Mixed examination practice 1

Short questions

1. Seven athletes take part in the 100 m final of the Olympic games. In how many ways can three medals be awarded? *[4 marks]*

2. In how many ways can five different letters be put into five different envelopes? *[5 marks]*

3. In how many ways can ten cartoon characters stand in a queue if Mickey, Bugs Bunny and Jerry must occupy the first three places in some order? *[5 marks]*

4. How many three digit numbers contain no zeros? *[6 marks]*

5. A committee of four children is chosen from eight children. The two oldest children cannot both be chosen. Find the number of ways the committee may be chosen. *[6 marks]*

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6. Solve the equation $(n+1)! = 30(n-1)!$ for $n \in \mathbb{N}$.
(Remember: $\mathbb{N}$ is the set of natural (whole non-negative) numbers.) *[5 marks]*

7. How many permutations of the word 'CAROUSEL' start and end in a consonant? *[5 marks]*

8. Solve the equation $\binom{n}{2} = 105$. *[6 marks]*

9. A group of 15 students contains seven boys and eight girls. In how many ways can a committee of five be selected if it must contain at least one boy? *[6 marks]*

10. Abigail, Bahar, Chris, Dasha, Eustace and Franz are sitting next to each other in six seats in a cinema. Bahar and Eustace cannot sit next to each other. In how many different ways can they permute themselves? *[6 marks]*

11. A committee of five is to be selected from a group of 12 children. The two youngest cannot both be on the committee. In how many ways can the committee be selected? *[6 marks]*

12. A car registration number consists of three different letters followed by five digits chosen from 1–9 (the digits can be repeated). How many different registration numbers can be made? *[6 marks]*

13. A van has eight seats, two at the front, a row of three in the middle and a row of three at the back. If only 5 out of a group of 8 people can drive, in how many different ways can they be arranged in the car? *[6 marks]*

14. Ten people are to travel in one car (taking four people) and one van (taking six people). Only two of the people can drive. In how many ways can they be allocated to the two vehicles? (The permutation of the passengers in each vehicle is not important.) *[7 marks]*

Long questions

1. Five girls, Anya, Beth, Carol, Dasha and Elena, stand in a line. How many possible permutations are there in which

(a) Anya is at one end of the line?

(b) Anya is not at either end?

(c) Anya is at the left of the line or Elena is on the right, or both? *[9 marks]*

2. In how many ways can five different sweets be split amongst two people if

(a) each person must have at least one sweet?

(b) one person can take all of the sweets?

(c) one of the sweets is split into two to be shared, and each person gets two of the remaining sweets? *[9 marks]*

3. In a doctor's waiting room, there are 14 seats in a row. Eight people are waiting to be seen.

(a) In how many ways can they be seated?

(b) Three of the people are all in the same family and they want to sit together. How many ways can this happen?

(c) The family no longer have to sit together, but there is someone with a very bad cough who must sit at least one seat away from anyone else. How many ways can this happen? *[8 marks]*

4. (a) Explain why the number of ways of arranging the letters RRDD, given that all the R's and all the D's are indistinguishable is $\binom{4}{2}$.

(b) How many ways are there of arranging n R's and n D's?

(c) A miner is digging a tunnel on a four by four grid. He starts in the top left box and wants to get to the gold in the bottom right box. He can only tunnel directly right or directly down one box at a time. How many different routes can he take?

(d) What will be the general formula for the number of routes when digging on an n by m grid? *[10 marks]*

5. 12 people need to be split up into teams for a quiz.

(a) Show that the number of ways of splitting them into two groups of the same size is $\frac{1}{2}\binom{12}{6}$.

(b) How many ways are there of splitting them into two groups of any size (but there must be at least one person in each group)?

(c) How many ways are there of splitting them into three groups of four people? *[9 marks]*

6. (a) How many different ways are there to select a group of three from a class of 31 people?

(b) In another class there are 1540 ways of selecting a group of three people. How many people are there in the class?

(c) In another class the teacher noted that the number of ways to select a group of size three is 100 times larger than the number of people in the class. How many people are in the class? *[9 marks]*

Mixed examination practice 2

Short questions

1. Solve $\log_5\left(\sqrt{x^2+49}\right)=2$. *[4 marks]*

2. If $a=\log x$, $b=\log y$ and $c=\log z$ (all logs base 10) find in terms of a, b, c and integers:

(a) $\log\dfrac{x^2\sqrt{y}}{z}$ (b) $\log\sqrt{0.1x}$ (c) $\log_{100}\left(\dfrac{y}{z}\right)$ *[6 marks]*

3. Solve the simultaneous equations:

$$\ln x+\ln y^2=8$$

$$\ln x^2+\ln y=6$$ *[6 marks]*

4. If $y=\ln x-\ln(x+2)+\ln(4-x^2)$, express x in terms of y. *[6 marks]*

5. Find the exact value of x satisfying the equation

$$2^{3x-2}\times 3^{2x-3}=36^{x-1}$$

giving your answer in simplified form $\dfrac{\ln p}{\ln q}$, where $p,q\in\mathbb{Z}$. *[5 marks]*

6. Given $\log_a b^2=c$ and $\log_b a=c-1$ for some value c, where $0<a<b$, express a in terms of b. *[6 marks]*

7. Solve the equation $9\log_5 x=25\log_x 5$, expressing your answers in the form $5^{\frac{p}{q}}$, where $p,q\in\mathbb{Z}$. *[6 marks]*

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8. Find the exact solution to the equation $\ln x=4\log_x \mathrm{e}$. *[5 marks]*

Long questions

1. The speed of a parachutist (V) in metres per second, t seconds after jumping is modelled by the expression:

$$V=42(1-\mathrm{e}^{-0.2t})$$

(a) Sketch a graph of V against t.

(b) What is the initial speed?

(c) What is the maximum speed that the parachutist could reach?

When the parachutist reaches 22 ms^{-1} he opens the parachute.

(d) How long is he falling before he opens his parachute? *[9 marks]*

2. Scientists think that the global population of tigers is falling exponentially. Estimates suggest that in 1970 there were 37 000 tigers but by 1980 the number had dropped to 22 000.

(a) Form a model of the form $T = ka^n$ connecting the number of tigers (T) with the number of years after 1970 (n).

(b) What does the model predict that the population will be in 2020?

(c) When the population reaches 1000 the tiger population will be described as 'near extinction'. In which year will this happen?

In the year 2000 a worldwide ban on the sale of tiger products was implemented, and it is believed that by 2010 the population of tigers had recovered to 10 000.

(d) If the recovery has been exponential find a model of the form $T = ka^m$ connecting the number of tigers (T) with the number of years after 2000 (m).

(e) If in each year since 2000 the rate of growth has been the same, find the percentage increase each year. *[12 marks]*

3. (a) If $\ln y = 2\ln x + \ln 3$ find y in terms of x.

(b) If the graph of $\ln y$ against $\ln x$ is a straight line with gradient 4 and y-intercept 6, find the relationship between x and y.

(c) If the graph of $\ln y$ against x is a straight line with gradient 3 and it passes through the point (1, 2), express y in terms of x.

