Name:

Mathematics IB HL Test 3

December 1, 2021

1 hour 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Calculators are **not allowed** for this examination paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is [**72 marks**].
- Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to **show all working**.
- Write your solutions in the space provided.

1. [Maximum mark: 6]

The polynomial $P(x) = 2x^3 + 9x^2 + ax + b$ is divisible by $(x+2)$ and leaves remainder of -6 when divided by $(x+1)$.

a) Find the values of a and b. [3 *points*]

b) Solve the inequality $P(x) \geqslant 0$. [3 *points*]

2. [Maximum mark: 5]

Tomasz and Maria need to arrange a set of 11 books on a shelf.

Of the 11 books 5 are Maths books, 4 are Philosophy books and 2 are Computer Science books.

4 books have red covers, 4 have blue covers, 2 have green covers and 1 has a black one.

Tomasz wants to arrange the books according to their subjects. Let T be the number of ways in which it can be done. Maria wants to arrange the books according to the colour of the cover. Let M be the number of ways this can be done. Find $\frac{T}{M}$ and hence decide who has more ways to arrange the books.

3. [Maximum mark: 5]

Consider the quadratic equation:

$$2x^2 - 7x + 13 = 0$$

Let α and β be the roots of this equation. Find an equation with integer coefficients whose roots are:

a) $\alpha + 1$ and $\beta + 1$, [2 *points*]

b) $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$. [3 *points*]

4. [Maximum mark: 5]

Find the coefficient of x^7 in the expansion of $(2 - \frac{1}{x^2})(x^3 + 3)^8$.

5. [Maximum mark: 8]

a) Prove the identity:

$$\sin A + \sin(A + B) + \sin(A + 2B) \equiv \sin(A + B)(1 + 2\cos B)$$

[5 *points*]

b) Hence, or otherwise, solve:

$$\frac{1}{2} + \sin\left(x + \frac{\pi}{6}\right) + \sin\left(2x + \frac{\pi}{6}\right) = 0$$

[3 *points*]

6. [Maximum mark: 7]

Consider the equation $z^3 - 1 = 0$.

a) State the real root of this equation. [1 *points*]

b) Find the remaining complex roots of the equation. [3 *points*]

c) Let A, B and C be points on the complex plane corresponding to the roots of this equation. Find the area of the triangle ABC. [3 *points*]

7. [Maximum mark: 5]

Let $f(x) = \arccos(\frac{x}{\pi})$.

a) Sketch the graph of $y = f(x)$. [2 *points*]

b) On the same diagram sketch the graph of $y = \frac{\pi}{f(x)}$. [3 *points*]

Clearly indicate any axes intercepts and asymptotes on each graph.

8. [Maximum mark: 9]

Let p_n be the product of all entries in the n-th row of the Pascal's triangle:

$$p_n = \binom{n}{0} \times \binom{n}{1} \times \binom{n}{2} \times ... \times \binom{n}{n}$$

a) Calculate p_2 and p_3. [2 *points*]

b) Show that:

$$\frac{p_{n+1}}{p_n} = \frac{(n+1)^n}{n!}$$

[4 *points*]

c) Hence, or otherwise, show that:

$$\frac{p_{n+1} \times p_{n-1}}{p_n^2} = \left(1 + \frac{1}{n}\right)^n$$

[3 *points*]

9. [Maximum mark: 22]

Consider the polynomial $P(x) = x^3 + Ax^2 + Bx + C$.

a) Consider first the case when $A = -3$, $C = 3$ and the roots of $P(x)$ are consecutive terms of an **arithmetic** sequence. [6 *points*]

i) Find B.

ii) Solve the inequality $P(x) \geqslant 0$.

b) Now consider the case when $A = -3$, $C = 8$ and the roots of $P(x)$ are consecutive terms of a **geometric** sequence. [5 *points*]

i) Find B.

ii) State the solutions to the equation $P(|x|) = 0$.

c) Now consider the case when $A = -3$, $C = -6$ and one of the roots of $P(x)$ is purely imaginary. Find B, given that $B \in \mathbb{R}$. [4 *points*]

d) Now let $A = 1 - 3i$, $C = -2$ and i is a root of $P(x)$. [7 *points*]

i) Explain why $-i$ is not necessarily a root of $P(x)$.

ii) Find B and the other two roots of $P(x)$.