \ln y^3 + \dots$ where $y > 0$.
b) Hence, find expressions for the nth term and the sum to n terms of the following arithmetic series, $\ln(xy) + \ln(xy^2) + \ln(xy^3) + \dots$ where $x > 0$ and $y > 0$.

4 Solve, for x, the equation $\log_2(5x^2 - x - 2) = 2 + 2\log_2 x$.

5 If $\log_2 4\sqrt{2} = x$, $\log_z y = 4$, and $y = 4x^2 - 2x - 6 + z$, find y.

6 Find the **exact** values of t for which $2e^{3t} - 7e^{2t} + 7e^{t} = 2$.

7 Find the **exact** solution(s) to the equation $8e^2 - 2e\ln x = (\ln x)^2$.

8 Find the exact value of x for each equation.

a) $\log_3 x - 4\log_x 3 + 3 = 0$

b) $\log_2(x - 5) + \log_2(x + 2) = 3$

9 Express each as a single logarithm.

a) $2\log a + 3\log b - \log c$

b) $3\ln x - \frac{1}{2}\ln y + 1$

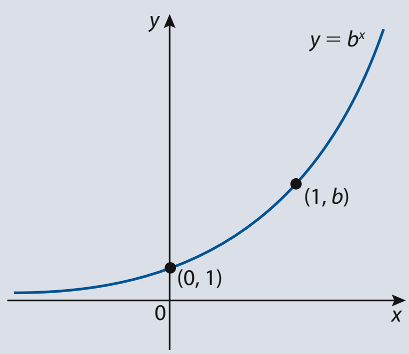
10 A piece of wood is recovered from an ancient building during an archaeological excavation. The formula $A(t) = A_0 e^{-0.000\,124t}$ is used to determine the age of the wood, where A_0 is the amount of carbon in any living tree, $A(t)$ is the amount of carbon in the wood being dated and t is the age of the wood in years. For the ancient piece of wood it is found that $A(t)$ is 79% of the amount of the carbon in a living tree. How old is the piece of wood, to the nearest 100 years?

11 The graph of the equation $y = \log_3(2x - 3) - 4$ intersects the x-axis at the point $(c, 0)$. Without using your GDC, find the exact value of c.

12 The graph of $y = b^x$, $b > 1$ is shown. On separate coordinate planes, sketch the graphs of

a) $y = b^{-x}$

b) $y = b^{1-x}$



13 Radium decays exponentially and its half-life is 1600 years. If A_0 represents the initial amount of radium in a sample and $A(t)$ represents the amount remaining after t years, then $A(t) = A_0 e^{-kt}$.

a) Find the value of k approximated to four significant figures.

b) Find what percentage of the original amount of radium will be remaining after 4000 years.

14 Solve the equation $e^{-x} - x + 1 = 0$.

15 Find the set of values of x for which $|0.1x^2 - 2x + 3| < \log_{10} x$.

16 Determine the values of x that satisfy the inequality $\dfrac{xe^x}{x^2 - 1} \geqslant 1$.

17 a) Solve the equation $2(4^x) + 4^{-x} = 3$.

b) (i) Solve the equation $a^x = e^{2x+1}$ where $a > 0$, giving your answer for x in terms of a.

(ii) For what value of a does the equation have no solution?

18 The solution of $2^{2x+3} = 2^{x+1} + 3$ can be expressed in the form $a + \log_2 b$ where $a, b \in \mathbb{Z}$. Find the value of a and of b.

19 Solve $2(\ln x)^2 = 3\ln x - 1$ for x. Give your answers in **exact** form.

20 A sum of \$100 is invested.

a) If the interest is compounded annually at a rate of 5% per year, find the total value V of the investment after 20 years.

b) If the interest is compounded monthly at a rate of $\frac{5}{12}$% per month, find the minimum number of months for the value of the investment to exceed V.

21 Solve the equation $9\log_5 x = 25\log_x 5$ expressing your answer in the form $5^{\frac{p}{q}}$, where $p, q \in \mathbb{Z}$.

22 Solve $|\ln(x + 3)| = 1$. Give your answers in **exact** form.

23 Solve the equation $\left|e^{2x} - \frac{1}{x+2}\right| = 2$.

24 An experiment is carried out in which the number n of bacteria in a liquid, is given by the formula $n = 650e^{kt}$, where t is the time in minutes after the beginning of the experiment and k is a constant. The number of bacteria doubles every 20 minutes. Find the exact value of k.

25 The function f is defined for $x > 2$ by $f(x) = \ln x + \ln(x - 2) - \ln(x^2 - 4)$.

a) Express $f(x)$ in the form $\ln\left(\frac{x}{x+a}\right)$.

b) Find an expression for $f^{-1}(x)$.

26 a) The function f is defined by $f : x \mapsto e^x - 1 - x$.

(i) Use your GDC to find the minimum value of f.

(ii) Prove that $e^x \geqslant 1 + x$ for all real values of x.

b) Use mathematical induction to prove that

$$(1 + 1)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{n}\right) = n + 1 \text{ for all integers } n \geqslant 1$$

c) Use the results of parts a) and b) to prove that

$$e^{\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)} > n$$

d) Find a value of n for which

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > 100$$

7 Trigonometric Functions and Equations

Assessment statements

2.1 Odd and even functions (also see Chapter 3).

3.1 The circle: radian measure of angles; length of an arc; area of a sector.

3.2 The circular functions $\sin x$, $\cos x$ and $\tan x$: their domains and ranges; their periodic nature; and their graphs.

Definition of $\cos\theta$ and $\sin\theta$ in terms of the unit circle.

Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$.

Exact values of sin, cos and tan of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples.

Definition of the reciprocal trigonometric ratios $\sec\theta$, $\csc\theta$ and $\cot\theta$.

Pythagorean identities: $\cos^2\theta + \sin^2\theta = 1$; $1 + \tan^2\theta = \sec^2\theta$; $1 + \cot^2\theta = \csc^2\theta$.

3.3 Compound angle identities.
Double angle identities.

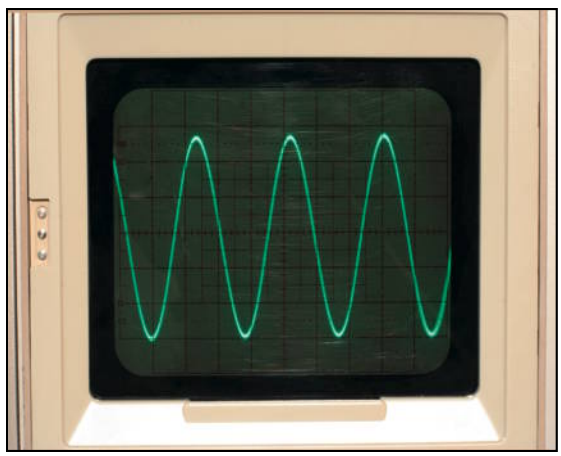
3.4 Composite functions of the form $f(x) = a\sin(b(x + c)) + d$.

3.5 The inverse functions $x \mapsto \arcsin x$, $x \mapsto \arccos x$, $x \mapsto \arctan x$; their domains and ranges; their graphs.

3.6 Algebraic and graphical methods of solving trigonometric equations in a finite interval including the use of trigonometric identities and factorization.

Introduction

The word *trigonometry* comes from two Greek words, *trigonon* and *metron*, meaning 'triangle measurement'. Trigonometry developed out of the use and study of triangles, in surveying, navigation, architecture and astronomy, to find relationships between lengths of sides of triangles and measurement of angles. As a result, trigonometric functions were initially defined as functions of angles – that is, functions with angle measurements as their domains. With the development of calculus in the seventeenth century and the growth of knowledge in the sciences, the application of trigonometric functions grew to include a wide variety of periodic (repetitive) phenomena such as wave motion, vibrating strings, oscillating pendulums, alternating electrical current and biological cycles. These applications of trigonometric functions require their domains to be sets of real numbers without reference to angles or triangles. Hence, trigonometry can be approached from two different perspectives – **functions**



The oscilloscope shows the graph of pressure of sound wave versus time for a high-pitched sound. The graph is a repetitive pattern that can be expressed as the sum of different 'sine' waves. A sine wave is any transformation of the graph of the trigonometric function $y = \sin x$ and takes the form $y = a\sin[b(x + c)] + d$.

of angles, or **functions of real numbers**. The first perspective is the focus of the next chapter where trigonometric functions will be defined in terms of the **ratios of sides of a right triangle**. The second perspective is the focus of this chapter, where trigonometric functions will be defined in terms of a real number that is the **length of an arc along the unit circle**. While it is possible to define trigonometric functions in these two different ways, they assign the same value (interpreted as an angle, an arc length, or simply a real number) to a particular real number. Although this chapter will not refer much to triangles, it seems fitting to begin by looking at angles and arc lengths – geometric objects indispensable to the two different ways of viewing trigonometry.

7.1 Angles, circles, arcs and sectors

Angles

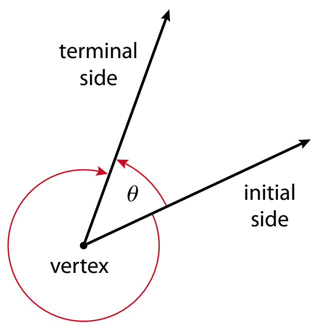


Figure 7.1 Components of an angle.

An **angle** in a plane is made by rotating a ray about its endpoint, called the **vertex** of the angle. The starting position of the ray is called the **initial side** and the position of the ray after rotation is called the **terminal side** of the angle (Figure 7.1). An angle having its vertex at the origin and its initial side lying on the positive x-axis is said to be in **standard position** (Figure 7.2a). A **positive angle** is produced when a ray is rotated in an anticlockwise direction, and a **negative angle** when a ray is rotated in a clockwise direction. Two angles in standard position whose terminal sides are in the same location – regardless of the direction or number of rotations – are called **coterminal angles**. Greek letters are often used to represent angles, and the direction of rotation is indicated by an arc with an arrow at its endpoint. The x- and y-axes divide the coordinate plane into four quadrants (numbered with Roman numerals). Figure 7.2b shows a positive angle α (alpha) and a negative angle β (beta) that are coterminal in quadrant III.

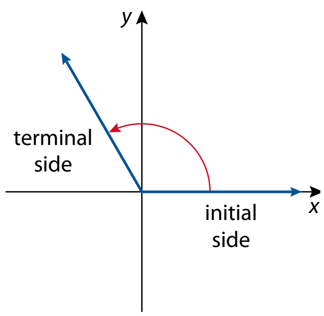


Figure 7.2a Standard position of an angle.

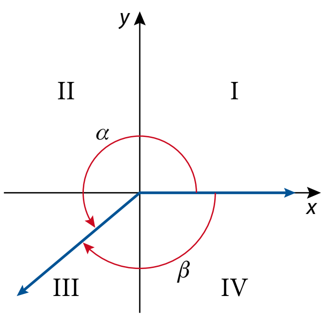


Figure 7.2b Coterminal angles.

Measuring angles: degree measure and radian measure

Perhaps the most natural unit for measuring large angles is the **revolution**. For example, most cars have an instrument (a tachometer) that indicates the number of revolutions per minute (rpm) at which the engine is operating. However, to measure smaller angles, we need a smaller unit. A common unit for measuring angles is the **degree**, of which there are 360 in one revolution. Hence, the unit of one degree (1°) is defined to be 1/360 of one anticlockwise revolution about the vertex.

The convention of having 360 degrees in one revolution can be traced back around 4000 years to ancient Babylonian civilizations. The number system most widely used today is a base 10, or **decimal**, system. Babylonian mathematics used a base 60, or **sexagesimal**, number system. Although 60 may seem to be an awkward number to have as a base, it does have certain advantages. It is the smallest number that has 2, 3, 4, 5 and 6 as factors – and it also has factors of 10, 12, 15, 20 and 30. But why 360 degrees? We're not certain but it may have to do with the Babylonians assigning 60 divisions to each angle in an equilateral triangle and exactly six equilateral triangles can be arranged around a single point. That makes $6 \times 60 = 360$ equal divisions in one full revolution. There are few numbers as small as 360 that have so many different factors. This makes the degree a useful unit for dividing one revolution into an equal number of parts. 120 degrees is $\frac{1}{3}$ of a revolution, 90 degrees is $\frac{1}{4}$ of a revolution, 60 degrees is $\frac{1}{6}$, 45 degrees is $\frac{1}{8}$, and so on.

There is another method of measuring angles that is more natural. Instead of dividing a full revolution into an arbitrary number of equal divisions (e.g. 360), consider an angle that has its vertex at the centre of a circle (a **central angle**) and subtends (or intercepts) a part of the circle, called an **arc of the circle**. Figure 7.3 shows three circles with radii of different lengths ($r_1 < r_2 < r_3$) and the same central angle θ (theta) subtending (intercepting) the arc lengths s_1, s_2 and s_3. Regardless of the size of the circle (i.e. length of the radius), the ratio of arc length (s) to radius (r) for a given circle will be constant. For the angle θ in Figure 7.3, $\frac{s_1}{r_1} = \frac{s_2}{r_2} = \frac{s_3}{r_3}$. Because this ratio is an arc length divided by another length (radius), it is just an ordinary real number and has no units.

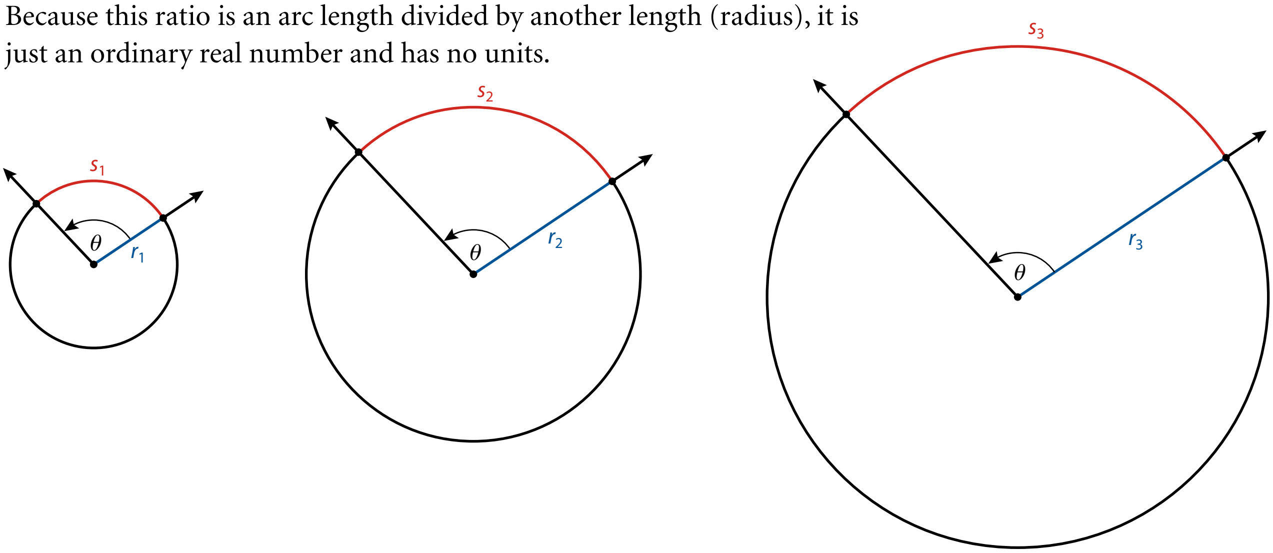


Figure 7.3 Different circles with the same central angle θ subtending different arcs, but the ratio of arc length to radius remains constant.

Minor and major arcs

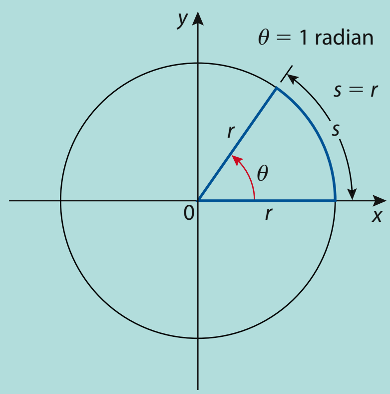
If a central angle is **less** than 180°, the subtended arc is referred to as a **minor arc**. If a central angle is **greater** than 180°, the subtended arc is referred to as a **major arc**.

The ratio $\frac{s}{r}$ indicates how many radius lengths, r, fit into the length of the arc s. For example, if $\frac{s}{r} = 2$, the length of s is equal to two radius lengths. This accounts for the name **radian** and leads to the following definition.

When the measure of an angle is, for example, 5 radians, the word 'radians' does not indicate units (as when writing centimetres, seconds or degrees) but indicates the *method* of angle measurement. If the measure of an angle is in units of degrees, we must indicate this by word or symbol. For example, $\theta = 5$ degrees or $\theta = 5°$. However, when radian measure is used it is customary to write no units or symbol. For example, a central angle θ that subtends an arc equal to five radius lengths (radians) is simply given as $\theta = 5$.

Radian measure

One **radian** is the measure of a central angle θ of a circle that subtends an arc s of the circle that is exactly the same length as the radius r of the circle. That is, when $\theta = 1$ radian, arc length = radius.



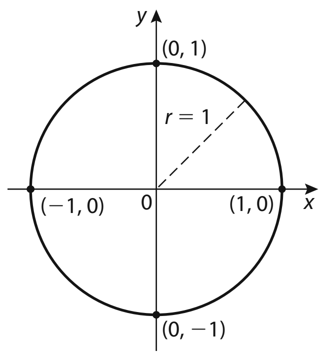


Figure 7.4 The unit circle.

The unit circle

When an angle is measured in radians it makes sense to draw it, or visualize it, so that it is in standard position. It follows that the angle will be a central angle of a circle whose centre is at the origin, as shown above. As Figure 7.3 illustrated, it makes no difference what size circle is used. The most practical circle to use is the circle with a radius of one unit so the radian measure of an angle will simply be equal to the length of the subtended arc.

$$\text{Radian measure: } \theta = \frac{s}{r} \qquad \text{If } r = 1\text{, then } \theta = \frac{s}{1} = s.$$

The circle with a radius of one unit and centre at the origin $(0, 0)$ is called the **unit circle** (Figure 7.4). The equation for the unit circle is $x^2 + y^2 = 1$. Because the circumference of a circle with radius r is $2\pi r$, a central angle of one full anticlockwise revolution (360°) subtends an arc on the unit circle equal to 2π units. Hence, if an angle has a degree measure of 360°, its radian measure is exactly 2π. It follows that an angle of 180° has a radian measure of exactly π. This fact can be used to convert between degree measure and radian measure, and vice versa.

Conversion between degrees and radians

Because $180° = \pi$ radians, $1° = \frac{\pi}{180}$ radians, and 1 radian $= \frac{180°}{\pi}$. An angle with a radian measure of 1 has a degree measure of approximately 57.3° (to 3 significant figures).

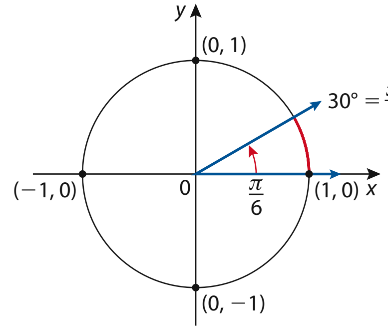
Example 1

The angles of 30° and 45°, and their multiples, are often encountered in trigonometry. Convert 30° and 45° to radian measure and sketch the corresponding arc on the unit circle. Use these results to convert 60° and 90° to radian measure.

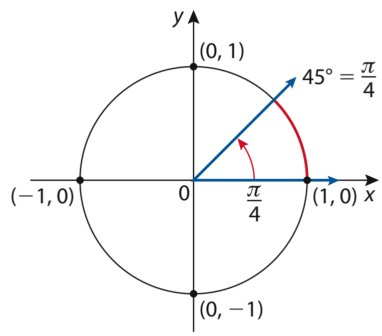
Solution

(Note that the 'degree' units cancel.)

$$30° = 30°\left(\frac{\pi}{180°}\right) = \frac{30°}{180°}\pi = \frac{\pi}{6}$$



$$45° = 45°\left(\frac{\pi}{180°}\right) = \frac{45°}{180°}\pi = \frac{\pi}{4}$$



Since $60° = 2(30°)$ and $30° = \frac{\pi}{6}$, then $60° = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$. Similarly, $90° = 2(45°)$ and $45° = \frac{\pi}{4}$, so $90° = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$.

● **Hint:** It is very helpful to be able to quickly recall the results from Example 1:
$30° = \frac{\pi}{6}$, $45° = \frac{\pi}{4}$, $60° = \frac{\pi}{3}$ and $90° = \frac{\pi}{2}$. Of course, not all angles are multiples of 30° or 45° when expressed in degrees, and not all angles are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$ when expressed in radians. However, these 'special' angles often appear in problems and applications. Knowing these four facts can help you to quickly convert mentally between degrees and radians for many common angles. For example, to convert 225° to radians, apply the fact that $225° = 5(45°)$. Since $45° = \frac{\pi}{4}$, then $225° = 5(45°) = 5\left(\frac{\pi}{4}\right) = \frac{5\pi}{4}$.
As another example, convert $\frac{11\pi}{6}$ to degrees: $\frac{11\pi}{6} = 11\left(\frac{\pi}{6}\right) = 11(30°) = 330°$.

Example 2

a) Convert the following radian measures to degrees. Express exactly, if possible. Otherwise, express accurate to 3 significant figures.

(i) $\frac{4\pi}{3}$ (ii) $-\frac{3\pi}{2}$ (iii) 5 (iv) 1.38

b) Convert the following degree measures to radians. Express exactly, if possible. Otherwise, express accurate to 3 significant figures.

(i) 135° (ii) −150° (iii) 175° (iv) 10°

Solution

a) (i) $\frac{4\pi}{3} = 4\left(\frac{\pi}{3}\right) = 4(60°) = 240°$

(ii) $-\frac{3\pi}{2} = -\frac{3}{2}(\pi) = -\frac{3}{2}(180°) = -270°$

(iii) $5\left(\frac{180°}{\pi}\right) \approx 286.479° \approx 286°$

(iv) $1.38\left(\frac{180°}{\pi}\right) \approx 79.068° \approx 79.1°$

● **Hint:** All GDCs will have a degree mode and a radian mode. Before doing any calculations with angles on your GDC, be certain that the mode setting for angle measurement is set correctly. Although you may be more familiar with degree measure, as you progress further in mathematics – and especially in calculus – radian measure is far more useful.

b) (i) $135° = 3(45°) = 3\left(\frac{\pi}{4}\right) = \frac{3\pi}{4}$

(ii) $-150° = -5(30°) = -5\left(\frac{\pi}{6}\right) = -\frac{5\pi}{6}$

(iii) $175°\left(\frac{\pi}{180°}\right) \approx 3.0543 \approx 3.05$

(iv) $10°\left(\frac{\pi}{180°}\right) \approx 0.174\,53 \approx 0.175$

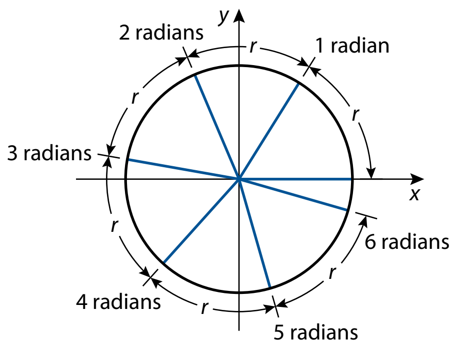


Figure 7.5 Arcs with lengths equal to the radius placed along circumference of a circle.

Because 2π is approximately 6.28 (3 significant figures), there are a little more than six radius lengths in one revolution, as shown in Figure 7.5.

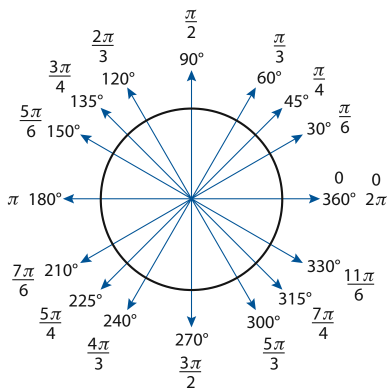


Figure 7.6 Degree measure and radian measure for common angles.

Figure 7.6 shows all of the angles between 0° and 360° inclusive, that are multiples of 30° or 45°, and their equivalent radian measure. You will benefit by being able to convert quickly between degree measure and radian measure for these common angles.

Arc length

For any angle θ, its radian measure is given by $\theta = \frac{s}{r}$. Simple rearrangement of this formula leads to another formula for computing arc length.

Arc length

For a circle of radius r, a central angle θ subtends an arc of the circle of length s given by

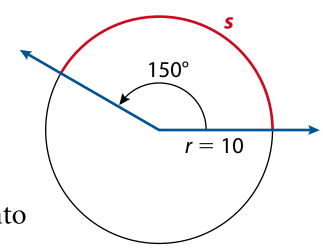
$$s = r\theta$$

where θ is in radian measure.

Example 3

A circle has a radius of 10 cm. Find the length of the arc of the circle subtended by a central angle of 150°.

Solution



To use the formula $s = r\theta$, we must first convert 150° to radian measure.

$$150° = 150°\left(\frac{\pi}{180°}\right) = \frac{150\pi}{180} = \frac{5\pi}{6}$$

Given that the radius, r, is 10 cm, substituting into the formula gives

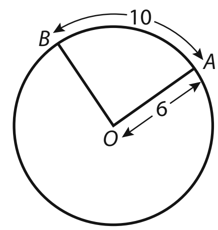
$$s = r\theta \Rightarrow s = 10\left(\frac{5\pi}{6}\right) = \frac{25\pi}{3} \approx 26.179\,94 \text{ cm}$$

The length of the arc is approximately 26.18 cm (4 significant figures).

Note that the units of the product $r\theta$ are the same as the units of r because in radian measure θ has no units.

Example 4

The diagram shows a circle of centre O with radius $r = 6$ cm. Angle AOB subtends the minor arc AB such that the length of the arc is 10 cm. Find the measure of angle AOB in degrees to 3 significant figures.



Solution

From the arc length formula, $s = r\theta$, we can state that $\theta = \frac{s}{r}$. Remember that the result for θ will be in radian measure. Therefore, angle $AOB = \frac{10}{6} = \frac{5}{3}$ or $1.\overline{6}$ radians. Now, we convert to degrees: $\frac{5}{3}\left(\frac{180°}{\pi}\right) \approx 95.49297°$. The degree measure of angle AOB is approximately 95.5°.

Geometry of a circle

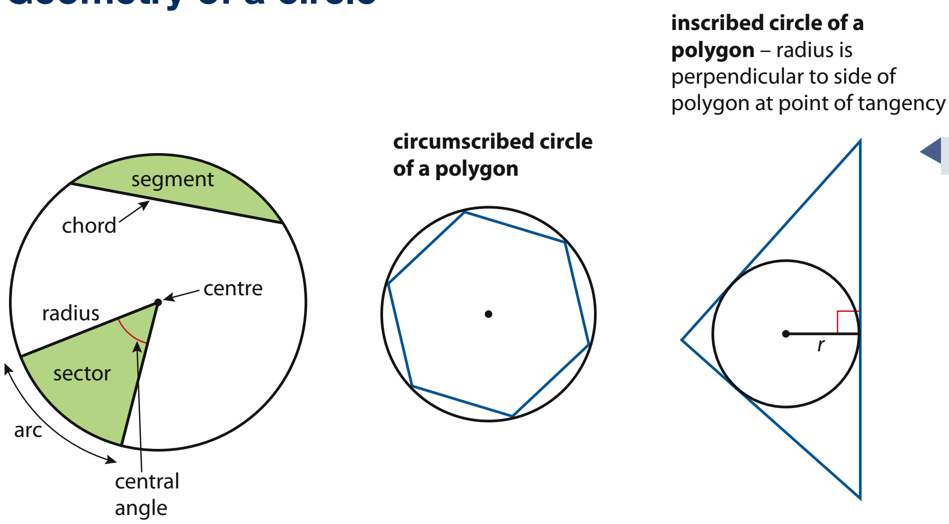


Figure 7.7 Circle terminology.

Sector of a circle

A **sector of a circle** is the region bounded by an arc of the circle and the two sides of a central angle (Figure 7.8). The ratio of the area of a sector to the area of the circle (πr^2) is equal to the ratio of the length of the subtended arc to the circumference of the circle ($2\pi r$). If s is the arc length and A is the area of the sector, we can write the following proportion: $\frac{A}{\pi r^2} = \frac{s}{2\pi r}$. Solving for A gives $A = \frac{\pi r^2 s}{2\pi r} = \frac{1}{2}rs$. From the formula for arc length we have $s = r\theta$, with θ the radian measure of the central angle. Substituting $r\theta$ for s gives the area of a sector to be $A = \frac{1}{2}rs = \frac{1}{2}r(r\theta) = \frac{1}{2}r^2\theta$. This result makes sense because, if the sector is the entire circle, $\theta = 2\pi$ and area $A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2(2\pi) = \pi r^2$, which is the formula for the area of a circle.

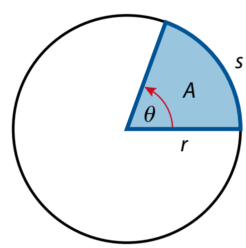


Figure 7.8 Sector of a circle.

Area of a sector

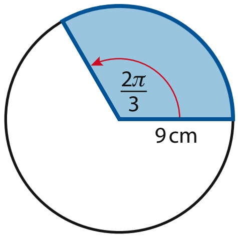
In a circle of radius r, the area of a sector with a central angle θ measured in radians is

$$A = \frac{1}{2}r^2\theta$$

Example 5

A circle of radius 9 cm has a sector whose central angle has radian measure $\frac{2\pi}{3}$. Find the exact values of the following: a) the length of the arc subtended by the central angle, and b) the area of the sector.

Solution



a) $s = r\theta \Rightarrow s = 9\left(\frac{2\pi}{3}\right) = 6\pi$

The length of the arc is exactly 6π cm.

b) $A = \frac{1}{2}r^2\theta \Rightarrow A = \frac{1}{2}(9)^2\left(\frac{2\pi}{3}\right) = 27\pi$

The area of the sector is exactly 27π cm^2.

● **Hint:** The formula for arc length, $s = r\theta$, and the formula for area of a sector, $A = \frac{1}{2}r^2\theta$, are true only when θ is in radians.

Exercise 7.1

In questions 1–9, find the exact radian measure of the angle given in degree measure.

1 60° **2** 150° **3** −270°

4 36° **5** 135° **6** 50°

7 −45° **8** 400° **9** −480°

In questions 10–18, find the degree measure of the angle given in radian measure. If possible, express exactly. Otherwise, express accurate to 3 significant figures.

10 $\frac{3\pi}{4}$ **11** $-\frac{7\pi}{2}$ **12** 2

13 $\frac{7\pi}{6}$ **14** −2.5 **15** $\frac{5\pi}{3}$

16 $\frac{\pi}{12}$ **17** 1.57 **18** $\frac{8\pi}{3}$

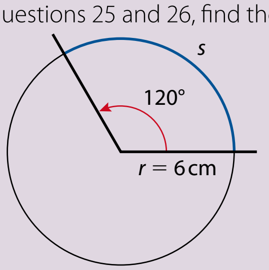
In questions 19–24, the measure of an angle in standard position is given. Find two angles – one positive and one negative – that are coterminal with the given angle. If no units are given, assume the angle is in radian measure.

19 30° **20** $\frac{3\pi}{2}$ **21** 175°

22 $-\frac{\pi}{6}$ **23** $\frac{5\pi}{3}$ **24** 3.25

In questions 25 and 26, find the length of the arc s in the figure.

25



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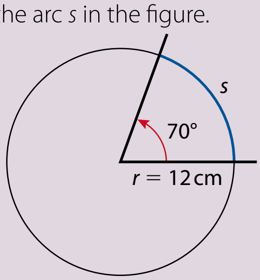
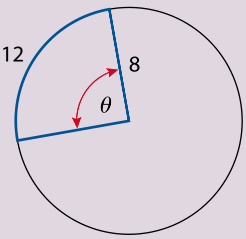
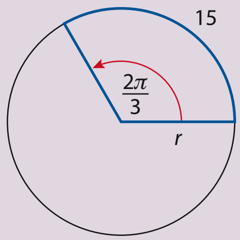


27 Find the angle θ in the figure in both radian measure and degree measure.

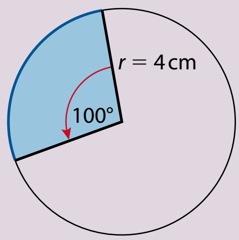


28 Find the radius r of the circle in the figure.



In questions 29 and 30, find the area of the sector in each figure.

29



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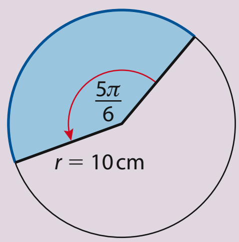


31 An arc of length 60 cm subtends a central angle α in a circle of radius 20 cm. Find the measure of α in both degrees and radians, approximate to 3 significant figures.

32 Find the length of an arc that subtends a central angle with radian measure of 2 in a circle of radius 16 cm.

33 The area of a sector of a circle with a central angle of 60° is 24 cm^2. Find the radius of the circle.

34 A bicycle with tyres 70 cm in diameter is travelling such that its tyres complete one and a half revolutions every second. That is, the **angular velocity** of a wheel is 1.5 revolutions per second.

a) What is the angular velocity of a wheel in radians per second?

b) At what speed (in km/hr) is the bicycle travelling along the ground? (This is the **linear velocity** of the bicycle.)

35 A bicycle with tyres 70 cm in diameter is travelling along a road at 25 km/hr. What is the angular velocity of a wheel of the bicycle in radians per second?

36 Given that ω is the angular velocity in radians/second of a point on a circle with radius r cm, express the linear velocity, v, in cm/second, of the point as a function in terms of ω and r.

37 A chord of 26 cm is in a circle of radius 20 cm. Find the length of the arc the chord subtends.

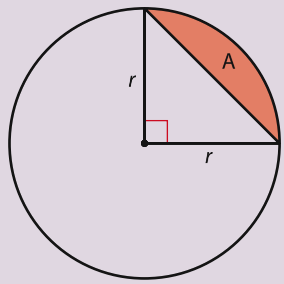
38 A circular irrigation system consists of a 400 metre pipe that is rotated around a central pivot point. If the irrigation pipe makes one full revolution around the pivot point in a day, then how much area, in square metres, does it irrigate each hour?

39 a) Find the radius of a circle circumscribed about a regular polygon of 64 sides if one side is 3 cm.

b) What is the difference between the circumference of the circle and the perimeter of the polygon?

40 What is the area of an equilateral triangle that has an inscribed circle with an area of 50π cm^2, and a circumscribed circle with an area of 200π cm^2?

41 In the diagram, the sector of a circle is subtended by two perpendicular radii. If the area of the sector is **A** square units, then find an expression for the area of the circle in terms of **A**.



7.2 The unit circle and trigonometric functions

Several important functions can be described by mapping the coordinates of points on the real number line onto the points of the unit circle. Recall from the previous section that the unit circle has its centre at $(0, 0)$, it has a radius of one unit and its equation is $x^2 + y^2 = 1$.

A wrapping function: the real number line and the unit circle

Suppose that the real number line is tangent to the unit circle at the point $(1, 0)$ – and that zero on the number line matches with $(1, 0)$ on the circle, as shown in Figure 7.9. Because of the properties of circles, the real number line in this position will be perpendicular to the x-axis. The scales on the number line and the x- and y-axes need to be the same. Imagine that the real number line is flexible like a string and can wrap around the circle, with zero on the number line remaining fixed to the point $(1, 0)$ on the unit circle. When the top portion of the string moves along the circle, the wrapping is anticlockwise ($t > 0$), and when the bottom portion of the string moves along the circle, the wrapping is clockwise ($t < 0$). As the string wraps around the unit circle, each real number t on the string is mapped onto a point (x, y) on the circle. Hence, the real number line from 0 to t makes an arc of length t starting on the circle at $(1, 0)$ and ending at the point (x, y) on the circle. For example, since the circumference of the unit circle is 2π, the number $t = 2\pi$ will be wrapped anticlockwise around the circle to the point $(1, 0)$. Similarly, the number $t = \pi$ will be wrapped anticlockwise halfway around the circle to the point $(-1, 0)$ on the circle. And the number $t = -\frac{\pi}{2}$ will be wrapped clockwise one-quarter of the way around the circle to the point $(0, -1)$ on the circle. Note that each number t on the real number line is mapped (corresponds) to *exactly one* point on the unit circle, thereby satisfying the definition of a function (Section 2.1) – consequently this mapping is called a **wrapping function**.

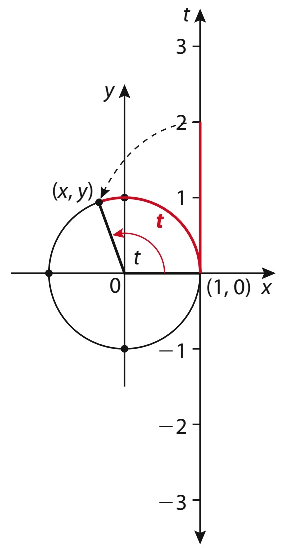


Figure 7.9 The wrapping function.

Before we leave our mental picture of the string (representing the real number line) wrapping around the unit circle, consider any pair of points on the string that are exactly 2π units from each other. Let these two points represent the real numbers t_1 and $t_1 + 2\pi$. Because the circumference of the unit circle is 2π, these two numbers will be mapped to the same point on the unit circle. Furthermore, consider the infinite number of points whose distance from t_1 is any integer multiple of 2π, i.e. $t_1 + k \cdot 2\pi$, $k \in \mathbb{Z}$, and again all of these numbers will be mapped to the same point on the unit circle. Consequently, the wrapping function is not a one-to-one function as defined in Section 2.3. Output for the function (points on the unit circle) are unchanged by the addition of any integer multiple of 2π to any input value (a real number). Functions that behave in such a repetitive (or cyclic) manner are called **periodic**.

We are surrounded by periodic functions. A few examples include: the average daily temperature variation during the year; sunrise and the day of the year; animal populations over many years; the height of tides and the position of the Moon; and an electrocardiogram, which is a graphic tracing of the heart's electrical activity.

Definition of a periodic function

A function f such that $f(x) = f(x + p)$ is a **periodic function**. If p is the least positive constant for which $f(x) = f(x + p)$ is true, p is called the **period** of the function.

Trigonometric functions

From our discussions about functions in Chapter 2, any function will have a domain (input) and range (output) that are sets having individual numbers as elements. We use the individual coordinates x and y of the points on the unit circle to define six **trigonometric functions**: the **sine**, **cosine**, **tangent**, **cosecant**, **secant** and **cotangent** functions. The names of these functions are often abbreviated in writing (but not speaking) as **sin**, **cos**, **tan**, **csc**, **sec**, **cot**, respectively.

When the real number t is wrapped to a point (x, y) on the unit circle, the value of the y-coordinate is assigned to the sine function; the x-coordinate is assigned to the cosine function; and the ratio of the two coordinates $\frac{y}{x}$ is assigned to the tangent function. Sine, cosine and tangent are often referred to as the **basic trigonometric functions**. The other three, cosecant, secant and cotangent, are each a reciprocal of one of the basic trigonometric functions and thus, are often referred to as the **reciprocal trigonometric functions**. All six are defined by means of the length of an arc on the unit circle as follows.

● **Hint:** To help you remember these definitions, note that the functions in the bottom row are the reciprocals of the function directly above in the top row.

Definition of the trigonometric functions

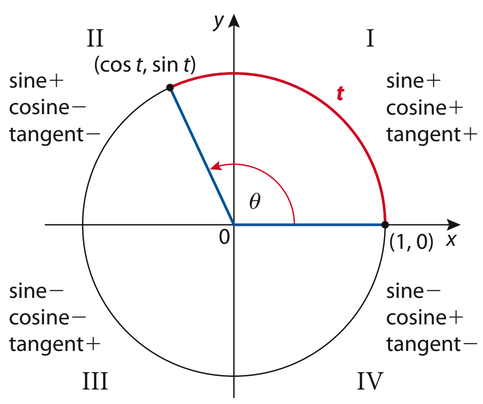
Let t be any real number and (x, y) a point on the unit circle to which t is mapped. Then the function definitions are:

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x},\ x \neq 0$$

$$\csc t = \frac{1}{y},\ y \neq 0 \qquad \sec t = \frac{1}{x},\ x \neq 0 \qquad \cot t = \frac{x}{y},\ y \neq 0$$

● **Hint:** Most calculators do not have keys for cosecant, secant and cotangent. You have to use the sine, cosine or tangent keys and the appropriate quotient. Because cosecant is the reciprocal of sine, to evaluate $\csc\frac{\pi}{3}$, for example, you need to evaluate $\frac{1}{\sin\frac{\pi}{3}}$. There is a key on your GDC labelled $\sin^{-1}$. It is **not** the reciprocal of sine but represents the inverse of the sine function, also denoted as the arcsine function (abbreviated arcsin). This is the same for $\cos^{-1}$ and $\tan^{-1}$. We will learn about these three inverse trigonometric functions in the last section of this chapter.

Figure 7.10 Signs of the trigonometric functions depend on the quadrant where the arc t terminates.



On the unit circle: $x = \cos t$, $y = \sin t$.

● **Hint:** When sine, cosine and tangent are defined as circular functions based on the unit circle, radian measure is used. The values for the domain of the sine and cosine functions are real numbers that are arc lengths on the unit circle. As we know from the previous section, the arc length on the unit circle subtends an angle in standard position, whose radian measure is equivalent to the arc length (see Figure 7.10).

Because the definitions for the sine, cosine and tangent functions given here do not refer to triangles or angles, but rather to a real number representing an arc length on the unit circle, the name **circular functions** is also given to them. In fact, from this chapter's perspective that these functions are *functions of real numbers* rather than *functions of angles*, 'circular' is a more appropriate adjective than 'trigonometric'. Nevertheless, trigonometric is the more common label and will be used throughout the book.