(d) If the graph of e^y against x^2 is a straight line through the origin with gradient 4, find the gradient of the graph of y against $\ln x$. *[10 marks]*

Mixed examination practice 3

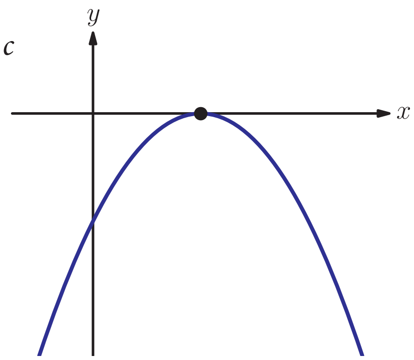
Short questions

1. A quadratic graph passes through the points $(k, 0)$ and $(k+4, 0)$. Find in terms of k the x-coordinates of the turning point. *[4 marks]*

2. The diagram shows the graph of $y = ax^2 + bx + c$

Complete the table to show whether each expression is positive, negative or zero.

expression	positive	negative	zero
a			
c			
$b^2 - 4ac$			
b			



[6 marks]
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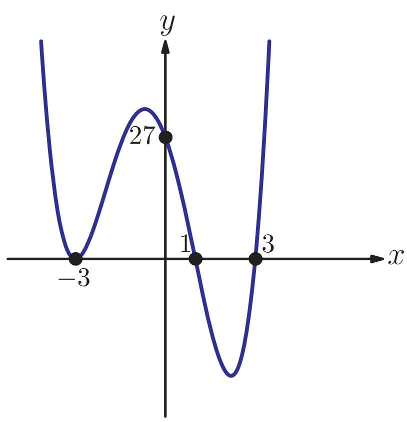
3. The diagram shows the graph with equation $y = ax^4 + bx^3 + cx^2 + dx + e$.

Find the values of a, b, c, d and e. *[6 marks]*



4. The remainder when $(ax+b)^3$ is divided by $(x-2)$ is 8 and the remainder when it is divided by $(x+3)$ is −27. Find the values of a and b. *[5 marks]*

5. (a) Show that $(x-2)$ is a factor of $f(x) = x^3 - 4x^2 + x + 6$.

(b) Factorise $f(x)$.

(c) Sketch the graph of $y = f(x)$. *[7 marks]*

6. The remainder when $(ax + b)^4$ is divided by $(x - 2)$ is 16 and the remainder when it is divided by $(x + 1)$ is 81. Find the possible values of a and b. *[6 marks]*

7. Sketch the graph of $y = (x - a)^2 (x - b)(x - c)$ where $b < 0 < a < c$. *[5 marks]*

8. Find the exact values of k for which the equation $2kx^2 + (k + 1)x + 1 = 0$ has equal roots. *[5 marks]*

9. Find the set of values of k for which the equation $2x^2 + kx + 6 = 0$ has no real roots. *[6 marks]*

10. Find the range of values of k for which the quadratic function $x^2 - (2k + 1)x + 5$ has at least one real zero. *[6 marks]*

11. The polynomial $x^2 - 4x + 3$ is a factor of the polynomial $x^3 + ax^2 + 27x + b$. Find the values of a and b. *[6 marks]*

12. Let α and β denote the roots of the quadratic equation

$x^2 - kx + (k - 1) = 0$.

(a) Express α and β in terms of the real parameter k.

(b) Given that $\alpha^2 + \beta^2 = 17$, find the possible values of k. *[7 marks]*

13. Let $q(x) = kx^2 + (k - 2)x - 2$. Show that the equation $q(x) = 0$ has real roots for all values of k. *[7 marks]*

14. Find the range of values of k such that for all x, $kx - 2 \le x^2$. *[7 marks]*

Long questions

1. (a) Find the coordinates of the point where the curve $y = x^2 + bx - a$ crosses the y-axis, giving your answer in terms of a and/or b.

(b) State the equation of the axis of symmetry of $x^2 + bx - a$, giving your answer in terms of a and/or b.

(c) Show that the remainder when $x^2 + bx - a$ is divided by $x - \frac{a}{b}$ is always positive.

(d) The remainder when $x^2 + bx - a$ is divided by $x - a$ is -9. Find the possible values that b can take.

[14 marks]

2. (a) Show that for all values of p, $(x-2)$ is a factor of

$$f(x)=x^3+(p-2)x^2+(5-2p)x-10.$$

(b) By factorising $f(x)$, or otherwise, find the exact values of p for which the equation

$$x^3+(p-2)x^2+(5-2p)x-10=0$$

has exactly two real roots.

(c) For the smaller of the two values of p found above, sketch the graph of $y=f(x)$.

[10 marks]

3. (a) On the graph of $y=\dfrac{x^2+4x+5}{x+2}$ prove that there is no value of x for which $y=0$.

(b) Find the equation of the vertical asymptote of the graph.

(c) Rearrange the equation to find x in terms of y.

(d) Hence show that y cannot take values between -2 and 2.

(e) Sketch the graph of $y=\dfrac{x^2+4x+5}{x+2}$.

[18 marks]

4. Let $f(x)=x^4+x^3+x^2+x+1$.

(a) Evaluate $f(1)$.

(b) Show that $(x-1)f(x)\equiv x^5-1$.

(c) Sketch $y=x^5-1$.

(d) Hence show that $f(x)$ has no real roots.

[10 marks]

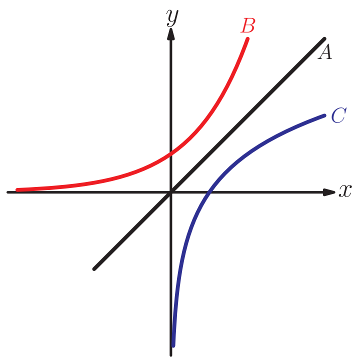
Mixed examination practice 5

Short questions

1. Find the inverses of the following functions:

(a) $f(x) = \log_3(x+3),\ x > 0$ (b) $g(x) = 3e^{x^3-1}$ *[5 marks]*

2. The diagram shows three graphs.



A is part of the graph of $y = x$.
B is part of the graph of $y = 2^x$.
C is the reflection of graph B in line A.
Write down:

(a) The equation of C in the form $y = f(x)$

(b) The coordinates of the point where C cuts the x-axis. *[5 marks]*

3. (a) Write down the equations of all asymptotes of the graph of $y = \frac{4x-3}{5-x}$.

(b) Find the inverse function of $f(x) = \frac{4x-3}{5-x}$. *[6 marks]*

4. The function f is given by $f(x) = x^2 - 6x + 10$, for $x \geq 3$.

(a) Write $f(x)$ in the form $(x-p)^2 + q$.

(b) Find the inverse function $f^{-1}(x)$.

(c) State the domain of $f^{-1}(x)$. *[6 marks]*

5. If $h(x) = x^2 - 6x + 2$:

(a) Write $h(x)$ in the form $(x-p)^2 + q$.

(b) Hence or otherwise find the range of $h(x)$.

(c) By using the largest possible domain of the form $x > k$ where, find the inverse function $h^{-1}(x)$. *[7 marks]*

6. The function $f(x)$ is defined by $f(x)=\dfrac{3-x}{x+1}, x\neq -1$.

(a) Find the range of *f*.

(b) Sketch the graph of $y=f(x)$.

(c) Find the inverse function of *f* in the form $f^{-1}(x)=\dfrac{ax+b}{cx+d}$. State its domain and range. *[11 marks]*

7. A function is defined by:

$$f(x)=\begin{cases}5-x, & x<0\\ pe^{-x}, & x\geq 0\end{cases}$$

(a) Given that $p=3$,

(i) Find the range of $f(x)$.