Let's use the definitions for these three trigonometric, or circular, functions to evaluate them for some 'easy' values of t.

Example 6

Evaluate the sine, cosine and tangent functions for the following values of t.

a) $t = 0$ b) $t = \frac{\pi}{2}$ c) $t = \pi$

d) $t = \frac{3\pi}{2}$ e) $t = 2\pi$

Solution

Evaluating the sin, cos and tan functions for any value of t involves finding the coordinates of the point on the unit circle where the arc of length t will 'wrap to' (or terminate), starting at the point $(1, 0)$. It is useful to remember that an arc of length π is equal to one-half of the circumference of the unit circle. All of the values for t in this example are positive, so the arc length will wrap along the unit circle in an anticlockwise direction.

a) An arc of length $t = 0$ has no length so it 'terminates' at the point $(1, 0)$. By definition:

$\sin 0 = y = 0$ $\cos 0 = x = 1$

$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$ $\csc 0 = \frac{1}{y} = \frac{1}{0}$ is undefined

$\sec 0 = \frac{1}{x} = \frac{1}{1} = 1$ $\cot 0 = \frac{x}{y} = \frac{1}{0}$ is undefined

b) An arc of length $t = \frac{\pi}{2}$ is equivalent to one-quarter of the circumference of the unit circle (Figure 7.11) so it terminates at the point $(0, 1)$.
By definition:

$\sin \frac{\pi}{2} = y = 1$ $\cos \frac{\pi}{2} = x = 0$

$\tan \frac{\pi}{2} = \frac{y}{x} = \frac{1}{0}$ is undefined $\csc \frac{\pi}{2} = \frac{1}{y} = 1$

$\sec \frac{\pi}{2} = \frac{1}{x}$ is undefined $\cot \frac{\pi}{2} = \frac{x}{y} = 0$

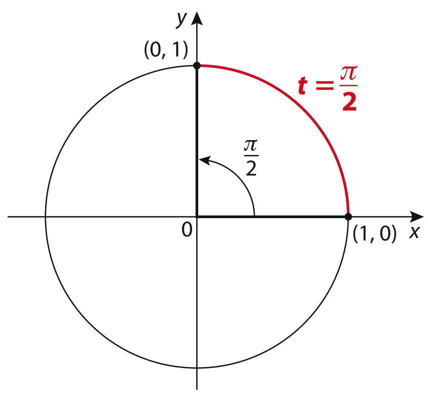


Figure 7.11 Arc length of $\frac{\pi}{2}$ or one-quarter of an anticlockwise revolution.

c) An arc of length $t = \pi$ is equivalent to one-half of the circumference of the unit circle (Figure 7.12) so it terminates at the point $(-1, 0)$. By definition:

$\sin \pi = y = 0$ $\cos \pi = x = -1$

$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$ $\csc \pi = \frac{1}{y}$ is undefined

$\sec \pi = \frac{1}{-1} = -1$ $\cot \pi = \frac{x}{y}$ is undefined

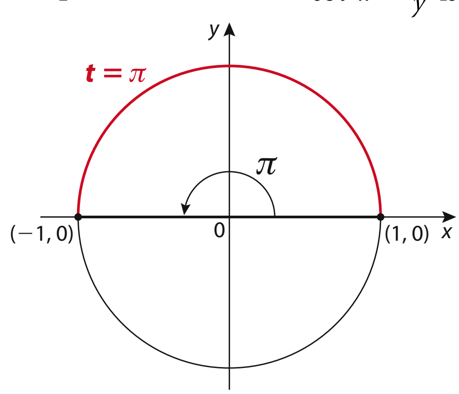


Figure 7.12 Arc length of π, one-half of an anticlockwise revolution.

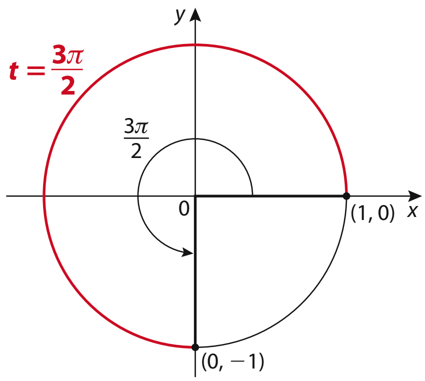
d) An arc of length $t = \frac{3\pi}{2}$ is equivalent to three-quarters of the circumference of the unit circle (Figure 7.13), so it terminates at the point $(0, -1)$. By definition:

$\sin\frac{3\pi}{2} = y = -1$ $\quad\quad$ $\cos\frac{3\pi}{2} = x = 0$

$\tan\frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0}$ is undefined $\quad\quad$ $\csc\frac{3\pi}{2} = \frac{1}{y} = -1$

$\sec\frac{3\pi}{2} = \frac{1}{x}$ is undefined $\quad\quad$ $\cot\frac{3\pi}{2} = \frac{x}{y} = 0$

Figure 7.13 Arc length of $\frac{3\pi}{2}$, three-quarters of an anticlockwise revolution.



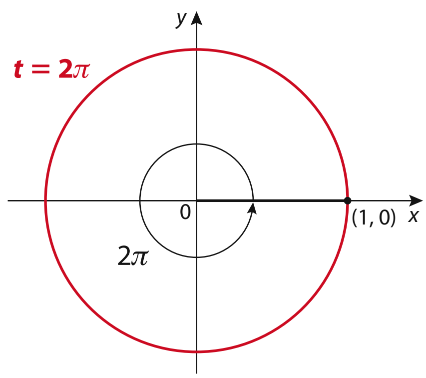
e) An arc of length $t = 2\pi$ terminates at the same point as arc of length $t = 0$ (Figure 7.14), so the values of the trigonometric functions are the same as found in part a):

$\sin 0 = y = 0$ $\quad\quad$ $\cos 0 = x = 1$

$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$ $\quad\quad$ $\csc 0 = \frac{1}{y}$ is undefined

$\sec 0 = \frac{1}{x} = 1$ $\quad\quad$ $\cot 0 = \frac{x}{y}$ is undefined

Figure 7.14 Arc length of 2π, one full anticlockwise revolution.



Domain and range of trigonometric functions

If s and t are coterminal arcs (i.e. terminate at the same point), then the trigonometric functions of s are equal to those of t. That is, $\sin s = \sin t$, $\cos s = \cos t$, etc.

Because every real number t corresponds to exactly one point on the unit circle, the domain for both the sine function and the cosine function is the set of all real numbers. In Example 6, the tangent function and the three reciprocal trigonometric functions were sometimes undefined. Hence, the domain for these functions cannot be all real numbers. From the definitions of the functions, it is clear that the tangent and secant functions will be undefined when the x-coordinate of the arc's terminal point is zero. Therefore, the domain of the tangent and secant functions is all real numbers but **not** including the infinite set of numbers generated by adding any integer multiple of π to $\frac{\pi}{2}$. For example, $\frac{\pi}{2} + \pi = \frac{3\pi}{2}$ and $\frac{\pi}{2} - \pi = -\frac{\pi}{2}$ (see Figure 7.15), thus the tangent and secant of $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$ are undefined. Similarly, the cotangent and cosecant functions will be undefined when the y-coordinate of the arc's terminal point is zero. Therefore, the domain of the cotangent and cosecant functions is all real numbers but **not** including all of the integer multiples of π.

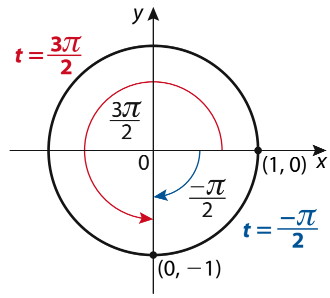
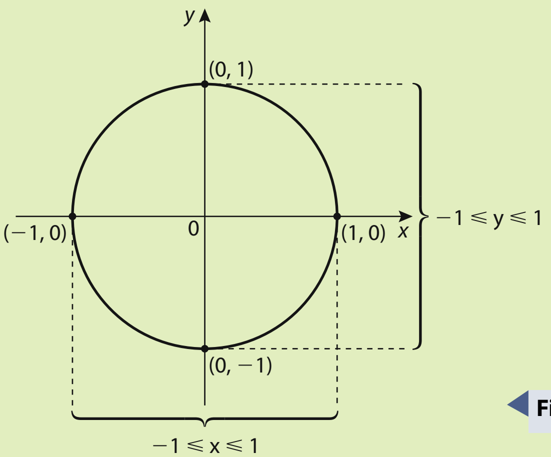


Figure 7.15

Domains of the six trigonometric functions



$f(t) = \sin t$ and $f(t) = \cos t$

domain: $\{t : t \in \mathbb{R}\}$

$f(t) = \tan t$ and $f(t) = \sec t$

domain: $\left\{t : t \in \mathbb{R}, t \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$

$f(t) = \cot t$ and $f(t) = \csc t$

domain: $\{t : t \in \mathbb{R}, t \neq k\pi, k \in \mathbb{Z}\}$

Figure 7.16

To determine the range of the sine and cosine functions, consider the unit circle shown in Figure 7.16. Because $\sin t = y$ and $\cos t = x$ and (x, y) is on the unit circle, we can see that $-1 \leqslant y \leqslant 1$ and $-1 \leqslant x \leqslant 1$. Therefore, $-1 \leqslant \sin t \leqslant 1$ and $-1 \leqslant \cos t \leqslant 1$. The range for the tangent function will not be bounded as for sine and cosine. As t approaches values where $x = \cos t = 0$, the value of $\frac{y}{x} = \tan t$ will become very large – either negative or positive, depending on which quadrant t is in. Therefore, $-\infty < \tan t < \infty$; or, in other words, $\tan t$ can be any real number.

Domain and range of sine, cosine and tangent functions

$f(t) = \sin t$	domain: $\{t : t \in \mathbb{R}\}$	range: $-1 \leqslant f(t) \leqslant 1$
$f(t) = \cos t$	domain: $\{t : t \in \mathbb{R}\}$	range: $-1 \leqslant f(t) \leqslant 1$
$f(t) = \tan t$	domain: $\left\{t : t \in \mathbb{R}, t \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$	range: $f(t) \in \mathbb{R}$

From our previous discussion of periodic functions, we can conclude that all three of these trigonometric functions are periodic. Given that the sine and cosine functions are generated directly from the wrapping function, the period of each of these functions is 2π. That is,

$$\sin t = \sin(t + k \cdot 2\pi),\ k \in \mathbb{Z} \text{ and } \cos t = \cos(t + k \cdot 2\pi),\ k \in \mathbb{Z}$$

Since the cosecant and secant functions are reciprocals, respectively, of sine and cosine, the period of cosecant and secant will also be 2π.

Initial evidence from Example 6 indicates that the period of the tangent function is π. That is,

$$\tan t = \tan(t + k \cdot \pi),\ k \in \mathbb{Z}$$

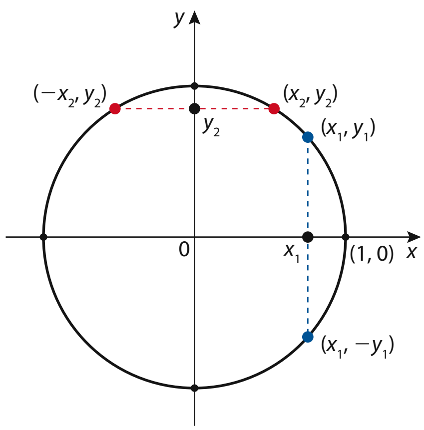
We will establish these results graphically in the next section. Also note that since these functions are periodic then they are not one-to-one functions.

This is an important fact with regard to establishing inverse trigonometric functions (Section 7.6).

Evaluating trigonometric functions

In Example 6, the unit circle was divided into four equal arcs corresponding to t values of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π. Let's evaluate the sine, cosine and tangent functions for further values of t that would correspond to dividing the unit circle into eight equal arcs. The symmetry of the unit circle dictates that any points on the unit circle which are reflections about the x-axis will have the same x-coordinate (same value of sine), and any points on the unit circle which are reflections about the y-axis will have the same y-coordinate, as shown in Figure 7.17.

Figure 7.17



Example 7

Evaluate the sine, cosine and tangent functions for $t = \frac{\pi}{4}$, and then use that result to evaluate the same functions for $t = \frac{3\pi}{4}$, $t = \frac{5\pi}{4}$ and $t = \frac{7\pi}{4}$.

Solution

When an arc of length $t = \frac{\pi}{4}$ is wrapped along the unit circle starting at $(1, 0)$, it will terminate at a point (x_1, y_1) in quadrant I that is equidistant from $(1, 0)$ and $(0, 1)$. Since the line $y = x$ is a line of symmetry for the unit circle, (x_1, y_1) is on this line. Hence, the point (x_1, y_1) is the point of intersection of the unit circle $x^2 + y^2 = 1$ with the line $y = x$. Let's find the coordinates of the intersection point by solving this pair of simultaneous equations by substituting x for y into the equation $x^2 + y^2 = 1$.

$x^2 + y^2 = 1 \Rightarrow x^2 + x^2 = 1 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

Rationalizing the denominator gives $x = \pm\frac{\sqrt{2}}{2}$ and, since the point is in the first quadrant, $x = \frac{\sqrt{2}}{2}$. Given that the point is on the line $y = x$ then $y = \frac{\sqrt{2}}{2}$. Therefore, the arc of length $t = \frac{\pi}{4}$ will terminate at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle. Using the symmetry of the unit circle, we can also determine the points on the unit circle where arcs of length $t = \frac{3\pi}{4}$, $t = \frac{5\pi}{4}$ and $t = \frac{7\pi}{4}$ terminate. These arcs and the coordinates of their terminal points are given in Figure 7.18.

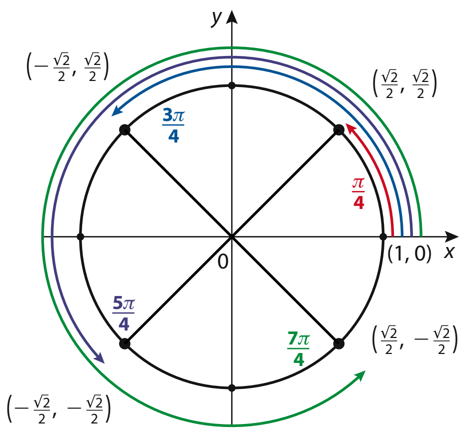


Figure 7.18

Using the coordinates of these points, we can now evaluate the trigonometric functions for $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$. By definition:

$$t = \frac{\pi}{4}: \quad \sin\frac{\pi}{4} = y = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = x = \frac{\sqrt{2}}{2} \qquad \tan\frac{\pi}{4} = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$t = \frac{3\pi}{4}: \sin\frac{3\pi}{4} = y = \frac{\sqrt{2}}{2} \qquad \cos\frac{3\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \tan\frac{3\pi}{4} = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$$

$$t = \frac{5\pi}{4}: \sin\frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2} \qquad \cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \tan\frac{5\pi}{4} = \frac{y}{x} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

$$t = \frac{7\pi}{4}: \sin\frac{7\pi}{4} = y = -\frac{\sqrt{2}}{2} \qquad \cos\frac{7\pi}{4} = x = \frac{\sqrt{2}}{2} \qquad \tan\frac{7\pi}{4} = \frac{y}{x} = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$$

We can use a method similar to that of Example 7 to find the point on the unit circle where an arc of length $t = \frac{\pi}{6}$ terminates in the first quadrant. Then we can again apply symmetry about the line $y = x$ and the y- and x-axes to find points on the circle corresponding to arcs whose lengths are multiples of $\frac{\pi}{6}$, e.g. $\frac{2\pi}{6} = \frac{\pi}{3}$, $\frac{4\pi}{6} = \frac{2\pi}{3}$, etc. Arcs whose lengths are multiples of $\frac{\pi}{4}$ and $\frac{\pi}{6}$ correspond to eight equally spaced points and twelve equally spaced points, respectively, around the unit circle, as shown in Figures 7.19 and 7.20. The coordinates of these points give us the sine, cosine and tangent values for common values of t.

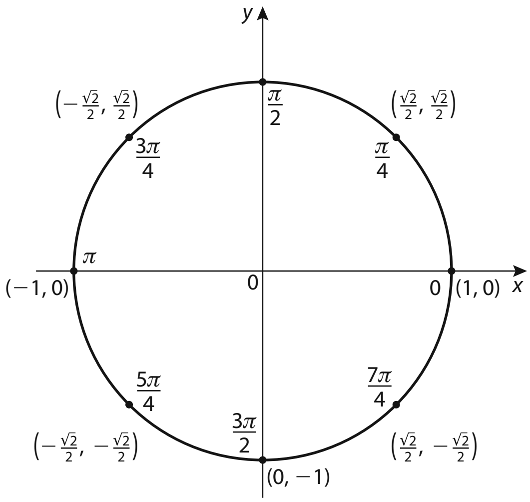


Figure 7.19 Arc lengths that are multiples of $\frac{\pi}{4}$ divide the unit circle into eight equally spaced points.

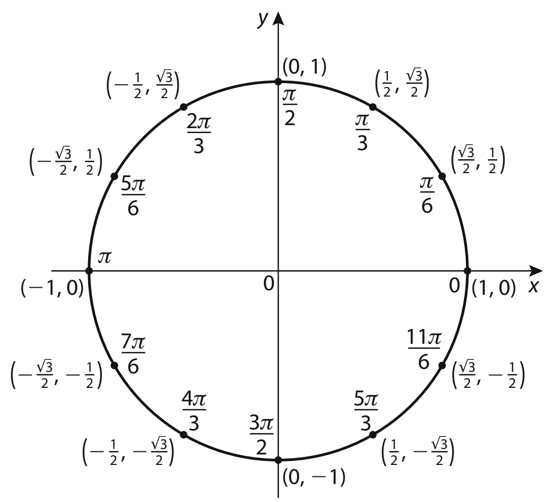


Figure 7.20 Arc lengths that are multiples of $\frac{\pi}{6}$ divide the unit circle into twelve equally spaced points.

The tangent, cosecant, secant and cotangent functions can all be expressed in terms of the sine and/or cosine functions. The following four identities follow directly from the definitions for the trigonometric functions.

$\tan t = \frac{\sin t}{\cos t}$ $\quad \csc t = \frac{1}{\sin t}$

$\sec t = \frac{1}{\cos t}$ $\quad \cot t = \frac{\cos t}{\sin t}$

You will find it very helpful to know from memory the exact values of sine and cosine for numbers that are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Use the unit circle diagrams shown in Figures 7.19 and 7.20 as a guide to help you do this and to visualize the location of the terminal points of different arc lengths. With the symmetry of the unit circle and a point's location in the coordinate plane telling us the sign of x and y (see Figure 7.10), we only need to remember the sine and cosine of common values of t in the first quadrant and on the positive x- and y-axes. These are organized in Table 7.1.

Table 7.1 The trigonometric functions evaluated for special values of t.

t	$\sin t$	$\cos t$	$\tan t$	$\csc t$	$\sec t$	$\cot t$
0	0	1	0	undefined	1	undefined
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	undefined	1	undefined	0

If t is not a multiple of one of these common values, the values of the trigonometric functions for that number can be found using your GDC.

• **Hint:** Memorize the values of $\sin t$ and $\cos t$ for the values of t that are highlighted in the red box in Table 7.1. These values can be used to derive the values of all six trigonometric functions for any multiple of $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$ or $\frac{\pi}{2}$.

For any arc s on the unit circle ($r = 1$) the arc length formula from the previous section, $s = r\theta$, shows us that each real number t not only measures an arc along the unit circle but also measures a central angle in radians. That is, $t = r\theta = 1 \cdot \theta = \theta$ in radian measure. Therefore, when you are evaluating a trigonometric function it does not make a difference whether the argument of the function is considered to be a real number (i.e. length of an arc) or an angle in radians.

Example 8

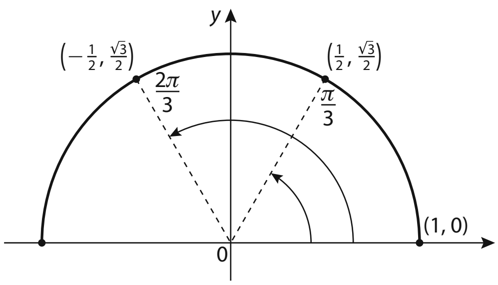
Find the following function values. Find the exact value, if possible. Otherwise, find the approximate value accurate to 3 significant figures.

a) $\sin\frac{2\pi}{3}$ b) $\cos\frac{5\pi}{4}$ c) $\tan\frac{11\pi}{6}$

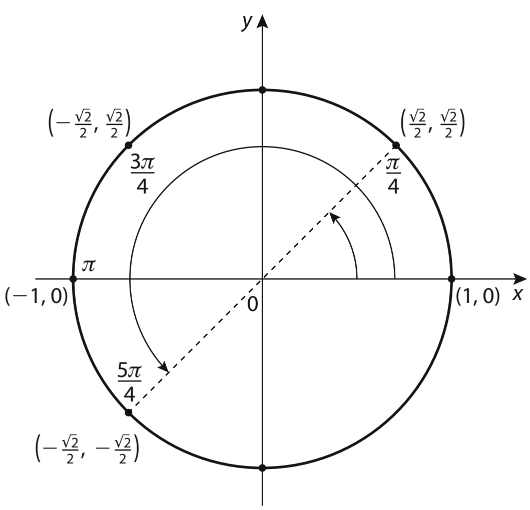
d) $\csc\frac{13\pi}{6}$ e) $\sec 3.75$

Solution

a) The terminal point for $\frac{2\pi}{3}$ is in the second quadrant and is the reflection in the y-axis of the terminal point for $\frac{\pi}{3}$, whose y-coordinate is $\frac{\sqrt{3}}{2}$. Therefore, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

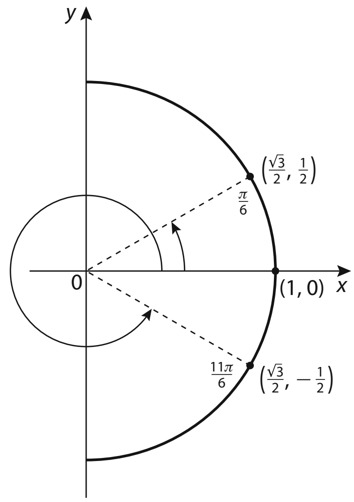


b) $\frac{5\pi}{4}$ is in the third quadrant. Hence, its x-coordinate and cosine must be negative. All of the odd multiples of $\frac{\pi}{4}$ have terminal points with x- and y-coordinates of $\pm\frac{\sqrt{2}}{2}$. Therefore, $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$.



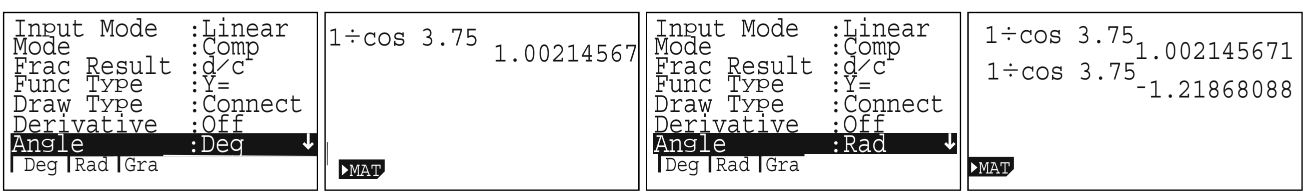
c) $\frac{11\pi}{6}$ is in the fourth quadrant, so its tangent will be negative. Its terminal point is the reflection in the x-axis of the terminal point for $\frac{\pi}{6}$, whose coordinates are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Therefore,

$$\tan\frac{11\pi}{6} = \frac{y}{x} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$



d) $\frac{13\pi}{6}$ is more than one revolution. Because $\frac{13\pi}{6} = \frac{\pi}{6} + 2\pi$ and the period of the cosecant function is 2π [i.e. $\csc t = \csc(t + k \cdot 2\pi)$, $k \in \mathbb{Z}$], then $\csc\frac{13\pi}{6} = \csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$.

e) To evaluate sec 3.75 you must use your GDC. An arc of length 3.75 will have its terminal point in the third quadrant since $\pi \approx 3.14$ and $\frac{3\pi}{2} \approx 4.71$, meaning $\pi < 3.75 < \frac{3\pi}{2}$. Hence, cos 3.75 must be negative, and because the secant function is the reciprocal of cosine, then sec 3.75 is also negative. This fact indicates that the result in the second GDC image below must be incorrect with the GDC wrongly set to 'degree' mode. Changing to 'radian' mode allows for the correct result to be computed. To an accuracy of three significant figures, $\sec 3.75 \approx -1.22$.



Have you ever wondered how your calculator computes a value for a trigonometric function – such as cos 0.75? Evaluating an algebraic function (Chapter 3) is relatively straightforward because, by definition, it consists of a finite number of elementary operations (i.e. addition, subtraction, multiplication, division, and extracting a root). It is not so straightforward to evaluate non-algebraic functions like exponential, logarithmic and trigonometric functions and efforts by mathematicians to do so have led to some sophisticated approximation techniques using **power series** that are studied in further calculus. A power series is an infinite series that can be thought of as a polynomial with an infinite number of terms. You will learn about the theory and application of power series if your Mathematics HL class covers the *Option: Infinite series and differential equations*. If you look in the Mathematics HL Information (Formulae) Booklet in the Topic 10 section (for series and differential equations) you will see the power series (infinite polynomial) approximation for some functions including the cosine function.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ where $n! = 1 \cdot 2 \cdot 3 \dots n$ [$n!$ is read 'n factorial']

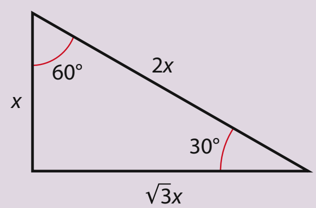
Exploiting the fact that polynomial functions are easy to evaluate, we can easily program a calculator to compute enough terms of the power series to obtain a result to the required accuracy. For example, if we use the first three terms of the power series for cosine to find cos 0.75, we get

$\cos 0.75 = 1 - \frac{0.75^2}{2!} + \frac{0.75^4}{4!} = 0.731\,933\,593\,75$. Compare this to the value obtained using your GDC.

Several important mathematicians in the 17th and 18th centuries, including Isaac Newton, James Gregory, Gottfried Leibniz, Leonhard Euler and Joseph Fourier, contributed to the development of using power series to represent non-algebraic functions. However, the two names most commonly associated with power series are the English mathematician Brook Taylor (1685–1731) and the Scottish mathematician Colin Maclaurin (1698–1746).

Exercise 7.2

1 a) By knowing the ratios of sides in any triangle with angles measuring 30°, 60° and 90° (see figure), find the coordinates of the points on the unit circle where an arc of length $t = \frac{\pi}{6}$ and $t = \frac{\pi}{3}$ terminate in the first quadrant.

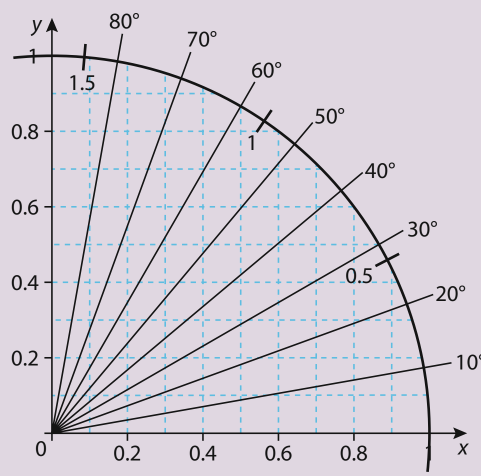


b) Using the result from a) and applying symmetry about the unit circle, find the coordinates of the points on the unit circle corresponding to arcs whose lengths are $\frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$.

Draw a large unit circle and label all of these points with their coordinates and the measure of the arc that terminates at each point.

Questions 2–9

The figure of quadrant I of the unit circle shown right indicates angles in intervals of 10 degrees and also indicates angles in radian measure of 0.5, 1 and 1.5. Use the figure and the definitions of the sine and cosine functions to approximate the function values to one decimal place in questions 2–9. Check your answers with your GDC (be sure to be in the correct angle measure mode).



2 $\cos 50°$ **3** $\sin 80°$ **4** $\cos 1$

5 $\sin 0.5$ **6** $\tan 70°$ **7** $\cos 1.5$

8 $\sin 20°$ **9** $\tan 1$

In questions 10–18, t is the length of an arc on the unit circle starting from (1, 0). a) State the quadrant in which the terminal point of the arc lies. b) Find the coordinates of the terminal point (x, y) on the unit circle. Give exact values for x and y, if possible. Otherwise, approximate values to 3 significant figures.

10 $t = \frac{\pi}{6}$ **11** $t = \frac{5\pi}{3}$ **12** $t = \frac{7\pi}{4}$

13 $t = \frac{3\pi}{2}$ **14** $t = 2$ **15** $t = -\frac{\pi}{4}$

16 $t = -1$ **17** $t = -\frac{5\pi}{4}$ **18** $t = 3.52$

In questions 19–27, state the exact value of the sine, cosine and tangent of the given real number.

19 $\frac{\pi}{3}$ **20** $\frac{5\pi}{6}$ **21** $-\frac{3\pi}{4}$

22 $\frac{\pi}{2}$ **23** $-\frac{4\pi}{3}$ **24** 3π

25 $\frac{3\pi}{2}$ **26** $-\frac{7\pi}{6}$ **27** $t = 1.25\pi$

In questions 28–31, use the periodic properties of the sine and cosine functions to find the exact value of $\sin x$ and $\cos x$.

28 $x = \frac{13\pi}{6}$ **29** $x = \frac{10\pi}{3}$

30 $x = \frac{15\pi}{4}$ **31** $x = \frac{17\pi}{6}$

32 Find the exact function values, if possible. Do not use your GDC.

a) $\cos \frac{5\pi}{6}$ b) $\sin 315°$ c) $\tan \frac{3\pi}{2}$

d) $\sec \frac{5\pi}{3}$ e) $\csc 240°$

33 Find the exact function values, if possible. Otherwise, use your GDC to find the approximate value accurate to three significant figures.

a) $\sin 2.5$ b) $\cot 120°$ c) $\cos \frac{5\pi}{4}$

d) $\sec 6$ e) $\tan \pi$

In questions 34–41, specify in which quadrant(s) an angle θ in standard position could be given the stated conditions.

34 $\sin \theta > 0$

35 $\sin \theta > 0$ and $\cos \theta < 0$

36 $\sin \theta < 0$ and $\tan \theta > 0$

37 $\cos \theta < 0$ and $\tan \theta < 0$

38 $\cos \theta > 0$

39 $\sec \theta > 0$ and $\tan \theta > 0$

40 $\cos \theta > 0$ and $\csc \theta < 0$

41 $\cot \theta < 0$

7.3 Graphs of trigonometric functions

The graph of a function provides a useful visual image of its behaviour. For example, from the previous section we know that trigonometric functions are periodic, i.e. their values repeat in a regular manner. The graphs of the trigonometric functions should provide a picture of this periodic behaviour. In this section, we will graph the sine, cosine and tangent functions and transformations of the sine and cosine functions.

```
sin(2.53)
          .5741721484
sin(2.53+2π)
          .5741721484
sin(2.53+4π)
          .5741721484
```

Graphs of the sine and cosine functions

Since the period of the sine function is 2π, we know that two values of t (domain) that differ by 2π (e.g. $\frac{\pi}{6}$ and $\frac{13\pi}{6}$ in Example 8d) will produce the same value for y (range). This means that any portion of the graph of $y = \sin t$ with a t-interval of length 2π (called one **period** or **cycle** of the graph) will repeat. Remember that the domain of the sine function is all real numbers, so one period of the graph of $y = \sin t$ will repeat indefinitely in the positive and negative direction. Therefore, in order to construct a complete graph of $y = \sin t$, we need to graph just one period of the function, that is, from $t = 0$ to $t = 2\pi$, and then repeat the pattern in both directions.

The period of $y = \sin x$ is 2π.

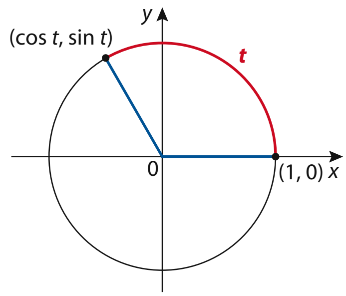
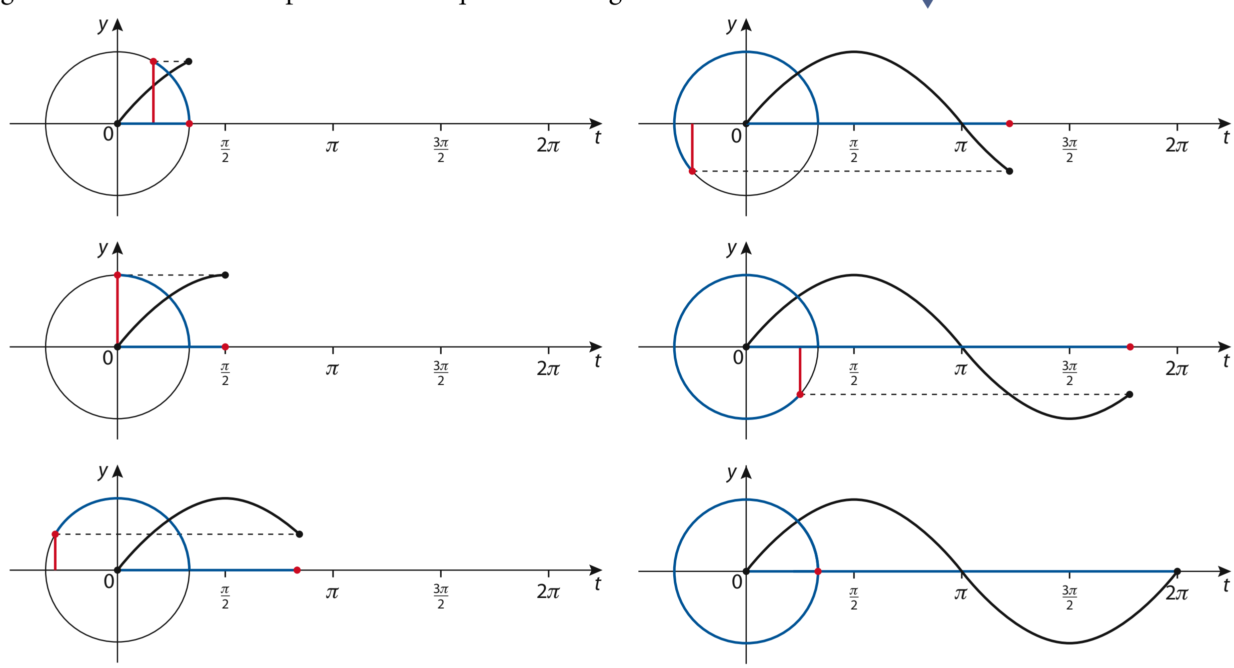


Figure 7.21 Coordinates of terminal point of arc t gives the values of $\cos t$ and $\sin t$.

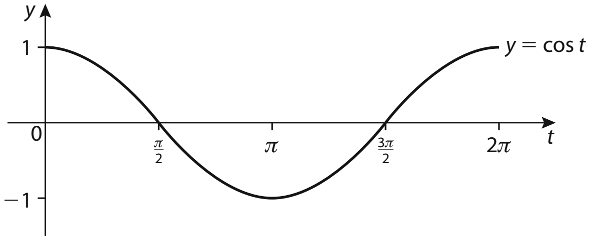
We know from the previous section that $\sin t$ is the y-coordinate of the terminal point on the unit circle corresponding to the real number t (Figure 7.21). In order to generate one period of the graph of $y = \sin t$, we need to record the y-coordinates of a point on the unit circle and the corresponding value of t as the point travels anticlockwise one revolution, starting from the point $(1, 0)$. These values are then plotted on a graph with t on the horizontal axis and y (i.e. $\sin t$) on the vertical axis. Figure 7.22 illustrates this process in a sequence of diagrams.

Figure 7.22 Graph of the sine function for $0 \leqslant t \leqslant 2\pi$ generated from a point travelling along the unit circle.



As the point ($\cos t$, $\sin t$) travels along the unit circle, the x-coordinate (i.e. $\cos t$) goes through the same cycle of values as the y-coordinate ($\sin t$). The only difference is that the x-coordinate begins at a different value in the cycle – when $t = 0$, $y = 0$, but $x = 1$. The result is that the graph of $y = \cos t$ is the exact same shape as $y = \sin t$ but it has been shifted to the left $\frac{\pi}{2}$ units. The graph of $y = \cos t$ for $0 \leqslant t \leqslant 2\pi$ is shown in Figure 7.23.

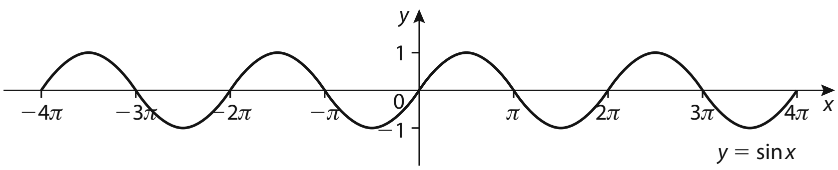
Figure 7.23 Graph of $y = \cos t$ for $0 \leqslant t \leqslant 2\pi$.

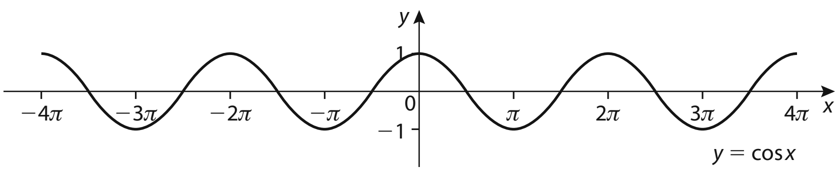


The convention is to use the letter x to denote the variable in the domain of the function. Hence, we will use the letter x rather than t and write the trigonometric functions as $y = \sin x$, $y = \cos x$ and $y = \tan x$.

Because the period for both the sine function and cosine function is 2π, to graph $y = \sin x$ and $y = \cos x$ for wider intervals of x we simply need to repeat the shape of the graph that we generated from the unit circle for $0 \leqslant x \leqslant 2\pi$ (Figures 7.22 and 7.23). Figure 7.24 shows the graphs of $y = \sin x$ and $y = \cos x$ for $-4\pi \leqslant x \leqslant 4\pi$.

Figure 7.24 $y = \sin x$ and $y = \cos x$, $0 \leqslant x \leqslant 4\pi$.





Aside from their periodic behaviour, these graphs reveal further properties of the graphs of $y = \sin x$ and $y = \cos x$. Note that the sine function has a maximum value of $y = 1$ for all $x = \frac{\pi}{2} + k \cdot 2\pi$, $k \in \mathbb{Z}$, and has a minimum value of $y = -1$ for all $x = -\frac{\pi}{2} + k \cdot 2\pi$, $k \in \mathbb{Z}$. The cosine function has a maximum value of $y = 1$ for all $x = k \cdot 2\pi$, $k \in \mathbb{Z}$, and has a minimum value of $y = -1$ for all $x = \pi + k \cdot 2\pi$, $k \in \mathbb{Z}$. This also confirms – as established in the previous section – that both functions have a domain of all real numbers and a range of $-1 \leqslant y \leqslant 1$.

Closer inspection of the graphs, in Figure 7.24, shows that the graph of $y = \sin x$ has rotational symmetry about the origin – that is, it can be rotated one-half of a revolution about (0, 0) and it remains the same. This graph symmetry can be expressed with the identity: $\sin(-x) = -\sin x$. For example, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\left[\sin\left(\frac{\pi}{6}\right)\right] = -\left[\frac{1}{2}\right] = -\frac{1}{2}$. A function that is symmetric about the origin is called an **odd function**. The graph of $y = \cos x$ has line symmetry in the y-axis – that is, it can be reflected in the line $x = 0$ and it remains the same. This graph symmetry can be expressed with the identity: $\cos(-x) = \cos x$. For example, $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. A function that is symmetric about the y-axis is called an **even function**.

Recall that odd and even functions were first discussed in Section 3.1.

Odd and even functions

A function is **odd** if, for each x in the domain of f, $f(-x) = -f(x)$.

The graph of an odd function is symmetric with respect to the origin (rotational symmetry).

A function is **even** if, for each x in the domain of f, $f(-x) = f(x)$.

The graph of an even function is symmetric with respect to the y-axis (line symmetry).

Graphs of transformations of the sine and cosine functions

In Section 2.4, we learned how to transform the graph of a function by horizontal and vertical translations, by reflections in the coordinate axes, and by stretching and shrinking – both horizontal and vertical. The following is a review of these transformations.

Review of transformations of graphs of functions

Assume that a, b, c and d are real numbers.

To obtain the graph of:	**From the graph of $y = f(x)$:**
$y = f(x) + d$	Translate d units up for $d > 0$, d units down for $d < 0$.
$y = f(x + c)$	Translate c units left for $c > 0$, c units right for $c < 0$.
$y = -f(x)$	Reflect in the x-axis.
$y = af(x)$	Vertical stretch ($a > 1$) or shrink ($0 < a < 1$) of factor a.
$y = f(-x)$	Reflect in the y-axis.
$y = f(bx)$	Horizontal stretch ($0 < b < 1$) or shrink ($b > 1$) of factor $\frac{1}{b}$.

In this section, we will look at the composition of sine and cosine functions of the form

$$f(x) = a\sin[b(x + c)] + d \quad \text{and} \quad f(x) = a\cos[b(x + c)] + d$$

Example 9

Sketch the graph of each function on the interval $-\pi \leqslant x \leqslant 3\pi$.

a) $f(x) = 2\cos x$

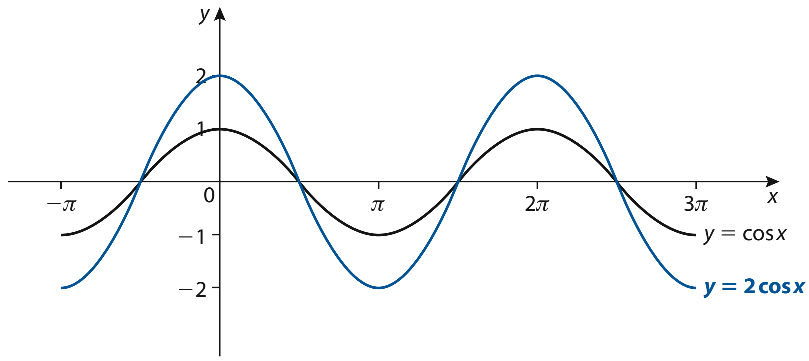
b) $g(x) = \cos x + 3$

c) $h(x) = 2\cos x + 3$

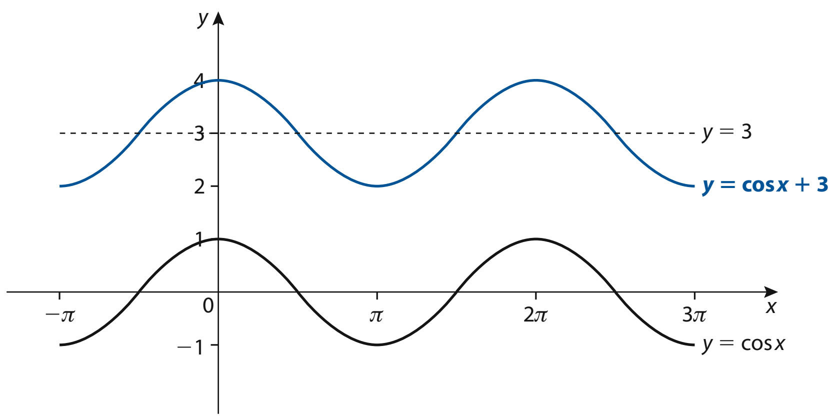
d) $p(x) = \frac{1}{2}\sin x - 2$

Solution

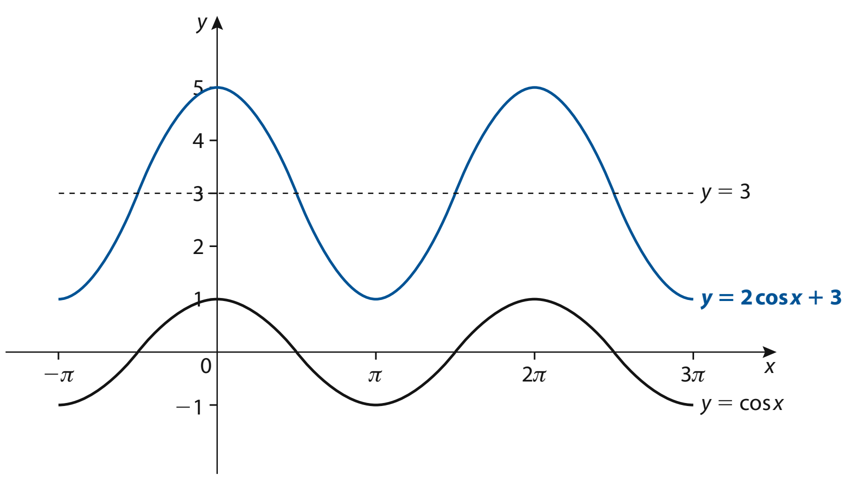
a) Since $a = 2$, the graph of $y = 2\cos x$ is obtained by vertically stretching the graph of $y = \cos x$ by a factor of 2.



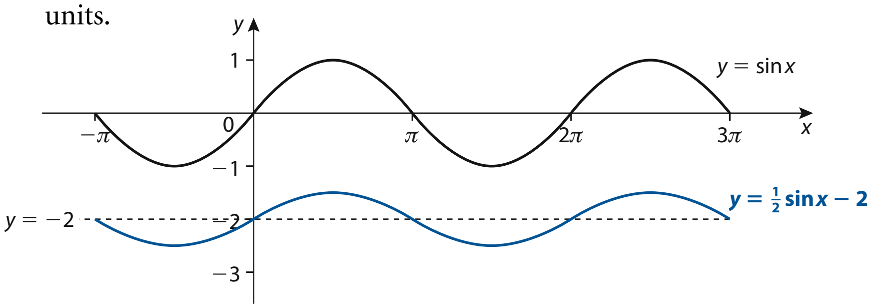
b) Since $d = 3$, the graph of $y = \cos x + 3$ is obtained by translating the graph of $y = \cos x$ three units up.



c) We can obtain the graph of $y = 2\cos x + 3$ by combining both of the transformations to the graph of $y = \cos x$ performed in parts a) and b) – namely, a vertical stretch of factor 2 and a translation three units up.



d) The graph of $y = \frac{1}{2}\sin x - 2$ can be obtained by vertically shrinking the graph of $y = \sin x$ by a factor of $\frac{1}{2}$ and then translating it down two units.



In part a), the graph of $y = 2\cos x$ has many of the same properties as the graph of $y = \cos x$: same period, and the maximum and minimum values occur at the same values of x. However, the graph ranges between -2 and 2 instead of -1 and 1. This difference is best described by referring to the **amplitude** of each graph. The amplitude of $y = \cos x$ is 1 and the amplitude of $y = 2\cos x$ is 2. The amplitude of a sine or cosine graph is not always equal to its maximum value. In part b), the amplitude of $y = \cos x + 3$ is 1; in part c), the amplitude of $y = 2\cos x + 3$ is 2; and the amplitude of $y = \frac{1}{2}\sin x - 2$ is $\frac{1}{2}$. For all three of these, the graphs oscillate about the horizontal line $y = d$. How *high* and *low* the graph oscillates with respect to the mid-line, $y = d$, is the graph's amplitude. With respect to the general form $y = af(x)$, changing the amplitude is equivalent to a vertical stretching or shrinking. Thus, we can give a more precise definition of amplitude in terms of the parameter a.

Amplitude of the graph of sine and cosine functions

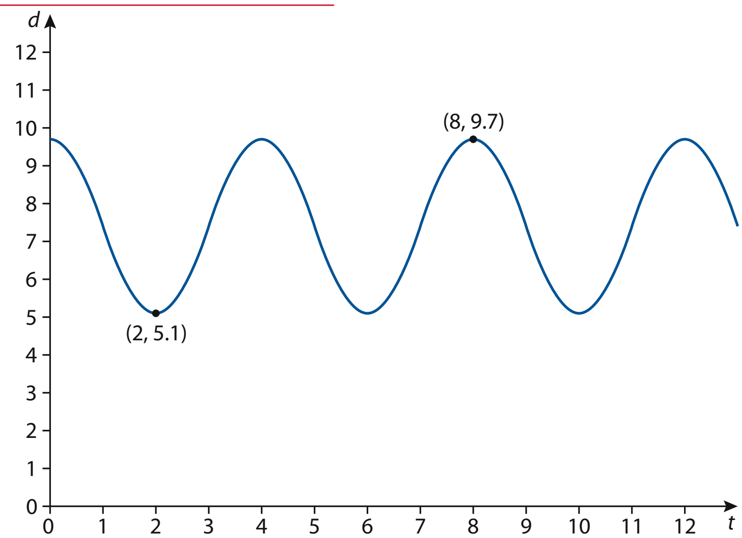
The graphs of $f(x) = a\sin[b(x + c)] + d$ and $f(x) = a\cos[b(x + c)] + d$ have an **amplitude** equal to $|a|$.

Example 10

Waves are produced in a long tank of water. The depth of the water, d metres, at t seconds, at a fixed location in the tank, is modelled by the function $d(t) = M\cos\left(\frac{\pi}{2}t\right) + K$, where M and K are positive constants. On the right is the graph of $d(t)$ for $0 \leqslant t \leqslant 12$ indicating that the point (2, 5.1) is a minimum and the point (8, 9.7) is a maximum.

a) Find the value of K and the value of M.

b) After $t = 0$, find the first time when the depth of the water is 9.7 metres.



Solution

a) The constant K is equivalent to the constant d in the general form of a cosine function: $f(x) = a\cos[b(x + c)] + d$. To find the value of K and the equation of the horizontal mid-line, $y = K$, find the average of the function's maximum and minimum value: $K = \frac{9.7 + 5.1}{2} = 7.4$.

The constant M is equivalent to the constant a whose absolute value is the amplitude. The amplitude is the difference between the function's maximum value and the mid-line: $|M| = 9.7 - 7.4 = 2.3$. Thus, $M = 2.3$ or $M = -2.3$. Try $M = 2.3$ by evaluating the function at one of the known values:

$d(2) = 2.3\cos\left(\frac{\pi}{2}(2)\right) + 7.4 = 2.3\cos\pi + 7.4 = 2.3(-1) + 7.4 = 5.1.$

This agrees with the point $(2, 5.1)$ on the graph. Therefore, $M = 2.3$.

b) Maximum values of the function ($d(8) = 9.7$) occur at values of t that differ by a value equal to the period. From the graph, we can see that the difference in t-values from the minimum $(2, 5.1)$ to the maximum $(8, 9.7)$ is equivalent to one-and-a-half periods. Therefore, the period is 4 and the first time after $t = 0$ at which $d = 9.7$ is $t = 4$.

All four of the functions in Example 9 had the same period of 2π, but the function in Example 10 had a period of 4. Because $y = \sin x$ completes one period from $x = 0$ to $x = 2\pi$, it follows that $y = \sin bx$ completes one period from $bx = 0$ to $bx = 2\pi$. This implies that $y = \sin bx$ completes one period from $x = 0$ to $x = \frac{2\pi}{b}$. This agrees with the period for the function $d(t) = 2.3\cos\left(\frac{\pi}{2}t\right) + 7.4$ in Example 10: period $= \frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{2}} = \frac{2\pi}{1} \cdot \frac{2}{\pi} = 4$.

Note that the change in amplitude and vertical translation had no effect on the period. We should also expect that a horizontal translation of a sine or cosine curve should not affect the period. The next example looks at a function that is horizontally translated (shifted) and has a period different from 2π.

Example 11

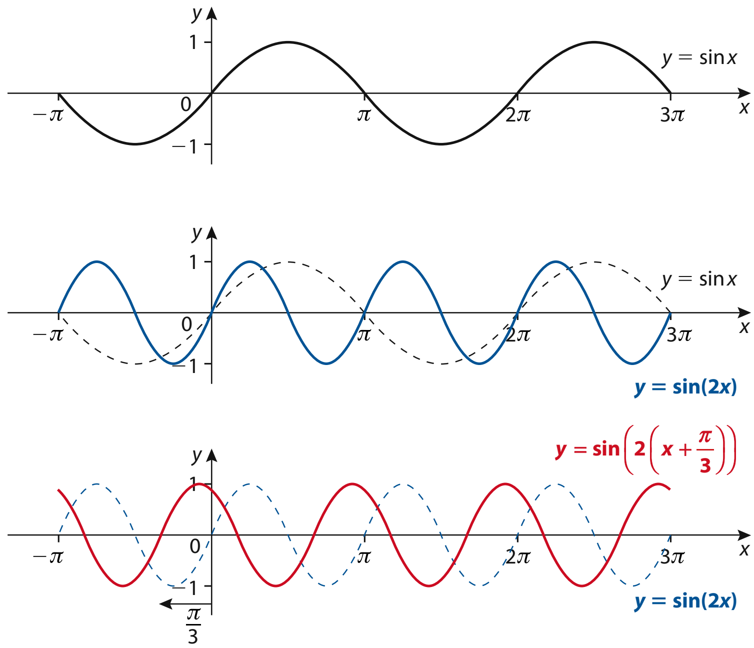
Sketch the function $f(x) = \sin\left(2x + \frac{2\pi}{3}\right)$.

Solution

To determine how to transform the graph of $y = \sin x$ to obtain the graph of $y = \sin\left(2x + \frac{2\pi}{3}\right)$, we need to make sure the function is written in the form $f(x) = a\sin[b(x + c)] + d$. Clearly, $a = 1$ and $d = 0$, but we will need to factorize a 2 from the expression $2x + \frac{2\pi}{3}$ to get $f(x) = \sin\left[2\left(x + \frac{\pi}{3}\right)\right]$. According to our general transformations from Chapter 2, we expect that the graph of f is obtained by first performing a horizontal shrinking of factor $\frac{1}{2}$ to the graph of $y = \sin x$ and then a translation to the left $\frac{\pi}{3}$ units (see Section 2.4).

Transformations of the graphs of trigonometric functions follow the same rules as for other functions. The rules were established in Section 2.4 and summarized on page 84.

The graphs on the next page illustrate the two-stage sequence of transforming $y = \sin x$ to $y = \sin\left[2\left(x + \frac{\pi}{3}\right)\right]$.



Note: A horizontal translation of a sine or cosine curve is often referred to as a **phase shift**. The equations $y = \sin\left(x + \frac{\pi}{3}\right)$ and $y = \sin\left[2\left(x + \frac{\pi}{3}\right)\right]$ both underwent a phase shift of $-\frac{\pi}{3}$.

Period and horizontal translation (phase shift) of sine and cosine functions

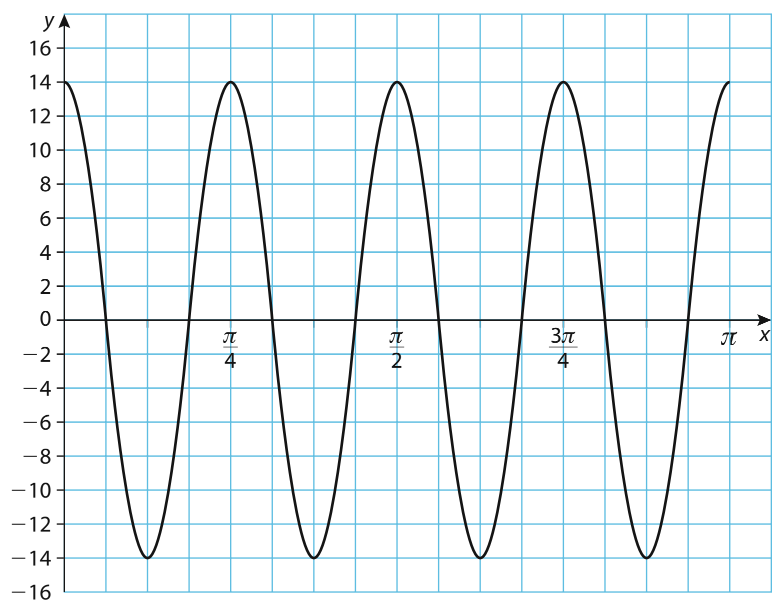
Given that b is a positive real number, $y = a\sin[b(x + c)] + d$ and $y = a\cos[b(x + c)] + d$ have a **period** of $\frac{2\pi}{b}$ and a horizontal translation (**phase shift**) of $-c$.

Example 12

The graph of a function in the form $y = a\cos bx$ is given in the diagram right.

a) Write down the value of a.

b) Calculate the value of b.



Solution

a) The amplitude of the graph is 14. Therefore, $a = 14$.

b) From inspecting the graph we can see that the period is $\frac{\pi}{4}$.

$$\text{Period} = \frac{2\pi}{b} = \frac{\pi}{4}$$

$$b\pi = 8\pi \Rightarrow b = 8.$$

Example 13

For the function $f(x) = 2\cos\left(\frac{x}{2}\right) - \frac{3}{2}$:

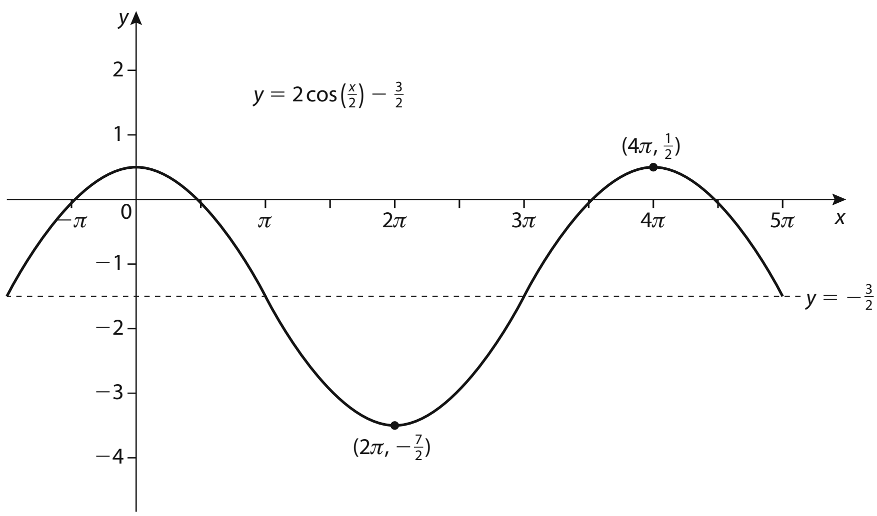
a) Sketch the function for the interval $-\pi \leqslant x \leqslant 5\pi$. Write down its amplitude and period.

b) Determine the domain and range for $f(x)$.

c) Write $f(x)$ as a trigonometric function in terms of sine rather than cosine.

Solution

a) $a = 2 \Rightarrow$ amplitude $= 2$; $b = \frac{1}{2} \Rightarrow$ period $= \dfrac{2\pi}{\frac{1}{2}} = 4\pi$. To obtain the graph of $y = 2\cos\left(\frac{x}{2}\right) - \frac{3}{2}$, we perform the following transformations on $y = \cos x$: (i) a horizontal stretch by factor $\dfrac{1}{\frac{1}{2}} = 2$, (ii) a vertical stretch by factor 2, and (iii) a vertical translation down $\frac{3}{2}$ units.



b) The domain is all real numbers. The function will reach a maximum value of $d + a = -\frac{3}{2} + 2 = \frac{1}{2}$, and a minimum value of $d - a = -\frac{3}{2} - 2 = -\frac{7}{2}$.

Hence, the range is $-\frac{7}{2} \leqslant y \leqslant \frac{1}{2}$.

c) The graph of $y = \cos x$ can be obtained by translating the graph of $y = \sin x$ to the left $\frac{\pi}{2}$ units. Thus, $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, or, in other words, any cosine function can be written as a sine function with a phase shift $= -\frac{\pi}{2}$. Therefore, $f(x) = 2\cos\left(\frac{x}{2}\right) - \frac{3}{2} = 2\sin\left(\frac{x}{2} + \frac{\pi}{2}\right) - \frac{3}{2}$.

Horizontal translation (phase shift) identities

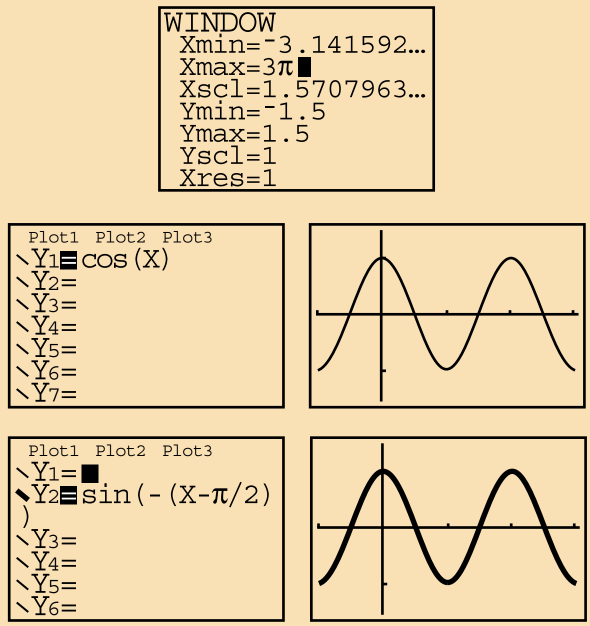
The following are true for all values of x:

$$\cos x = \sin\left(x + \frac{\pi}{2}\right) \qquad \sin x = \cos\left(x - \frac{\pi}{2}\right)$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \qquad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

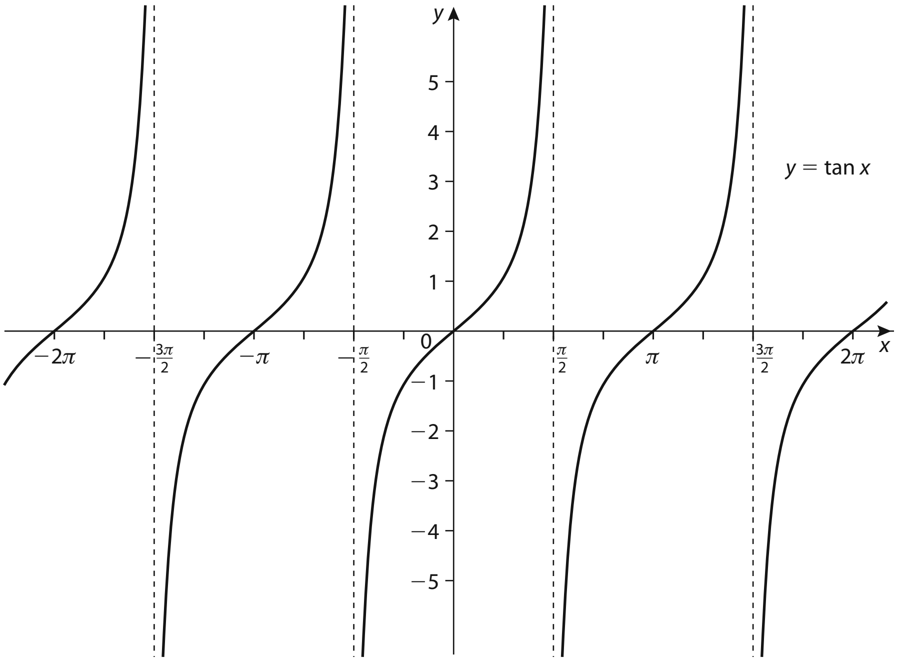
The identity $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ is equivalent to the identity $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ because $\sin\left(\frac{\pi}{2} - x\right) = \sin\left[-\left(x - \frac{\pi}{2}\right)\right]$ and the graph of $y = \sin\left[-\left(x - \frac{\pi}{2}\right)\right]$ can be obtained by first translating $y = \sin x$ to the right $\frac{\pi}{2}$ units, and then reflecting the graph in the y-axis. This produces the same graph as $y = \cos x$. This can be confirmed nicely on your GDC as shown. Therefore, $\cos x = \sin\left(\frac{\pi}{2} - x\right)$. In fact, it is also true that $\sin x = \cos\left(\frac{\pi}{2} - x\right)$. Clearly, $x + \left(\frac{\pi}{2} - x\right) = \frac{\pi}{2}$. If the domain ($x$) values were being treated as angles, then x and $\frac{\pi}{2} - x$ would be complementary angles.

This is why cosine is considered the co-function of sine. Two trigonometric functions f and g are co-functions if the following are true for all x: $f(x) = g\left(\frac{\pi}{2} - x\right)$ and $f\left(\frac{\pi}{2} - x\right) = g(x)$.



Graph of the tangent function

From work done earlier in this chapter, we expect that the behaviour of the tangent function will be significantly different from that of the sine and cosine functions. In Section 7.2, we concluded that the function $f(x) = \tan x$ has a domain of all real numbers such that $x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$, and that its range is all real numbers. Also, the results for Example 6 in Section 7.2 led us to speculate that the period of the tangent function is π. This makes sense since the identity $\tan x = \frac{\sin x}{\cos x}$ informs us that $\tan x$ will be zero whenever $\sin x = 0$, which occurs at values of x that differ by π (visualize arcs on the unit circle whose terminal points are either $(1, 0)$ or $(-1, 0)$). The values of x for which $\cos x = 0$ cause $\tan x$ to be undefined ('gaps' in the domain) also differ by π (the points $(0, 1)$ or $(0, -1)$ on the unit circle). As x approaches these values where $\cos x = 0$, the value of $\tan x$ will become very large – either very large negative or very large positive. Thus, the graph of $y = \tan x$ has vertical asymptotes at $x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$. Consequently, the graphical behaviour of the tangent function will not be a wave pattern such as that produced by the sine and cosine functions, but rather a series of separate curves that repeat every π units. Figure 7.25 shows the graph of $y = \tan x$ for $-2\pi \leqslant x \leqslant 2\pi$.



The graph gives clear confirmation that the period of the tangent function is π, that is, $\tan x = \tan(x + k \cdot \pi)$, $k \in \mathbb{Z}$.

The graph of $y = \tan x$ has rotational symmetry about the origin – that is, it can be rotated one-half of a revolution about (0, 0) and it remains the same. Hence, like the sine function, tangent is an odd function and $\tan(-x) = -\tan x$.

Figure 7.25 $y = \tan x$ for $-2\pi \leqslant x \leqslant 2\pi$.

Although the graph of $y = \tan x$ can undergo a vertical stretch or shrink, it is meaningless to consider its amplitude since the tangent function has no maximum or minimum value. However, other transformations can affect the period of the tangent function.

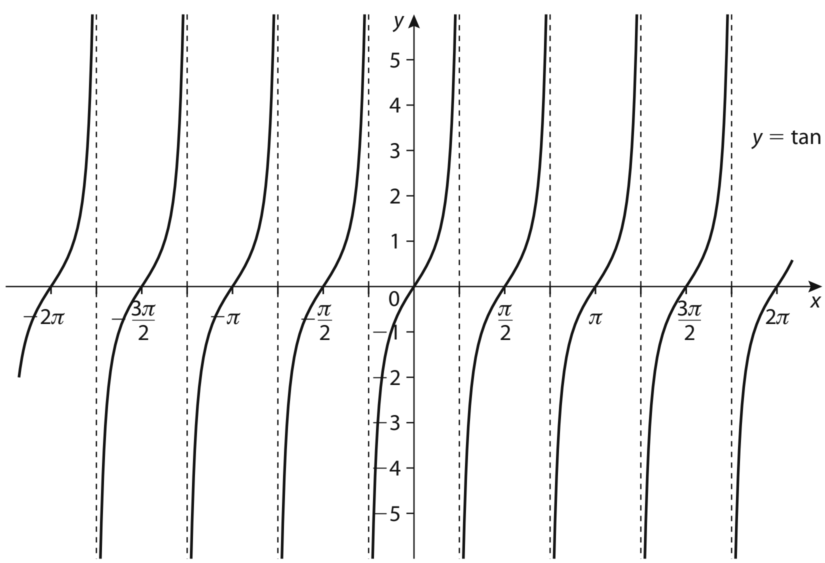
Example 14

Sketch each function.

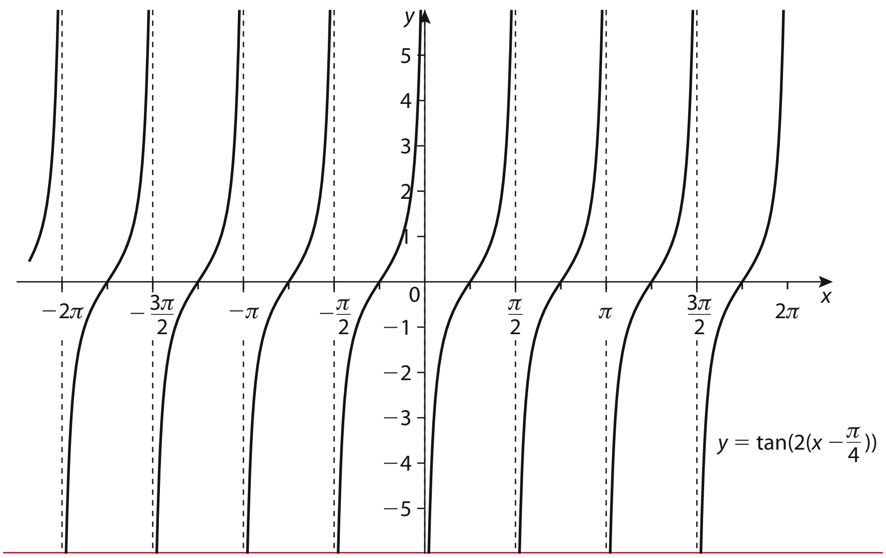
a) $f(x) = \tan 2x$ b) $g(x) = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$

Solution

a) An equation in the form $y = f(bx)$ indicates a horizontal shrinking of $f(x)$ by a factor of $\frac{1}{b}$. Hence, the period of $y = \tan 2x$ is $\frac{1}{2} \cdot \pi = \frac{\pi}{2}$.



b) The graph of $y = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$ is obtained by first performing a horizontal shrinking of the graph of $y = \tan x$ by a factor of $\frac{1}{2}$ and then translating the graph to the right $\frac{\pi}{4}$ units. As for $f(x) = \tan 2x$ in part a), the period of $g(x) = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$ is $\frac{\pi}{2}$.



Exercise 7.3

In questions 1–9, without using your GDC, sketch a graph of each equation on the interval $-\pi \leqslant x \leqslant 3\pi$.

1 $y = 2\sin x$

2 $y = \cos x - 2$

3 $y = \frac{1}{2}\cos x$

4 $y = \sin\left(x - \frac{\pi}{2}\right)$

5 $y = \cos(2x)$

6 $y = 1 + \tan x$

7 $y = \sin\left(\frac{x}{2}\right)$

8 $y = \tan\left(x + \frac{\pi}{2}\right)$

9 $y = \cos\left(2x - \frac{\pi}{4}\right)$

For each function in questions 10–12:

a) Sketch the function for the interval $-\pi \leqslant x \leqslant 5\pi$. Write down its amplitude and period.

b) Determine the domain and range for $f(x)$.

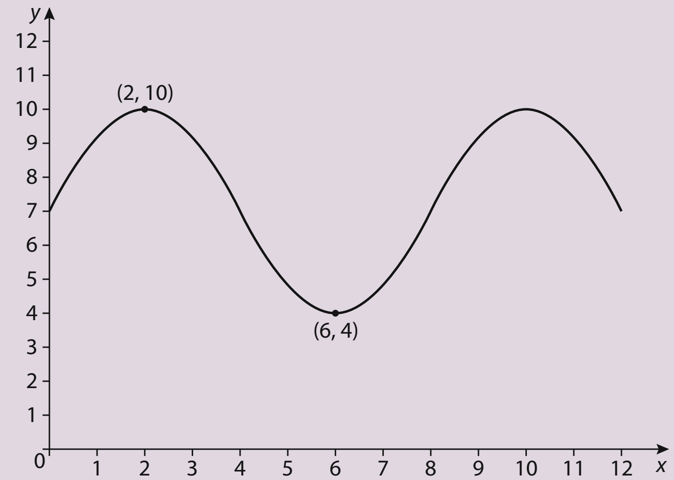
10 $f(x) = \frac{1}{2}\cos x - 3$

11 $g(x) = 3\sin(3x) - \frac{1}{2}$

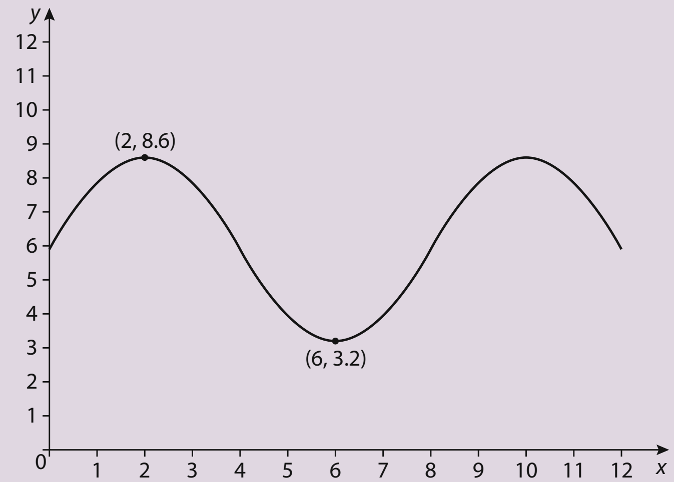
12 $g(x) = 1.2\sin\left(\frac{x}{2}\right) + 4.3$

In questions 13 and 14, a graph of a trigonometric equation is shown, on the interval $0 \leqslant x \leqslant 12$, that can be written in the form $y = A\sin\left(\frac{\pi}{4}x\right) + B$. Two points – one a minimum and the other a maximum – are indicated on the graph. Find the value of A and B for each.

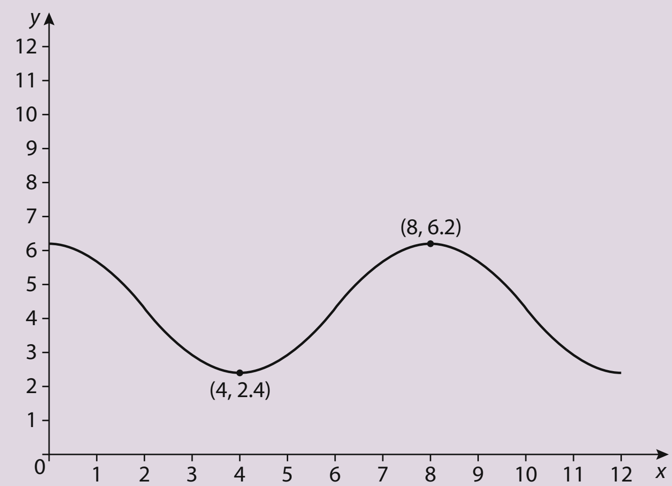
13



14



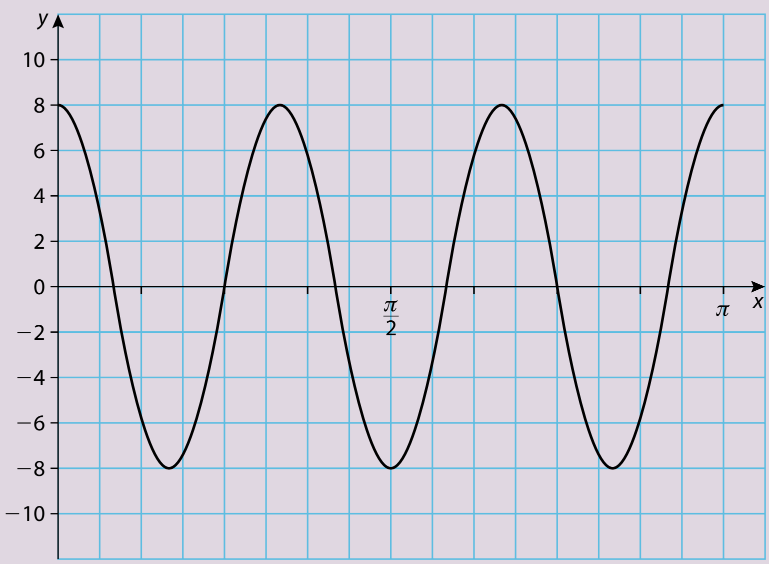
15 A graph of a trigonometric equation is shown below, on the interval $0 \leqslant x \leqslant 12$, that can be written in the form $y = A\cos\left(\frac{\pi}{4}x\right) + B$. Two points – one a minimum and the other a maximum – are indicated on the graph. Find the value of A and of B for each.



16 The graph of a function in the form $y = p\cos qx$ is given in the diagram below.

a) Write down the value of p.

b) Calculate the value of q.



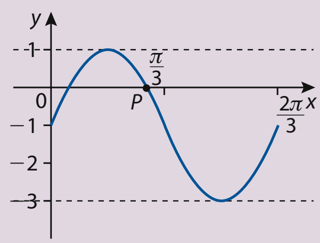
17 a) With help from your GDC, sketch the graphs of the three reciprocal trigonometric functions $y = \csc x$, $y = \sec x$ and $y = \cot x$ for the interval $0 \leqslant x \leqslant 2\pi$. Include any vertical asymptotes as dashed lines.

b) The domain of all of the trigonometric functions is stated in Section 7.2. State the range for each of the three reciprocal trigonometric functions.

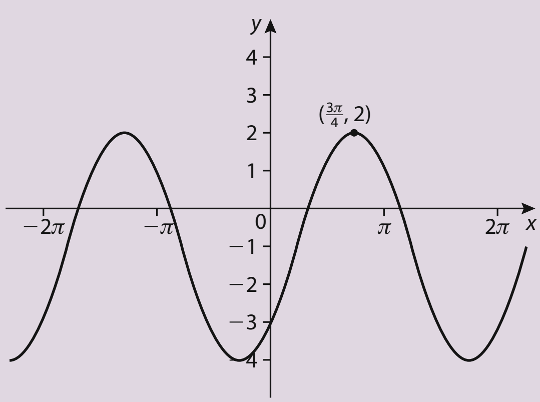
18 The diagram shows part of the graph of a function whose equation is in the form $y = a\sin(bx) + c$.

a) Write down the values of a, b and c.

b) Find the exact value of the x-coordinate of the point P, the point where the graph crosses the x-axis as shown in the diagram.



19 The graph below represents $y = a\sin(x + b) + c$, where a, b, and c are constants. Find values for a, b, and c.



7.4 Trigonometric equations

The primary focus of this section is to give an overview of concepts and strategies for solving **trigonometric equations**. In general, we will look at finding solutions by means of applying algebraic techniques (analytic solution) and/or by analyzing a graph (graphical solution). The following are all examples of trigonometric equations:

$$\csc x = 2, \sin^2\theta + \cos^2\theta = 1,\ 2\cos(3x - \pi) = 1,$$
$$\sec^2\alpha - 2\tan\alpha - 4 = 0, \tan 2\theta = \frac{2\tan}{1 - \tan^2\theta}$$

The mathematical symbol $\equiv$ is used to indicate that an equation has the special property of being an **identity**. It is not consistently used. You will notice that it is not used in the identities listed in the IB Information (Formulae) Booklet for Mathematics HL. The trigonometric identities required for this course are covered in the next section of this chapter.

The equations $\sin^2\theta + \cos^2\theta = 1$ and $\tan 2\theta = \frac{2\tan}{1 - \tan^2\theta}$ are examples of special equations called **identities** (Section 7.5). As we learned in Section 1.6, an identity is an equation that is true for all possible values of the variable. The other equations are true for only certain values or for none. Trigonometric identities will be covered thoroughly in the next section. They will prove to be an indispensable tool for obtaining analytic solutions to certain trigonometric equations. In this chapter, however, we will be applying methods similar to that used to solve equations encountered earlier in this book

The unit circle and exact solutions to trigonometric equations

When you are asked to solve a trigonometric equation, there are two important questions you need to consider:

1. Is it possible, or required, to express any solution(s) exactly?
2. For what interval of the variable are all solutions to be found?

With regard to the first question, exact solutions are only attainable, in most cases, if they are an integer multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$. Although we are primarily interested in finding numerical solutions (rather than angles in degrees), the language of angles is convenient. Recall from the first section of this chapter that if angles are given using radian measure, then angles between 0 and $\frac{\pi}{2}$ have their terminal sides in quadrant I, angles between $\frac{\pi}{2}$ and π have their terminal sides in quadrant II, and so on. Consequently, we will sometimes refer to a solution of an equation being, for example, a 'number in quadrant I', meaning a number that can be interpreted as either the length of an arc on the unit circle or a central angle in radian measure between 0 and $\frac{\pi}{2}$. As explained in Section 7.2, trigonometric domain values that are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$ commonly occur and it is important to be familiar with the exact trigonometric function values for these numbers (Table 7.1).

Concerning the second question, for most trigonometric equations there are infinitely many solutions. For example, the solutions to the equation

$\sin x = \frac{1}{2}$ are any number (arc or central angle) in quadrants I or II positioned so that the terminal point on the unit circle has a y-coordinate of $\frac{1}{2}$ (Figure 7.26). There are an infinite set of numbers that do this, being $\frac{\pi}{6}$ plus any multiple of 2π (quadrant I) or $\frac{5\pi}{6}$ plus any multiple of 2π (quadrant II). This infinite set is concisely written as $x = \frac{\pi}{6} + k \cdot 2\pi$ or $x = \frac{5\pi}{6} + k \cdot 2\pi$, $k \in \mathbb{Z}$. However, for this course the number of solutions to any trigonometric equation will be limited to a finite set by the fact that the solution set will always be restricted to a specified interval. For the equation $\sin x = \frac{1}{2}$, if the solution set is restricted to the interval $0 \leqslant x < 2\pi$, then the solutions are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. If the solution set is restricted to the interval $-2\pi < x < 2\pi$, then the solutions are $-\frac{11\pi}{6}$, $-\frac{7\pi}{6}$, $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. If the solution set is restricted to the interval $0 \leqslant x < 4\pi$, then the solutions are $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{13\pi}{6}$ and $\frac{17\pi}{6}$. Figure 7.27 illustrates how the graph of $y = \sin x$ can be used to locate the solutions for the equation $\sin x = \frac{1}{2}$ for different intervals of x. When asked to solve a trigonometric equation, a solution interval will always be given, as in the example below.

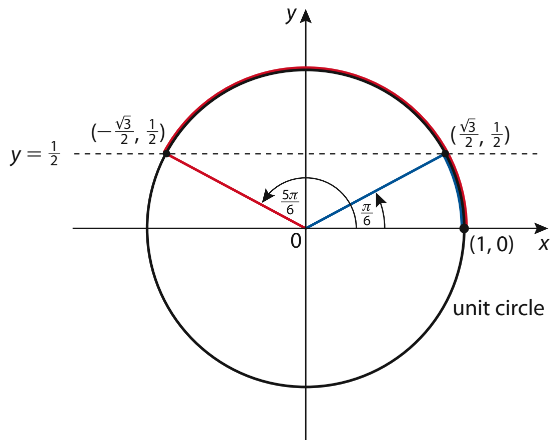


Figure 7.26 Solution to $\sin x = \frac{1}{2}$, $0 \leqslant x < 2\pi$.

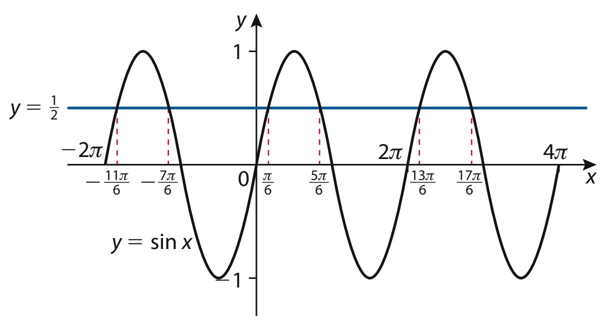


Figure 7.27 Points of intersection between $y = \sin x$ and $y = \frac{1}{2}$.