(ii) Find an expression for $f^{-1}(x)$ and state its domain.

(b) Find the value of p for which $f(x)$ is continuous. *[7 marks]*

8. The functions $f(x)$ and $g(x)$ are given by $f(x)=\sqrt{x-2}$ and $g(x)=x^2+x$. The function $f\circ g(x)$ is defined for $x\in\mathbb{R}$ except for the interval $]\,a,b[$.

(a) Calculate the value of *a* and of *b*.

(b) Find the range of $f\circ g$. *[7 marks]*

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Long questions

1. If $f(x)=x^2+1,\ x>3$ and $g(x)=5-x$:

(a) evaluate $f(3)$.

(b) Find and simplify an expression for $gf(x)$.

(c) State the geometric relationship between the graphs of $y=f(x)$ and $y=f^{-1}(x)$.

(d) (i) Find an expression for $f^{-1}(x)$.

(ii) Find the range of $f^{-1}(x)$.

(iii) Find the domain of $f^{-1}(x)$.

(e) Solve the equation $f(x)=g(3x)$. *[10 marks]*

2. If $f(x)=2x+1$ and $g(x)=\dfrac{x+3}{x-1},\ x\neq 1$

(a) find and simplify

(i) $f(7)$ (ii) the range of $f(x)$

(iii) $fg(x)$ (iv) $ff(x)$

(b) Explain why $gf(x)$ does not exist.

(c) (i) Find the form of $g^{-1}(x)$.

(ii) State the domain of $g^{-1}(x)$.

(iii) State the range of $g^{-1}(x)$. *[9 marks]*

3. The functions f and g are defined over the domain of all real numbers, $g(x) = e^x$.

(a) Write $f(x) = x^2 + 4x + 9 \;\; x \in \mathbb{R}$ in the form $f(x) = (x+p)^2 + q$.

(b) Hence sketch the graph of $y = x^2 + 4x + 9$, labelling carefully all axes intercepts and the coordinates of the turning point.

(c) State the range of $f(x)$ and $g(x)$.

(d) Hence or otherwise find the range of $h(x) = e^{2x} + 4e^x + 9$. *[10 marks]*

4. Given that $(2x+3)(4-y) = 12$ for $x, y \in \mathbb{R}$:

(a) Write y in terms of x, giving your answer in the form $y = \dfrac{ax+b}{cx+d}$.

(b) Sketch the graph of y against x.

(c) Let $g(x) = 2x + k$ and $h(x) = \dfrac{8x}{2x+3}$.

(i) Find $h(g(x))$.

(ii) Write down the equations of the asymptotes of the graph of $y = h(g(x))$.

(iii) Show that when $k = -\dfrac{19}{2}$, $h(g(x))$ is a self-inverse function. *[17 marks]*

5. **(a)** Show that if $g(x) = \dfrac{1}{x}$ then $gg(x) = x$.

(b) A function satisfies the identity $f(x) + 2f\left(\dfrac{1}{x}\right) = 2x + 1$.

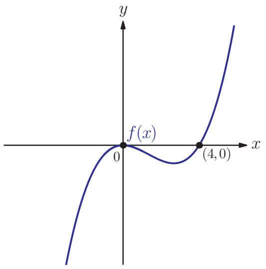
By replacing all instances of x with $\dfrac{1}{x}$, find another identity that $f(x)$ satisfies.

(c) By solving these two identities simultaneously, express $f(x)$ in terms of x. *[10 marks]*

Mixed examination practice 6

Short questions

1. The graph of $y = f(x)$ is shown.



Sketch on separate diagrams the graphs of

(a) $y = 3f(x-2)$

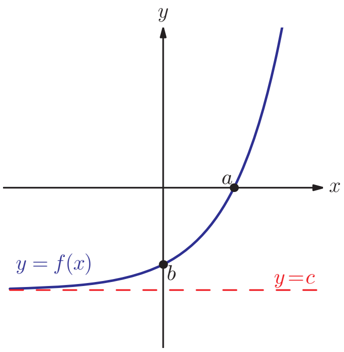
(b) $\frac{1}{f(x)}$

Indicate clearly the positions of any x-intercepts and asymptotes. *[6 marks]*

2. The graph of $y = x^3 - 1$ is transformed by applying a translation with vector $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ followed by a vertical stretch with scale factor 2. Find the equation of the resulting graph in the form $y = ax^3 + bx^2 + cx + d$. *[4 marks]*

3. Solve the inequality $|2x-1| < x$. *[6 marks]*

4. The diagram shows the graph of $y = f(x)$.



On separate diagrams sketch the following graphs, labelling appropriately.

(a) $y = |f(x)|$

(b) $y = f(|x|) - 1$ *[5 marks]*

5. (a) Sketch the graph of $y = -\frac{3}{x}$.

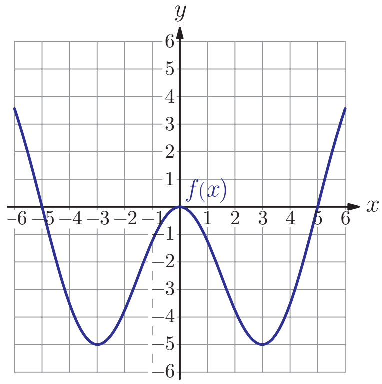
(b) Describe two transformations which transform the graph of $y = \frac{1}{x}$ to the graph of $y = -\frac{3}{x}$.

(c) Let $f(x) = -\frac{3}{x}, x \neq 0$. Write down an equation for $f^{-1}(x)$. *[4 marks]*

6. The graph of $y = f(x)$ is shown.

(a) On the same diagram sketch the graph of $y = \dfrac{1}{f(x)}$.

(b) State the coordinates of the maximum points. *[5 marks]*



7. Find two transformations whose composition transforms the graph of $y = (x-1)^2$ to the graph of $y = 3(x+2)^2$. *[4 marks]*

8. (a) Describe two transformations whose composition transforms the graph of $y = f(x)$ to the graph of $y = 3f\left(\dfrac{x}{2}\right)$.

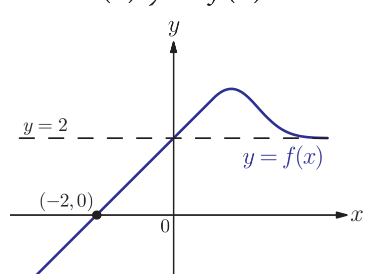
(b) Sketch the graph of $y = 3\ln\left(\dfrac{x}{2}\right)$.

(c) Sketch the graph of $y = 3\ln\left(\dfrac{x}{2}+1\right)$ marking clearly the positions of any asymptotes and x-intercepts. *[7 marks]*

9. The diagram shows a part of the graph of $y = f(x)$

On separate diagrams sketch the graphs of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = xf(x)$ *[6 marks]*



10. For which values of the real number x is $|x+k| = |x| + k$, where k is a positive real number? *[4 marks]*

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Long questions

1 (a) Describe two transformations which transform the graph of $y = x^2$ to the graph of $y = 3x^2 - 12x + 12$.

(b) Describe two transformations which transform the graph of $y = x^2 + 6x - 1$ to the graph of $y = x^2$.

(c) Hence describe a sequence of transformations which transform the graph of $y = x^2 + 6x - 1$ to the graph of $y = 3x^2 - 12x + 12$.