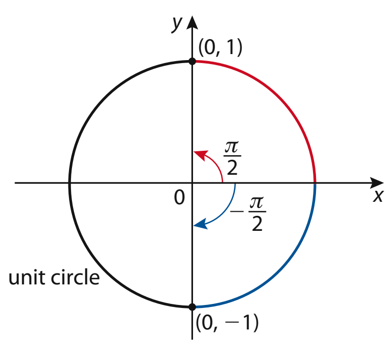
- **Hint:** As explained here, if the solution set for the equation $\sin x = \frac{1}{2}$ is not restricted, then the **general solution** is $x = \frac{\pi}{6} + k \cdot 2\pi$ or $x = \frac{5\pi}{6} + k \cdot 2\pi$, $k \in \mathbb{Z}$. This infinite solution corresponds to all of the points of intersection between the graphs of $y = \sin x$ and $y = \frac{1}{2}$ as they will repeatedly intersect as the graphs extend indefinitely in both directions (Figure 7.27). It is recommended that you are familiar with how to use a parameter (k in this case) to write the general solution for an equation with an infinite solution set, though it is not required for this course.

Example 15

Find the exact solution(s) to the equation $\sin x \cos x = 2\cos x$ for $-\pi < x < \pi$.

Solution

There is a temptation to divide both sides by $\cos x$, but as pointed out in Section 3.5, this can result in losing a solution to the equation. In fact, for this equation, both solutions would be lost. Instead, set the equation equal to zero and factorize out the common factor of $\cos x$.

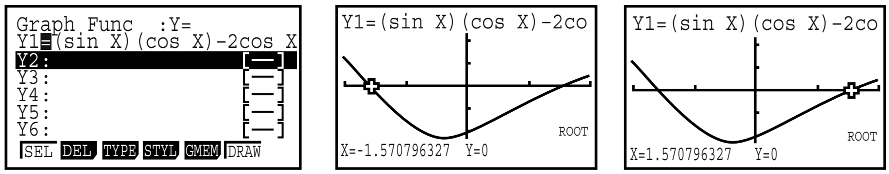


$$\sin x \cos x - 2\cos x = 0$$
$$\cos x(\sin x - 2) = 0$$
$$\cos x = 0 \quad \text{or} \quad \sin x = 2$$

2 is outside the range of the sine function so there is no solution to $\sin x = 2$. Solutions to $\cos x = 0$ occur for arcs (angles) that terminate where the x-coordinate is 0. For the solution interval $-\pi < x < \pi$, this

occurs where the unit circle intersects the y-axis as shown in the diagram. Therefore this analytic solution gives the exact solutions of $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$.

Your GDC can be a very effective tool for searching for solutions graphically. However, it can be limited when exact solutions are requested. The sequence of GDC images below show a graphical solution for the equation in Example 15.



The GDC gives the two solutions in the interval $-\pi < x < \pi$ as $x = -1.570\,796\,327$ and $x = 1.570\,796\,327$. These values are approximations (to 10 significant figures) of the irrational numbers, $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$, and confirms that they are the correct solutions. If exact solutions are required then you need to first attempt an analytic solution, and then a graphical confirmation can be performed.

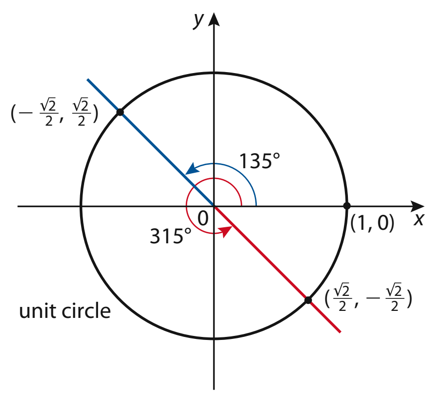
Example 16

Find the exact solution(s) to the equation $\tan(\theta) + 1 = 0$ for $0 \leqslant x < 360°$.

● **Hint:** The expression $\tan x + 1$ is not equivalent to $\tan(x + 1)$. In the first expression, x alone is the argument of the function, and in the second expression, $x + 1$ is the argument of the function. It is a good habit to use brackets to make it absolutely clear what is, or is not, the argument of a function. For example, there is no ambiguity if $\tan x + 1$ is written as $\tan(x) + 1$, or as $1 + \tan x$.

Solution

Since the solution interval is expressed in terms of degrees, it is necessary to give any solution as an angle in degree measure. Solutions to this equation are values of θ such that $\tan\theta = -1$. Applying the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we have $\frac{\sin\theta}{\cos\theta} = -1$. We need to find any angles θ such that $\sin\theta$ and $\cos\theta$ have opposite signs. This occurs in quadrant II at $\theta = 135°$ and in quadrant IV at $\theta = 315°$ as shown in the diagram.



It is possible to arrive at exact answers that are not multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$, as the next example illustrates.

Example 17

Find the exact solution(s) to the equation $\cos^2\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$ for $0 \leqslant x < 2\pi$.

Solution

The expression $\cos^2\left(x - \frac{\pi}{3}\right)$ can also be written as $\left[\cos\left(x - \frac{\pi}{3}\right)\right]^2$. The first step is to take the square root of both sides – remembering that every positive number has two square roots – which gives $\cos\left(x - \frac{\pi}{3}\right) = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$. All of the odd integer multiples of $\frac{\pi}{4}\left(\ldots -\frac{3\pi}{4}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{3\pi}{4}, \ldots\right)$ have a cosine equal to either $\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$.

That is, $x - \frac{\pi}{3} = \frac{\pi}{4} + k \cdot \frac{\pi}{2}$. Now, solve for x.

$x = \frac{\pi}{3} + \frac{\pi}{4} + k \cdot \frac{\pi}{2} = \frac{7\pi}{12} + k \cdot \frac{6\pi}{12}$. The last step is to substitute in different integer values for k to generate all the possible values for x so that $0 \leqslant x < 2\pi$.

When $k = 0$: $x = \frac{7\pi}{12}$; when $k = 1$: $x = \frac{7\pi}{12} + \frac{6\pi}{12} = \frac{13\pi}{12}$;

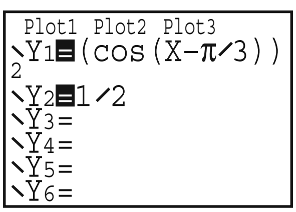
when $k = 2$: $x = \frac{7\pi}{12} + \frac{12\pi}{12} = \frac{19\pi}{12}$;

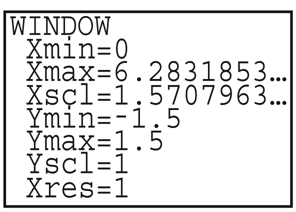
when $k = 3$: $x = \frac{7\pi}{12} + \frac{18\pi}{12} = \frac{25\pi}{12}$; however, $\frac{25\pi}{12} > 2\pi \ldots$ but,

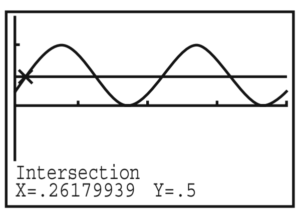
when $k = -1$: $x = \frac{7\pi}{12} - \frac{6\pi}{12} = \frac{\pi}{12}$.

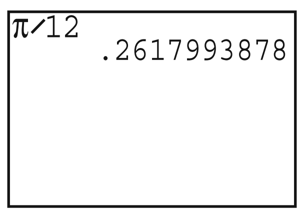
Therefore, there are four exact solutions in the interval $0 \leqslant x < 2\pi$, and they are: $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$ or $\frac{19\pi}{12}$.

● **Hint:** As we did at the end of Example 15, check the solutions to trigonometric equations with your GDC. The sequence of GDC images here verifies that $x = \frac{\pi}{12}$ is the first solution to the equation in Example 17.









When entering the equation $x = \cos^2\left(x - \frac{\pi}{3}\right)$ into your GDC (as shown in the first GDC image), you will have to enter it in the form $y = \left[\cos\left(x - \frac{\pi}{3}\right)\right]^2$. Be aware that $\cos^2\left(x - \frac{\pi}{3}\right)$ is not equivalent to $\cos\left(x - \frac{\pi}{3}\right)^2$. The expression $\cos\left(x - \frac{\pi}{3}\right)^2$ indicates that the quantity $x - \frac{\pi}{3}$ is squared first and then the cosine of the resulting value is found. However, the expression $y = \cos\left(x - \frac{\pi}{3}\right)$. indicates that the cosine of $x - \frac{\pi}{3}$ is found first and then that value is squared.

Graphical solutions to trigonometric equations

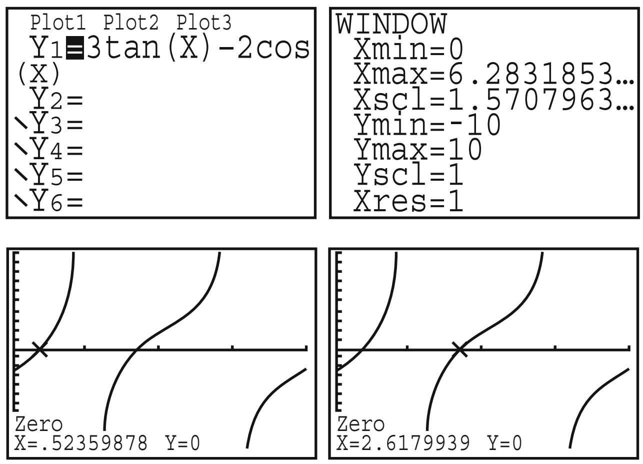
If exact solutions are not required then a graphical solution using your GDC is a very effective way to find approximate solutions to trigonometric equations. Unless instructed to do otherwise, you should give approximate solutions to an accuracy of three significant figures.

Example 18

Find all solutions to the equation $3\tan x = 2\cos x$ in the interval $0 \leqslant x < 2\pi$.

Solution

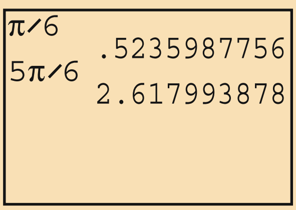
Graph the equation $y = 3\tan x - 2\cos x$ and find all of its zeros (x-intercepts) in the interval $0 \leqslant x < 2\pi$. Because the domain of the tangent function is $\left\{x : x \in \mathbb{R}, x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$, then we expect there to be 'gaps' (and vertical asymptotes) in the graph at $x = \frac{\pi}{2}$ and at $x = \frac{3\pi}{2}$.



This sequence of GDC images indicates approximate solutions of $x \approx 0.524$ and $x \approx 2.62$ to an accuracy of three significant figures.

It is possible to solve the equation in Example 18 analytically. See Exercise 7.4, question 30.

The exact solutions are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. The GDC image shows their approximate values agree with the solutions found in the example.



A graphical approach is effective and appropriate when it is very difficult, or not possible, to find exact solutions.

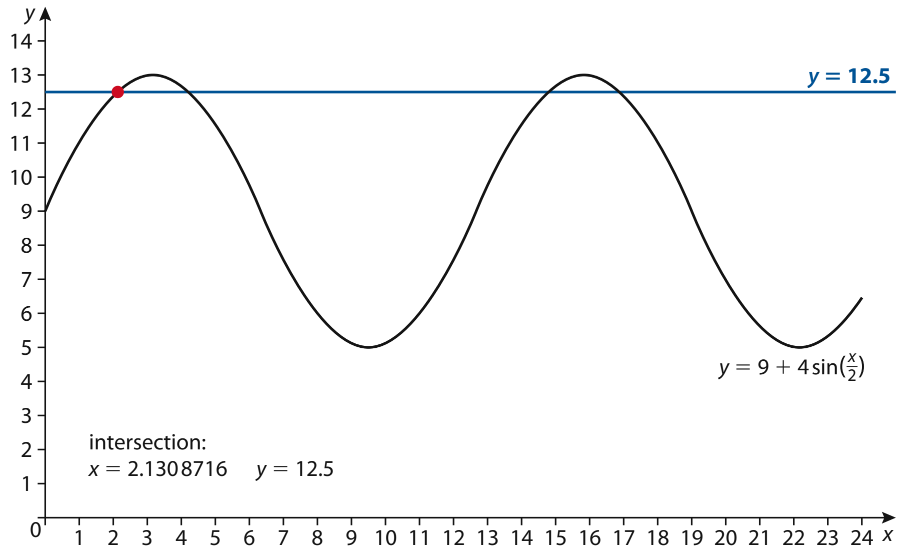
Example 19

The peak height, h metres, of ocean waves during a storm is given by the equation $h = 9 + 4\sin\left(\frac{t}{2}\right)$, where t is the number of hours after midnight.

A tsunami alarm is triggered when the peak height goes above 12.5 metres. Find the value of t when the alarm first sounds.

Solution

Graph the equations $y = 9 + 4\sin\left(\frac{x}{2}\right)$ and $y = 12.5$ and find the first point of intersection for $x > 0$.



Using the Intersect command on the GDC indicates that the first point of intersection has an x-coordinate of approximately 2.13. Therefore, the alarm will first sound when $t \approx 2.13$ hours.

Analytic solutions to trigonometric equations

An analytical approach requires you to devise a solution strategy utilizing algebraic methods that you have applied to other types of equations – such as quadratic equations. Trigonometric equations that demand an analytic approach will often, but not always, result in exact solutions. Although our approach for equations in this section focuses on algebraic techniques, it is important to use graphical methods to support or confirm our analytical solutions.

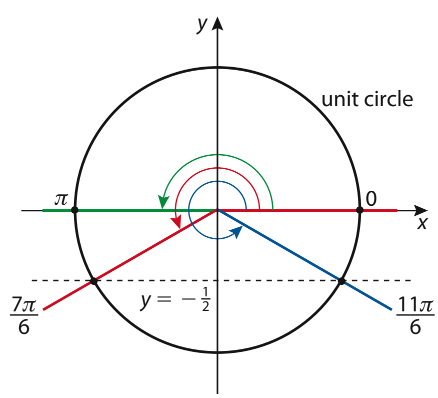
Example 20

Solve $2\sin^2 x + \sin x = 0$ for $0 \leqslant x < 2\pi$.

Solution

Factorizing gives $\quad \sin x(2\sin x + 1) = 0$

$$\sin x = 0 \text{ or } \sin x = -\tfrac{1}{2}$$

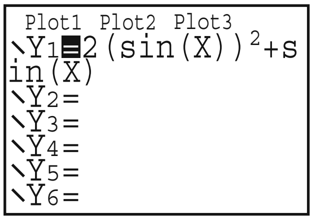


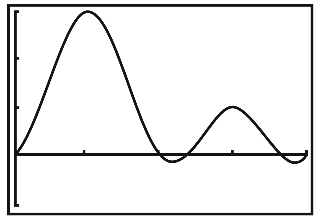
Solutions to $\sin x = 0$ are where the angle is on the x-axis; and solutions to $\sin x = -\frac{1}{2}$ are angles in quadrant III and IV such that their intersection point with the unit circle has y-coordinate of $-\frac{1}{2}$.

$$\text{for } \sin x = 0\text{: } x = 0, \pi \qquad \text{for } \sin x = -\tfrac{1}{2}\text{: } x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Therefore, the solutions are $x = 0, \pi, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$.

● **Hint:** Although exact answers were not demanded in Example 20, given our knowledge of the unit circle and familiarity with the sine of common values $\left(\text{i.e. multiples of } \frac{\pi}{6} \text{ and } \frac{\pi}{4}\right)$, we are able to give exact answers without any difficulty. It would have been acceptable to give approximate solutions using your GDC, but it is worth recognizing that this would have required considerable more effort than providing exact solutions. Entering and graphing the equation $y = 2\sin^2 x + \sin x$ on your GDC (see GDC images) would not be the most efficient or appropriate solution method, but if sufficient time is available it is an effective way to confirm your exact solutions. [Note that $\sin^2 x$ must be entered in a GDC as $(\sin x)^2$.]





● **Hint:** As we will see in the next section, it is often the case that an analytic solution is not possible unless a substitution is made using a suitable trigonometric identity.

The next example illustrates how the application of a trigonometric identity can be helpful to rewrite the equation in a way that allows us to solve it algebraically. The next section will introduce many further trigonometric identities and examples of using them to assist in solving trigonometric equations.

Example 21

Solve $3\cos x + \cot x = 0$ for $0 \leqslant x \leqslant 2\pi$.

Solution

Since the structure of this equation is such that an expression is set equal to zero, it would be nice to be able to use the same algebraic technique as the previous example – that is, factorize and solve for when each factor is zero. However, it is not possible to factorize the expression $3\cos x + \cot x$, and rewriting the equation as $3\cos x = -\cot x$ does not help. Are there any expressions in the equation for which we can substitute an equivalent expression that will make the equation accessible to an algebraic solution? We do not have any equivalent expressions for $\cos x$, but we do have an identity for $\cot x$. Since $\cot x$ is the reciprocal of $\tan x$ we know that $\cot x = \dfrac{\cos x}{\sin x}$. Let's see what happens when we substitute $\dfrac{\cos x}{\sin x}$ for $\cot x$.

$$3\cos x + \frac{\cos x}{\sin x} = 0$$

Now, get a common denominator.

$$\frac{3\sin x\cos x}{\sin x} + \frac{\cos x}{\sin x} = 0$$

$$\frac{3\sin x\cos x + \cos x}{\sin x} = 0$$

Noting that $\sin x \neq 0$, multiply both sides by $\sin x$. A fraction equals zero when the denominator equals zero.

$$3\sin x\cos x + \cos x = 0$$

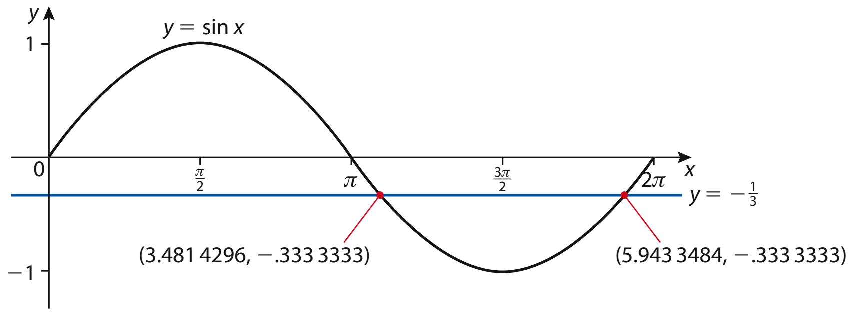
$$\cos x(3\sin x + 1) = 0$$

Factorize.

$$\cos x = 0 \quad \text{or} \quad \sin x = -\tfrac{1}{3}$$

For $\cos x = 0$: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

We know that solutions to $\cos x = 0$ are angles on the y-axis giving the two exact solutions of $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. Although we know solutions to $\sin x = -\frac{1}{3}$ are angles in quadrants III and IV, we do not know their exact values. So, we will need to use our GDC to find approximate solutions to $\sin x = -\frac{1}{3}$ for $0 \leqslant x \leqslant 2\pi$.



Thus, for $\sin x = -\frac{1}{3}$: $x \approx 3.48$ or $x \approx 5.94$ (3 significant figures)

Therefore, the full solution set for the equation is $x = \frac{\pi}{2}, \frac{3\pi}{2}$; $x \approx 3.48, 5.94$.

● **Hint:** A strategy that often proves fruitful is to try and rewrite a trigonometric equation in terms of just one trigonometric function. If that is not possible, then try and rewrite it in terms of only the sine and cosine functions. This strategy was used in Example 21.

Exercise 7.4

In questions 1–12, find the exact solution(s) for $0 \leqslant x < 2\pi$. Verify your solution(s) with your GDC.

1 $\cos x = \frac{1}{2}$

2 $2\sin x + 1 = 0$

3 $1 - \tan x = 0$

4 $\sqrt{3} = 2\sin x$

5 $2\sin^2 x = 1$

6 $4\cos^2 x = 3$

7 $\tan^2 x - 1 = 0$

8 $4\cos^2 x = 1$

9 $\tan x(\tan x + 1) = 0$

10 $\sin x \cos x = 0$

11 $5 - \sec x = 3$

12 $\csc^2 x = 2$

In questions 13–20, use your GDC to find approximate solution(s) for $0 \leqslant x < 2\pi$. Express solutions accurate to 3 significant figures.

13 $\sin x = 0.4$

14 $3\cos x + 1 = 0$

15 $\tan x = 2$

16 $\sec 2x = 3.46$

17 $\cos(x - 1) = -0.38$

18 $3\tan^2 x = 1$

19 $\csc(2x - 3) = \frac{3}{2}$

20 $3\cot x = 10$

In questions 21–24, given that k is any integer, list all of the possible values for x that are in the specified interval.

21 $\frac{\pi}{2} + k \cdot \pi$, $-3\pi \leqslant x \leqslant 3\pi$

22 $\frac{\pi}{6} + k \cdot 2\pi$, $-2\pi \leqslant x \leqslant 2\pi$

23 $\frac{7\pi}{12} + k \cdot \pi$, $0 \leqslant x < 2\pi$

24 $\frac{\pi}{4} + k \cdot \frac{\pi}{4}$, $0 \leqslant x < 4\pi$

In questions 25–32, find the **exact** solutions for the indicated interval. The interval will also indicate whether the solutions are given in degree or radian measure. Write a complete analytic solution.

25 $\cos\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}, 0 \leqslant x < 2\pi$

26 $\tan(\theta + \pi) = 1, -\pi \leqslant \theta \leqslant \pi$

27 $\sin 2x = \frac{\sqrt{3}}{2}, 0 \leqslant x < 360°$

28 $\sin^2\left(\alpha + \frac{\pi}{2}\right) = \frac{3}{4}, -\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}$

29 $2\cos^2\theta - 5\cos\theta - 3 = 0, 0 \leqslant \theta < 2\pi$

30 $3\tan x = 2\cos x, 0 \leqslant x < 2\pi$

31 $2\cos(x + 90°) = \sqrt{2}, 0 \leqslant x < 360°$

32 $9\sec^2\theta = 12, 0 \leqslant \theta < \pi$

33 The number, N, of empty birds' nests in a park is approximated by the function $N = 74 + 42\sin\left(\frac{\pi}{12}t\right)$, where t is the number of hours after midnight. Find the value of t when the number of empty nests first equals 90. Approximate the answer to 1 decimal place.

34 In Edinburgh, the number of hours of daylight on day D is modelled by the function $H = 12 + 7.26\sin\left[\frac{2\pi}{365}(D - 80)\right]$, where D is the number of days after December 31 (e.g. January 1 is $D = 1$, January 2 is $D = 2$, and so on). Do not use your GDC on part a).

a) Which days of the year have 12 hours of daylight?

b) Which days of the year have about 15 hours of daylight?

c) How many days of the year have more than 17 hours of daylight?

In questions 35–42, solve the equation for the stated solution interval. Find exact solutions when possible, otherwise give solutions to three significant figures. Verify solutions with your GDC.

35 $2\cos^2 x + \cos x = 0, 0 \leqslant x < 2\pi$

36 $2\sin^2\theta - \sin\theta - 1 = 0, 0 \leqslant \theta < 2\pi$

37 $\tan^2 x - \tan x = 2, -90° \leqslant x \leqslant 90°$

38 $3\cos^2 x - 6\cos x = 2, -\pi < x \leqslant \pi$

39 $2\sin\beta = 3\cos\beta, 0 \leqslant \beta \leqslant 180°$

40 $\sin^2 x = \cos^2 x, 0 \leqslant x \leqslant \pi$

41 $\sec^2 x + 2\sec x + 4 = 0, 0 \leqslant x < 2\pi$

42 $\sin x \tan x = 3\sin x, 0 \leqslant x < 360°$

7.5 Trigonometric identities

The **co-function identities** for sine and cosine were established in Section 7.3 by means of investigating horizontal shifts of graphs of the sine and cosine functions. Similarly we can prove co-function identities for secant and cosecant, and for tangent and cotangent. These appear in Table 7.2 on the next page.

You will recall that an identity is an equation that is true for all values of the variable for which the expressions in the equation are defined. Several trigonometric identities have been introduced earlier in this chapter. They are reviewed here (Table 7.2) and a number of important new identities are presented and proved in this section.

Trigonometric identities are used in a variety of ways. For example, one of the reciprocal identities is applied whenever the cosecant, secant or cotangent function is evaluated on a calculator. The following uses of trigonometric identities will be illustrated in this section.

1. Evaluate trigonometric functions.

2. Simplify trigonometric expressions.
3. Prove other trigonometric identities.
4. Solve trigonometric equations.

The first portion of this section is devoted to developing some further trigonometric identities that are organized into three groups: Pythagorean identities, compound angle identities, and double angle identities.

Table 7.2 Summary of fundamental trigonometric identities.

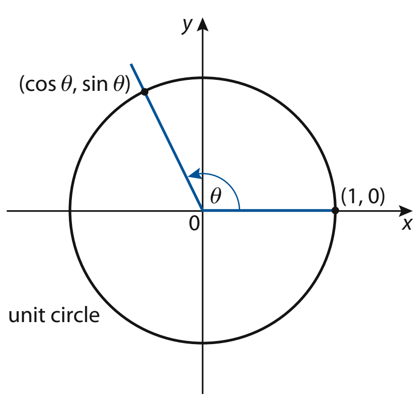
Reciprocal identities:		
$\csc x = \frac{1}{\sin x}$	$\sec x = \frac{1}{\cos x}$	$\cot x = \frac{1}{\tan x}$
Tangent and cotangent identities:		
$\tan x = \frac{\sin x}{\cos x}$	$\cot x = \frac{\cos x}{\sin x}$	
Odd/even function identities:		
$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$	$\tan(-x) = -\tan x$
$\csc(-x) = -\csc x$	$\sec(-x) = \cos x$	$\cot(-x) = -\tan x$
Co-function identities:		
$\sin\left(\frac{\pi}{2} - x\right) = \cos x$	$\sec\left(\frac{\pi}{2} - x\right) = \csc x$	$\tan\left(\frac{\pi}{2} - x\right) = \cot x$
$\cos\left(\frac{\pi}{2} - x\right) = \sin x$	$\csc\left(\frac{\pi}{2} - x\right) = \sec x$	$\cot\left(\frac{\pi}{2} - x\right) = \tan x$

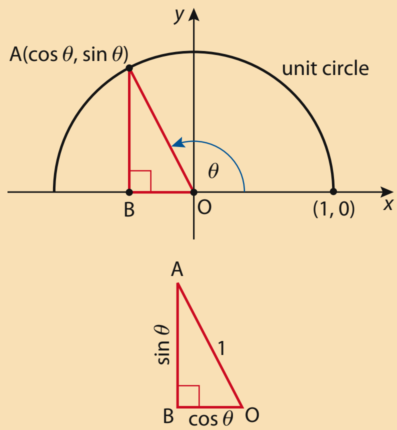
It was confirmed in Section 7.3 that sine and tangent are **odd functions** and that cosine is an **even function**. We will accept without proof that if a function is odd, then its reciprocal is also odd; and the same is true for even functions. Therefore, cosecant and cotangent are odd functions, and secant is an even function.

Pythagorean identities

At the start of the previous section, it was stated that the equation $\sin^2\theta + \cos^2\theta = 1$ is an identity; that is, it's true for all possible values of θ. Let's prove that this is the case.

Recall from Section 7.1 that the equation for the unit circle is $x^2 + y^2 = 1$. That is, the coordinates (x, y) of any point on the circle satisfy the equation $x^2 + y^2 = 1$. As we learned in Section 7.2, if θ is any real number that represents a central angle (in radian measure) of the unit circle that terminates at (x, y), then $x = \cos\theta$ and $y = \sin\theta$. Substituting directly into the equation for the circle gives $\sin^2\theta + \cos^2\theta = 1$. Therefore, the equation $\sin^2\theta + \cos^2\theta = 1$ is true for any real number x.





The identity $\sin^2\theta + \cos^2\theta = 1$ is referred to as a *Pythagorean* identity because it can be derived directly from Pythagoras' theorem. As Figure 7.28 illustrates, for any point angle θ with its terminal side intersecting the unit circle at point **A** (except for a point on the x- or y-axis), a perpendicular segment can be drawn to a point **B** on the y-axis thereby constructing right triangle **ABO**. Side **AB** is equal to $\sin\theta$ and side **OB** is equal to $\cos\theta$. The hypotenuse **AO** is a radius of the unit circle so its length is one. Hence, by Pythagoras' theorem: $\sin^2\theta + \cos^2\theta = 1$.

Figure 7.28

Hint: Graph the equation $y = \sin^2 x + \cos^2 x$ on your GDC with the y-axis ranging from -2 to 2 and the x-axis ranging from -2π to 2π (radian mode) or $-360°$ to $360°$ (degree mode). What do you observe?

Phrases such as 'prove the identity' and 'verify the identity' are often used. Both mean, 'prove that the given equation is an identity'. We do this by performing a series of algebraic manipulations to show that the expression on one side of the equation can be transformed into the expression on the other side, or that both expressions can be transformed into some third expression. When verifying that an equation is an identity, you should not perform an operation to both sides of the equation; for example, multiplying both sides of the equation by a quantity. This can only be done if it is known that the two sides of the equation are equal, but this is exactly what we are trying to verify in the process of 'proving an identity.'

Example 22

Prove that $1 + \tan^2\theta = \sec^2\theta$ is an identity.

Solution

There is more of an opportunity to perform algebraic manipulations on the left side than the right side. Thus, our task is to transform the expression $1 + \tan^2\theta$ into the expression $\sec^2\theta$.

$$1 + \tan^2\theta = \sec^2\theta$$
Using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ substitute $\frac{\sin^2\theta}{\cos^2\theta}$ for $\tan^2\theta$.

$$1 + \frac{\sin^2\theta}{\cos^2\theta} =$$
Find a common denominator.

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} =$$

$$\frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} =$$
Apply the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$.

$$\frac{1}{\cos^2\theta} =$$
Because $\frac{1}{\cos\theta} = \sec\theta$, then $\frac{1}{\cos^2\theta} = \sec^2\theta$.

$$\sec^2\theta = \sec^2\theta$$
Q.E.D.

Q.E.D. is an abbreviation for the Latin phrase '*quod erat demonstrandum*' which means 'that which was to be proved (or demonstrated)'. It is often written at the end of a proof to indicate that its conclusion has been reached.

Another identity than can be proved in a manner similar to the identity in Example 22 is $1 + \cot^2 \theta = \csc^2 \theta$.

Pythagorean identities

$\sin^2 \theta + \cos^2 \theta = 1 \qquad 1 + \tan^2 \theta = \sec^2 \theta \qquad 1 + \cot^2 \theta = \csc^2 \theta$

The Pythagorean identities are sometimes used in radical forms such as $\sin \theta = \pm\sqrt{1 - \cos^2 \theta}$ or $\tan \theta = \pm\sqrt{\sec^2 \theta - 1}$ where the sign (+ or −) depends on θ (which quadrant it is in).

Example 23

a) Express $2\cos^2 x + \sin x$ in terms of $\sin x$ only.

b) Solve the equation $2\cos^2 x + \sin x = -1$ for x in the interval $0 \leqslant x \leqslant 2\pi$, expressing your answer(s) exactly.

Solution

a) $2\cos^2 x + \sin x = 2(1 - \sin^2 x) + \sin x$ Using Pythagorean identity:
$= 2 - 2\sin^2 x + \sin x$ $\cos^2 x = 1 - \sin^2 x$.

b) $2\cos^2 x + \sin x = -1$

$2 - 2\sin^2 x + \sin x = -1$ Substitute result from a).

$2\sin^2 x - \sin x - 3 = 0$ (Alternatively: let $\sin x = y$, then $2y^2 - y - 3 = 0$)

$(2\sin x - 3)(\sin x + 1) = 0$ Factorize. (alt: $(2y - 3)(y + 1) = 0$)

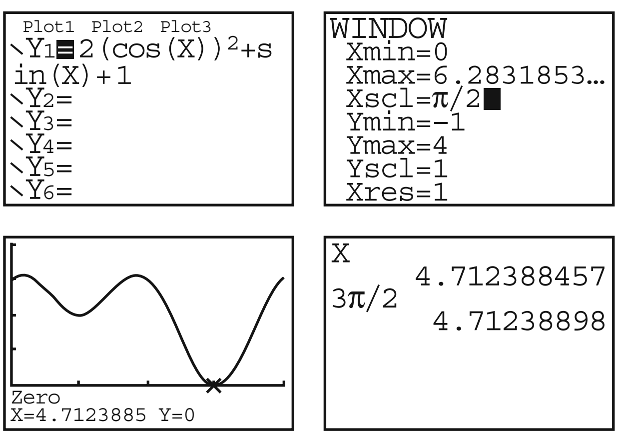
$\sin x = \frac{3}{2}$ or $\sin x = -1$ (Alt: $y = \frac{3}{2}$ or $y = -1 \Rightarrow \sin x = \frac{3}{2}$ or $\sin x = -1$)

For $x = \frac{3}{2}$: no solution because $\frac{3}{2}$ is not in the range of the sine function.

For $\sin x = -1$: $x = \dfrac{3\pi}{2}$.

Therefore, there is only one solution in $0 \leqslant x \leqslant 2\pi$: $x = \dfrac{3\pi}{2}$.

Use your GDC to check this result by rewriting $2\cos^2 x + \sin x = -1$ as $2\cos^2 x + \sin x + 1 = 0$ and then graph $y = 2\cos^2 x + \sin x + 1$; confirming a single zero at $x = \dfrac{3\pi}{2}$ in the interval $x \in [0, 2\pi]$.



Compound angle identities (sum and difference identities)

In this section we develop trigonometric identities known as the compound angle identities for sine, cosine and tangent. These contain the expressions $\sin(\alpha+\beta)$, $\sin(\alpha-\beta)$, $\cos(\alpha+\beta)$, $\cos(\alpha-\beta)$, $\tan(\alpha+\beta)$ and $\tan(\alpha-\beta)$. We first find a formula for $\cos(\alpha+\beta)$.

On first reaction you might wonder whether $\cos(\alpha+\beta)=\cos\alpha+\cos\beta$. Often it is easier to prove a mathematical statement false than to prove it true. One counter-example is sufficient to prove a statement false. Let $\alpha=\frac{\pi}{3}$ and $\beta=\frac{\pi}{6}$. Does $\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\cos\frac{\pi}{3}+\cos\frac{\pi}{6}$?

● **Hint:** As will occur in Chapter 8, Greek letters such as α (alpha), β (beta), or θ (theta) are frequently used to name angles. In the development of the formula for $\cos(\alpha+\beta)$, α and β are arcs along the unit circle, but they could just as well be representing the central angle (in radian measure) that cuts off (subtends) the arc.

$\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\cos\left(\frac{2\pi}{6}+\frac{\pi}{6}\right)=\left(\frac{3\pi}{6}\right)=\cos\left(\frac{\pi}{2}\right)=0$

and $\cos\frac{\pi}{3}+\cos\frac{\pi}{6}=\frac{1}{2}+\frac{\sqrt{3}}{2}=\frac{1+\sqrt{3}}{2}$.

Thus, the answer is 'no'; $\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\neq\cos\frac{\pi}{3}+\cos\frac{\pi}{6}$.

Although $\cos(\alpha+\beta)=\cos\alpha+\cos\beta$ may be true for some values (e.g. it's true for $\alpha=\frac{\pi}{2}$ and $\beta=\frac{3\pi}{4}$), it's not true for **all** possible values of α and β, and therefore, it is **not** an identity.

Derivation of identity for the cosine of the sum of two numbers

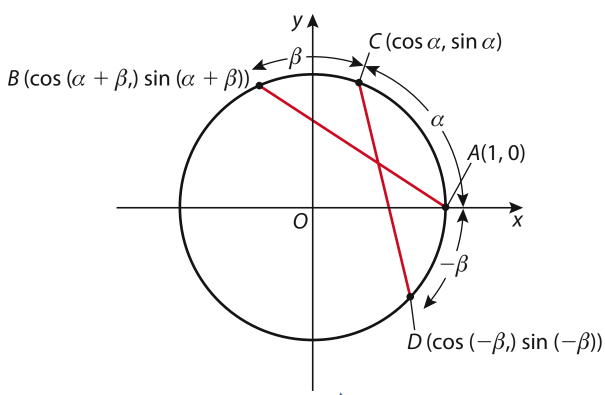


Figure 7.29

To find a formula for $\cos(\alpha+\beta)$, we use Figure 7.29 showing the four points A, B, C and D on the unit circle and the two chords AB and CD. The arc lengths α, β and $-\beta$ have been marked. The coordinates of A, B, C and D in terms of sines and cosines of the arcs are also indicated. The coordinates of point D are $(\cos(-\beta), \sin(-\beta))$, but we can apply the odd/even identities to write the coordinates of D more simply as $(\cos\beta, -\sin\beta)$. Observe that the arc length from A to B is equal to the arc length from D to C because they both have a length equal to $\alpha+\beta$. Since equal arcs on a circle determine equal chords, it must follow that $AB=CD$. Using the respective coordinates for A, B, C and D, we can express $AB=CD$ using the distance formula as

$$\sqrt{(\cos(\alpha+\beta)-1)^2+\sin^2(\alpha+\beta)}=\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha+\sin\beta)^2}$$

Squaring both sides and expanding, gives

$$\cos^2(\alpha+\beta)-2\cos(\alpha+\beta)+1+\sin^2(\alpha+\beta)$$
$$=\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha+2\sin\alpha\sin\beta+\sin^2\beta$$
$$[\cos^2(\alpha+\beta)+\sin^2(\alpha+\beta)]-2\cos(\alpha+\beta)+1$$
$$=(\cos^2\alpha+\sin^2\alpha)+(\sin^2\beta+\cos^2\beta)-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta$$

Applying the Pythagorean identity $\sin^2\theta+\cos^2\theta=1$, we can replace three expressions with 1:

$$1-2\cos(\alpha+\beta)+1=1+1-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta$$

Subtracting 2 from each side and dividing both sides by -2, gives

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

This is the **identity for the cosine of the sum of two numbers**.

Previously we were only able to find exact values of a trigonometric function for certain 'special' numbers, i.e. multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$.

Example 24 – Using the sum identity for cosine

Find the exact values for a) $\cos\frac{5\pi}{12}$, and b) $\cos 75°$.

Solution

a) $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$

Applying the identity $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$ with $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{6}$, gives $\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

Therefore, $\cos\frac{5\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$.

Derivation of identity for the cosine of the difference of two numbers

We can use the identity for the cosine of the sum of two numbers and the fact that cosine is an even function and sine is an odd function to derive the formula for $\cos(\alpha + \beta)$.

Let's replace β with $-\beta$ in $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

$$\cos[\alpha + (-\beta)] = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$$

Substituting $-\sin\beta$ for $\sin(-\beta)$, and $\cos\beta$ for $\cos(-\beta)$, gives

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

This is the **identity for the cosine of the difference of two numbers.**

Example 25 – Using the sum and difference identities for cosine

Given that A and B are numbers representing arcs or angles that are in the first quadrant, and $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, find the exact values of a) $\cos(A + B)$ and b) $\cos(A - B)$.

Solution

We are given the exact values for $\sin A$ and $\cos B$, but we also need exact values for $\sin B$ and $\cos A$ in order to use the sum and difference identities for cosine.

Since B is in the first quadrant then $B > 0$ and re-arranging one of the Pythagorean identities, we have

$$\sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{\frac{25}{169}} = \frac{5}{13}.$$

Similarly, $\cos A = \sqrt{1 - \sin^2 A} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}.$

a) Substituting into the identity for the cosine of the sum of two numbers, gives

$$\cos(A + B) = \cos A \cos B - \sin A \sin B = \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) = \frac{16}{65}.$$

Therefore, $\cos(A + B) = \frac{16}{65}$.

b) Substituting into the identity for the cosine of the difference of two numbers, gives

$$\cos(A - B) = \cos A \cos B + \sin A \sin B = \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{5}{13}\right) = \frac{56}{65}.$$

Therefore, $\cos(A - B) = \frac{56}{65}$.

• **Hint:** Notice that in Example 25, we obtained $\cos(A + B)$ and $\cos(A - B)$ without finding the actual values of A and B.

Derivation of identities for the sine of the sum/difference of two numbers

The identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ can be used to derive an identity for $\sin(\alpha + \beta)$. Substituting $\frac{\pi}{2}$ for α and $(\alpha + \beta)$ for β, gives

$$\cos\left[\frac{\pi}{2} - (\alpha + \beta)\right] = \cos\left[\left(\frac{\pi}{2} - \alpha\right) - \beta\right]$$
$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta$$

Now using the co-function identities $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ and $\sin\left(\frac{\pi}{2} - x\right) = \cos x$, we have,

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

This is the **identity for the sine of the sum of two numbers.**

By replacing β with $-\beta$, in the identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, we get

$$\sin(\alpha - \beta) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)$$

Applying the odd/even identities for $\cos(-\beta)$ and $\sin(-\beta)$, produces

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

This is the **identity for the sine of the difference of two numbers.**

Derivation of identities for the tangent of the sum/difference of two numbers

To produce an identity for $\sin(\alpha + \beta)$ in terms of $\tan\alpha$ and $\tan\beta$, we start with the fundamental identity that the tangent is the quotient of sine and cosine. We have

$$\tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \quad \text{given } \cos(\alpha + \beta) \neq 0$$
$$= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

So that the identity involves $\tan\alpha$ and $\tan\beta$, we divide the numerator and denominator by $\cos\alpha\cos\beta$, with the assumption that $\cos\alpha\cos\beta \neq 0$.

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

This is the **identity for the tangent of the sum of two numbers.**

If in this identity β is replaced with $-\beta$, we get

$$\tan[\alpha + (-\beta)] = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$$

Tangent is an odd function, so $\tan(-\beta) = -\tan\beta$. Making this substitution, gives

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

This is the **identity for the tangent of the difference of two numbers.**

Compound angle identities

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

● **Hint:** The compound angle identities are also referred to as the 'sum and difference identities', or the 'addition and subtraction identities'.

Example 26 – Using the sum identity for tangent

If $\tan(A+B) = \frac{1}{7}$ and $\tan A = 3$, find the value of $\tan B$.

Solution

Using the identity for the tangent of the sum of two numbers, we write

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Substituting $\frac{1}{7}$ for $\tan(A+B)$, and 3 for $\tan A$.

$$\frac{1}{7} = \frac{3 + \tan B}{1 - 3\tan B}$$

Cross-multiply and solve for $\tan B$.

$$21 + 7\tan B = 1 - 3\tan B$$

$$10\tan B = -20$$

$$\tan B = -2$$

Note that, similar to Example 25, we found the exact value of $\tan B$ without finding the actual value of B. In fact, we're not even certain which quadrant B is in, only that it must be in either quadrant II or IV since $\tan B < 0$.

Double angle identities

Is $\sin 2\theta = 2\sin\theta$ an identity? Clearly, it is not – as the counter-example $\theta = \frac{\pi}{6}$ shows.

$$\sin\left(2\cdot\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \text{ and } 2\sin\left(\frac{\pi}{6}\right) = 2\left(\frac{1}{2}\right) = 1$$

A direct consequence of the compound angle identities developed in the past few pages are formulas for $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$, that is, **double angle identities**. For example, the formula for $\sin 2\theta$ can be derived by taking the identity for the sine of two numbers and by letting $\alpha = \beta = \theta$.

$$\sin 2\theta = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$$

Similarly, for $\cos 2\theta$ we have,

$$\cos 2\theta = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$

By applying the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$, we can write the double angle identity for $\cos 2\theta$ in two other useful ways.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$$

Hint: The double angle identity for the tangent function does not hold if $\theta = \frac{\pi}{4} + k\cdot\frac{\pi}{2}$, where k is any integer, because for these values of θ the denominator is zero. The identity also does not hold if $\theta = \frac{\pi}{2} + k\cdot\pi$, where k is any integer, because for these values $\tan\theta$ does not exist. Nevertheless, the equation is still an identity because it is true for all values of θ for which both sides are defined.

To derive the formula for expressing $\tan 2\theta$ in terms of $\tan\theta$, we take the same approach and start with the identity for the tangent of the sum of two numbers and let $\alpha = \beta = \theta$.

$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \frac{2\tan\theta}{1 - \tan^2\theta}$$

We now have a useful set of identities for the sine, cosine and tangent of twice an angle (or number).

Double angle identities

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \begin{cases} \cos^2\theta - \sin^2\theta \\ 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{cases}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Now let's look at some further applications of the trigonometric identities we have established, especially for solving more sophisticated equations.

Example 27

Solve the equation $\cos 2x + \cos x = 0$ for $0 \leqslant x \leqslant 2\pi$.

Solution

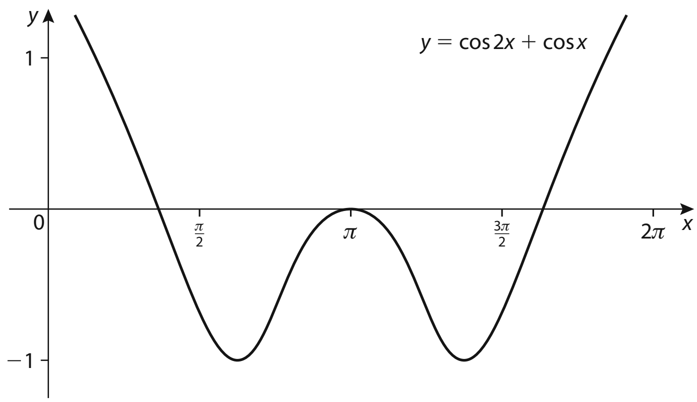
Taking an initial look at the graph of $y = \cos 2x + \cos x$ suggests that there are possibly three solutions in the interval $x \in [0, 2\pi]$. Although the expression $\cos 2x + \cos x$ contains terms with only the cosine function, it is not possible to perform any algebraic operations on them because they have different arguments. In order to solve algebraically, we need both cosine functions to have arguments of x (rather than $2x$). There are three different double angle identities for $\cos 2x$. It is best to have the equation in terms of one trigonometric function, so we choose to substitute $2\cos^2 x - 1$ for $\cos 2x$.

$$\cos 2x + \cos x = 0 \Rightarrow 2\cos^2 x - 1 + \cos x = 0 \Rightarrow 2\cos^2 x + \cos x - 1 = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0 \Rightarrow \cos x = \tfrac{1}{2} \text{ or } \cos x = -1$$

For $\cos x = \frac{1}{2}$: $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$; for $\cos x = -1$: $x = \pi$.

Therefore, all of the solutions in the interval $0 \leqslant x \leqslant 2\pi$ are: $x = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$.



Example 28

Solve the equation $2\sin 2x = 3\cos x$ for $0 \leqslant x \leqslant \pi$.

Solution

$2\sin 2x = 3\cos x$

$2(2\sin x\cos x) = 3\cos x$ Using double angle identity for sine.

$4\sin x\cos x = 3\cos x$ Do not divide by $\cos x$; solution(s) may be eliminated.

$4\sin x\cos x - 3\cos x = 0$ Set equal to zero to prepare for solving by factorization.

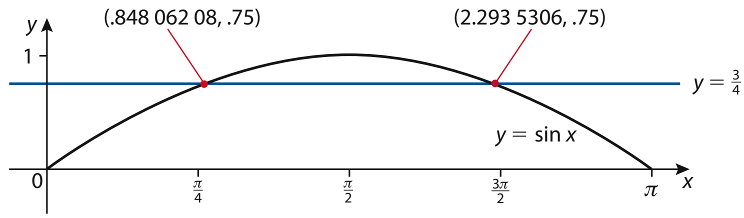
$\cos x(4\sin x - 3) = 0$ Factorize.

$\cos x = 0$ or $\sin x = \frac{3}{4}$

For $\cos x = 0$: $x = \dfrac{\pi}{2}$.

For $\sin x = \frac{3}{4}$: $x \approx 0.848$ or 2.29.

Approximate solutions are found using the Intersect command on the GDC.
All solutions in interval $0 \leqslant x \leqslant \pi$ are: $x = \dfrac{\pi}{2}$; $x \approx 0.848, 2.29$.



The next example illustrates how trigonometric identities can be applied to find exact values to trigonometric expressions.

Example 29

Given that $\cos x = \frac{1}{4}$ and that $0 < x < \frac{\pi}{2}$, find the *exact* values of

a) $\sin x$ b) $\sin 2x$

Solution

a) Given $0 < x < \frac{\pi}{2}$ it follows that $\sin x > 0$, because the arc with length x will terminate in the first quadrant. The Pythagorean identity is useful when relating $\sin x$ and $\cos x$.

$$\sin^2 x = 1 - \cos^2 x \Rightarrow \sin x = \sqrt{1 - \cos^2 x}$$

$$\Rightarrow \sin x = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$$

b) $\sin 2x = 2 \sin x \cos x = 2\left(\frac{\sqrt{15}}{4}\right)\left(\frac{1}{4}\right) = \frac{\sqrt{15}}{8}$

Example 30

Prove the following identity.

$$\frac{\cos A}{\cos A - \sin A} + \frac{\sin A}{\cos A + \sin A} = 1 + \tan 2A$$

● **Hint:** An effective approach to proving identities is to try and work exclusively on one side of the equation. Choosing the side that has an expression that is more 'complicated' is often an efficient path to transform the expression to the one on the other side by means of algebraic manipulations and substitutions. If you do choose to simplify both sides, be careful to work on each side independent of the other. In other words, as mentioned previously, do not perform an operation to both sides (e.g. multiplying both sides by the same quantity). This is only valid if it is known that both sides are equal but this is precisely what you are trying to prove.

Solution

Although we could apply a double angle identity to $\tan 2A$ on the right side it would not help to simplify the expression. The left side appears riper for simplification given that the common denominator of the two fractions is $\cos^2 A - \sin^2 A$ which is equivalent to $\cos 2A$.

$$\frac{\cos A}{\cos A - \sin A} \cdot \frac{\cos A + \sin A}{\cos A + \sin A} + \frac{\sin A}{\cos A + \sin A} \cdot \frac{\cos A - \sin A}{\cos A - \sin A} = \text{RHS}$$

Find a common denominator.

$$\frac{\cos^2 A + \sin A \cos A}{\cos^2 A - \sin^2 A} + \frac{\sin A \cos A - \sin^2 A}{\cos^2 A - \sin^2 A} = \text{RHS}$$

Multiply conjugates $(a + b)(a - b) = a^2 - b^2$.

$$\frac{\cos^2 A - \sin^2 A + 2 \sin A \cos A}{\cos^2 A - \sin^2 A} = \text{RHS}$$

$$\frac{\cos 2A + 2 \sin A \cos A}{\cos 2A} = \text{RHS}$$ Substitute $\cos 2A$ for $\cos^2 A - \sin^2 A$.

Observing that the right-hand side (RHS) has a term equal to 1 directs us to split the left side into two fractions since one of the terms in the numerator is equal to the denominator.

$$\frac{\cos 2A}{\cos 2A} + \frac{2 \sin A \cos A}{\cos 2A} = \text{RHS}$$

$$1 + \frac{\sin 2A}{\cos 2A} = \text{RHS}$$ Substitute $\sin 2A$ for $2 \sin A \cos A$.

$$1 + \tan 2A = 1 + \tan 2A \quad \text{Q.E.D.}$$ Apply tangent identity $\tan x = \frac{\sin x}{\cos x}$.

Reciprocal identities		
$\csc\theta = \dfrac{1}{\sin\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$
Tangent and cotangent identities		
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	
Odd/even function identities		
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \cos\theta$	$\cot(-\theta) = -\tan\theta$
Co-function identities		
$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$	$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$	$\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$
Pythagorean identities		
$\sin^2\theta + \cos^2\theta = 1$	$1 + \tan^2\theta = \sec^2\theta$	$1 + \cot^2\theta = \csc^2\theta$
Compound angle identities		
$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$		
Double angle identities		
$\sin 2\theta = 2\sin\theta\cos\theta$ $\cos 2\theta = \begin{cases} \cos^2\theta - \sin^2\theta \\ 2\cos^2\theta - 1 \\ 1 - 2\sin^2\theta \end{cases}$ $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$		

Table 7.3 Summary of trigonometric identities.

Exercise 7.5

In questions 1–6, use a compound angle identity to find the **exact** value of the expression.

1 $\cos\frac{7\pi}{12}$ **2** $\sin 165°$

3 $\tan\frac{\pi}{12}$ **4** $\sin\left(-\frac{5\pi}{12}\right)$

5 $\cos 255°$ **6** $\cot 75°$

7 a) Find the **exact** value of $\cos\frac{\pi}{12}$.

b) By writing $\cos\frac{\pi}{12}$ as $\cos\left(2 \cdot \frac{\pi}{24}\right)$ and using a double angle identity for cosine, find the **exact** value of $\cos\frac{\pi}{24}$.

In questions 8–10, prove the co-function identity using the compound angle identities.

8 $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$ **9** $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ **10** $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

11 Given that $\sin x = \frac{3}{5}$ and that $0 < x < \frac{\pi}{2}$, find the exact values of

a) $\cos x$ b) $\cos 2x$ c) $\sin 2x$

12 Given that $\cos x = -\frac{2}{3}$ and that $\frac{\pi}{2} < x < \pi$, find the exact values of

a) $\sin x$ b) $\sin 2x$ c) $\cos 2x$

In questions 13–16, find the exact values of $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$ subject to the given conditions.

13 $\sin\theta = \frac{2}{3}, \frac{\pi}{2} < \theta < \pi$

14 $\cos\theta = -\frac{4}{5}, \pi < \theta < \frac{3\pi}{2}$

15 $\tan\theta = 2, 0 < \theta < \frac{\pi}{2}$

16 $\sec\theta = -4, \csc\theta > 0$

In questions 17–20, use a compound angle identity to write the given expression as a function of x alone.

17 $\cos(x - \pi)$

18 $\sin\left(x - \frac{\pi}{2}\right)$

19 $\tan(x + \pi)$

20 $\cos\left(x + \frac{\pi}{2}\right)$

In questions 21–24, use identities to find an equivalent expression involving only sines and cosines, and then simplify it.

21 $\sec\theta + \sin\theta$

22 $\frac{\sec\theta\csc\theta}{\tan\theta\sin\theta}$

23 $\frac{\sec\theta + \csc\theta}{2}$

24 $\frac{1}{\cos^2\theta} + \frac{1}{\cot^2\theta}$

In questions 25–32, simplify each expression.

25 $\cos\theta - \cos\theta\sin^2\theta$

26 $\frac{1 - \cos^2\theta}{\sin^2\theta}$

27 $\cos 2\theta + \sin^2\theta$

28 $\frac{\sin^2\theta}{\cos^2\theta} + \frac{1}{\cot^2\theta}$

29 $\sin(\alpha + \beta) + \sin(\alpha - \beta)$

30 $\frac{1 + \cos 2A}{2}$

31 $\cos(\alpha + \beta) + \cos(\alpha - \beta)$

32 $2\cos^2\theta - \cos 2\theta$

In questions 33–46, prove each identity.

33 $\frac{\cos 2\theta}{\cos\theta + \sin\theta} = \cos\theta - \sin\theta$

34 $(1 - \cos\alpha)(1 + \sec\alpha) = \sin\alpha\tan\alpha$

35 $\frac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$

36 $\cos^4\theta - \sin^4\theta = \cos 2\theta$

37 $\cot\theta - \tan\theta = 2\cot 2\theta$

38 $\frac{\cos\beta - \sin\beta}{\cos\beta + \sin\beta} = \frac{\cos 2\beta}{1 + \sin 2\beta}$

39 $\dfrac{1}{\sec\theta(1-\sin\theta)} = \sec\theta + \tan\theta$

40 $(\tan A - \sec A)^2 = \dfrac{1-\sin A}{1+\sin A}$

41 $\dfrac{\tan 2x \tan x}{\tan 2x - \tan x} = \sin 2x$

42 $\dfrac{\sin 2\theta - \cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 1} = \tan\theta$

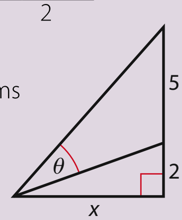
43 $\dfrac{1+\cos\alpha}{\sin\alpha} = 2\csc\alpha - \dfrac{\sin\alpha}{1+\cos\alpha}$

44 $\dfrac{1+\cos\beta}{\sin\beta} + \dfrac{\sin\beta}{1+\cos\beta} = 2\csc\beta$

45 $\dfrac{\cot x - 1}{1-\tan x} = \dfrac{\csc x}{\sec x}$

46 $\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$

47 Given the figure shown right, find an expression in terms of x for the value of $\tan\theta$.



● **Hint:** For question 46, first prove that $\sin^2 x = 1 - \dfrac{\cos 2x}{2}$, then make a suitable substitution for x. This identity is called the **half-angle identity** for sine. Can you find the corresponding half-angle identity for cosine?

In questions 48–57, solve each equation for x in the given interval. Give answers exactly, if possible. Otherwise, give answers accurate to three significant figures.

48 $2\sin^2 x - \cos x = 1, 0 \leqslant x < 2\pi$

49 $\sec^2 x = 8\cos x, -\pi < x \leqslant \pi$

50 $2\cos x + \sin 2x = 0, -180° < x \leqslant 180°$

51 $2\sin x = \cos 2x, 0 \leqslant x < 2\pi$

52 $\cos 2x = \sin^2 x, 0 \leqslant x < 2\pi$

53 $2\sin x\cos x + 1 = 0, 0 \leqslant x < 2\pi$

54 $\cos^2 x - \sin^2 x = -\frac{1}{2}, 0 \leqslant x \leqslant \pi$

55 $\sec^2 x - \tan x - 1 = 0, 0 \leqslant x < 2\pi$

56 $\tan 2x + \tan x = 0, 0 \leqslant x < 2\pi$

57 $2\sin 2x\cos 3x + \cos 3x = 0, 0 \leqslant x \leqslant 180°$

58 Find an identity for $\sin 3x$ in terms of $\sin x$.

59 a) By squaring $\sin^2 x + \cos^2 x$, prove that $\sin^4 x + \cos^4 x = \frac{1}{4}(\cos 4x + 3)$.

b) Hence, or otherwise, solve the equation $\sin^4 x + \cos^4 x = \frac{1}{2}$ for $0 \leqslant x < 2\pi$.

7.6 Inverse trigonometric functions

In Section 2.3, we learned that if a function f is one-to-one then f has an inverse f^{-1}. A defining characteristic of a one-to-one function is that it is always increasing or always decreasing in its domain. Also, recall that no horizontal line can pass through the graph of a one-to-one function at more than one point. It is evident that none of the trigonometric functions are one-to-one functions given their periodic nature. Therefore, the inverse of any of the trigonometric functions over their domain is not a function.

Defining the inverse sine function

Recall that the domain of $y = \sin x$ is all real numbers ($\mathbb{R}$) and its range is the set of all real numbers in the closed interval $-1 \leqslant y \leqslant 1$. The sine function is not one-to-one and hence its inverse is not a function, since more than one value of x corresponds to the same value of y. For example, $\sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{13\pi}{6} = \frac{1}{2}$. That is, for $y = \sin x$ there are an infinite number of ordered pairs with a y-coordinate of $\frac{1}{2}$ (see Figure 7.30).

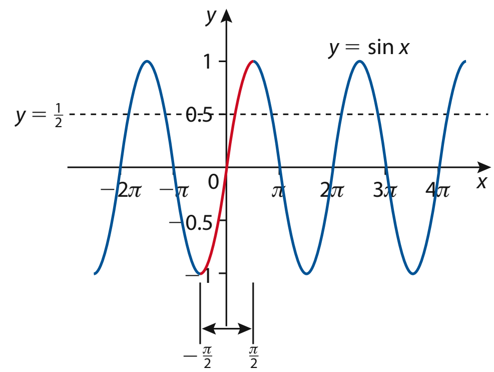


Figure 7.30 A horizontal line, $y = \frac{1}{2}$ shown here, can intersect the graph of $y = \sin x$ more than once, thus indicating that the inverse of $y = \sin x$ is not a function. The portion of the graph (in red) from $-\frac{\pi}{2}$ to $-\frac{\pi}{2}$ is used to define the inverse and only intersects a horizontal line once.

Examples 13 and 15 in Section 2.3, showed us that a function that is not one-to-one can often be made so by restricting its domain. Consequently, even though there is no inverse function for the sine function for all $\mathbb{R}$, we can define the inverse sine function if we restrict its domain so that it is one-to-one (and passes the horizontal line test). We have an unlimited number of ways of restricting the domain but it seems sensible to select an interval of x including zero, and it's standard to restrict the domain to the 'largest' set possible. Consider restricting the domain of $y = \sin x$ to the interval $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$. In this interval, $y = \sin x$ is always increasing and takes on every value from -1 to 1 exactly once. Thus, the function $y = \sin x$ with domain $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$ is one-to-one and its inverse is a function. We have the following definition:

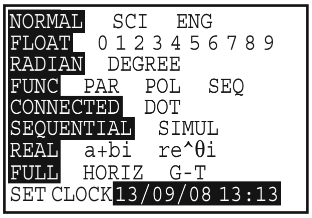
The equation $y = \arcsin x$ is interpreted, 'y is the arc whose sine is x', or 'y is the angle whose sine is x,' or 'y is the real number whose sine is x.' Any GDC labels the inverse sine function as $\sin^{-1} x$. The symbols $y = \arcsin x$ and $y = \sin^{-1} x$ are both commonly used to indicate the inverse sine function, but a disadvantage of writing $y = \sin^{-1} x$ is that it can be confused with $y = (\sin x)^{-1} = \frac{1}{\sin x} = \csc x$.

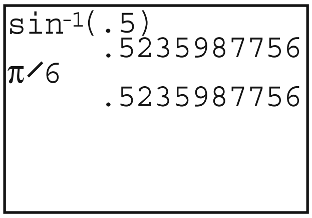
Inverse sine function

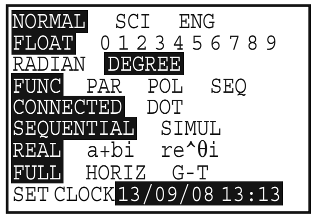
The inverse sine function, denoted by $x = \arcsin x$ or $y = \sin^{-1} x$, is the function with a domain of $-1 \leqslant x \leqslant 1$ and a range of $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ defined by

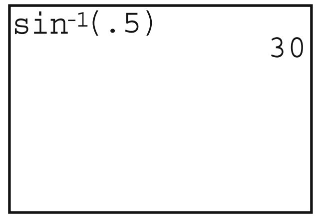
$$y = \arcsin x \quad \text{if and only if} \quad x = \sin y$$

Thus, $\arcsin x$ (or $\sin^{-1} x$) is the number in the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x. For example, $\arcsin\frac{1}{2} = \frac{\pi}{6}$ because the one number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{1}{2}$ is $\frac{\pi}{6}$. Your GDC is programmed such that it will give the same result. If your GDC is in radian mode it will give the approximate value of $\frac{\pi}{6}$ to several significant figures, and if it is in degree mode, it will give the exact result of 30°. See the GDC images on the next page.









From the graphical symmetry of inverse functions, the graph of $y = \arcsin x$ is a reflection of $y = \sin x$ about the line $y = x$, as shown in Figures 7.31 and 7.32.

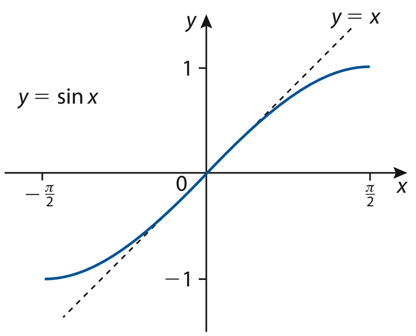


Figure 7.31 The graph of $y = \sin x$ with domain restricted to $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$.

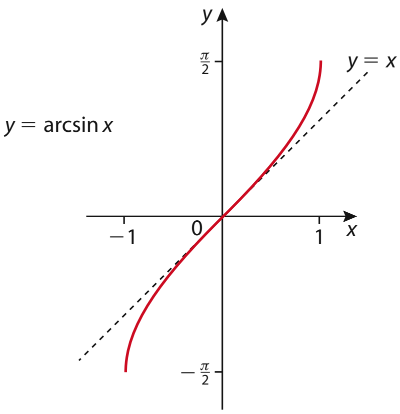
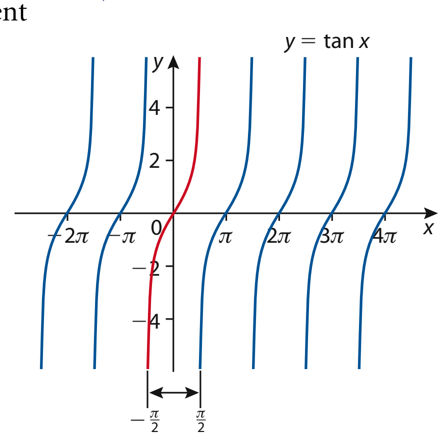


Figure 7.32 The graph of $y = \arcsin x$.

Defining the inverse cosine and inverse tangent functions

The inverse cosine function and inverse tangent function can be defined by following a parallel procedure to that used for defining the inverse sine function. The graphs of $y = \cos x$ and $y = \tan x$ (Figures 7.33 and 7.34) clearly show that neither function is one-to-one and consequently their inverses are not functions. Consider restricting the domain of the cosine function to the closed interval $0 \leqslant x \leqslant \pi$ (Figure 7.33) and restricting the domain of the tangent function to the open interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (Figure 7.34). The interval for tangent cannot include the endpoints, $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, because tangent is undefined for these values. For these domain restrictions cosine and tangent will attain each of its function values exactly once. Hence, with these restrictions, both cosine and tangent will be one-to-one and their inverses will be functions.

Figure 7.34 The graph of $y = \tan x$ with the portion of the graph (in red) from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ (exclusive) used to define its inverse.



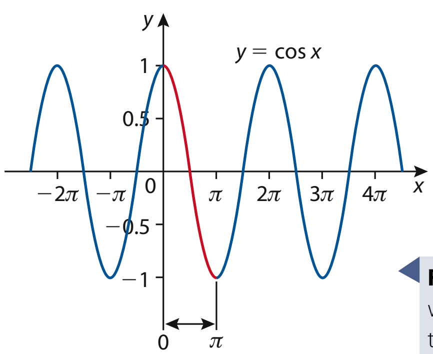


Figure 7.33 The graph of $y = \cos x$ with portion of the graph (in red) from 0 to π (inclusive) used to define its inverse.

Inverse cosine function

The inverse cosine function, denoted by $y = \arccos x$, or $y = \cos^{-1} x$, is the function with a domain of $-1 \leqslant x \leqslant 1$ and a range of $0 \leqslant y \leqslant \pi$ defined by

$$y = \arccos x \quad \text{if and only if} \quad x = \cos y$$

Inverse tangent function

The inverse tangent function, denoted by $y = \arctan x$, or $y = \tan^{-1} x$, is the function with a domain of $\mathbb{R}$ and a range of $-\frac{\pi}{2} < y < \frac{\pi}{2}$ defined by

$$y = \arctan x \quad \text{if and only if} \quad x = \tan y$$

The inverse cotangent, secant and cosecant functions are rarely used (and are not in the Maths Higher Level syllabus) so definitions will not be given for them here.

The graphs of $y = \cos x$ (for the appropriate interval) and $y = \arccos x$ are shown in Figures 7.35 and 7.36.

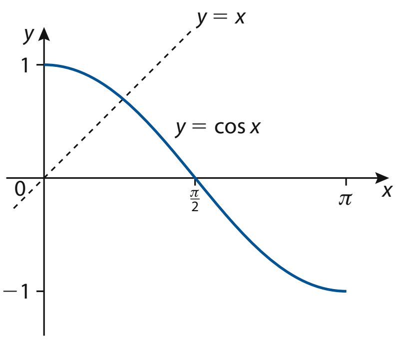


Figure 7.35 The graph of $y = \cos x$ with domain restricted to $0 \leqslant x \leqslant \pi$.

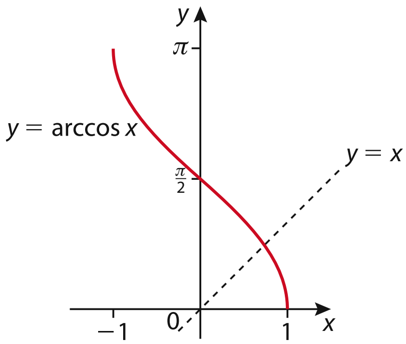


Figure 7.36 The graph of $y = \arccos x$.

The graphs of $y = \tan x$ (for the appropriate interval) and $y = \arctan x$ are shown in Figures 7.37 and 7.38.

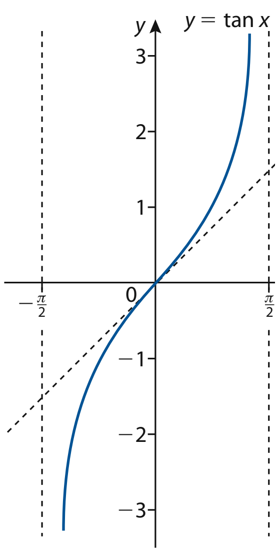


Figure 7.37 The graph of $y = \tan x$ with domain restricted to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

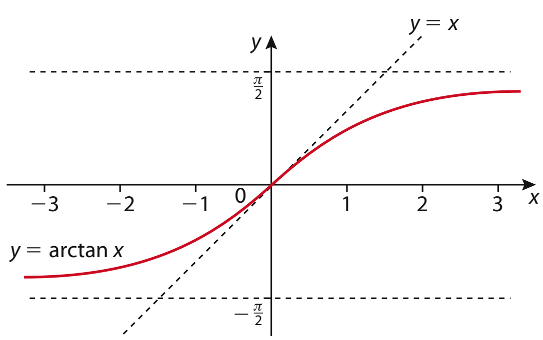


Figure 7.38 The graph of $y = \arctan x$.

● **Hint:** Unless specifically instructed otherwise, we will assume that the result of evaluating an inverse trigonometric function will be a real number that can be interpreted as either an arc length on the unit circle or an angle in radian measure. If the result is to be an angle in degree measure then the instructions will explicitly request this.

Example 30

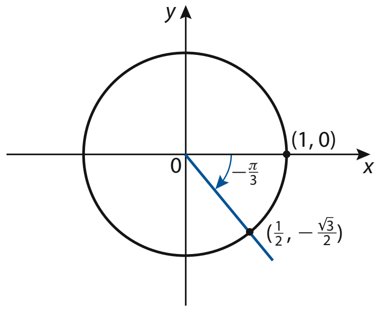
Without using your GDC, find the exact value of each expression.

a) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ b) $\arccos 1$ c) $\arctan\sqrt{3}$ d) $\arcsin\frac{3}{2}$

Solution

a) The expression $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ can be interpreted as 'the number y such that $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ whose sine is $-\frac{\sqrt{3}}{2}$' or 'the number in quadrant I or IV whose sine is $-\frac{\sqrt{3}}{2}$.' We know sine function values are negative in quadrants III and IV, so the number we are looking for is in quadrant IV. The diagram shows that the required number is $-\frac{\pi}{3}$. An angle of $-\frac{\pi}{3}$ in standard position will intersect the unit circle at a point whose y-coordinate is $-\frac{\sqrt{3}}{2}$.

Therefore, $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.



b) The range of the function $y = \arccos x$ is $0 \leqslant y \leqslant \pi$. Thus we are looking for a number in quadrant I or II whose cosine is 1. The number we are looking for is 0, because an angle of measure 0 in standard position will intersect the unit circle at a point whose x-coordinate is 1. Therefore, $\arccos 1 = 0$.

c) The range of the function $y = \arctan x$ is $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Thus we are looking for a number in quadrant I or IV for which the ratio $\frac{\text{sine}}{\text{cosine}}$ is equal to $\sqrt{3}$. It must be in quadrant I because in quadrant IV tangent values are negative. Familiarity with the sine and cosine values for common angles covered earlier in this chapter helps us to recognize that the required ratio will be $\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$. The required number is $\frac{\pi}{3}$ because it is in the first quadrant with $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$.

Therefore, $\arctan\sqrt{3} = \frac{\pi}{3}$.

d) The domain of the function $y = \arccos x$ is $-1 \leqslant x \leqslant 1$, but $\frac{3}{2}$ is not in this interval. There is no number whose sine is $\frac{3}{2}$. Therefore, $\arcsin\frac{3}{2}$ is not defined.

Compositions of trigonometric and inverse trigonometric functions

Recall from Chapter 2 that for a pair of inverse functions the following two properties hold true.

$f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}; and $f^{-1}(f(x)) = x$ for all x in the domain of f.

It follows that the following properties hold true for the inverse sine, cosine and tangent functions.

● **Hint:** Note that the inverse property arcsin(sin β) $= \beta$ does **not** hold true when $\beta = \frac{3\pi}{4}$.

$\arcsin\left(\sin\frac{3\pi}{4}\right) = \arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$

and

$\arcsin\left(\sin\frac{5\pi}{4}\right) = \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

The property arcsin(sin β) $= \beta$ is not valid for values of β outside the interval $-\frac{\pi}{2} \leqslant \beta \leqslant \frac{\pi}{2}$. Similarly, the property arccos(cos β) $= \beta$ is not valid for values of β outside the interval $0 \leqslant \beta \leqslant \pi$; and arctan(tan β) $= \beta$ is not valid for values of β outside the interval $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$.

Inverse properties

If $-1 \leqslant \alpha \leqslant 1$, then $\sin(\arcsin\alpha) = \alpha$; and if $-\frac{\pi}{2} \leqslant \beta \leqslant \frac{\pi}{2}$, then $\arcsin(\sin\beta) = \beta$.

If $-1 \leqslant \alpha \leqslant 1$, then $\cos(\arccos\alpha) = \alpha$; and if $0 \leqslant \beta \leqslant \pi$ then $\arccos(\cos\beta) = \beta$.

If $\alpha \in \mathbb{R}$, then $\tan(\arctan\alpha) = \alpha$; and if $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$, then $\arctan(\tan\beta) = \beta$.

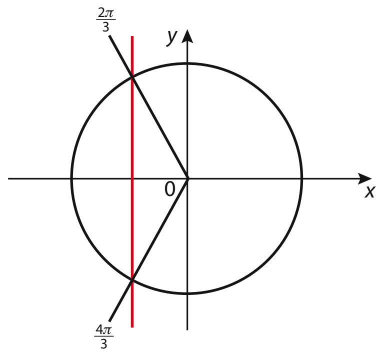
Example 31

Find the exact values, if possible, for the following expressions.

a) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ b) $\tan(\arctan(-7))$ c) $\sin(\arcsin\sqrt{3})$

Solution

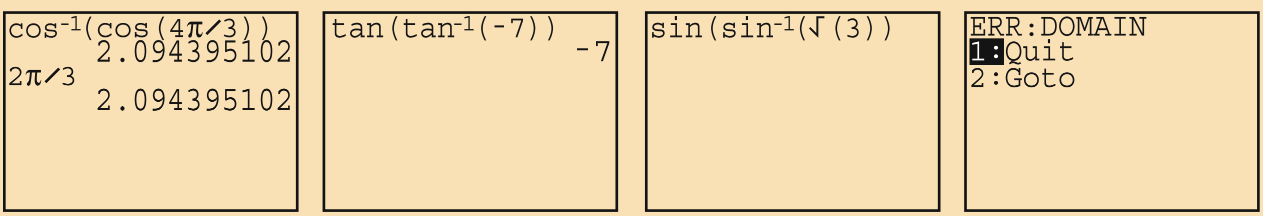
a) $\frac{4\pi}{3}$ is not in the range of the $\cos^{-1}$, or arccos, function $0 \leqslant \beta \leqslant \pi$. However, using the symmetry of the unit circle we know that $\frac{4\pi}{3}$ has the same cosine as $\frac{2\pi}{3}$ (see figure) which is in the interval $0 \leqslant \beta \leqslant \pi$. Thus, $\cos^{-1}\left(\cos\frac{4\pi}{3}\right) = \cos^{-1}\left(\cos\frac{2\pi}{3}\right) = \frac{2\pi}{3}$.



b) -7 is in the range of the tangent function (and in the domain of the arctangent function), so the inverse property applies. Therefore, $\tan(\arctan(-7)) = -7$.

c) $\sqrt{3}$ is not in the range of the sine function $-1 \leqslant \alpha \leqslant 1$, so $\arcsin\sqrt{3}$ is not defined. It follows that $\sin(\arcsin\sqrt{3})$ is not defined.

All of the results in Example 31 can be quickly verified on your GDC as shown below. Be sure to be in radian mode.



Example 32

Without using your GDC, find the exact value of each expression.

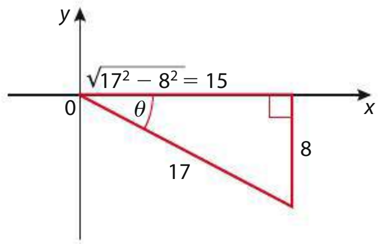
a) $\cos\left[\sin^{-1}\left(-\frac{8}{17}\right)\right]$

b) $\arcsin\left(\tan\frac{3\pi}{4}\right)$

c) $\sec\left[\arctan\left(\frac{3}{5}\right)\right]$

Solution

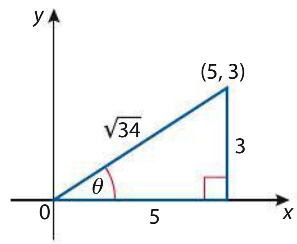
a) If we let $\theta = \sin^{-1}\left(-\frac{8}{17}\right)$, then $\sin\theta = -\frac{8}{17}$. Because $\sin\theta$ is negative, then θ must be an angle (arc) in quadrant IV. From a simple sketch of an appropriately labeled triangle in quadrant IV, we can determine $\cos\theta = \cos\left(\sin^{-1}\left(-\frac{8}{17}\right)\right)$.



Therefore, $\cos\left(\sin^{-1}\left(-\frac{8}{17}\right)\right) = \frac{15}{17}$.

b) $\arcsin\left(\tan\frac{3\pi}{4}\right) = \arcsin(-1) = -\frac{\pi}{2}$

c) If we let $\theta = \arctan\left(\frac{3}{5}\right)$ then $\tan\theta = \frac{3}{5}$. Because $\tan\theta > 0$ then θ must be in quadrant I. Consequently, we can construct a right triangle containing θ in quadrant I by drawing a line from the origin to the point $(5, 3)$, as shown in the diagram. The hypotenuse is $\sqrt{25 + 9} = \sqrt{34}$.



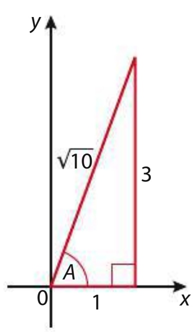
Therefore, $\sec\left[\arctan\left(\frac{3}{5}\right)\right] = \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{5}{\sqrt{34}}} = \frac{\sqrt{34}}{5}$.

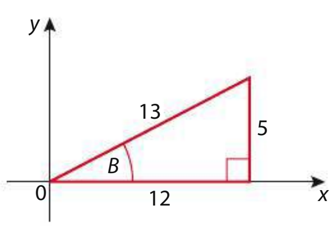
Example 33

If $C = \arctan 3 + \arcsin\left(\frac{5}{13}\right)$, find the exact value of $\cos C$.

Solution

Let $A = \arctan 3$ and $B = \arcsin\left(\frac{5}{13}\right)$. Thus, $C = A + B$ and a strategy for finding $\cos C$ is to use the following compound angle identity: $\cos C = \cos(A + B) = \cos A\cos B - \sin A\sin B$. We know that $\sin B = \frac{5}{13}$. We need to find exact values for $\cos A$, $\cos B$ and $\sin A$. The range for $\arctan x$ is $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and the range for $\arcsin x$ is $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$, and since $\tan A = 3 > 0$ and $\sin B = \frac{5}{13} > 0$, both A and B are in quadrant I.





$\sin A = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$ $\qquad$ $\sin B = \frac{5}{13}$

$\cos A = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$ $\qquad$ $\cos B = \frac{12}{13}$

Hence, $\cos C = \cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \left(\frac{\sqrt{10}}{10}\right)\left(\frac{12}{13}\right) - \left(\frac{3\sqrt{10}}{10}\right)\left(\frac{5}{13}\right)$$

$$= \frac{(12 - 15)\sqrt{10}}{130}$$

$$= \frac{-3\sqrt{10}}{130}$$

Therefore, $\cos C = \dfrac{-3\sqrt{10}}{130}$.

Example 34

Find all solutions, accurate to three significant figures, to the equation $3 \sin 2\theta = 1$ in the interval $0 \leqslant \theta < 2\pi$.

Solution

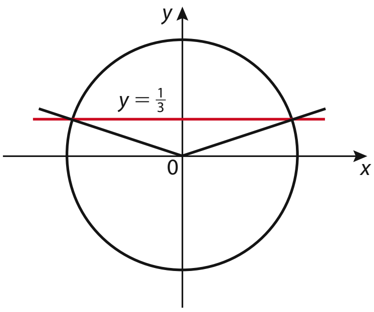
A reasonable idea is to apply a double angle identity and substitute $2 \sin \theta \cos \theta$ for $\sin 2\theta$. Although a substitution like this proved to be an effective technique in the previous section, it is not always the best strategy. In this case, the transformed equation becomes $6 \sin \theta \cos \theta = 1$ which would prove difficult to solve. A better approach is

$$3 \sin 2\theta = 1$$

$$\sin 2\theta = \tfrac{1}{3}$$

$$2\theta = \arcsin(\tfrac{1}{3})$$

$$\theta = \frac{1}{2}\arcsin(\tfrac{1}{3})$$



There is one angle in quadrant I with a sine equal to $\frac{1}{3}$ and one angle in quadrant II with a sine equal to $\frac{1}{3}$ (see figure). None of the common angles has a sine equal to $\frac{1}{3}$, so we will need to use the inverse sine ($\sin^{-1}$) on our GDC to obtain an approximate answer. Since the range of the inverse sine function, $\sin^{-1}$, is $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ your GDC's computation of $\sin^{-1}(\frac{1}{3})$ will only give the angle (arc) in quadrant I. From the symmetry of the unit circle, we can obtain the angle in quadrant II by subtracting the angle in quadrant I from π. The GDC images below show the computation to find both answers – and a check of the two answers.

```
sin⁻¹(1/3)
           .3398369095
.5*Ans
           .1699184547
Ans→A
           .1699184547
```

```
sin⁻¹(1/3)
           .3398369095
.5(π-Ans)
           1.400877872
Ans→B
           1.400877872
```

```
3sin(2A)
                     1
3sin(2B)
                     1
```

Therefore, $\theta \approx 0.170$ or $\theta \approx 1.40$ accurate to 3 significant figures.

To an observer, the apparent size of an object depends on the distance from the observer to the object. The farther an object is from an observer, the smaller its apparent size. For example, although the Sun's diameter is 400 times wider than our Moon's diameter, the two objects appear to have the same diameter as viewed from the Earth (see Figure 7.39). Thus, during a total solar eclipse, the Moon blocks out the Sun. Also, if an object is sufficiently above or below the horizontal position of the observer, the apparent size of the object will also decrease if you move close to the object. Thus for this situation, there will be a distance for which the angle subtended at the eye of the observer is a maximum (Example 35).