(d) Sketch the graph of $y = \dfrac{1}{3x^2 - 12x + 12}$. *[12 marks]*

2. Given that $f(x) = \dfrac{3x-5}{x-2}$

(a) Write down the equation of the horizontal asymptote of the graph of $y = f(x)$.

(b) Find the value of constants p and q such that $f(x) = p + \dfrac{q}{x-2}$.

(c) Hence describe a single transformation which transforms the graph of $y = \dfrac{1}{x}$ to the graph of $y = f(x)$.

(d) Find an expression for $f^{-1}(x)$ and state its domain.

(e) Describe the transformation which transforms the graph of $y = f(x)$ to the graph of $y = f^{-1}(x)$. *[11 marks]*

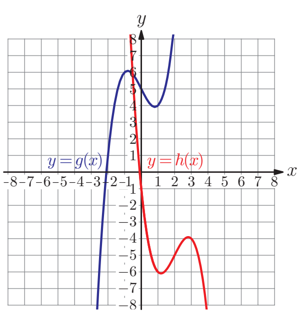
3. (a) Describe a transformation which transforms the graph of $y = f(x)$ to the graph of $y = f(x+2)$.

(b) Sketch on the same diagram the graphs of

(i) $y = \ln(x+2)$ (ii) $y = \dfrac{1}{\ln(x+2)}$.

Mark clearly any asymptotes and x-intercepts on your sketches.

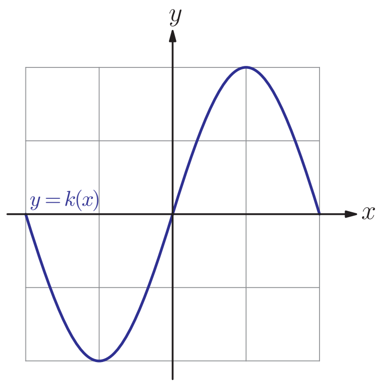
(c) The graph of the function $y = g(x)$ has been translated and then reflected in the x-axis to produce the graph of $y = h(x)$.



(i) State the translation vector.

(ii) If $g(x) = x^3 - 2x + 5$, find constants a, b, c and d such that $h(x) = ax^3 + bx^2 + cx + d$.

(d) The diagram below shows the graph of $y = k(x)$.



On the same diagram, sketch the graph of $y = (k(x))^2$. *[14 marks]*

4. $f(x) = x^2 - 7x + 10$ $g(x) = x^2 - 7|x| + 10$

(a) Sketch the graph of $y = f(x)$.

(b) Show that $g(x) = f(|x|)$.

(c) Sketch the graph of $y = g(x)$.

(d) Solve the equation $g(x) = x^2$.

(e) Solve the equation $g(x) = -2$. *[12 marks]*

5. If $f(x) = 3x^2 + bx + 10$ and the graph $y = f(x)$ has a line of symmetry when $x = 3$

(a) find b.

(b) If $f(x) = f(d - x)$ for all x, find the value of d.

(c) $g(x) = f(x + p) + q$ and $g(x)$ is an even function which passes through the origin. Find p and q. *[14 marks]*

(d) Find the set values which satisfy $g(x) = g(|x|)$.

6. (a) Sketch the graph of $y = e^x - 2$, including the coordinates of all axes intercepts.

(b) On separate axes sketch the graphs of

(i) $y = |e^x - 2|$

(ii) $y = e^{|x|} - 2$

(c) Hence solve the equation $e^{|x|} - 2 = |e^x - 2|$. *[10 marks]*

Mixed examination practice 7

Short questions

1. The fourth term of an arithmetic sequence is 9.6 and the ninth term is 15.6.
Find the sum of the first nine terms. *[5 marks]*

2. The sum of the first n terms of a series is given by:

$$S_n = 2n^2 - n, \text{ where } n \in \mathbb{Z}^+.$$

(a) Find the first three terms of the series.

(b) Find an expression for the nth term of the series, giving your answer in terms of n. *[6 marks]*

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3. Which is the first term of this sequence which is less than 10^{-6}?

$$\frac{1}{3}, \frac{1}{9} \ldots \frac{1}{3^n}$$ *[5 marks]*

4. The fifth term of an arithmetic sequence is three times larger than the second term. Find the ratio: $\dfrac{\text{common difference}}{\text{first term}}$ *[6 marks]*

5. A geometric sequence and an arithmetic sequence both start with a first term of 1. The third term of the arithmetic sequence is the same as the second term of the geometric sequence. The fourth term of the arithmetic sequence is the same as the third term of the geometric sequence. Find the possible values of the common difference of the arithmetic sequence. *[7 marks]*

6. Evaluate $\displaystyle\sum_{i=0}^{i=\infty} \frac{(2^i + 4^i)}{6^i}$. *[6 marks]*

7. Find the sum of all the integers between 300 and 600 which are divisible by 7. *[7 marks]*

8. Find an expression for the sum of the first 23 terms of the series

$$\ln\frac{a^3}{\sqrt{b}} + \ln\frac{a^3}{b} + \ln\frac{a^3}{b\sqrt{b}} + \ln\frac{a^3}{b^2} + \ldots$$

giving your answer in the form $\ln\dfrac{a^m}{b^n}$, where $m, n \in \mathbb{Z}$. *[7 marks]*

Long questions

1. Kenny is offered two investment plans, each requiring an initial investment of \$10 000:

Plan A offers a fixed return of \$800 per year.

Plan B offers a return of 5% each year, reinvested in the plan.

(a) Find an expression for the amount in plan A after n years.

(b) Find an expression for the amount in plan B after n years.

(c) Over what period of time is plan A better than plan B? *[10 marks]*

2. Ben builds a pyramid out of toy bricks. The top row contains one brick, the second row contains three bricks and each row after that contains two more bricks than the previous row.

(a) How many bricks are in the nth row?

(b) If a total of 36 bricks are used how many rows are there?

(c) In Ben's largest ever pyramid he noticed that the total number of bricks was four more than four times the number of bricks in the bottom row. What is the total number of bricks? *[10 marks]*

3. A pupil writes '1' on the first line of a page, then the next two integers '2, 3' on the second line of the page then the next three integers '4, 5, 6' on the third line. He continues this pattern.

(a) How many integers are on the nth line?

(b) What is the last integer on the nth line?

(c) What is the first integer on the nth line?

(d) Show that the sum of all the integers on the nth line is $\frac{n}{2}(n^2+1)$.

(e) The sum of all the integers on the last line of the page is 16 400. How many lines are on the page? *[10 marks]*

4. Selma has a mortgage of £150 000. At the end of each year 6% interest is added before Selma pays £10 000.

(a) Show that at the end of the third year the amount owing is

$$£150000\times(1.06)^3-10000\times(1.06)^2-10000\times1.06-10000$$

(b) Find an expression for how much is owed at the end of the nth year.

(c) After how many years will the mortgage be paid off? *[10 marks]*

Mixed examination practice 8

Short questions

1. Find the coefficient of x^5 in the expansion of $(2-x)^{12}$. *[5 marks]*

2. $a=2-\sqrt{2}$. Using the binomial theorem or otherwise, express a^5 in the form $m+n\sqrt{2}$. *[5 marks]*

3. (a) Find the expansion of $(2+x)^5$, giving your answer in ascending powers of x.

(b) By letting $x=0.01$ or otherwise, find the exact value of 2.01^5. *[7 marks]*

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4. Determine the first 3 terms in the expansion of $(1-2x)^3(3+4x)^5$. *[7 marks]*

5. Fully expand and simplify $\left(x^2-\frac{2}{x}\right)^4$. *[6 marks]*

6. The coefficient of x in the expansion of $\left(x+\frac{1}{ax^2}\right)^7$ is $\frac{7}{3}$. Find the possible values of a. *[3 marks]*

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7. Given that $(1+x)^6(1+mx)^5 \equiv 1+nx+415x^2+\ldots+m^5x^{11}$, find the possible values of m and n. *[8 marks]*

Long questions

1. (a) Sketch the graph of $y=(x+2)^3$.

(b) Find the binomial expansion of $(x+2)^3$.

(c) Find the exact value of 2.001^3.

(d) Solve the equation $x^3+6x^2+12x+16=0$. *[12 marks]*

2. $f(x)=(1+x)^5$ and $g(x)=(2+x)^4$.

(a) Write down the vertical asymptote and axes intercepts of the graph $y=\frac{f(x)}{g(x)}$.

(b) Write down binomial expansions for $f(x)$ and $g(x)$.

(c) (i) Show that $\frac{f(x)}{g(x)}=x-k+\frac{ax^3+50x^2+85x+49}{g(x)}$, where k and a are constants to be found.

(ii) Hence explain why the graph of $y=\dfrac{f(x)}{g(x)}$ approaches a straight line when x is large, and write down the equation of this straight line.

(d) Sketch the curve $y=\dfrac{f(x)}{g(x)}$ for $-10\le x\le 10$. *[12 marks]*

3. (a) Write $\left(1+\sqrt{2}\right)^3$ in the form $p+q\sqrt{2}$ where $p,\ q\in\mathbb{Z}$.

(b) Write down the general term in the binomial expansion of $\left(1+\sqrt{2}\right)^n$.

(c) Hence show that $\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n$ is always an integer.

(d) What is the smallest value of n such that $\left(1+\sqrt{2}\right)^n$ is within 10^{-9} of a whole number? *[12 marks]*

4. The expansion of $(a+x)^n$ where $n\in\mathbb{N}$ has the form:

$$a^n+\ldots+\alpha x^r+\beta x^{r+1}+\gamma x^{r+2}+\ldots+x^n$$

(a) Show that the ratio of $\dfrac{\alpha}{\beta}$ is $\dfrac{r+1}{n-r}a$.

(b) If $a=1$ show that the expansion will contain two consecutive terms with the same coefficient as long as n is odd.

(c) Using the result of part (a) deduce an expression for $\dfrac{\beta}{\gamma}$

(d) Prove that there are no values for a such that in the expansion of $(a+x)^n$, $n\in\mathbb{N}$, three consecutive terms have the same coefficient. *[16 marks]*

Mixed examination practice 9

Short questions

EXAM HINT

Many examination questions combine ideas from this chapter with further trigonometry techniques you will meet in chapters 10–12.

1. The height of a wave at a distance x metres from a buoy is modelled by the function:

$f(x) = 1.4\sin(3x - 0.1) - 0.6$

(a) State the amplitude of the wave.

(b) Find the distance between consecutive peaks of the wave. *[4 marks]*

2. Sketch the graph of $y = \sin(2x) + 2\sin(6x)$ and hence find the exact period of the function. *[4 marks]*

3. A runner is jogging around a level circular track. His distance north of the centre of the track in metres is given by $60\cos 0.08t$ where t is measured in seconds.

(a) How long does is take the runner to complete one lap?

(b) What is the length of the track?

(c) At what speed is the runner jogging? *[7 marks]*

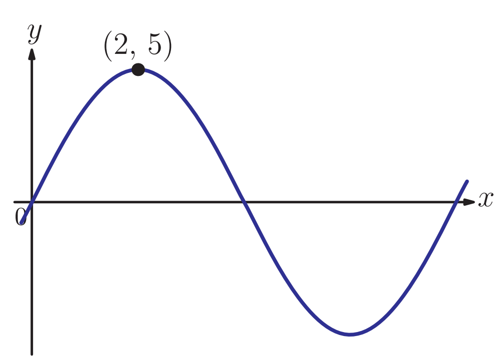
4. Let $f(x) = 3\sin 2\left(x - \frac{\pi}{3}\right)$.

(a) State the period of the function.

(b) Find the coordinates of the zeros of $f(x)$ for $x \in [0, 2\pi]$.

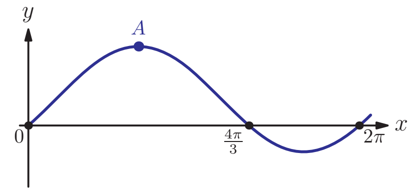
(c) Hence sketch the graph of $y = f(x)$ for $x \in [0, 2\pi]$, showing the coordinates of the maximum and minimum points. *[7 marks]*

5. The diagram shows the graph of the function $f(x) = a\sin(bx)$. Find the values of a and b. *[4 marks]*



Long questions

1. The graph shows the function $f(x) = \sin(x - k) + c$.



(a) (i) Write down the coordinates of A.

(ii) Hence find the values of k and c.

(b) Find all the zeros of the function in the interval $[-4\pi, 0]$.

(c) Consider the equation $f(x) = k$ with $-0.5 < k < 0$.

(i) Write down the number of solutions of this equation in the interval $[0, 9\pi]$.

(ii) Given that the smallest positive solution is α, write the next two solutions in terms of α. *[11 marks]*

2. (a) (i) Sketch the graph of $y = \tan x$ for $0 \le x \le 2\pi$.

(ii) On the same graph, sketch the line $y = \pi - x$.

(b) Consider the equation $x + \tan x = \pi$. Denote by x_0 the solution of this equation in the interval $]0, \frac{\pi}{2}[$.

(i) Find, in terms of x_0 and π, the remaining solutions of the equation in the interval $[0, 2\pi]$.

(ii) How many solutions does the equation $x + \tan x = \pi$ have for $x \in \mathbb{R}$?

(c) Given that $\cos A = c$ and $\sin A = s$,

(i) Write down the values of $\cos\left(\frac{\pi}{2} - A\right)$ and $\sin\left(\frac{\pi}{2} - A\right)$.

(ii) Hence show that $\tan\left(\frac{\pi}{2} - A\right) = \frac{1}{\tan A}$.

(iii) Given that $\tan A + \tan\left(\frac{\pi}{2} - A\right) = \frac{4}{\sqrt{3}}$, find the possible values of $\tan A$.

(iv) Hence find the values of $x \in]0, \frac{\pi}{2}[$ for which $\tan A + \tan\left(\frac{\pi}{2} - A\right) = \frac{4}{\sqrt{3}}$. *[16 marks]*

3. (a) Write down the minimum value of $\cos x$ and the smallest positive value of x (in radians) for which the minimum occurs.

(b) (i) Describe two transformations which transform the graph of $y = \cos x$ to the graph of $y = 2\cos\left(x + \frac{\pi}{6}\right)$.

(ii) Hence state the minimum value of $2\cos\left(x + \frac{\pi}{6}\right)$ and find the value of $x \in [0, 2\pi]$ for which the minimum occurs.

(c) The function f is defined for $x \in [0, 2\pi]$ by $f(x) = \dfrac{5}{2\cos\left(x + \frac{\pi}{6}\right) + 3}$.

(i) State, with a reason, whether f has any vertical asymptotes.