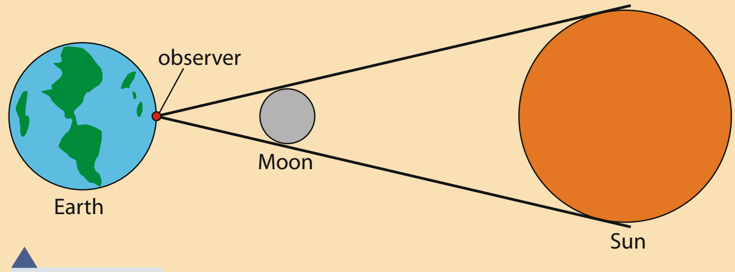


Figure 7.39

On the surface of the Earth the angle subtended by the moon and the Sun is nearly the same. It is approximately 0.54 degrees for the Moon and 0.52 degrees for the Sun. The Sun is 400 times wider than the Moon and coincidentally 400 times further from the Earth than the Moon.

Example 35

A painting that is 125 cm from top to bottom is hanging on the wall of a gallery such that it's base is 250 cm from the floor. Pablo is standing x cm from the wall from which the painting is hung. Pablo's eyes are 170 cm from the floor and from where he stands the painting subtends an angle α degrees. a) Write a function for α in terms of x. b) Find α, accurate to four significant figures, for the following values of x: (i) $x = 75$ cm; (ii) $x = 125$ cm; and (iii) $x = 175$ cm. c) Using a GDC, approximate to the nearest cm, how far Pablo should stand from the wall so that the subtended angle α is a maximum.

Solution

a) The figure shows α, the angle subtended by the painting, and β, the angle subtended by the part of the wall above eye level and below the painting. Let θ be the sum of these two angles. Hence, $\theta = \alpha + \beta$ and $\alpha = \theta - \beta$. From the compound angle identity for tangent, we have

$$\tan\alpha = \frac{\tan\theta - \tan\beta}{1 + \tan\theta\tan\beta}$$

From the right triangles in the figure, we can determine that

$$\tan\beta = \frac{80}{x} \quad \text{and} \quad \tan\theta = \frac{205}{x}$$

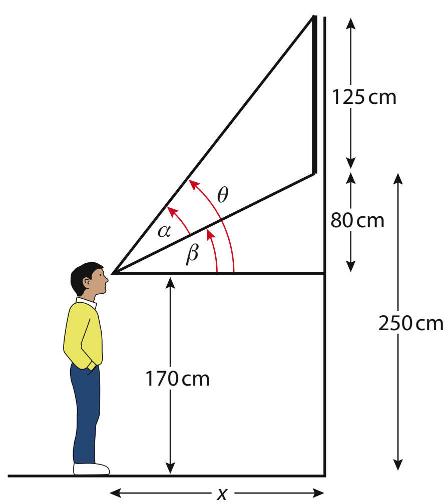
Substituting these into the expression for $\tan\alpha$, gives

$$\tan\alpha = \frac{\frac{205}{x} - \frac{80}{x}}{1 + \left(\frac{205}{x}\right)\left(\frac{80}{x}\right)}$$

$$\tan\alpha = \frac{\frac{125}{x}}{1 + \left(\frac{205}{x}\right)\left(\frac{80}{x}\right)} \cdot \frac{x^2}{x^2}$$

$$\tan\alpha = \frac{125x}{x^2 + 16\,400}$$

Therefore, $\alpha = \tan^{-1}\left(\frac{125x}{x^2 + 16\,400}\right)$.

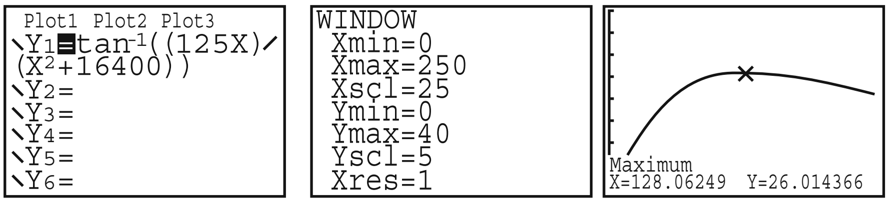


b) (i) For $x = 75$ cm: $\alpha = \tan^{-1}\left(\dfrac{125 \cdot 75}{75^2 + 16\,400}\right) \approx \tan^{-1}(0.425\,6527)$
$\approx 23.06°$.

(ii) For $x = 125$ cm: $\alpha = \tan^{-1}\left(\dfrac{125 \cdot 125}{125^2 + 16\,400}\right) \approx \tan^{-1}(0.487\,9001)$
$\approx 26.01°$.

(iii) For $x = 175$ cm: $\alpha = \tan^{-1}\left(\dfrac{125 \cdot 175}{175^2 + 16\,400}\right) \approx \tan^{-1}(0.465\,1781)$
$\approx 24.95°$.

c) Graph the function found in a). On the GDC, it will be entered as $y = \tan^{-1}\left(\dfrac{125x}{x^2 + 16\,400}\right)$. Find the value of x that gives the maximum value for y (subtended angle α) by either tracing or using a 'maximum' command on the calculator. See the GDC images below.



Therefore, if Pablo stands 128 cm away from the wall the painting will subtend the widest possible angle at his eye – or, in other words, give him the 'best' view of the painting.

Exercise 7.6

In questions 1–6, find the exact value (in radian measure) of each expression without using your GDC.

1 arcsin 1 **2** $\arccos\left(\dfrac{1}{\sqrt{2}}\right)$ **3** $\arctan(-\sqrt{3})$

4 $\arccos\left(-\dfrac{1}{2}\right)$ **5** arctan 0 **6** $\arcsin\left(\dfrac{-\sqrt{3}}{2}\right)$

In questions 7–20, without using your GDC, find the exact value, if possible, for each expression. Verify your result with your GDC.

7 $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$ **8** $\cos^{-1}\left(\cos\dfrac{3}{2}\right)$

9 tan(arctan 12) **10** $\cos\left(\arccos\dfrac{2\pi}{3}\right)$

11 $\arctan\left(\tan\left(-\dfrac{3\pi}{4}\right)\right)$ **12** $\sin(\arcsin \pi)$

13 $\sin\left(\arctan\dfrac{3}{4}\right)$ **14** $\cos\left(\arcsin\left(\dfrac{7}{25}\right)\right)$

15 $\arcsin\left(\tan\dfrac{\pi}{3}\right)$ **16** $\tan^{-1}\left(2\sin\dfrac{\pi}{3}\right)$

17 $\cos\left(\arctan\left(\dfrac{1}{2}\right)\right)$ **18** $\cos(\sin^{-1}(0.6))$

19 $\sin\left(\arccos\left(\dfrac{3}{5}\right) + \arctan\left(\dfrac{5}{12}\right)\right)$ **20** $\cos\left(\tan^{-1}3 + \sin^{-1}\left(\dfrac{1}{3}\right)\right)$

In questions 21–26, rewrite the expression as an algebraic expression in terms of x.

21 $\cos(\arcsin x)$ **22** $\tan(\arccos x)$

23 $\cos(\tan^{-1} x)$

24 $\sin(2\cos^{-1} x)$

25 $\tan\left(\frac{1}{2}\arccos x\right)$

26 $\sin(\arcsin x + 2\arctan x)$

27 Show that $\arcsin\frac{4}{5} + \arcsin\frac{5}{13} = \arccos\frac{16}{65}$.

28 Show that $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$.

29 Find x if $\tan^{-1} x + \tan^{-1}(1 - x) = \tan^{-1}\frac{4}{3}$.

In questions 30–37, solve for x in the indicated interval.

30 $5\cos(2x) = 2, 0 \leqslant x \leqslant \pi$

31 $\tan\left(\frac{x}{2}\right) = 2, 0 < x \leqslant 2\pi$

32 $2\cos x - \sin x = 0, 0 < x \leqslant 2\pi$

33 $3\sec^2 x = 2\tan x + 4, 0 < x \leqslant 2\pi$

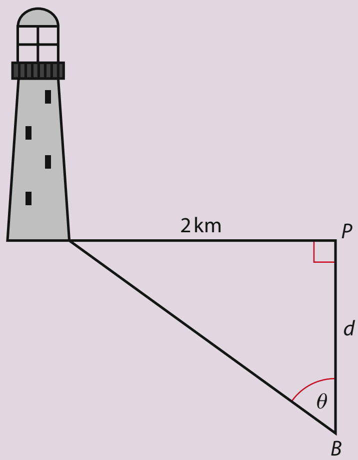
34 $2\tan^2 x - 3\tan x + 1 = 0, 0 \leqslant x \leqslant \pi$

35 $\tan x \csc x = 5, 0 < x \leqslant 2\pi$

36 $\tan 2x + 3\tan x = 0, 0 < x \leqslant 2\pi$

37 $2\cos^2 x - 3\sin 2x = 2, 0 \leqslant x \leqslant \pi$

38 An offshore lighthouse is located 2 km from a straight coastline. The lighthouse has a revolving light. Let θ be the angle that the beam of light from the lighthouse makes with the coastline; and P is the point on the coast the shortest distance from the lighthouse (see figure). If d is the distance in km from P to the point B where the beam of light is hitting the coast, express θ as a function of d. Sketch a complete graph of this function and indicate the portion of the graph that sufficiently represents the given situation.



39 The screen in a movie cinema is 7 metres from top to bottom and is positioned 3 metres above the horizontal floor of the cinema. The first row of seats is 2.5 metres from the wall that the screen is on and the rows are each 1 metre apart. You decide to sit in the row where you get the 'best' view, that is, where the angle subtended at your eyes by the screen is a maximum. When you are sitting in one of the cinema's seats your eyes are 1.2 metres above the horizontal floor.

a) Let x be the distance that you are from the wall that the screen is on, and θ is the angle subtended at your eyes by the screen.
   (i) Draw a clear diagram to represent all the information given.
   (ii) Find a function for θ in terms of x.
   (iii) Sketch a graph of the function.
   (iv) Use your GDC to find the value of x that gives a maximum for θ. In which row should you sit?

b) Suppose that, starting with the first row of seats, the floor of the cinema is sloping upwards at an angle of 20° above the horizontal. Again, the first row of seats is 2.5 metres from the wall that the screen is on and the rows are each 1 metre apart measured along the sloping floor. Let x be the distance from where the first row starts and your seat in the cinema.
   (i) Draw a clear diagram to represent all the information given.
   (ii) Find a function for θ in terms of x.
   (iii) Sketch a graph of the function.
   (iv) Use your GDC to find the value of x that gives a maximum for θ. In which row should you sit?

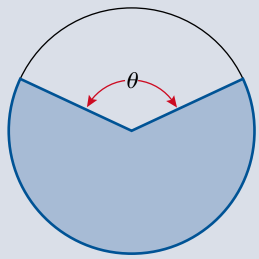
Practice questions

1 A toy on an elastic string is attached to the top of a doorway. It is pulled down and released, allowing it to bounce up and down. The length of the elastic string, L centimetres, is modelled by the function $L = 110 + 25\cos(2\pi t)$, where t is time in seconds after release.

a) Find the length of the elastic string after 2 seconds.
b) Find the minimum length of the string.
c) Find the first time after release that the string is 85 cm.
d) What is the period of the motion?

2 Find the exact solution(s) to the equation $2\sin^2 x - \cos x + 1 = 0$ for $0 \leqslant x \leqslant 2\pi$.

3 The diagram shows a circle of radius 6 cm.
The perimeter of the shaded sector is 25 cm.
Find the radian measure of the angle θ.



4 Consider the two functions $f(x) = \cos 4x$ and $g(x) = \cos\left(\frac{x}{2}\right)$.

a) Write down: **(i)** the minimum value of the function f
(ii) the period of g.
b) For the equation $f(x) = g(x)$, find the number of solutions in the interval $0 \leqslant x \leqslant \pi$.

5 A reflector is attached to the spoke of a bicycle wheel. As the wheel rolls along the ground, the distance, d centimetres, that the reflector is above the ground after t seconds is modelled by the function

$$d = p + q\cos\left(\frac{2\pi}{m}t\right), \text{ where } p, q \text{ and } m \text{ are constants.}$$

The distance d is at a maximum of 64 cm at $t = 0$ seconds and at $t = 0.5$ seconds, and is at a minimum of 6 cm at $t = 0.25$ seconds and at $t = 0.75$ seconds. Write down the value of:

a) p **b)** q **c)** m.

6 Find all solutions to $1 + \sin 3x = \cos(0.25x)$ such that $x \in [0, \pi]$.

7 Find all solutions to both trigonometric equations in the interval $x \in [0, 2\pi]$. Express the solutions exactly.

a) $2\cos^2 x + 5\cos x + 2 = 0$ **b)** $\sin 2x - \cos x = 0$

8 The value of x is in the interval $\frac{\pi}{2} < x < \pi$ and $\cos^2 x = \frac{8}{9}$. Without using your GDC, find the exact values for the following:

a) $\sin x$ **b)** $\cos 2x$ **c)** $\sin 2x$

9 The depth, d metres, of water in a harbour varies with the tides during each day. The first high (maximum) tide after midnight occurs at 5:00 a.m. with a depth of 5.8 m. The first low (minimum) tide occurs at 10:30 a.m. with a depth of 2.6 m.

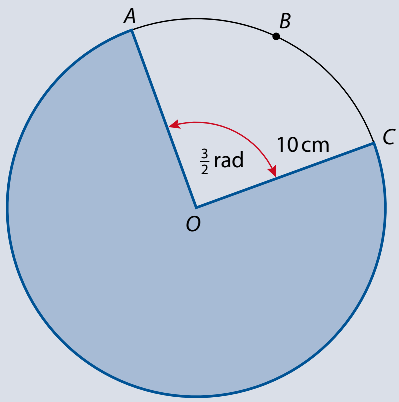
a) Find a trigonometric function that models the depth, d, of the water t hours after midnight.
b) Find the depth of the water at 12 noon.
c) A large boat needs at least 3.5 m of water to dock in the harbour. During what time interval after 12 noon can the boat dock safely?

10 Solve the equation $\tan^2 x + 2\tan x - 3 = 0$ for $0 \leqslant x \leqslant \pi$. Give solutions exactly, if possible. Otherwise, give solutions to 3 significant figures.

11 The following diagram shows a circle of centre O and radius 10 cm. The arc ABC subtends an angle of $\frac{3}{2}$ radians at the centre O.

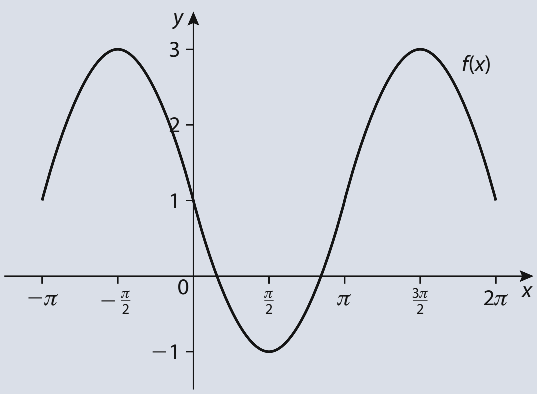
a) Find the length of the arc ACB.

b) Find the area of the shaded region.



12 Consider the function $f(x) = \frac{5}{2}\cos\left(2x - \frac{\pi}{2}\right)$. For what values of k will the equation $f(x) = k$ have no solutions?

13 A portion of the graph of $y = k + a\sin x$ is shown below. The graph passes through the points $(0, 1)$ and $\left(\frac{3\pi}{2}, 3\right)$. Find the value of k and a.



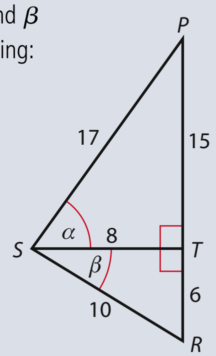
14 The angle α satisfies the equation $2\tan^2\alpha - 5\sec\alpha - 10 = 0$ where α is in the second quadrant. Find the **exact** value of $\sec\alpha$.

15 Triangles PTS and RTS are right-angled at T with angles α and β as shown in the diagram. Find the exact values of the following:

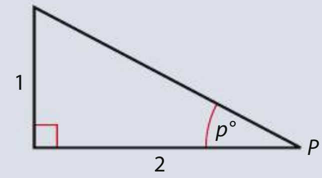
a) $\sin(\alpha + \beta)$

b) $\cos(\alpha + \beta)$

c) $\tan(\alpha + \beta)$



16 The diagram shows a right triangle with legs of length 1 unit and 2 units as shown. The angle at vertex P has a degree measure of $p°$. Find the exact values of $\sin 2p°$ and $\sin 3p°$.



17 The obtuse angle B is such that $\tan B = -\frac{5}{12}$. Find the values of

a) $\sin B$ **b)** $\cos B$ **c)** $\sin 2B$ **d)** $\cos 2B$

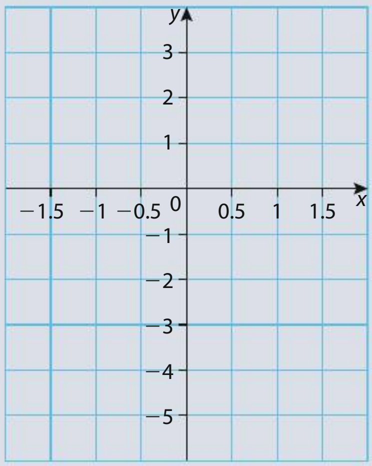
18 Given that $\tan 2\theta = \frac{3}{4}$, find the possible values of $\tan\theta$.

19 If $\sin(x - \alpha) = k\sin(x + \alpha)$ express $\tan x$ in terms of k and α.

20 Solve $\tan^2 2\theta = 1$, in the interval $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$.

21 Let f be the function $f(x) = x\arccos x + \frac{1}{2}x$ for $-1 \leqslant x \leqslant 1$ and g the function $g(x) = \cos 2x$ for $-1 \leqslant x \leqslant 1$.

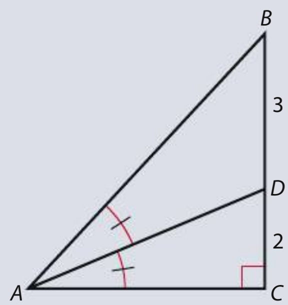
a) On the grid below, sketch the graph of f and of g.



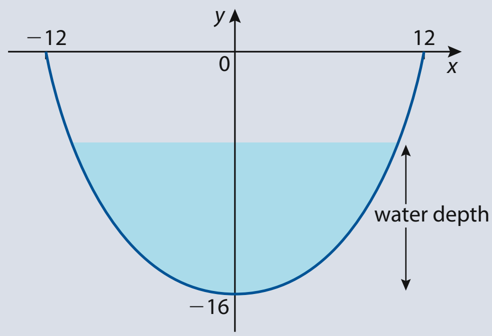
b) Write down the solution of the equation $f(x) = g(x)$.

c) Write down the range of g.

22 Let ABC be a right-angled triangle, where $\hat{C} = 90°$. The line (AD) bisects $B\hat{A}C$, $BD = 3$, and $DC = 2$, as shown in the diagram. Find $D\hat{A}C$.



23 The diagram below shows the boundary of the cross section of a water channel.



The equation that represents this boundary is $y = 16\sec\left(\frac{\pi x}{36}\right) - 32$ where x and y are both measured in cm. The top of the channel is level with the ground and has a width of 24 cm. The maximum depth of the channel is 16 cm. Find the width of the water surface in the channel when the water depth is 10 cm. Give your answer in the form $a \arccos b$, where $a, b \in \mathbb{R}$.

Questions 17–23 © International Baccalaureate Organization

40 a) sum $= -3$, product $= -\frac{5}{2}$

b) sum $= -3$, product $= -1$

c) sum $= 0$, product $= -\frac{3}{2}$

d) sum $= a$, product $= -2a$

e) sum $= 6$, product $= -4$

f) sum $= \frac{1}{3}$, product $= -\frac{2}{3}$

41 $4x^2 + 5x + 4 = 0$

42 a) $\frac{1}{9}$ b) $\frac{1}{12}$ c) $\frac{55}{27}$

43 a) -2 and -6 b) $k = 12$

44 a) $-\frac{1}{4}$ b) $4x^2 + x + 1 = 0$

45 a) $x^2 - 19x + 25 = 0$ b) $25x^2 + 72x - 5 = 0$

Exercise 3.3

1 $3x^2 + 5x - 5 = (x+3)(3x-4) + 7$

2 $3x^4 - 8x^3 + 9x + 5 = (x-2)(3x^3 - 2x^2 - 4x + 1) + 7$

3 $x^3 - 5x^2 + 3x - 7 = (x-4)(x^2 - x - 1) - 11$

4 $9x^3 + 12x^2 - 5x + 1 = (3x-1)(3x^2 + 5x) + 1$

5 $x^5 + x^4 - 8x^3 + x + 2 = (x^2 + x - 7)(x^3 - x + 1) + (-7x + 9)$

6 $(x-7)(x-1)(2x-1)$ **7** $(x-2)(2x+1)(3x+2)$

8 $(x-2)^2(x+4)(3x+2)$ **9** $Q(x) = x - 2,\ R = -2$

10 $Q(x) = x^2 + 2,\ R = -3$ **11** $Q(x) = 3,\ R(x) = 20x + 5$

12 $Q(x) = x^4 + x^3 + 4x^2 + 4x + 4,\ R = -2$

13 $P(2) = 5$ **14** $P(-1) = -17$

15 $P(-7) = -483$ **16** $P\left(\frac{1}{4}\right) = \frac{49}{64}$

17 $x = 2 + i$ or $x = 2 - i$ **18** $x = \frac{1+\sqrt{5}}{2}$ or $x = \frac{1-\sqrt{5}}{2}$

19 $k = \sqrt{1 - x\sqrt{3}}$ or $k = -\sqrt{1 - x\sqrt{3}}$

20 $a = 5,\ b = 12$

21 $x^3 - 3x^2 - 6x + 8$ **22** $x^4 - 3x^3 - 7x^2 + 15x + 18$

23 $x^3 - 6x^2 + 12x - 8$ **24** $x^3 - x^2 + 2$

25 $x^4 + 2x^3 + x^2 + 18x - 72$ **26** $x^4 - 8x^3 + 27x^2 - 50x + 50$

27 $x = 2 + 3i,\ x = 3$

28 a) $a = -1,\ b = -2$ b) $3x + 2$

29 $a = \frac{4}{3},\ b = \frac{1}{3}$

30 $x = 3,\ x = -1,\ x = -\frac{1}{4} + \frac{\sqrt{3}}{4}i,\ x = -\frac{1}{4} - \frac{\sqrt{3}}{4}i$

31 $a = -1,\ b = -4,\ c = 4$ **32** $p = -5,\ q = 23,\ r = -51$

33 $a = -5$ **34** $m = -2,\ n = -6$

35 $b = 18$ **36** b) $R = 3$

37 a) sum $= \frac{2}{3}$, product $= 5$ b) sum $= 1$, product $= 7$

c) sum $= \frac{1}{3}$, product $= -\frac{1}{2}$

39 $-9, 3, 6$

40 $2, -4, 8$

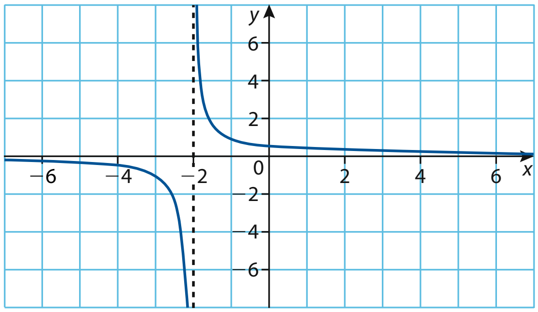
41 $3 + 2i,\ 2 + i,\ 2 - i$

42 $k = 3$

43 $k = -8$

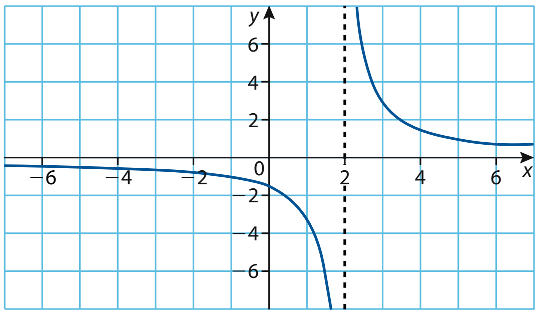
Exercise 3.4

1



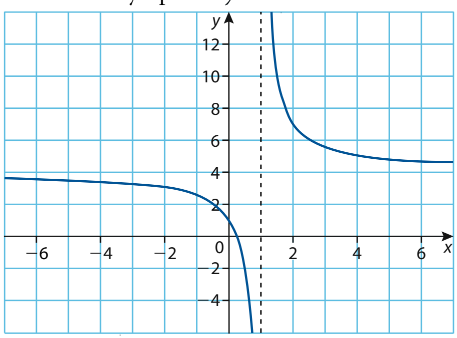
vertical asymptote: $x = -2$
horizontal asymptote: $y = 0$

2



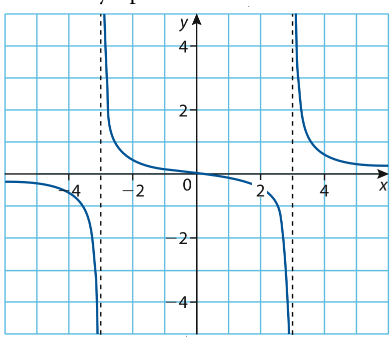
vertical asymptote: $x = 2$
horizontal asymptote: $y = 0$

3



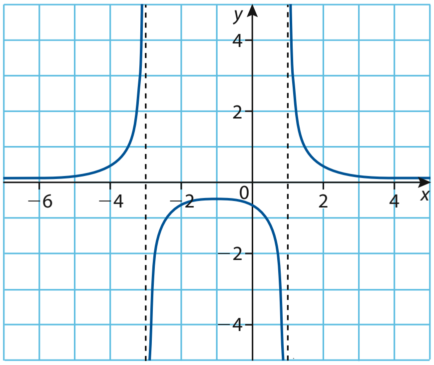
x-intercept: $\left(\frac{1}{4}, 0\right)$, y-intercept: $(0, 1)$
vertical asymptote: $x = 1$ horizontal asymptote: $y = 4$

4



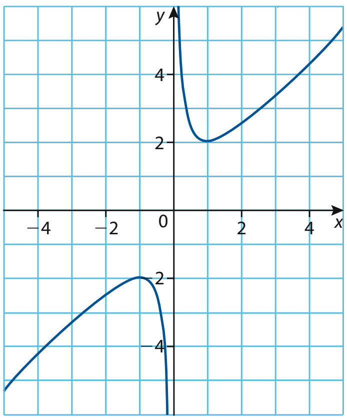
x- and y-intercept: $(0, 0)$
vertical asymptotes: $x = -3,\ x = 3$
horizontal asymptote: $y = 0$

5



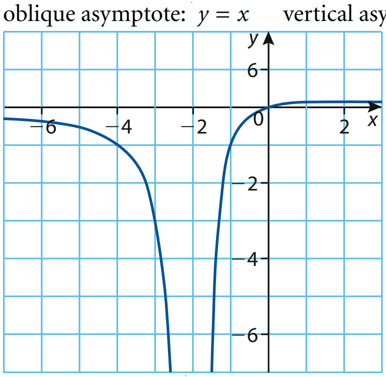
x-intercept: none, y-intercept: $\left(0, -\frac{2}{3}\right)$
vertical asymptotes: $x = -3$, $x = 1$
horizontal asymptote: $y = 0$

6



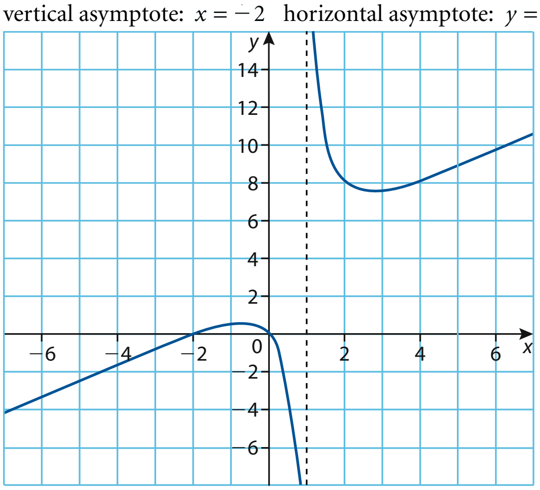
oblique asymptote: $y = x$ vertical asymptote: $x = 0$

7



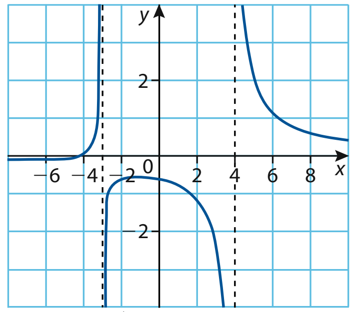
x- and y-intercept: $(0, 0)$
vertical asymptote: $x = -2$ horizontal asymptote: $y = 0$

8



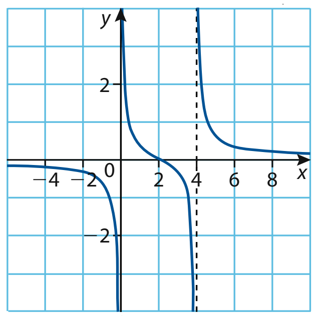
x- and y-intercept: $(0, 0)$
vertical asymptote: $x = 1$ oblique asymptote: $y = x + 3$

9



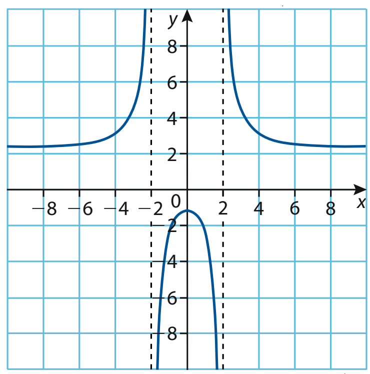
x-intercept: $(-4, 0)$ y-intercept: $\left(0, -\frac{2}{3}\right)$
vertical asymptotes: $x = -3$ and $x = 4$
horizontal asymptote: $y = 0$

10



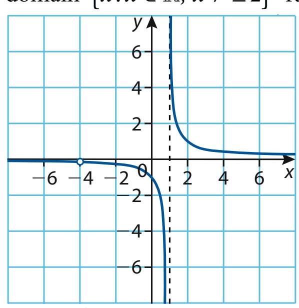
x-intercept: $(2, 0)$ y-intercept: none
vertical asymptotes: $x = 0$ and $x = 4$
horizontal asymptote: $y = 0$

11



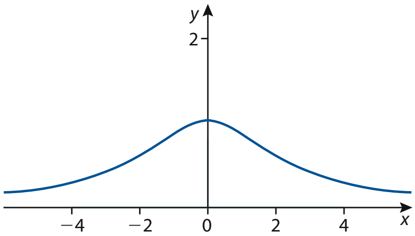
domain $\{x : x \in \mathbb{R},\ x \neq \pm 2\}$ range $\left\{y : y \leqslant -\frac{5}{4} \text{ or } y > 2\right\}$

12



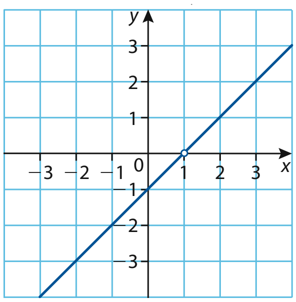
domain $\{x : x \in \mathbb{R},\ x \neq -4, 1\}$ range $\{y : y \in \mathbb{R},\ y \neq 0\}$

13



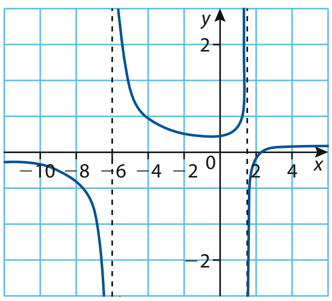
domain $\{x : x \in \mathbb{R}\}$ range $\{y : 0 < y \leqslant 1\}$

14



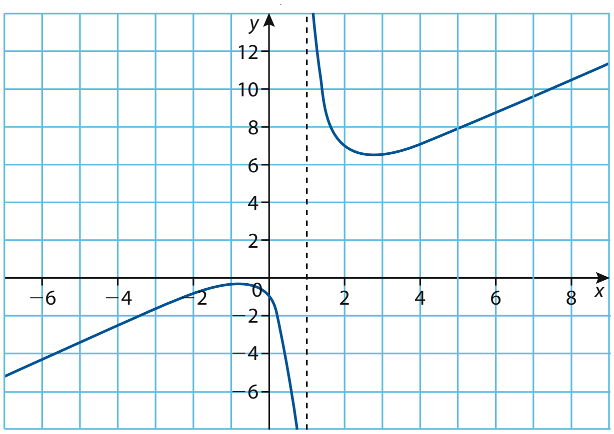
domain $\{x : x \in \mathbb{R},\ x \neq 1\}$
range $\{y : y \in \mathbb{R},\ y \neq 0\}$

15



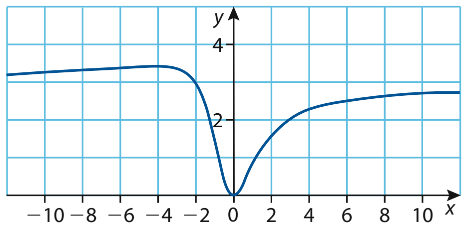
x-intercept: $\left(\frac{5}{2}, 0\right)$ y-intercept: $\left(0, \frac{5}{18}\right)$
vertical asymptotes: $x = -6$ and $x = \frac{3}{2}$
horizontal asymptote: $y = 0$

16



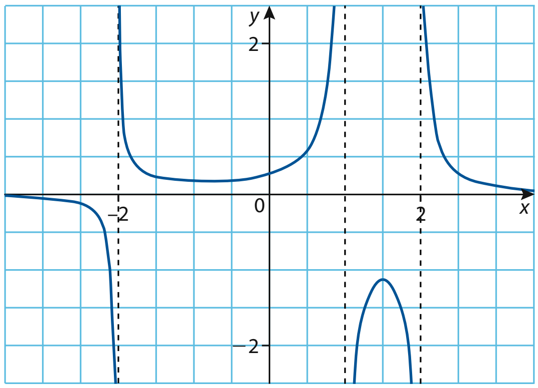
x-intercept: none
y-intercept: $(0, -1)$
vertical asymptote: $x = 1$
oblique asymptote: $y = x + 2$

17



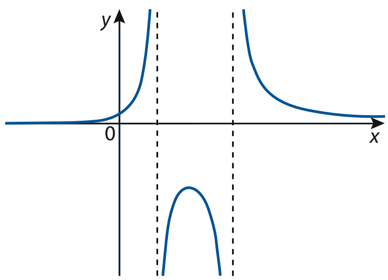
x- and y-intercept: $(0, 0)$
horizontal asymptote: $y = 3$

18

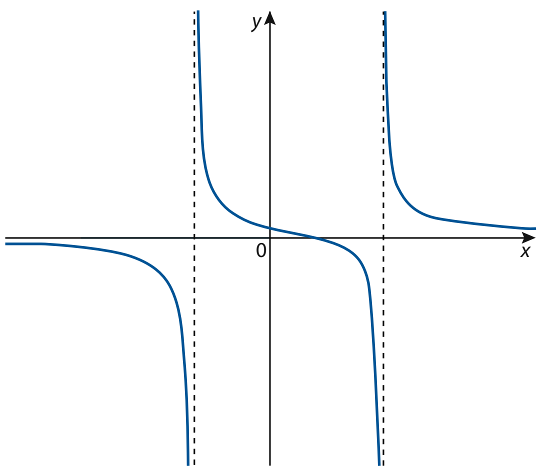


x-intercept: none y-intercept: $\left(0, \frac{1}{4}\right)$
vertical asymptotes: $x = -2$, $x = 1$ and $x = 2$
horizontal asymptote: $y = 0$

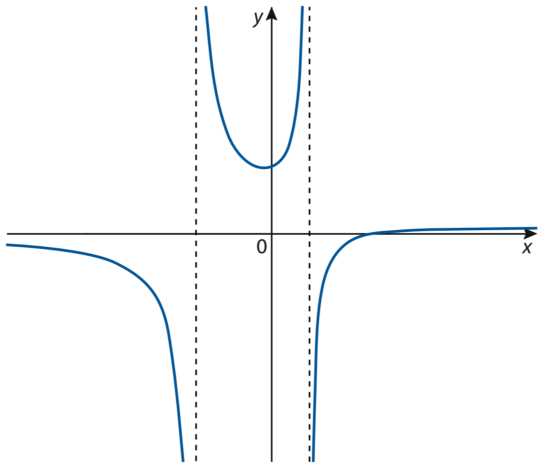
19 a)



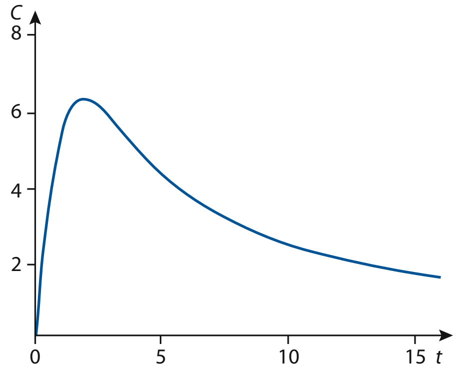
b)



c)



20 a)



b) At $t = 2$ minutes, concentration is 6.25 mg/l.
c) It continues to decrease and approaches zero as amount of time increases.
d) 50 minutes (49 minutes 55 seconds)

Exercise 3.5

1 $x = 3$
2 $x = 9$
3 $x = 5$ or $x = -2$
4 $x = 11$ or $x = 3$
5 $x = -5$
6 $x = 1$ or $x = -2$
7 $x = 2$ or $x = -2$
8 $x = \frac{1}{2}$
9 $x = \pm\sqrt{5}$
10 $x = 27$ or $x = -\frac{125}{8}$
11 $x = 3$ or $x = 2$
12 $x = \frac{1 \pm \sqrt{41}}{4}$
13 $x = \frac{4}{3}$ or $x = -4$
14 $x = 15$ or $x = \frac{9}{2}$
15 No solution
16 $x = 2$ or $x = -1$
17 $x = 2$ or $x = \frac{1}{2}$
18 $x = 9$
19 $x = -5$
20 $x = \frac{\pm 2\sqrt{5}}{5}$ or $x = \pm 1$
21 $x = \frac{1 + \sqrt{41}}{2}$
22 $x = \frac{49}{4}$ or $x = \frac{64}{9}$
23 $-\frac{2}{3} < x < 2$
24 $x < -2, x \geqslant 3$
25 $-10 \leqslant x \leqslant 6$
26 $x < \frac{3}{2}, x > 2$
27 $x > \frac{17}{2}$
28 $-4 \leqslant x \leqslant -1, 1 \leqslant x \leqslant 4$
29 $x < -1, x > 2$
30 $x < -1, -\frac{2}{3} < x < 3, x > 4$
31 a) $p = \frac{9}{4}$ b) $p < \frac{9}{4}$ c) $p > \frac{9}{4}$
32 $x < -1, x > \frac{1}{3}$

33 a) $m + \frac{1}{n} > 2 \Rightarrow mn + 1 > 2n \Rightarrow mn - 2n + 1 > 0$; since $m > n \Rightarrow mn > n^2$ it follows that $mn - 2n + 1 > n^2 - 2n + 1$ and since $n^2 - 2n + 1 = (n-1)^2 > 0$ then $mn - 2n + 1 > 0$ $\Rightarrow m + \frac{1}{n} > 2$

b) $(m+n)\left(\frac{1}{m} + \frac{1}{n}\right) > 4 \Rightarrow (m+n)\left(\frac{1}{m} + \frac{1}{n}\right)mn > 4mn \Rightarrow$ $(m+n)(n+m) > 4mn \Rightarrow m^2 + 2mn + n^2 > 4mn \Rightarrow$ $m^2 - 2mn + n^2 > 0 \Rightarrow (m-n)^2 > 0$ which is true for all x and is equivalent to original inequality – thus, $(m+n)\left(\frac{1}{m} + \frac{1}{n}\right) > 4$ is true for all x.

34 $x = \frac{-1 \pm \sqrt{13}}{2}$, $x = 1$ or $x = -2$

35 $(a+b+c)^2 < 3(a^2 + b^2 + c^2)$
$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2ac + 2bc < 3a^2 + 3b^2 + 3c^2$
$\Rightarrow 0 < 2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc$
$\Rightarrow a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + a^2 - 2ac + c^2 > 0$
$\Rightarrow (a-b)^2 + (b-c)^2 + (c-a)^2 > 0.$
Since all the numbers are unequal, the squares of their differences are strictly larger than zero therefore their sum too is strictly larger than zero.

36 a) $1 < x < 3$ b) $x < -2, -1 < x < 1, x > 3$

37 If a and b have the same sign, then $|a+b| = |a| + |b|$; and if a and b are of opposite sign, then $|a+b| < |a| + |b|$.

Practice questions

1 $x = a$ or $x = 3b$
2 $x \leqslant 4$
3 $c = 5$
4 $a = -\frac{1}{2}, b = 4, c = -2$
5 $\omega = -2, p = 2, q = -8$
6 a) $m > -2$ b) $-2 < m < 0$
7 $a = 2, b = -1, c = -2$
8 $x < 5, x > \frac{15}{2}$
9 $-1 < k < 15$
10 a) $f(x) = 2 - \frac{3}{(x+2)^2 + 1}$
b) (i) $\lim_{x \to +\infty} f(x) = 2$ (ii) $\lim_{x \to -\infty} f(x) = 2$
c) $(-2, -1)$
11 $k \in \mathbb{R}$
12 $a = -1$
13 $a = \frac{7}{4}, b = -\frac{1}{4}$
14 $a = -6$
15 $a = 4$
16 $a = -2, b = 6$
17 $a = 1$
18 $k = 6$
19 $k = 6$
20 $-2.80 < k < 0.803$ (3 s.f.)
21 $-3 \leqslant k \leqslant 4.5$
22 $-4 \leqslant m \leqslant 0$
23 $1 \leqslant x \leqslant 3$
24 $-2.30 < x < 0$ or $1 < x < 1.30$
25 $-3 \leqslant x \leqslant \frac{1}{3}$
26 $x < -1$ or $4 < x \leqslant 14$
27 $x \leqslant 3$ or $x \geqslant 27$
28 $x = 2 - i$ and $x = 2$
29 $x < \frac{1}{3}$

Chapter 4

Exercise 4.1

1 $-1, 1, 3, 5, 7$
2 $-1, 1, 5, 13, 29$
3 $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}$
4 $5, 8, 11, 14, 17$
5 $1, 7, -5, 19, -29$
6 $3, 7, 13, 21, 31$
7 $-1, 1, 3, 5, 7, 97$
8 $2, 6, 18, 54, 162, 4.786 \times 10^{23}$

9 $\frac{2}{3}, -\frac{2}{3}, \frac{6}{11}, -\frac{4}{9}, \frac{10}{27}, \frac{50}{1251}$
10 1, 2, 9, 64, 625, 1.776×10^{83}
11 3, 11, 27, 59, 123, 4.50×10^{15}
12 0, 3, $\frac{3}{7}, \frac{21}{13}, \frac{39}{55}$, approx. 1
13 2, 6, 18, 54, 162, 4.786×10^{23}
14 −1, 1, 3, 5, 7, 97
15 $u_n = \frac{1}{4}u_{n-1}, u_1 = \frac{1}{3}$
16 $u_n = \frac{4a^2}{3}u_{n-1}, u_1 = \frac{1}{2}a$
17 $u_n = u_{n-1} + a - k, u_1 = a - 5k$
18 $u_n = n^2 + 3$
19 $u_n = 3n - 1$
20 $u_n = \frac{2n-1}{n^2}$
21 $u_n = \frac{2n-1}{n+3}$
22 a) $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}$
23 a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

Exercise 4.2

1 $3, \frac{19}{5}, \frac{23}{5}, \frac{27}{5}, \frac{31}{5}, 7$
2 a) Arithmetic, $d = 2, a_{50} = 97$
b) Arithmetic, $d = 1, a_{50} = 52$
c) Arithmetic, $d = 2, a_{50} = 97$
d) Not arithmetic, *no common difference*
e) Not arithmetic, *no common difference*
f) Arithmetic, $d = -7, a_{50} = -341$
3 a) 26
b) $a_n = -2 + 4(n-1)$
c) $a_1 = -2, a_n = a_{n-1} + 4$ for $n > 1$
4 a) 1
b) $a_n = 29 - 4(n-1)$
c) $a_1 = 29, a_n = a_{n-1} - 4$ for $n > 1$
5 a) 57
b) $a_n = -6 + 9(n-1)$
c) $a_1 = -6, a_n = a_{n-1} + 9$ for $n > 1$
6 a) 9.23
b) $a_n = 10.07 - 0.12(n-1)$
c) $a_1 = 10.07, a_n = a_{n-1} - 0.12$ for $n > 1$
7 a) 79
b) $a_n = 100 - 3(n-1)$
c) $a_1 = 100, a_n = a_{n-1} - 3$ for $n > 1$
8 a) $-\frac{27}{4}$
b) $a_n = 2 - \frac{5}{4}(n-1)$
c) $a_1 = 2, a_n = a_{n-1} - \frac{5}{4}$ for $n > 1$
9 13, 7, 1, −5, −11, −17, −23
10 299, $299\frac{1}{4}, 299\frac{1}{2}, 299\frac{3}{4}$, 300
11 $a_n = -10 + 4(n-1) = 4n - 14$
12 $a_n = -\frac{142}{3} + \frac{11}{3}(n-1) = -51 + \frac{11}{3}n$
13 88
14 36
15 11
16 16
17 11
18 9, 3, −3, −9, −15
19 99.25, 99.50, 99.75
20 $a_n = 4n - 1$
21 $a_n = \frac{19n - 277}{3}$
22 $a_n = 4n + 27$
23 Yes, 3271th term
24 Yes, 1385th term
25 No

Exercise 4.3

1 Geometric, $r = 3^a, g_{10} = 3^{9a+1}$
2 Arithmetic, $d = 3, a_{10} = 27$
3 Geometric, $r = 2, b_{10} = 4096$
4 Neither, not geometric, $r = 2, c_{10} = -1534$
5 Geometric, $r = 3, u_{10} = 78\,732$
6 Geometric, $r = 2.5, a_{10} = 7629.394\,531\,25$
7 Geometric, $r = -2.5, a_{10} = -7629.394\,531\,25$
8 Arithmetic, $d = 0.75, a_{10} = 8.75$
9 Geometric, $r = -\frac{2}{3}, a_{10} = -\frac{1024}{2187}$
10 Arithmetic, $d = 3$
11 Geometric, $r = -3$
12 Geometric, $r = 2$
13 Neither
14 Neither
15 Arithmetic, $d = 1.3$
16 a) 32 b) $-3 + 5(n-1)$
c) $a_1 = -3, a_n = a_{n-1} + 5$ for $n > 1$
17 a) −9 b) $19 - 4(n-1)$
c) $a_1 = 19, a_n = a_{n-1} - 4$ for $n > 1$
18 a) 69 b) $-8 + 11(n-1)$
c) $a_1 = -8, a_n = a_{n-1} + 11$ for $n > 1$
19 a) 9.35 b) $10.05 - 0.1(n-1)$
c) $a_1 = 10.05, a_n = a_{n-1} - 0.1$ for $n > 1$
20 a) 93 b) $100 - (n-1)$
c) $a_1 = 100, a_n = a_{n-1} - 1$ for $n > 1$
21 a) $-\frac{17}{2}$ b) $2 - 1.5(n-1)$
c) $a_1 = 2, a_n = a_{n-1} - 1.5$ for $n > 1$
22 a) 384 b) $3 \times 2^{n-1}$
c) $a_1 = 3, a_n = 2a_{n-1}$ for $n > 1$
23 a) 8748 b) $4 \times 3^{n-1}$
c) $a_1 = 4, a_n = 3a_{n-1}$ for $n > 1$
24 a) −5 b) $5 \times (-1)^{n-1}$
c) $a_1 = 5, a_n = -a_{n-1}$ for $n > 1$
25 a) −384 b) $3 \times (-2)^{n-1}$
c) $a_1 = 3, a_n = -2a_{n-1}$ for $n > 1$
26 a) $-\frac{4}{9}$ b) $972 \times (-\frac{1}{3})^{n-1}$
c) $a_1 = 972, a_n = (-\frac{1}{3})a_{n-1}$ for $n > 1$
27 a) $\frac{2187}{64}$ b) $a_n = -2\left(-\frac{3}{2}\right)^{n-1}$
c) $a_1 = -2, a_n = -\frac{3}{2}a_{n-1}, n > 1$
28 a) $\frac{390\,625}{117\,649}$ b) $a_n = 35\left(\frac{5}{7}\right)^{n-1}$
c) $a_1 = 35, a_n = \frac{5}{7}a_{n-1}, n > 1$
29 a) $-\frac{3}{64}$ b) $a_n = -6\left(\frac{1}{2}\right)^{n-1}$
c) $a_n = -6, a_n = \frac{1}{2}a_{n-1}, n > 1$
30 a) 1216 b) $9.5 \times 2^{n-1}$
c) $a_1 = 9.5, a_n = 2a_{n-1}, n > 1$
31 a) $69.833\,729\,609\,375 = \frac{893\,871\,739}{12\,800\,000}$
b) $a_n = 100\left(\frac{19}{20}\right)^{n-1}$
c) $a_1 = 100, a_n = \frac{19}{20}a_{n-1}, n > 1$
32 a) $0.002\,085\,685\,73 = \frac{2187}{1\,048\,576}$
b) $a_n = 2\left(\frac{3}{8}\right)^{n-1}$ c) $a_1 = 2, a_n = \frac{3}{8}a_{n-1}, n > 1$
33 6, 12, 24, 48
34 35, 175, 875
35 36
36 21, 63, 189, 567
37 −24, 24
38 1.5, $a_n = 24\left(\frac{1}{2}\right)^{n-1}$
39 $a_4 = \pm 3, r = \pm\frac{1}{2}, a_n = 24\left(\pm\frac{1}{2}\right)^{n-1}$
40 $\frac{49}{3}$
41 10th term
42 Yes, 10th term
43 Yes, 10th term
44 2228.92
45 £945.23
46 €2968.79
47 7745 thousands
48 $\frac{98}{9}$

49 10th term

50 €3714.87

51 £2921.16

Exercise 4.4

1 11 280

2 $-\frac{10\,5469}{1024}$

3 0.7

4 $\frac{10}{7}$

5 $\frac{16 + 4\sqrt{3}}{39}$

6 a) $\frac{52}{99}$ b) $\frac{449}{990}$ c) $\frac{7459}{2475}$

7 13 026.135 (£13 026.14)

8 940

9 6578

10 42 625

11 $\frac{n(7 + 3n)}{2}$

12 17 terms

13 85 terms

14 $d = 4$

15 a) 250, 125 250, b) 83 501

16 $a = 1, d = 5$

17 2890

18 0.290

19 -2.065

20 11 400

21 1.191

22 49.2

23 $\frac{6}{5}$

24 $\frac{3 + \sqrt{6}}{2}$

25 $3, \frac{18}{5}, \frac{93}{25}, \frac{468}{125}, \frac{15}{4}\left(1 - \frac{1}{5^n}\right)$

26 $\frac{1}{6}, \frac{1}{4}, \frac{3}{10}, \frac{1}{3}; \frac{n}{2n + 4}$

27 $\sqrt{2} - 1, \sqrt{3} - 1, 1, \sqrt{5} - 1; \sqrt{n + 1} - 1$

28 1.945, 152.42

29 127, 128

30 $\frac{819}{128}, \frac{32}{5}$

31 11 866

32 763 517

33 14 348 906

34 ≈ 150

Exercise 4.5

1 a) 120 b) 120 c) 20 d) 336

2 a) 1 b) 1 c) 120 d) 120

3 a) 70 b) 70 c) 330 d) 330

4 a) 0 b) 39 916 800 c) 0 d) 10

5 a) F b) F c) T

6 24

7 72

8 312

9 16 777 216

10 262 144

11 1 757 600 000

12 81 000

13 a) 40 320 b) 384

14 a) 40 320 b) 720

15 JANE, JAEN, JNAE, JNEA, JEAN, JENA, AJNE, AJEN, ANJE, ANEJ, AEJN, AENJ, NJAE, NJEA, NEJA, NEAJ, NAJE, NAEJ, EJAN, EJNA, EAJN, EANJ, ENJA, ENAJ

16 Mag, Mga, Mai, …(60 of them)

17 a) 175 760 000 b) 174 790 000

18 a) 4080 b) 1680 c) 1050 d) 1980 e) 3150

19 a) 296 b) 1460 c) 504

20 a) 125 000 b) 117 600 c) 61 250 d) 176 400

21 768

22 a) 36 b) 256

23 a) 5985 b) 2376 c) 2475

24 a) 2280 b) 748 c) 770

25 a) 1 192 052 400 b) 4560, 0.000 38% c) 265 004 096, 22.2%

26 a) 74 613 b) 7560

27 54 867 456 000

Exercise 4.6

1 a) $x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

b) $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

c) $x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729$

d) $16 - 32x^3 + 24x^6 - 8x^9 + x^{12}$

e) $x^7 - 21bx^6 + 189b^2x^5 - 945b^3x^4 + 2835b^4x^3 - 5103b^5x^2 + 5103b^6x - 2187b^7$

f) $64n^6 + 192n^3 + 240 + \frac{160}{n^3} + \frac{60}{n^6} + \frac{12}{n^9} + \frac{1}{n^{12}}$

g) $\frac{81}{x^4} - \frac{216}{x^2\sqrt{x}} + \frac{216}{x} - 96\sqrt{x} + 16x^2$

2 a) 56 b) 0 c) 1225 d) 32 e) 0

3 a) $x^7 + 14x^6y + 84x^5y^2 + 280x^4y^3 + 560x^3y^4 + 672x^2y^5 + 448xy^6 + 128y^7$

b) $a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$

c) $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$

d) $x^{18} - 12x^{15} + 60x^{12} - 160x^9 + 240x^6 - 192x^3 + 64$

e) $x^7 - 21bx^6 + 189b^2x^5 - 945b^3x^4 + 2835b^4x^3 - 5103b^5x^2 + 5103b^6x - 2187b^7$

f) $64n^6 + 192n^3 + 240 + \frac{160}{n^3} + \frac{60}{n^6} + \frac{12}{n^9} + \frac{1}{n^{12}}$

g) $\frac{81}{x^4} - \frac{216}{x^2\sqrt{x}} + \frac{216}{x} - 96\sqrt{x} + 16x^2$

h) 112 i) $1792\sqrt{3}$

j) 16 k) $-23 + 10i\sqrt{2}$

4 a) $x^{45} - 90x^{43} + 3960x^{41}$

b) Does not exist as the powers of x decrease by 2's starting at 45. There is no chance for any expression to have zero exponent.

c) $\binom{45}{43}x^2\left(\frac{-2}{x}\right)^{43} + \binom{45}{44}x\left(\frac{-2}{x}\right)^{44} + \left(\frac{-2}{x}\right)^{45} = -\binom{45}{43}\frac{2^{43}}{x^{41}} + \binom{45}{44}\frac{2^{44}}{x^{43}} - \frac{2^{45}}{x^{45}}$

d) $\binom{45}{21}x^{24}\left(\frac{-2}{x}\right)^{21} = -\binom{45}{21} \cdot 2^{21}x^3$

5 $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$

6 $(1 + 1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}$

$2^n = 1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} \Rightarrow 2^n - 1 = \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{n}$

7 Answers vary

8 $\left(\frac{1}{3} + \frac{2}{3}\right)^6 = 1$

9 $\left(\frac{2}{5} + \frac{3}{5}\right)^8 = 1$

10 $\left(\frac{1}{7} + \frac{6}{7}\right)^n = 1$

11 15

12 90 720

13 16 128

14 1.1045, 0.9045

15 Proof

16 a) $\frac{7}{9}$ b) $\frac{38}{110}$ c) $\frac{31\,808}{9900}$

17 $-145\,152$

18 $35a^3$

19 96 096

20 $243n^5 - 810n^4m + 1080n^3m^2 - 720n^2m^3 + 240nm^4 - 32m^5$

21 7 838 208

Exercise 4.7

1 $2 + 4 + 6 + \ldots + 2n = n(n + 1)$

2–20 All proofs

Practice questions

1 $D = 5, n = 20$

2 €2098.63

3 a) Nick: 20 Charlotte: 17.6
b) Nick: 390 Charlotte: 381.3
c) Charlotte will exceed the 40 hours during week 14.
d) In week 12 Charlotte will catch up with Nick and exceed him.

4 a) Loss for the second month = 1060 g
Loss for the third month = 1123.6 g
b) Plan A loss = 1880 g
Plan B loss = 1898.3 g
c) (i) Loss due to plan A in all 12 months = 17 280 g
(ii) Loss due to Plan B in all 12 months = 16 869.9 g

5 a) €895.42 b) €6985.82

6 a) 142.5 b) 19 003.5

7 $1, \sqrt[3]{7}, 1, 1, \sqrt[3]{7}, 1, \ldots; 2, 0, 2, 0, 2, \ldots$

8 a) On the 37th day b) 407 km

9 a) 1.5 b) 207 595
c) 2009 d) 619 583
e) Market saturation

10 $-4, 3006$

11 a) $\sqrt{\frac{1}{4}+\frac{1}{4}} = \frac{\sqrt{2}}{2}$ b) $\frac{1}{2}$
c) (i) $\frac{1}{4}$ (ii) $\frac{1}{2}$ d) (i) $\frac{1}{512}$ (ii) 2

12 a) 1220 b) 36 920

13 a) Area A = 1, Area B = $\frac{1}{9}$ b) $\frac{1}{81}$
c) $1 + \frac{8}{9}, 1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2$ d) 0

14 a) Neither, geometric converging, arithmetic, geometric diverging
b) 6

15 a) (i) Kell: 18 400, 18 800; YBO: 18 190, 19 463.3
(ii) Kell: 198 000; YBO: 234 879.62
(iii) Kell: 21 600; YBO: 31 253.81
b) (i) After the second year
(ii) 4th year

16 a) 62 b) 936

17 a) $7000(1 + 0.0525)^t$ b) 7 years
c) Yes, since $10\,084.7 > 10\,015.0$

18 a) 11 b) 2 c) 15

19 $15, -8$ **20** $-2, -7$ **21** 10 300

22 Proof

23 a) $a_n = 8n - 3$ b) 50

24 2 099 520

25 $6n - 5$ **26** 72 **27** 559

28 $-3, 3$ **29** 9 **30** 62

31 $-\frac{36}{5}$

32 a) 4 b) $16(4^n - 1)$

33 a) $|x| < 1.5$ b) 5

34 3168

35 a) $\frac{n(3n+1)}{2}$ b) 30

36 -7

37 $1275 \ln 2$

38 a) 4, 8, 16
b) (i) $u_n = 2^n$ (ii) proof

39 a) $\frac{2}{3}$ b) 9

40 $2, -3$ **41** 55 **42** $-2, 4$

43 $\frac{\theta}{1 - \cos\theta}$

44 a) 1, 5, 9 b) $4n - 3$

45 a) $32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$
a) 32.808 040 1001

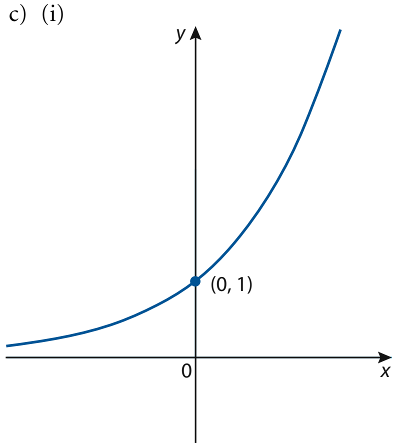
46 a) $5000(1.063)^n$ b) 6786.35
c) (i) $5000(1.063)^n > 1000$ (ii) 12

47 Proof **48** 7

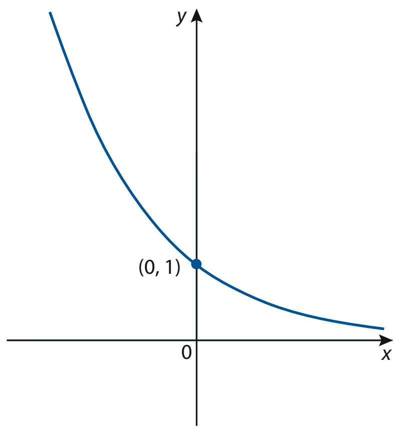
Chapter 5

Exercise 5.1 and 5.2

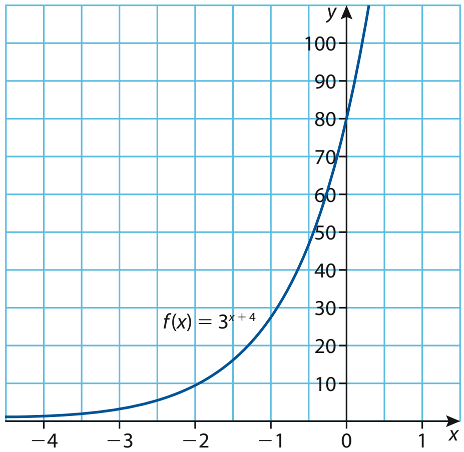
1 a) $y = b^x$
b) Domain $\{x : x \in \mathbb{R}\}$, range $\{y : y > 0\}$
c) (i)



(ii)

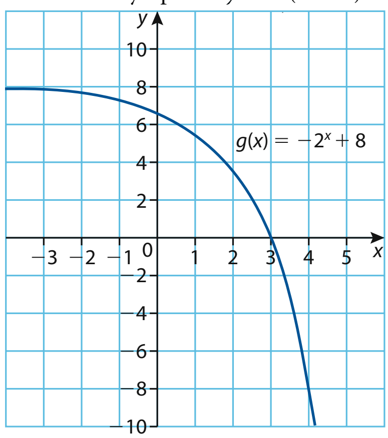


2



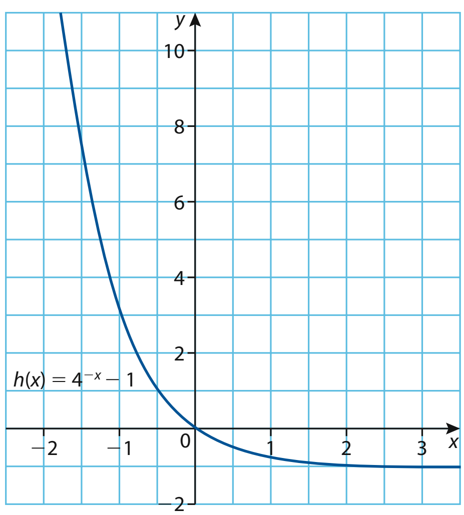
domain: $x \in \mathbb{R}$
range: $y > 0$
y-intercept: $(0, 81)$
horizontal asymptote: $y = 0$ (x-axis)

3



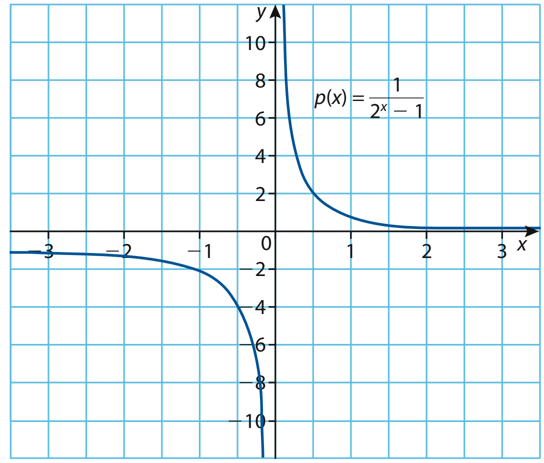
domain: $x \in \mathbb{R}$ range: $y < 8$
y-intercept: $(0, 7)$ horizontal asymptote: $y = 8$

4



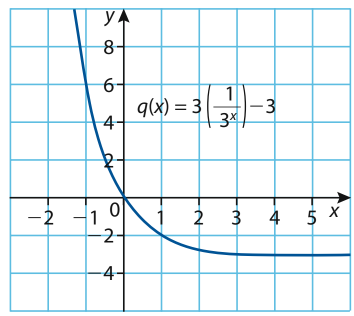
domain: $x \in \mathbb{R}$ range: $y > -1$
y-intercept: $(0, 0)$ horizontal asymptote: $y = -1$

5



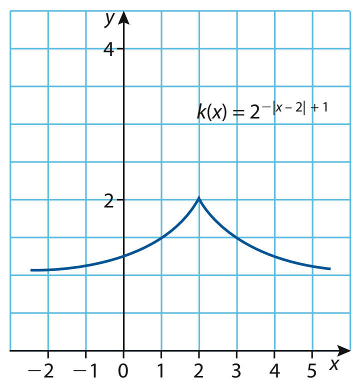
domain: $x \in \mathbb{R}, x \neq 0$ range: $y < -1$ or $y > 0$
y-intercept: none
horizontal asymptotes: $y = 0$ and $y = -1$

6



domain: $x \in \mathbb{R}$ range: $y > -3$
y-intercept: $(0, 0)$ horizontal asymptote: $y = -3$

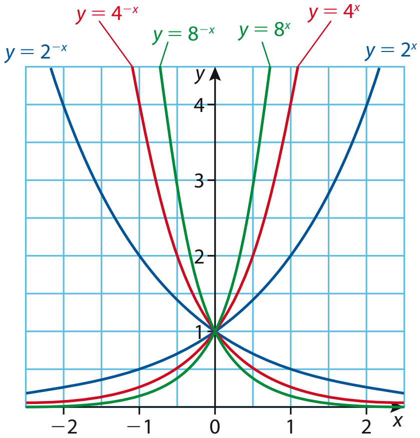
7



domain: $x \in \mathbb{R}$ range: $y > 1$
y-intercept: $\left(0, \frac{5}{4}\right)$ horizontal asymptote: $y = 1$

8 Domain: $x \in \mathbb{R}$
range: if $a > 0 \Rightarrow y > d$, if $a < 0 \Rightarrow y < d$
y-intercept: $\left(0, a(b)^{-c} + d\right)$ horizontal asymptote: $y = d$

9



10 a) $y = \left(\frac{1}{2}\right)^x$ b) $y = \left(\frac{1}{4}\right)^x$ c) $y = \left(\frac{1}{8}\right)^x$

11 $y = b^x$ is steeper

12 $P(t) = 100\,000(3)^{\frac{t}{25}}$ where t is number of years
a) 900 000 b) 2 167 402 c) 8 100 000

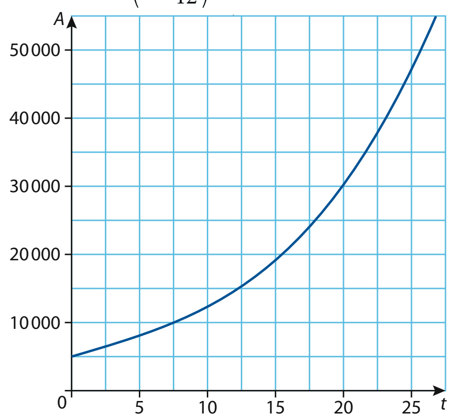
13 $N(t) = 10^4(2)^{\frac{t}{3}}$
a) 20 000 b) 80 000
c) 5 120 000 d) 10 485 760 000

14 a) $A(t) = A_0(2)^{\frac{t}{10}}$ b) 7.18%

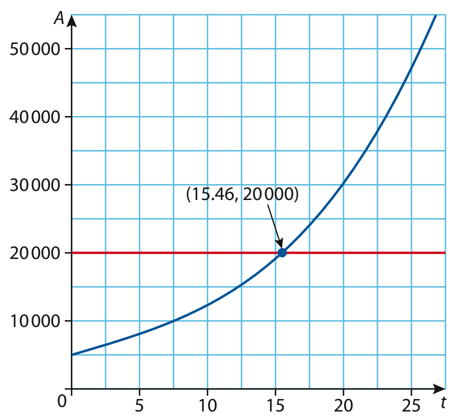
15 a) \$17 204.28 b) \$29 598.74
c) \$50 922.51

16 a) $A(t) = 5000\left(1 + \frac{.09}{12}\right)^{12t}$

b)



c)



minimum number of years is 16

17 a) \$16 850.58 b) \$17 289.16
c) \$17 331.09 d) \$17 332.47

18 a) \$2 b) \$2.61 c) \$2.71 d) \$2.72 e) \$2.72

19 a) 240 310 b) 192 759

20 8.90%

21 $0.0992A_0$ (or 9.92% of A_0 remains)

22 a) $A(w) = 1000(0.7)^w$ b) About 20 weeks

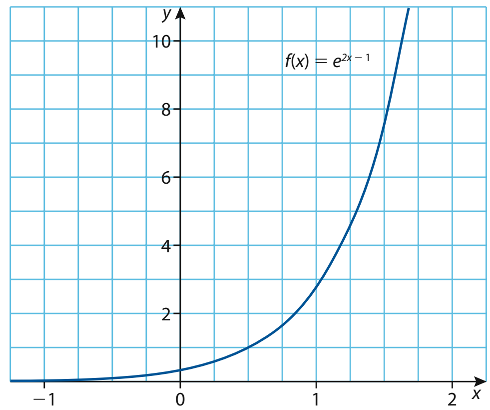
23 $b > 0$ because if $b = 0$ then the result is always zero, and if $b < 0$ then b^x gives a positive result when x is an even integer and a negative result when x is an odd integer.

24 Payment plan I: \$465; payment plan II: \$10 737 418.23

25 a) $a = 2, k = 3$ b) $a = \frac{1}{3}, k = 2$
c) $a = 3, k = -4$ d) $a = 10, k = \frac{3}{2}$

Exercise 5.3

1

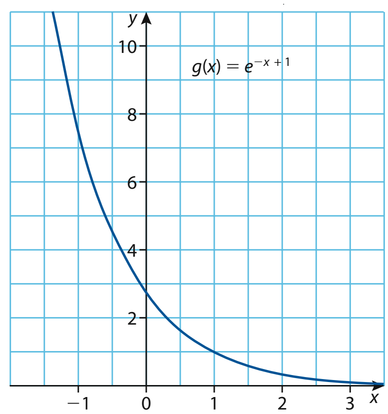


a) Domain: $x \in \mathbb{R}$, range: $y > 0$

b) x-intercept: none, y-intercept: $\left(0, \frac{1}{e}\right)$

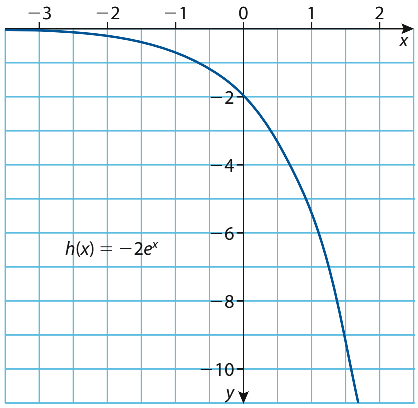
c) Horizontal asymptote: $y = 0$

2



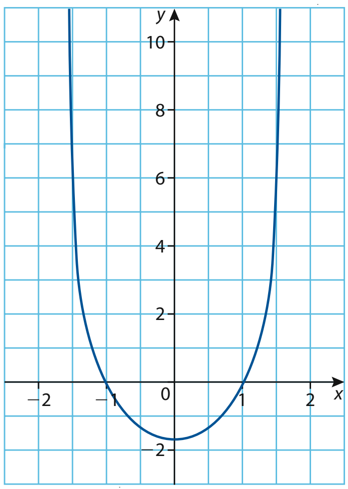
a) Domain: $x \in \mathbb{R}$, range: $y < 0$
b) x-intercept: none, y-intercept: $(0, e)$
c) Horizontal asymptote: $y = 0$

3



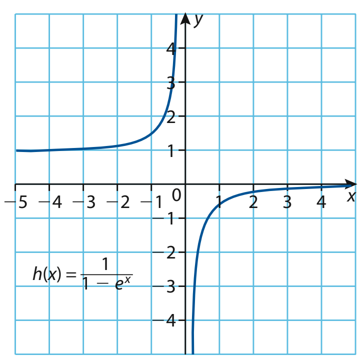
a) Domain: $x \in \mathbb{R}$, range: $y < 0$
b) x-intercept: none, y-intercept: $(0, -2)$
c) Horizontal asymptote: $y = 0$

4



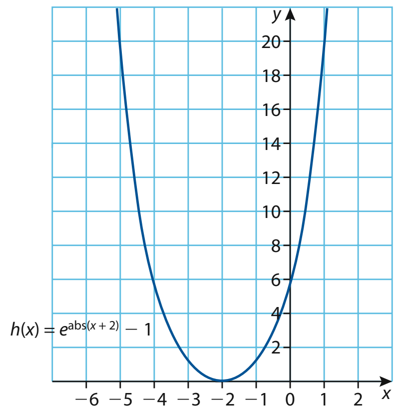
a) Domain: $x \in \mathbb{R}$, range: $y \geqslant 1 - e$
b) x-intercept: $(-1, 0)$ and $(1, 0)$, y-intercept: $(0, 1 - e)$
c) No asymptotes

5



a) Domain: $x \in \mathbb{R}, x \neq 0$, range: $y < 0, y > 1$
b) x-intercept: none, y-intercept: none
c) Horizontal asymptotes: $y = 0$ and $y = 0$

6



a) Domain: $x \in \mathbb{R}$, range: $y \geqslant 0$
b) x-intercept: $(-2, 0)$, y-intercept: $(0, e^2 - 1)$
c) No asymptotes

7 a) $e = \lim_{x \to \infty}\left(1 + \frac{1}{n}\right)^n$
b) 0.366 032 3413, 0.367 861 0464, 0.367 879 2572
c) 0.367 88; reciprocal of e, $\frac{1}{e} \approx 0.367\,879\,4412$

8 $y = \left(x + \frac{1}{x}\right)^x$ will not intersect $y = 2.72$ because

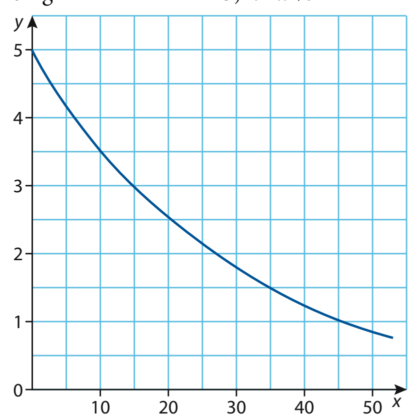
$\lim_{x \to \infty}\left(x + \frac{1}{x}\right)^x = e \approx 2.718\,281\,828\ldots < 2.72$

9 Bank A: earn 608.79 euros in interest.
Bank B: earn 609.16 euros in interest.
Bank B account earns 0.37 euros more in interest.

10 Blue Star has greater total of \$1358.42 which is \$11.93 more than the Red Star.

11 a) 0.976 kg b) 0.787 kg c) 0.0916 kg d) 0.002 54 kg

12 a) 5 kg b) 71.7%
c)



d) 20 days

13 a) $8\frac{1}{2}$% compounded semi-annually is the better investment.

14 a) $r \approx 1.070\,37$ (6 s.f.) b) 7.037% (4 s.f.)

15 a) Less than 1 b) Less than 1
c) Greater than 1 d) Greater than 1

16 a) £1568.31, £2459.60
b) 15.4 years
c) 15.4 years
d) Same; doubling time is independent of initial amount

Exercise 5.4

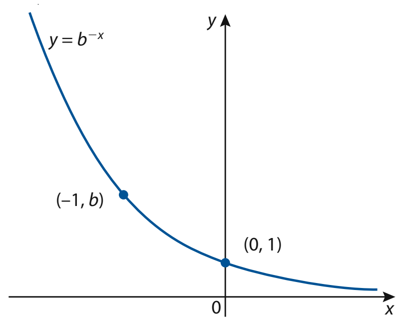
1 $2^4 = 16$ **2** $e^0 = 1$ **3** $10^2 = 100$
4 $10^{-2} = 0.01$ **5** $7^3 = 343$ **6** $e^{-1} = \frac{1}{e}$
7 $10^y = 50$ **8** $e^{12} = x$ **9** $e^3 = x + 2$
10 $\log_2 1024 = 10$ **11** $\log_{10} 0.0001 = -4$ **12** $\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$
13 $\log_3 81 = 4$ **14** $\log_{10} 1 = 0$ **15** $\ln 5 = x$
16 $\log_2 0.125 = -3$ **17** $\ln y = 4$
18 $\log_{10} y = x + 1$ **19** 6 **20** 3
21 -3 **22** 5 **23** $\frac{3}{4}$ **24** $\frac{1}{3}$
25 -3 **26** 13 **27** 0 **28** 6
29 -3 **30** $\sqrt{2}$ **31** 3 **32** $\frac{1}{2}$
33 -2 **34** 88 **35** $\frac{1}{2}$ **36** 18
37 $\frac{1}{3}$ **38** π **39** 1.6990 **40** 0.2386
41 3.912 **42** 0.5493 **43** 1.398 **44** 0.2090
45 4.605 **46** 13.82 **47** $x > 2$ **48** $x \in \mathbb{R}$
49 $x > 0$ **50** $x < \frac{8}{5}$ **51** $-2 \leqslant x < 3$ **52** $x < 0$
53 Domain $\{x : x > 0,\ x \neq 1\}$, range $\{y : y \in \mathbb{R},\ y \neq 0\}$
54 Domain $\{x : x > 1\}$, range $\{y : y \geqslant 0\}$
55 Domain: $x > 0, x \neq 1$, range: $y < 0$
56 $f(x) = \log_4 x$ **57** $f(x) = \log_2 x$
58 $f(x) = \log_{10} x$ **59** $f(x) = \log_3 x$
60 $\log_2 2 + \log_2 m = 1 + \log_2 m$ **61** $\log 9 - \log x$
62 $\frac{1}{5}\ln x$ **63** $\log a + 3\log b$
64 $\log 10x + \log(1+r)^t = \log 10 + \log x + t\log(1+r)$
65 $3\ln m - \ln n$ **66** $\log_b p + \log_b q + \log_b r$
67 $2\log_b p + 3\log_b q - \log_b r$ **68** $\frac{\log_b p}{4} + \frac{\log_b q}{4}$
69 $\frac{\log_b q}{2} + \frac{\log_b r}{2} - \frac{\log_b p}{2}$ **70** $\log_b p + \frac{1}{2}\log_b q - \log_b r$
71 $3\log_b p + 3\log_b q - \frac{1}{2}\log_b r$ **72** $\log x$
73 $\log_3 72$ **74** $\ln\left(\frac{y^4}{4}\right)$ **75** $\log_b 4$ **76** $\log\left(\frac{p}{qr}\right)$
77 $\ln\left(\frac{36}{e}\right)$ **78** 9.97 **79** -5.32 **80** 2.06
81 -0.179 **82** 4.32 **83** 1.86
84 $\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$ **85** $\log e = \frac{\ln e}{\ln 10} = \frac{1}{\ln 10}$
86 $dB = 10\log\left(\frac{I}{10^{-16}}\right) = 10(\log I - \log 10^{-16}) = 10(\log I + 16)$
$= 10\log 10^{-4} + 160 = 10(-4) + 160 = 120$ decibels

Exercise 5.5

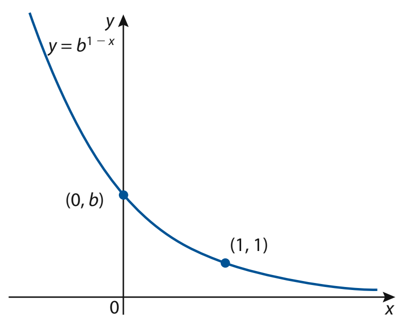
1 0.699 **2** 2.5 **3** 7.99 **4** 3.64
5 -1.92 **6** 2.71 **7** 0.434 **8** 2.12
9 4.42 **10** 0.225 **11** 0.642 **12** 22.0
13 3 **14** 0 or -1 **15** $\frac{\ln\left(\frac{3}{2}\right)}{\ln 6}$ or $\frac{\ln\left(\frac{4}{3}\right)}{\ln 6}$
16 1 or -1
17 a) \$6248.58 b) $9\frac{1}{4}$ years
18 12.9 years
19 20 hours (≈ 19.93)
20 a) 24 years (≈ 23.45) b) 12 years (≈ 11.9)
c) 9 years (≈ 8.04)
21 6 years
22 a) 99.7% b) 139 000 years
23 a) 37 dogs b) 9 years
24 a) 458 litres b) 8.89 minutes $\approx$ 8 min. 53 seconds
c) 39 minutes
25 a) 5 kg b) 17.7 days
26 $x = \frac{20}{3}$ **27** $x = 104$ **28** $x = \frac{1}{e^3}$
29 $x = 4$ **30** $x = 98$ **31** $x = \pm\sqrt{e^{16}} \approx \pm 2980.96$
32 $x = 2$ or $x = 4$ **33** $x = 9$ **34** $x = \frac{13}{5}$
35 $x = 3$ **36** $x = 1$ or $x = 100$
37 $x > \frac{1}{\sqrt[5]{100}}$ **38** $x < 2$ **39** $0 < x < \ln 6$
40 $0.161 < x < 1.14$ (approx. to 3 s.f.)

Practice questions

1 a) $(8, 0)$ b) $(0, 2)$ c) $\left(-\frac{2}{3}, 3\right)$
2 a) 183 g (3 s.f.) b) 154 years (3 s.f.)
3 a) $a_n = \ln(y^n)$, $S_n = \frac{n(n+1)}{2}\ln y$
b) $a_n = \ln(xy^n)$, $S_n = n\ln x + \frac{n(n+1)}{2}\ln y$
4 $x = 2$ **5** $y = 16$ **6** $x = 0$, $\ln\left(\frac{1}{2}\right)$ or $-\ln$
7 $x = e^{-4e}$ or e^{2e}
8 a) $x = 3$ b) $x = 6$
9 a) $\log\left(\frac{a^2b^3}{c}\right)$ b) $\ln\left(\frac{ex^3}{\sqrt{y}}\right)$
10 1900 years
11 $c = 22$
12 a)



b)



13 a) $k \approx 0.000\,4332$ b) 17.7% (3 s.f.)
14 $x \approx 1.28$
15 $1.52 < x < 1.79 \cup 17.6 < x < 19.1$
16 $-1 < x < -0.800 \cup x > 1$
17 a) $x = -\frac{1}{2}$ or $x = 0$
b) $x = \frac{1}{\ln a - 2}$ or $x = \frac{\log_a e}{1 - 2\log_a e}$
c) $a = e^2$
18 $a = -2, b = 3$

19 $x = \sqrt{e}$, $x = e$

20 a) $V = \$265.33$ b) 235 months

21 $x = 5^{\frac{5}{3}}$ or $x = 5^{\frac{-5}{3}}$

22 $x = e - 3$ or $x = \frac{1}{e} - 3$

23 $x = -2.50, -1.51$ or 0.440 (3 s.f.)

24 $k = \frac{\ln 2}{20}$

25 a) $f(x) = \ln\left(\frac{x}{x+2}\right)$ b) $f^{-1}(x) = -\frac{2e^x}{e^x - 1}$ or $\frac{2e^x}{1 - e^x}$

26 a) (i) Minimum value of f is 0.