(ii) Find the range of f. *[13 marks]*

Mixed examination practice 10

Short questions

1. Solve the equation $\tan x = -0.62$ for $x \in (-90°, 270°)$. *[4 marks]*

2. Prove the identity $\frac{2}{\cos^2 x} - \tan^2 x = 2 + \tan^2 x$. *[5 marks]*

3. Solve the equation $5\sin^2\theta = 4\cos^2\theta$ for $-\pi \le \theta \le \pi$. *[5 marks]*

4. Prove the identity $\frac{1}{1+\cos x} + \frac{1}{1-\cos x} = \frac{2}{\sin^2 x}$. *[5 marks]*

5. Solve the equation $\cos\theta - 2\sin^2\theta + 2 = 0$ for $\theta \in [0°, 360°]$ *[6 marks]*

6. Use an algebraic method to solve the equation $6\sin^2 x + \cos x = 4$ for $0° \le x \le 360°$. *[6 marks]*

7. Find the exact values of $x \in [-\pi, \pi]$ satisfying the equation

   $2\cos\left(2x + \frac{\pi}{3}\right) = \sqrt{2}$. *[6 marks]*

8. (a) Given that $\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{16}{3}$, find the possible values of $\sin x$.

   (b) Hence find the exact solutions of the equation $\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} = \frac{16}{3}$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. *[6 marks]*

Long questions

1. The shape of a small bridge can be modelled by the equation $y = 1.8\sin\left(\frac{x}{3}\right)$, where y is the height of the bridge above water, and x is the distance from one river bank, both measured in metres.

   (a) Find the width of the river.

   (b) A barge has height 1.2 metres above the water level. Find the maximum possible width of the barge so it can pass under the bridge.

   (c) Another barge has width 3.5 m. What is the maximum possible height of the barge so it can pass under the bridge? *[10 marks]*

2. (a) Sketch the graph of the function $C(x) = \cos x + \frac{1}{2}\cos 2x$ for $-2\pi \le x \le 2\pi$.

   (b) Prove that the function $C(x)$ is periodic and state its period.

   (c) For what values of x, $-2\pi \le x \le 2\pi$, is $C(x)$ a maximum?

(d) Let $x = x_0$ be the smallest positive value of x for which $C(x) = 0$. Find an approximate value of x_0 which is correct to two significant figures.

(e) (i) Prove that $C(x) = C(-x)$ for all x.

(ii) Let $x = x_1$ be that value of x, $\pi < x < 2\pi$, for which $C(x) = 0$. Find the value of x_1 in terms of x_0. *[16 marks]*

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3. (a) Find the value of k for which the equation $4x^2 - kx + 1 = 0$ has a repeated root.

(b) Show that the equation $4\sin^2 x = 5 - k\cos x$ can be written as $4\cos^2\theta - k\cos\theta + 1 = 0$.

(c) Let $f_k(\theta) = 4\cos^2\theta - k\cos\theta + 1$.

(i) State the number of values of $\cos\theta$ which satisfy the equation $f_4(\theta) = 0$.

(ii) Find all the values of $\theta \in [-2\pi, 2\pi]$ which satisfy the equation $f_4(\theta) = 0$.

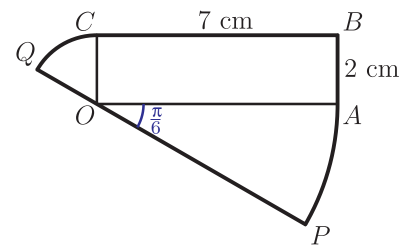
(iii) Find the value of k for which $x = 1$ is a solution of the equation $4x^2 - kx + 1 = 0$.

(iv) For this value of k, find the number of solutions of the equation $f_k(\theta) = 0$ for $\theta \in [-2\pi, 2\pi]$. *[14 marks]*

Mixed examination practice 11

Short questions

1. In the diagram, $OABC$ is a rectangle with sides 7 cm and 2 cm. PQ is a straight line. AP and CQ are circular arcs, and $A\hat{O}P = \frac{\pi}{6}$.



(a) Write down the size of $C\hat{O}Q$.

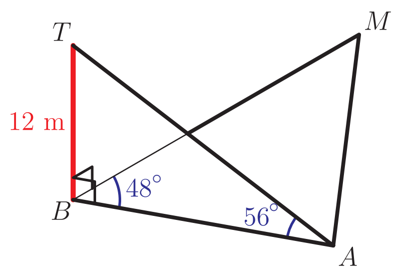
(b) Find the area of the whole shape.

(c) Find the perimeter of the whole shape. *[9 marks]*

2. A sector has perimeter 36 cm and radius 10 cm. Find its area. *[6 marks]*

3. In triangle ABC, $AB = 6.2$ cm, $CA = 8.7$ cm and $A\hat{C}B = 37.5°$. Find the two possible values of $A\hat{B}C$. *[6 marks]*

4. A vertical tree of height 12 m stands on horizontal ground. The bottom of the tree is at the point B. Observer A, standing on the ground, sees the top of the tree, T, at an angle of elevation of 56°.

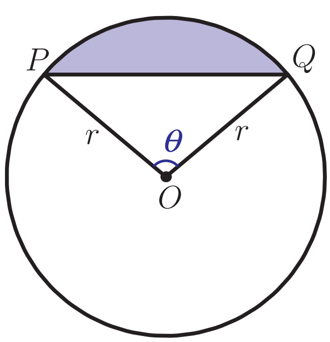


(a) Find the distance of A from the bottom of the tree.

Another observer, M, stands the same distance away from the tree, and $A\hat{B}M = 48°$.

(b) Find the distance AM. *[6 marks]*

5. The diagram shows a circle with centre O and radius $r = 7$ cm. The chord PQ subtends angle $\theta = 1.4$ radians at the centre of the circle.



Find:

(a) the area of the shaded region

(b) the perimeter of the shaded region. *[6 marks]*

6. In triangle ABC, $AB = 2\sqrt{3}$, $AC = 10$ and $B\hat{A}C = 150°$. Find the exact length of BC. *[6 marks]*

7. The perimeter of the sector shown in the diagram is 34 cm and its area is 52 cm^2. Find the radius of the sector. *[6 marks]*

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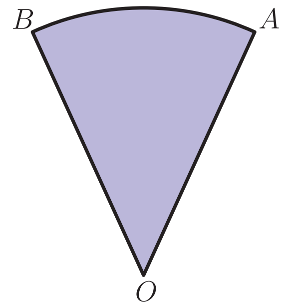
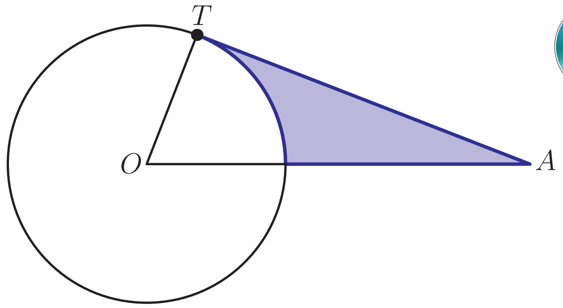


8. In the diagram, O is the centre of the circle and AT is the tangent to the circle at T.



Properties of circles and basic trigonometry are covered in the Prior learning Section W of the CD-ROM.

If $OA = 12$ cm, and the circle has a radius of 6 cm, find the area of the shaded region. *[4 marks]*

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9. The diagram shows a triangle and two arcs of circles.

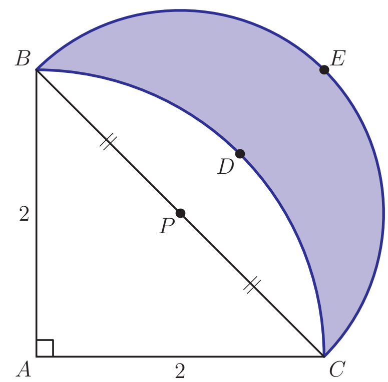
The triangle ABC is a right-angled isosceles triangle, with $AB = AC = 2$. The point P is the midpoint of BC.

The arc BDC is part of a circle with centre A.

The arc BEC is part of a circle with centre P.

(a) Calculate the area of the segment $BDCP$.

(b) Calculate the area of the shaded region $BECD$.



[6 marks]

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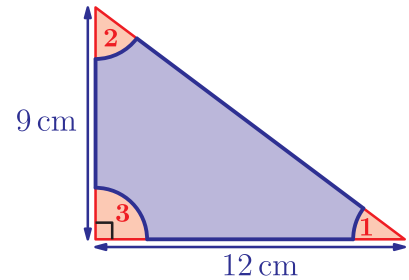
10. A right-angled triangle has sides 12 cm and 9 cm. At each vertex, a sector of radius 2 cm is cut out, as shown in the diagram. The angle at sector 1 is θ.



(a) Write down an expression for the area of sector 2 in terms of θ.

(b) Find the remaining area. *[6 marks]*

11. In the obtuse-angled triangle KLM, $LM = 6.1$ cm, $KM = 4.2$ cm and $K\hat{L}M = 42°$.

Find the area of the triangle. *[6 marks]*

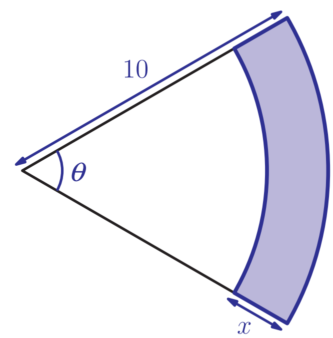
12. In triangle ABC, $AB = 10$ cm, $BC = 8$ cm and $CA = 7$ cm.

(a) Find the exact value of $\cos (A\hat{B}C)$.

(b) Find the exact value of $\sin (A\hat{B}C)$. *[8 marks]*

(c) Find the exact value of the area of the triangle.

13. The diagram shows two circular sectors with angle θ at the centre. The radius of the larger sector is 10 cm, the radius of the smaller sector is x cm smaller.



(a) Show that the area of the shaded region is given by $\frac{x(20-x)\theta}{2}$.

(b) If $\theta = 1.2$ find the value of x such that the area of the shaded region is equal to 54.6 cm^2. *[8 marks]*

Long questions

1. In triangle ABC, $AB = 5$, $AC = x$ and $\hat{A} = \theta$. M is the midpoint of the side AC.

(a) Use the cosine rule to find an expression for MB^2 in terms of x and θ.

(b) Given that $BC = MB$, show that $\cos , = \frac{3x}{20}$.

(c) If $x = 5$, find the value of the angle θ such that $MB = BC$. *[9 marks]*

2. Two circles have equal radius r and intersect at points S and T. The centres of the circles are A and B, and $A\hat{S}B = 90°$.

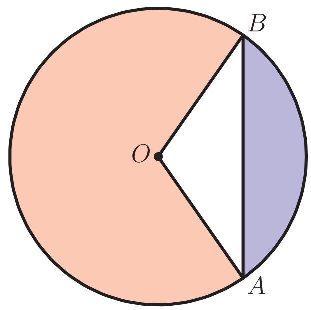
(a) Explain why $S\hat{A}T$ is also 90°.

(b) Find the length AB in terms of r.

(c) Find the area of the sector AST.

(d) Find the area of the overlap of the two circles. *[10 marks]*

3. The diagram shows a circle with centre O and radius r. Chord AB subtends an angle at the centre of size θ radians. The minor segment and the major sector are shaded.



(a) Show that the area of the minor segment is $\frac{1}{2}r^2(\theta - \sin\theta)$.

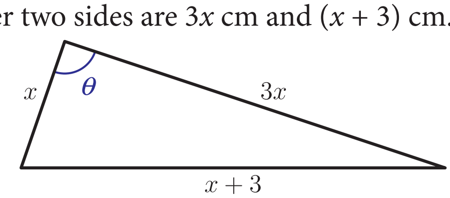
(b) Find the area of the major sector.

(c) Given that the ratio of the areas of the blue: pink regions is 1 : 2, show that:

$$\sin , = \frac{3}{2} , - \pi$$

(d) Find the value of θ. *[10 marks]*

4. The area of the triangle shown is 2.21 cm². The length of the shortest side is x cm and the other two sides are $3x$ cm and $(x + 3)$ cm.



(a) Using the formula for the area of the triangle, write down an expression for $\sin\theta$ in terms of x.

(b) Using the cosine rule, write down and simplify an expression for $\cos\theta$ in terms of x.

(c) (i) Using your answers to parts (a) and (b), show that:

$$\left(\frac{3x^2 - 2x - 3}{2x^2}\right)^2 = 1 - \left(\frac{4.42}{3x^2}\right)$$

(ii) Hence find the possible values of x and the corresponding values of θ, in radians, using your answer to part (b) above. *[10 marks]*

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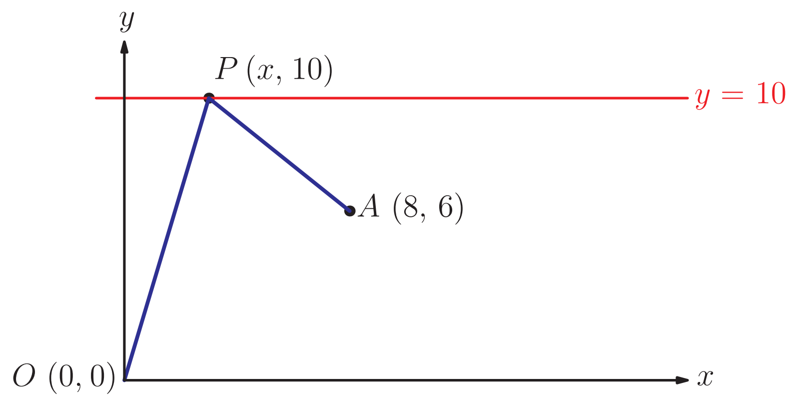
5. In triangle ABC, $AB = 10$, $BC = 5$, $CA = x$ and $B\hat{A}C = \theta°$.

(a) Show that $x^2 - 20x\cos\theta + 75 = 0$.

(b) Find the range of values of $\cos\theta$ for which the above equation has real roots.

(c) Hence find the set of values of θ for which it is possible to construct triangle ABC with the given measurements. *[8 marks]*

6. In the diagram below, the points O (0, 0) and A (8, 6) are fixed. The angle $O\hat{P}A$ varies as the point P (x, 10) moves along the horizontal line $y = 10$.



(a) (i) Show that $AP = \sqrt{x^2 - 16x + 80}$.

(ii) Write down a similar expression for OP in terms of x.