(ii) From part (i) $f(x) \geqslant 0 \Rightarrow e^x - 1 - x \geqslant 0 \Rightarrow e^x \geqslant 1 + x$

d) $n > e^{100}$

Chapter 6

Exercise 6.1 and 6.2

1 a) (i) $\begin{pmatrix} x-1 & x-3 \\ y+3 & y+1 \end{pmatrix}$ (ii) $\begin{pmatrix} -x-7 & 3x+3 \\ 3y-7 & 11-y \end{pmatrix}$

b) $x = -3, y = 5$ c) $x = 3, y = -3$

d) $AB = \begin{pmatrix} 2x-2 & xy-2x+6 \\ xy-x+y+11 & -3 \end{pmatrix}$;

$BA = \begin{pmatrix} -2x-3y+1 & x^2+x-9 \\ y^2-3y-6 & 4x+3y-6 \end{pmatrix}$

2 a) $x = 2, y = -10$ b) $p = 2, q = -4$

3 a) $\begin{bmatrix} 0&1&0&0&1&2&0 \\ 1&0&1&1&1&1&0 \\ 0&1&0&2&0&0&2 \\ 0&1&2&0&1&0&0 \\ 1&1&0&1&0&1&0 \\ 2&1&0&0&1&0&0 \\ 0&0&2&0&0&0&0 \end{bmatrix}$ b) $\begin{bmatrix} 6&3&1&2&3&2&0 \\ 3&5&2&3&3&3&2 \\ 1&2&9&1&3&1&0 \\ 2&3&1&6&1&2&4 \\ 3&3&3&1&4&3&0 \\ 2&3&1&2&3&6&0 \\ 0&2&0&4&0&0&4 \end{bmatrix}$

Matrix signifies the number of routes between each pair that go via one other city.

4 a) $A + C = \begin{pmatrix} x+1 & 10 & y+1 \\ 0 & -x-3 & y+3 \\ 2x+y+7 & x-3y & -x+2y-1 \end{pmatrix}$

b) $\begin{pmatrix} 17m+2 & -6 \\ 4-9m & 9 \\ 7m-2 & -17 \end{pmatrix}$

c) Not possible d) $x = 3, y = 1$

e) Not possible f) $m = 3$

5 $a = -3, b = 3, c = 2$

6 $x = 4, y = -3$

7 $m = 2, n = 3$

8 Shop A: €18.77

9 a) $\begin{pmatrix} 2 & 4 \\ -2 & 12 \end{pmatrix}$ b) associative

c) $\begin{pmatrix} -22 & 16 \\ 60 & -7 \end{pmatrix}$ d) associative

10 $AB = [88\ 142]$, which represents total profit.

11 $r = 3, s = -2$

12 a) (i) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ (iv) $\begin{pmatrix} 1^n & n \\ 0 & 3^n \end{pmatrix}$

b) (i) $\begin{pmatrix} 9 & 18 \\ 0 & 9 \end{pmatrix}$ (ii) $\begin{pmatrix} 27 & 81 \\ 0 & 27 \end{pmatrix}$

(iii) $\begin{pmatrix} 81 & 324 \\ 0 & 81 \end{pmatrix}$ (iv) $\begin{pmatrix} 3^n & 3^{n+1} \\ 0 & 3^n \end{pmatrix}$

13 $\left(\frac{11}{3}, \frac{8}{3}\right)$ **14** $(1, -4)$

15 5 **16** $(5, 1)$

Exercise 6.3

1 a) $\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$ b) $M = \begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$

c) $\begin{pmatrix} -39 & -44 \\ 17 & 19 \end{pmatrix}$

d) (i) $N = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$ (ii) $N = \begin{pmatrix} -14 & -11 \\ -7 & -6 \end{pmatrix}$

e) If $AB = C$ then $B = A^{-1}C$, while if $BA = C$, then $B = CA^{-1}$. Also, $A^{-1}C \neq CA^{-1}$.

2 $\begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & 0 \end{pmatrix}$

3 a) $|A| = -5 \neq 0$ b) $\begin{pmatrix} \frac{9}{5} & \frac{11}{5} & -\frac{8}{5} \\ \frac{6}{5} & \frac{9}{5} & -\frac{7}{5} \\ 1 & 1 & -1 \end{pmatrix}$ c) $\begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{5} \end{pmatrix}$

4 a) $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ b) $\begin{pmatrix} \frac{3}{a}+1 & -1 \\ -a-2 & a \end{pmatrix}$

5 $x = 2$ or $x = 3$

6 $n = 0.5$

7 a) $X = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{3}{4} & -\frac{7}{6} \end{pmatrix}$ b) $Y = \begin{pmatrix} 1 & \frac{13}{12} \\ -1 & -\frac{5}{3} \end{pmatrix}$

c) $X \neq Y$ — not commutative

8 a) $PQ = \begin{pmatrix} 5 & -4 & 3 \\ 33 & 5 & -1 \\ 2 & -3 & 2 \end{pmatrix}$, $QP = \begin{pmatrix} 4 & -5 & -8 \\ 8 & 0 & -4 \\ 7 & 10 & 8 \end{pmatrix}$

b) $P^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ -\frac{7}{5} & \frac{1}{5} & \frac{11}{5} \\ 1 & 0 & -2 \end{pmatrix}$, $Q^{-1} = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 1 & -1 & 1 \\ 2 & -\frac{7}{4} & 1 \end{pmatrix}$

$P^{-1}Q^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ \frac{23}{5} & -\frac{22}{5} & \frac{12}{5} \\ -4 & \frac{15}{4} & -2 \end{pmatrix}$

$Q^{-1}P^{-1} = \begin{pmatrix} -\frac{7}{20} & \frac{1}{20} & \frac{11}{20} \\ \frac{17}{5} & -\frac{1}{5} & -\frac{26}{5} \\ \frac{109}{20} & -\frac{7}{20} & -\frac{157}{20} \end{pmatrix}$

$(PQ)^{-1} = \begin{pmatrix} -\frac{7}{20} & \frac{1}{20} & \frac{11}{20} \\ \frac{17}{5} & -\frac{1}{5} & -\frac{26}{5} \\ \frac{109}{20} & -\frac{7}{20} & -\frac{157}{20} \end{pmatrix}$

$(QP)^{-1} = \begin{pmatrix} -2 & 2 & -1 \\ \frac{23}{5} & -\frac{22}{5} & \frac{12}{5} \\ -4 & \frac{15}{4} & -2 \end{pmatrix}$

9 a) $\begin{pmatrix} -7 \\ 3 \\ -2 \end{pmatrix}$ b) $\begin{pmatrix} -7 \\ 3 \\ -2 \end{pmatrix}$

10 $x = -1$ **11** $x = 1, y = 2$

12 $(0, 1)$ **13** $(-3, -29), (0, 1)$

14 $17x - 8y + 37 = 0$; $y + 2 = 0$; $x + 5 = 0$ **15** 165; 80; 136

16 $x = \frac{89}{2}$ or $x = \frac{129}{8}$; $x = -4$ or $x = -2$ or $x = -3 \pm\sqrt{21}$

17 $-3; 3$

18 a) -25
b) $x^2 - 7x - 25$, constant $= \det(A)$
c) $-(a + d)$
d) $f(A) = 0$
e) $ad - bc$; $x^2 - (a + d)x + (ad - bc)$,
constant $= \det(A)$; $f(A) = 0$

19 a) -22
b) $x^3 - x^2 - 22x + 22$, constant $= -\det(A)$
c) Opposite of the sum of the main diagonal
d) $f(A) = 0$

Exercise 6.4

1 $\mathbf{m} = 2$ or $\mathbf{m} = 3$

2 a) $a = 7, b = 2$ b) $(-1, 2, -1)$

3 $\mathbf{m} = 2$

4 a) $(-1, 3, 2)$ b) $(5, 8, -2)$
c) $\left(\frac{13}{16} + \frac{5}{16}t, \frac{11}{16} + \frac{19}{16}t, t\right)$ d) $(-7, 3, -2)$
e) $(-1 + 2t, 2 - 3t, t)$ f) inconsistent
g) $(-2, 4, 3)$ h) $(4, -2, 1)$

5 a) $k \neq \frac{-1 \pm \sqrt{33}}{4}$ b) $k = 1$
c) $\begin{pmatrix} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & 3 & 3 & -1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{pmatrix}$

6 a) $\frac{71 \pm i\sqrt{251}}{42}$ b) $k = 2$
c) $\begin{pmatrix} 1 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{2}{5} & \frac{4}{5} & \frac{-3}{5} \\ 0 & 0 & 1 & \frac{3}{5} & \frac{6}{5} & -1 \end{pmatrix}$

7 $\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{2}{3} & -\frac{5}{6} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$; $\begin{pmatrix} 1 & 0 & 0 & 2 & \frac{-16}{13} & \frac{-19}{13} \\ 0 & 1 & 0 & 1 & \frac{-11}{13} & \frac{-9}{13} \\ 0 & 0 & 1 & -1 & \frac{12}{13} & \frac{11}{13} \end{pmatrix}$
B is the inverse of A

8 a) $f(x) = 4x^2 - 6x - 5$
b) $f(x) = \frac{1}{2}(m - 27)x^2 + \frac{3}{2}(17 - m)x + m,\ m \in \mathbb{R}$
c) $f(x) = 3x^3 - 2x^2 - 7x + 3$
d) $f(x) = \frac{1}{6}(4 - m)x^3 + \frac{1}{3}(4 - m)x^2 - \frac{5}{6}(4 - m)x + m,\ m \in \mathbb{R}$

9 $m = 2$, $\begin{pmatrix} -t - \frac{3}{5} \\ -t - \frac{19}{5} \\ 5t \end{pmatrix}$ **10** $m = -1$, $\begin{pmatrix} 7t - \frac{9}{5} \\ \frac{3}{5} - 11t \\ 5t \end{pmatrix}$

11 a) 3 b) $\begin{pmatrix} 3 & -4 & -6 \\ 0 & -2 & -3 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$ c) 3
d) -1672 e) $\begin{pmatrix} 2 & 1 & -3 & 5 \\ 0 & 1 & 2 & -16 \\ 0 & 0 & 36 & -184 \\ 0 & 0 & 0 & -\frac{209}{9} \end{pmatrix}$ f) -1672

Practice questions

1 $x = -7$ or $x = 1$

2 a) $\begin{pmatrix} a^2 + 4 & 2a - 2 \\ 2a - 2 & 5 \end{pmatrix}$
b) $a = -1$; $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

3 $B = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$

4 $a = \frac{28}{33}$; $b = \frac{59}{33}$; $c = \frac{20}{33}$; $d = \frac{28}{33}$

5 a) $A^{-1} = \begin{pmatrix} \frac{1}{19} & \frac{2}{19} \\ \frac{-7}{19} & \frac{5}{19} \end{pmatrix}$
b) (i) $X = (C - B)A^{-1}$ (ii) $X = \begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$

6 a) $A + B = \begin{pmatrix} a + 1 & b + 2 \\ c + d & 1 + c \end{pmatrix}$
b) $AB = \begin{pmatrix} a + bd & 2a + bc \\ c + d & 3c \end{pmatrix}$

7 a) $\begin{pmatrix} 0.1 & 0.4 & 0.1 \\ -0.7 & 0.2 & 0.3 \\ -1.2 & 0.2 & 0.8 \end{pmatrix}$
b) $x = 1.2, y = 0.6, z = 1.6$

8 a) $Q = \begin{pmatrix} -3 & 2 \\ 1 & \frac{14 - a}{3} \end{pmatrix}$
b) $CD = \begin{pmatrix} -14 & -4 + 4a \\ -2 & 2 + 7a \end{pmatrix}$
c) $D^{-1} = \frac{1}{5a + 2}\begin{pmatrix} a & -2 \\ 1 & 5 \end{pmatrix}$

9 a) $(7, 2)$ b) $(-1, 2, -1)$

10 a) $B = A^{-1}C$ b) $DA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$
c) $(1, -1, 2)$

11 a) Det $= 0$ b) $\lambda = 5$ c) $(2 - 3t, 1 + t, t)$

12 No answer required – proof

Chapter 7

Exercise 7.1

1 $\frac{\pi}{3}$ **2** $\frac{5\pi}{6}$ **3** $-\frac{3\pi}{2}$ **4** $\frac{\pi}{5}$

5 $\frac{3\pi}{4}$ **6** $\frac{5\pi}{18}$ **7** $-\frac{\pi}{4}$ **8** $\frac{20\pi}{9}$

9 $-\frac{8\pi}{3}$

10 135° **11** -630° **12** 115° **13** 210°

14 -143° **15** 300° **16** 115° **17** $89.95^\circ \approx 90^\circ$

18 480° **19** $390^\circ, -330^\circ$ **20** $\frac{7\pi}{2}, -\frac{\pi}{2}$ **21** $535^\circ, -185^\circ$

22 $\frac{11\pi}{6}, -\frac{13\pi}{6}$ **23** $\frac{11\pi}{3}, -\frac{\pi}{3}$

24 $3.25 + 2\pi \approx 9.5,\ 3.25 - 2\pi \approx -3.03$

25 12.6 cm **26** 14.7 cm

27 1.5 radians, or approx. 85.9° **28** $r \approx 7.16$

29 Area $\approx 13.96 \approx 14.0\ \text{cm}^2$ **30** Area $\approx 131\ \text{cm}^2$

31 $\alpha = 3$ (radian measure), or $\alpha = 172^\circ$ **32** 32 cm

33 6.77 cm

34 a) 3π radians/second b) 11.9 km/hr

35 19.8 radians/second **36** $v = \omega \times r$

37 28.3 cm **38** 20 944 sq metres

39 a) $r \approx 30.6$ cm b) $r \approx 0.0771$ cm

40 $150\sqrt{3}\ \text{cm}^2$ **41** Area of circle $= \left(\frac{\pi - 2}{4\pi}\right) A$

Exercise 7.2

1 a) $t = \frac{\pi}{6} : \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$; $t = \frac{\pi}{3} : \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

2 0.6 **3** 1.0 **4** 0.5 **5** 0.5

6 2.7 **7** 0.1 **8** 0.3 **9** 1.6

10 a) I b) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

11 a) IV b) $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

12 a) IV b) $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

13 a) Negative x-axis b) $(0, -1)$

14 a) II b) $(-0.416, 0.909)$

15 a) I b) $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

16 a) IV b) $(0.540, 0.841)$

17 a) II b) $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

18 a) III b) $(-0.929, -0.369)$

19 $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{\pi}{3} = \frac{1}{2}$, $\tan\frac{\pi}{3} = \sqrt{3}$

20 $\sin\frac{5\pi}{6} = \frac{1}{2}$, $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$, $\tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$

21 $\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $\tan\left(-\frac{3\pi}{4}\right) = 1$

22 $\sin\frac{\pi}{2} = \frac{1}{2}$, $\cos\frac{\pi}{2} = 0$, $\tan\frac{\pi}{2}$ is undefined

23 $\sin\left(-\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(-\frac{4\pi}{3}\right) = -\frac{1}{2}$, $\tan\left(-\frac{4\pi}{3}\right) = -\sqrt{3}$

24 $\sin 3\pi = 0$, $\cos 3\pi = -1$, $\tan 3\pi = 0$

25 $\sin\frac{3\pi}{2} = -1$, $\cos\frac{3\pi}{2} = 0$, $\tan\frac{3\pi}{2}$ is undefined

26 $\sin\left(-\frac{7\pi}{6}\right) = \frac{1}{2}$, $\cos\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, $\tan\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{3}$

27 $\sin(1.25\pi) = -\frac{\sqrt{2}}{2}$, $\cos(1.25\pi) = -\frac{\sqrt{2}}{2}$, $\tan(1.25\pi) = 1$

28 $\sin\frac{13\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$; $\cos\frac{13\pi}{6} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

29 $\sin\frac{10\pi}{3} = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$; $\cos\frac{10\pi}{3} = \cos\frac{4\pi}{3} = -\frac{1}{2}$

30 $\sin\frac{15\pi}{4} = \sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$; $\cos\frac{15\pi}{4} = \cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$

31 $\sin\frac{17\pi}{6} = \sin\frac{5\pi}{6} = \frac{1}{2}$; $\cos\frac{17\pi}{6} = \cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

32 a) $-\frac{\sqrt{3}}{2}$ b) $-\frac{\sqrt{2}}{2}$ c) undefined

d) 2 e) $-\frac{2\sqrt{3}}{3}$

33 a) 0.598 b) $-\frac{\sqrt{3}}{3}$ c) $\frac{1}{2}$ d) 1.04 e) 0

34 I, II **35** II

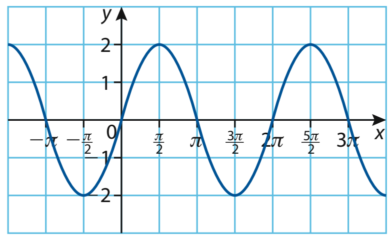
36 III **37** II

38 I, IV **39** I

40 IV **41** II, IV

Exercise 7.3

1



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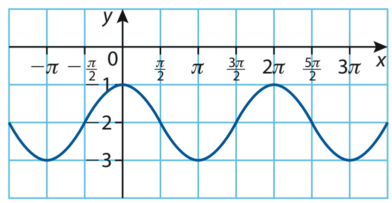

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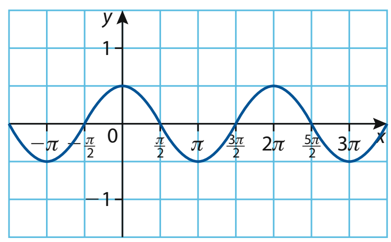

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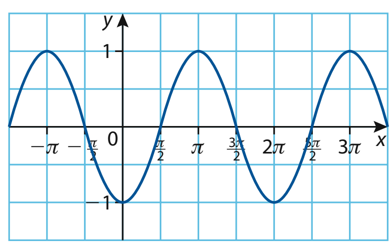

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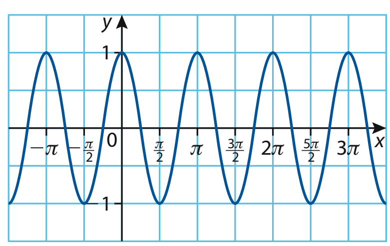

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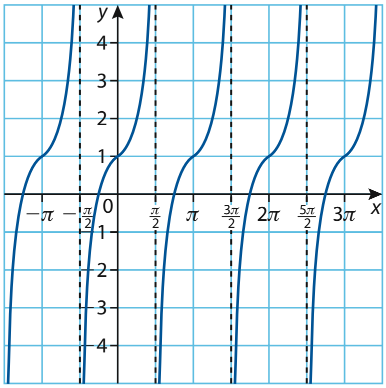

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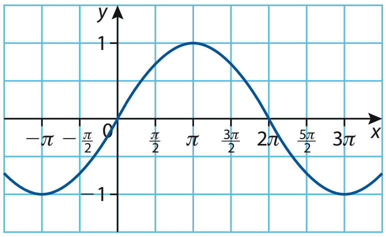

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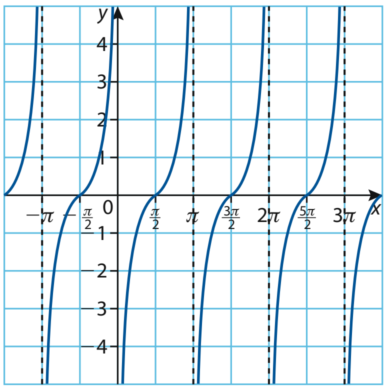

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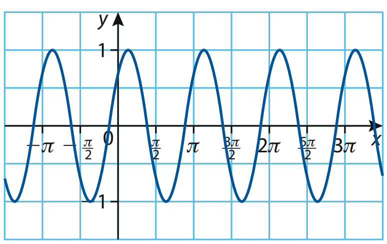
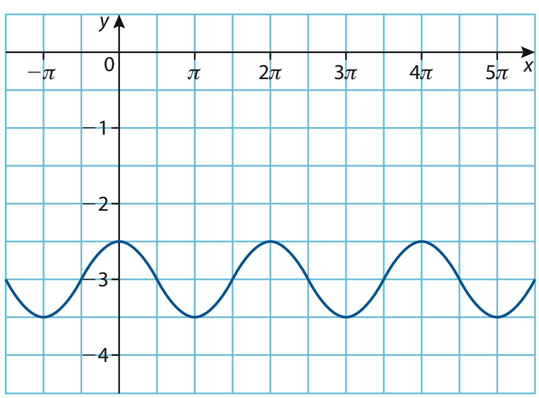


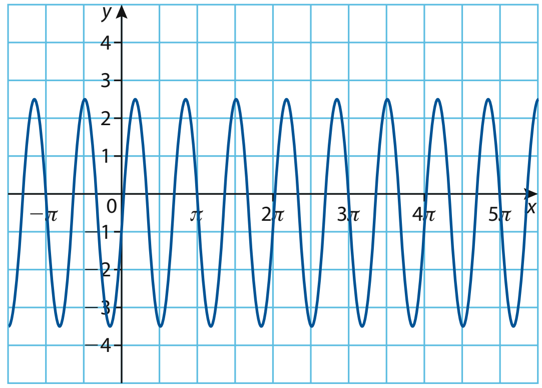
10 a)



amplitude $= \frac{1}{2}$, period $= 2\pi$

b) Domain: $x \in \mathbb{R}$, range: $-3.5 \leqslant y \leqslant -2.$

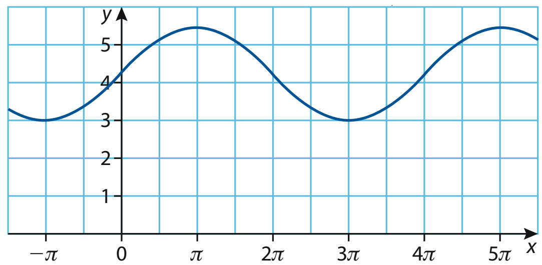
11 a)



amplitude $= 3$, period $= \frac{2\pi}{3}$

b) Domain: $x \in \mathbb{R}$, range: $-3.5 \leqslant y \leqslant 2.5$

12 a)



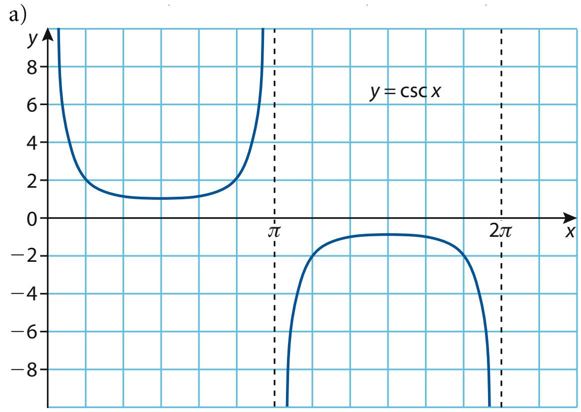
amplitude 1.2, period $= 4\pi$

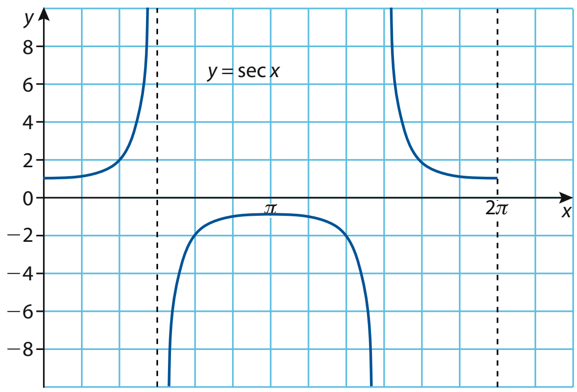
b) Domain: $x \in \mathbb{R}$, range: $3.1 \leqslant y \leqslant 5.5$

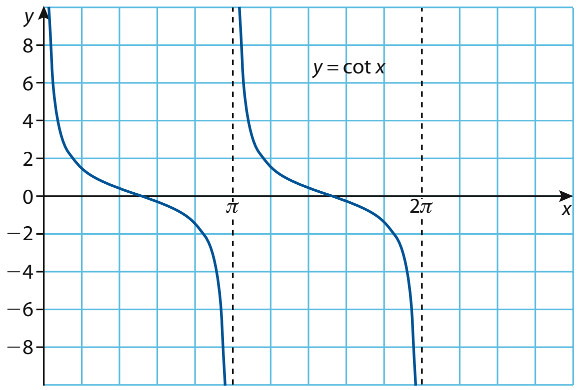
13 $A = 3$, $B = 7$ **14** $A = 2.7$, $B = 5.9$

15 $A = 1.9$, $B = 4.3$ **16** a) $p = 8$ b) $q = 6$

17 a)







b) $y = \sec x$, range: $y \geqslant 1$, $y \leqslant -1$;
$y = \csc x$, range: $y \geqslant 1$, $y \leqslant -1$; $y = \cot x$, range: $y \in \mathbb{R}$

18 a) $a = 2, b = 3, c = -1$ b) $\frac{5\pi}{18}$

19 $a = 3, b = -\frac{\pi}{4}, c = -1$

Exercise 7.4

1 $x = \frac{\pi}{3}, \frac{5\pi}{3}$ **2** $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

3 $x = \frac{\pi}{4}, \frac{5\pi}{4}$ **4** $x = \frac{\pi}{3}, \frac{2\pi}{3}$

5 $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **6** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

7 $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ **8** $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

9 $x = 0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi$ **10** $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

11 $x = \frac{\pi}{3}, \frac{5\pi}{3}$ **12** $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

13 $x \approx 0.412, 2.73$ **14** $x \approx 1.91, 4.37$

15 $x \approx 1.11, 4.25$ **16** $x \approx 5.64, 3.78, 2.50, 0.639$

17 $x \approx 2.96, 5.32$ **18** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

19 $x \approx 5.85, 5.01, 2.71, 1.86$ **20** $x \approx 3.43, 0.291, 2.71, 1.86$

21 $\frac{5\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}$

22 $\frac{\pi}{6}, -\frac{11\pi}{6}$ **23** $\frac{7\pi}{12}, \frac{19\pi}{12}$

24 $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$

25 $x = \frac{5\pi}{6}, \frac{3\pi}{2}$ **26** $\theta = -\frac{3\pi}{4}, \frac{\pi}{4}$

27 $x = 30^\circ, 60^\circ, 210^\circ, 240^\circ$ **28** $\alpha = -\frac{\pi}{6}, \frac{\pi}{6}$

29 $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$ **30** $x = \frac{\pi}{6}, \frac{5\pi}{6}$

31 $x = 225^\circ, 315^\circ$ **32** $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

33 $t \approx 1.5$ hours

34 a) 80th day (March 21) and approximately 263rd day (September 20)
b) 105th day (April 15) and approximately 238th day (August 26)
c) 94 days – from 125th day to 218th day

35 $x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2}, \frac{4\pi}{3}$ **36** $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

37 $x = -45^\circ, 63.4^\circ$ **38** $x \approx -1.87, 1.87$

39 $x \approx 56.3^\circ$ **40** $x = \frac{\pi}{4}, \frac{3\pi}{4}$

41 No solution **42** $x \approx 0^\circ, 71.6^\circ, 180^\circ, 252^\circ$

Exercise 7.5

1 $\frac{\sqrt{2}-\sqrt{6}}{4}$ **2** $\frac{\sqrt{6}-\sqrt{2}}{4}$ **3** $2-\sqrt{3}$

4 $\frac{-\sqrt{6}-\sqrt{2}}{4}$ **5** $\frac{\sqrt{2}-\sqrt{6}}{4}$ **6** $2-\sqrt{3}$

7 a) $\frac{\sqrt{6}+\sqrt{2}}{4}$ b) $\sqrt{\frac{\sqrt{6}+\sqrt{2}+4}{8}}$

8 $\tan\left(\frac{\pi}{2}-\theta\right) = \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)} = \frac{\sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta}{\cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta$

9 $\sin\left(\frac{\pi}{2}-\theta\right) = \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta = \cos\theta$

10 $\csc\left(\frac{\pi}{2}-\theta\right) = \frac{1}{\sin\left(\frac{\pi}{2}-\theta\right)} = \frac{1}{\sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta} = \frac{1}{\cos\theta} = \sec\theta$

11 a) $\frac{4}{5}$ b) $\frac{7}{25}$ c) $\frac{24}{25}$

12 a) $\frac{\sqrt{5}}{3}$ b) $-\frac{4\sqrt{5}}{9}$ c) $-\frac{1}{9}$

13 $\sin 2\theta = -\frac{4\sqrt{5}}{9}$, $\cos 2\theta = \frac{1}{9}$, $\tan 2\theta = -4\sqrt{5}$

14 $\sin 2\theta = \frac{24}{25}$, $\cos 2\theta = \frac{7}{25}$, $\tan 2\theta = \frac{24}{7}$

15 $\sin 2\theta = \frac{4}{5}$, $\cos 2\theta = -\frac{3}{5}$, $\tan 2\theta = -\frac{4}{3}$

16 $\sin 2\theta = -\frac{2\sqrt{15}}{16}\left(\text{or } \sin 2\theta = -\frac{\sqrt{15}}{8}\right)$, $\cos 2\theta = -\frac{7}{8}$, $\tan 2\theta = \frac{\sqrt{15}}{7}$

17 $-\cos x$ **18** $-\cos x$ **19** $\tan x$

20 $-\sin x$ **21** $\frac{1+\sin\theta\cos\theta}{\cos\theta}$ **22** $\frac{1}{\sin^3\theta}$

23 $\frac{\sin\theta+\cos\theta}{\sin 2\theta}$ **24** $\frac{1+\sin^2\theta}{\cos^2 x}$ **25** $\cos^3\theta$

26 1 **27** $\cos^2\theta$ **28** $2\tan^2\theta$

29 $2\sin\alpha\cos\beta$ **30** $\cos^2 A$ **31** $2\cos\alpha\cos\beta$

32 1 **33–46** No answers required (proofs)

47 $\tan\theta = \frac{5x}{x^2+14}$ **48** $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

49 $x = \frac{\pi}{3}, \frac{5\pi}{3}$ **50** $x = 90^\circ$ and -90°

51 $x \approx 0.375, 2.77$ **52** $x \approx 0.615, 2.53, 3.76, 5.67$

53 $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ **54** $x = \frac{\pi}{3}, \frac{2\pi}{3}$

55 $x = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$ **56** $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

57 $x = 30^\circ, 90^\circ, 105^\circ, 150^\circ, 165^\circ$ **58** $3\sin x - 4\sin^3 x$

59 b) $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Exercise 7.6

1 $\frac{\pi}{2}$ **2** $\frac{\pi}{4}$ **3** $-\frac{\pi}{3}$ **4** $\frac{2\pi}{3}$

5 0 **6** $-\frac{\pi}{3}$ **7** $\frac{\pi}{3}$ **8** $\frac{3}{2}$

9 12 **10** Not possible **11** $\frac{\pi}{4}$ **12** Not possible

13 $\frac{3}{5}$ **14** $\frac{24}{25}$ **15** Not possible **16** $\frac{\pi}{3}$

17 $\frac{2\sqrt{5}}{5}$ **18** $\frac{4}{5}$ **19** $\frac{63}{65}$

20 $\frac{2\sqrt{20}-3\sqrt{10}}{30}\left(\text{or } \frac{4\sqrt{5}-3\sqrt{10}}{30}\right)$

21 $\sqrt{1-x^2}$ **22** $\frac{\sqrt{1-x^2}}{x}$ **23** $\frac{1}{\sqrt{x^2+1}}$

24 $2x\sqrt{1-x^2}$ **25** $\sqrt{\frac{1-x}{1+x}}$

26 $\frac{-x^3+x+2x\sqrt{1-x^2}}{x^2+1}$

27 $\cos\left(\arcsin\frac{4}{5}+\arcsin\frac{5}{13}\right)=\cos\left(\arccos\frac{16}{65}\right)$

$\cos\left(\arcsin\frac{4}{5}\right)\cos\left(\arcsin\frac{5}{13}\right)-\sin\left(\arcsin\frac{4}{5}\right)\sin\left(\arcsin\frac{5}{13}\right)=\frac{16}{65}$

$\frac{3}{5}\cdot\frac{12}{13}-\frac{4}{5}\cdot\frac{5}{13}=\frac{36}{65}-\frac{20}{65}=\frac{16}{65}$ Q.E.D

28 $\sin\left(\arctan\frac{1}{2}+\arcsin\frac{1}{3}\right)=\sin\left(\frac{\pi}{4}\right)$

$\sin\left(\arctan\frac{1}{2}\right)\cos\left(\arctan\frac{1}{3}\right)+\cos\left(\arctan\frac{1}{2}\right)\sin\left(\arctan\frac{1}{3}\right)=\frac{\sqrt{2}}{2}$

$\frac{\sqrt{5}}{5}\cdot\frac{3\sqrt{10}}{10}+\frac{2\sqrt{5}}{5}\cdot\frac{\sqrt{10}}{10}=\frac{3\sqrt{50}}{50}+\frac{2\sqrt{50}}{50}=\frac{25\sqrt{2}}{50}=\frac{\sqrt{2}}{2}$ Q.E.D

29 $x=\frac{1}{2}$ **30** $x\approx 0.580,\ 2.56$

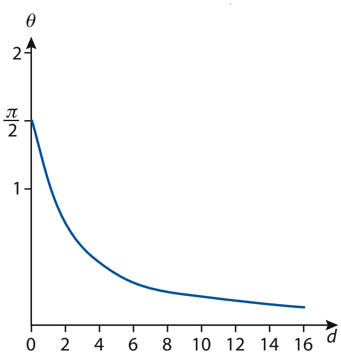
31 $x\approx 2.21$ **32** $x\approx 1.11,\ 4.25$

33 $x=\frac{\pi}{4},\frac{5\pi}{4};\ x\approx 2.82,\ 5.96$ **34** $x=\frac{\pi}{4};\ x\approx 0.464$

35 $x\approx 1.37,\ 4.91$

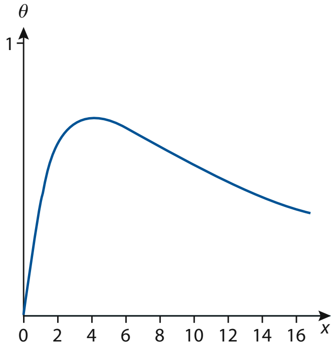
36 $x=\pi,\ 2\pi;\ x\approx 0.912,\ 2.23,\ 4.05,\ 5.37$

37 $x=0,\ \pi;\ x\approx 1.89,\ 5.03$ **38** $\theta=\arctan\left(\frac{2}{d}\right)$



39 a) (ii) $\theta=\arctan\left(\frac{7x}{x^2+15.84}\right)$

(iii)

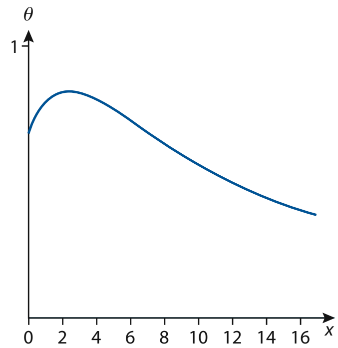


(iv) 3.98 m; sit in the 2nd row

b) (ii) $\theta=\arctan\left(\frac{7\left(x\cos\frac{\pi}{9}+2.5\right)}{\left(x\cos\frac{\pi}{9}+2.5\right)^2+\left(8.8-x\sin\frac{\pi}{9}\right)\left(1.8-x\sin\frac{\pi}{9}\right)}\right)$

$\left[\text{note: } 20^\circ=\frac{\pi}{9}\right]$

(iii)



(iv) 2.5 m; sit in the 3rd row

Practice questions

1 a) 135 cm b) 85 cm
c) $t=0.5$ sec d) 1 sec

2 $x=0,\ 2\pi$

3 $\theta\approx 2.12$ (radian measure)

4 a) (i) -1 (ii) 4π
b) four

5 a) $p=35$ b) $q=29$ c) $m=\frac{1}{2}$

6 $x=0,\ 1.06,\ 2.05$

7 a) $x=\frac{2\pi}{3},\frac{4\pi}{3}$ b) $x=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{3\pi}{2}$

8 a) $\sin x=\frac{1}{3}$ b) $\cos 2x=\frac{7}{9}$
c) $\sin 2x=-\frac{4\sqrt{2}}{9}$

9 a) $1.6\sin\left(\frac{2\pi}{11}\left(x-\frac{9}{4}\right)\right)+4.2$
b) Approximately 3.15 metres
c) Approximately 12:27 p.m. to 7:33 p.m.

10 $x\approx 0.785,\ 1.89$

11 a) 15 cm
b) area $\approx 239\ \text{cm}^2$

12 $k > 2.5, k < -2.5$

13 $k = 1, a = -2$

14 $\sec\theta = -\frac{3}{2}$

15 a) $\frac{84}{85}$ b) $-\frac{13}{85}$ c) $-\frac{84}{13}$

16 $\sin 2p^\circ = \frac{4}{5}, \sin 3p^\circ = \frac{11\sqrt{5}}{25}$

17 a) $-\frac{5}{13}$ b) $\frac{12}{13}$ c) $-\frac{120}{169}$ d) $\frac{119}{169}$

18 $\tan\theta = \frac{1}{3}$ or -3

19 $\tan x = \frac{-(k+1)}{k-1}\tan\alpha\left(\text{or } \tan x = \frac{\tan\alpha(k+1)}{1-k}\right)$

20 $\theta = \pm\frac{3\pi}{8}, \pm\frac{\pi}{8}$

21 b) $x \approx 0.412$

c) $\cos(2) \leqslant g(x) \leqslant 1$

22 24.1°

23 $\frac{72}{\pi}\arccos\frac{8}{13}$ cm

Chapter 8

Exercise 8.1

1 b) $\cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}, \cot\theta = \frac{4}{3}, \sec\theta = \frac{5}{4}, \csc\theta = \frac{5}{3}$

c) $\theta \approx 36.9^\circ; 53.1^\circ$

2 b) $\sin\theta = \frac{\sqrt{39}}{8}, \tan\theta = \frac{\sqrt{39}}{5}, \cot\theta = \frac{5\sqrt{39}}{39}, \sec\theta = \frac{8}{5}$,

$\csc\theta = \frac{8\sqrt{39}}{39}$

c) $\theta \approx 51.3^\circ; 38.7^\circ$

3 b) $\sin\theta = \frac{2\sqrt{5}}{5}, \cos\theta = \frac{\sqrt{5}}{5}, \cot\theta = \frac{1}{2}, \sec\theta = \sqrt{5}$,

$\csc\theta = \frac{\sqrt{5}}{2}$

c) $\theta \approx 63.4^\circ; 26.6^\circ$

4 b) $\sin\theta = \frac{\sqrt{51}}{10}, \tan\theta = \frac{\sqrt{51}}{7}, \cot\theta = \frac{7\sqrt{51}}{51}, \sec\theta = \frac{10}{7}$,

$\csc\theta = \frac{10\sqrt{51}}{51}$

c) $\theta \approx 45.6^\circ; 44.4^\circ$

5 b) $\sin\theta = \frac{3\sqrt{10}}{10}, \cos\theta = \frac{\sqrt{10}}{10}, \tan\theta = 3, \sec\theta = \sqrt{10}$,

$\csc\theta = \frac{\sqrt{10}}{3}$

c) $\theta \approx 71.6^\circ; 18.4^\circ$

6 b) $\cos\theta = \frac{3}{4}, \tan\theta = \frac{\sqrt{7}}{3}, \cot\theta = \frac{3\sqrt{7}}{7}, \sec\theta = \frac{4}{3}$,

$\csc\theta = \frac{4\sqrt{7}}{7}$

c) $\theta \approx 41.4^\circ; 48.6^\circ$

7 b) $\sin\theta = \frac{\sqrt{60}}{11}, \cos\theta = \frac{\sqrt{61}}{11}, \tan\theta = \frac{2\sqrt{915}}{61}$,

$\cot\theta = \frac{\sqrt{915}}{30}, \csc\theta = \frac{11\sqrt{60}}{60}$

c) $\theta \approx 44.8^\circ; 45.2^\circ$

8 b) $\sin\theta = \frac{9\sqrt{181}}{181}, \cos\theta = \frac{10\sqrt{181}}{181}, \cot\theta = \frac{10}{9}, \sec\theta = \frac{\sqrt{181}}{10}$,

$\csc\theta = \frac{\sqrt{181}}{9}$

c) $\theta \approx 42.0^\circ; 48.0^\circ$

9 b) $\sin\theta = \frac{7\sqrt{65}}{65}, \tan\theta = \frac{7}{4}, \cot\theta = \frac{4}{7}, \sec\theta = \frac{\sqrt{65}}{4}$,

$\csc\theta = \frac{\sqrt{65}}{7}$

c) $\theta \approx 60.3^\circ; 29.7^\circ$

10 $\theta = 60^\circ, \frac{\pi}{3}$

11 $\theta = 45^\circ, \frac{\pi}{4}$

12 $\theta = 60^\circ, \frac{\pi}{3}$

13 $\theta = 60^\circ, \frac{\pi}{3}$

14 $\theta = 45^\circ, \frac{\pi}{4}$

15 $\theta = 30^\circ, \frac{\pi}{6}$

16 $x \approx 86.6$

17 $x \approx 8.60$

18 $x \approx 20.6$

19 $x \approx 374$

20 $x = 18$

21 $x = 200$

22 $\alpha = 30^\circ, \beta = 60^\circ$

23 $\alpha \approx 67.4^\circ, \beta \approx 22.6^\circ$

24 $\alpha = 30^\circ, \beta = 60^\circ$

25 $\alpha \approx 20.0^\circ, \beta \approx 70.0^\circ$

26 114 metres

27 67.4°

28 4.05 metres

29 4105 m

30 $44^\circ, 68^\circ, 68^\circ$

31 5.76 km/hr

32 69.5 m

33 28.7 m

34 151 m

35 59.2 m

36 $3\sqrt{5}$

37 -0.6

38 $\frac{|ap + bq + c|}{\sqrt{a^2 + b^2}}$

39 Verify

40 14°

Exercise 8.2

1 $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}$

2 $\sin\theta = \frac{12}{37}, \cos\theta = -\frac{35}{37}, \tan\theta = -\frac{12}{35}$

3 $\sin\theta = -\frac{\sqrt{2}}{2}, \cos\theta = \frac{\sqrt{2}}{2}, \tan\theta = -1$

4 $\sin\theta = -\frac{1}{2}, \cos\theta = \frac{\sqrt{3}}{2}, \tan\theta = \frac{\sqrt{3}}{3}$