(b) Hence, show that:

$$\cos O\hat{P}A = \frac{x^2 - 8x + 40}{\sqrt{(x^2 - 16x + 80)(x^2 + 100)}}$$

(c) Find, in degrees, the angle $O\hat{P}A$ when $x = 8$.

(d) Find the positive value of x such that $O\hat{P}A = 60°$.

Let the function f be defined by

$$f(x) = \cos O\hat{P}A = \frac{x^2 - 8x + 40}{\sqrt{(x^2 - 16x + 80)(x^2 + 100)}}, \quad 0 \le x \le 15.$$

(e) Consider the equation $f(x) = 1$.

(i) Explain, in terms of the position of the points O, A and P, why this equation has a solution.

(ii) Find the exact solution to the equation.

[17 marks]

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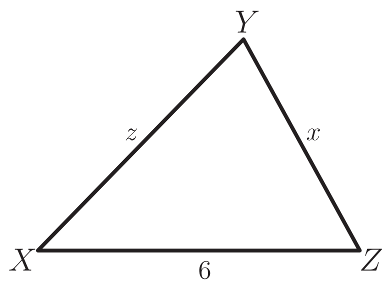
7. (a) Let $y = -16x^2 + 160x - 256$. Given that y has a maximum value, find:

(i) the value of x giving the maximum value of y

(ii) this maximum value of y.

The triangle XYZ has $XZ = 6$, $YZ = x$, $XY = z$ as shown. The perimeter of triangle XYZ is 16.



(b) (i) Express z in terms of x.

(ii) Using the cosine rule, express z^2 in terms of x and $\cos Z$.

(iii) Hence, show that $\cos Z = \dfrac{5x - 16}{3x}$.

Let the area of triangle XYZ be A.

(c) Show that $A^2 = 9x^2\sin^2 Z$.

(d) Hence, show that $A^2 = -16x^2 + 160x - 256$.

(e) (i) Hence, write down the maximum area for triangle XYZ.

(ii) What type of triangle is the triangle with maximum area?

[15 marks]

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8. Two circular cogs are connected by a chain as shown in diagram 1. The radii of the cogs are 3 cm and 8 cm and the distance between their centres is 25 cm.

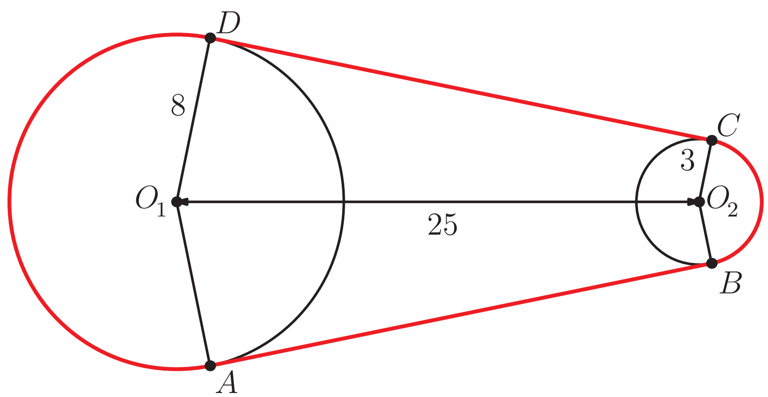

diagram 1

Diagram 2 shows the quadrilateral $O_1\hat{A}BO_2$. Line O_2P is drawn parallel to AB.

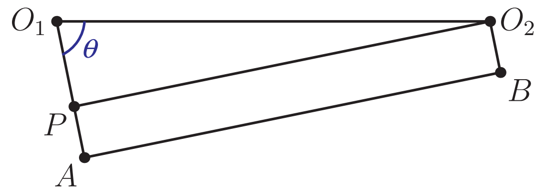

diagram 2

(a) Write down the size of $O_1\hat{A}B$ in radians, giving a reason for your answer.

(b) Explain why $PO_2 = AB$.

(c) Hence find the length AB.

(d) Find the size of the angle marked θ, giving your answer in radians correct to 4 significant figures.

(e) Calculate the length of the chain $ABCD$. *[12 marks]*

Mixed examination practice 12

Short questions

1. The angle θ satisfies the equation $2\tan^2\theta - 5\sec\theta - 10 = 0$, where θ is in the second quadrant. Find the *exact* value of $\sec\theta$.

[5 marks]

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2. (a) Write $\cos\left(x+\frac{\pi}{3}\right)$ in the form $a\cos x + b\sin x$.

(b) Hence find the exact values of $x \in [-2\pi, 2\pi]$ for which

$\cos\left(x+\frac{\pi}{3}\right) = \cos\left(x-\frac{\pi}{3}\right)$. *[6 marks]*

3. (a) Use the identity for $\cos(A+B)$ to show that $\cos 2\theta = 2\cos^2\theta - 1$.

(b) Solve the equation:

$$\frac{\sin\theta}{1+\cos\theta} = 3\cot\frac{\theta}{2} \text{ for } \theta \in (0, 2\pi):$$ *[6 marks]*

4. The angle θ satisfies the equation $\tan\theta + \cot\theta = 3$, where θ is in degrees. Find all the possible values of θ lying in the interval $[0°, 90°]$.

[5 marks]

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5. (a) Express $\sqrt{15}\sin(2x) + \sqrt{5}\cos(2x)$ in the form $R\sin(2x+\alpha)$.

(b) The function f is defined by:

$$f(x) = \frac{2}{5+\sqrt{15}\sin(2x)+\sqrt{5}\cos(2x)}$$

Using your answer to part (a), find:

(i) the maximum value of $f(x)$, giving your answer in the form $p + q\sqrt{5}$ where $p, q \in \mathbb{Q}$

(ii) the smallest positive value of x for which this maximum occurs, giving your answer exactly, in terms of π. *[7 marks]*

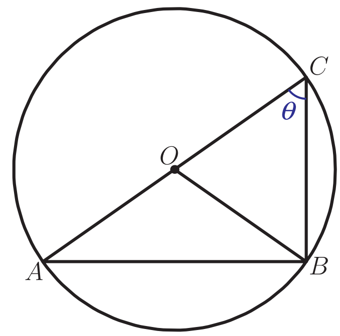
6. (a) Write down an expression for $\sin(\arcsin x)$.

(b) Show that $\sin(\arccos x) = \sqrt{1-x^2}$.

(c) Hence solve the equation $\arcsin x = \arccos x$ for $0 \le x \le 1$. *[6 marks]*

Long questions

1. The circle shown in the diagram has centre O and radius r.



(a) Write down the lengths of AB and BC in terms of r and θ.

(b) Write down an expression for the area of the triangle ABC.

(c) Write down an expression for the area of the triangle OBC.

(d) Hence find the ratio of the two areas in the form

$$\frac{\text{Area}(OBC)}{\text{Area}(ABC)} = k, \text{ where } k \in \mathbb{Q}.$$

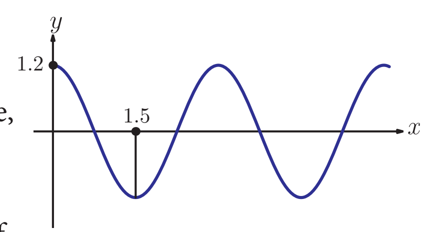
[10 marks]

2. (a) Use the identity for $\tan(A+B)$ to show that $\tan(2A) = \dfrac{2\tan A}{1-\tan^2 A}$.

(b) Write down the value of $\tan 135°$.

(c) Hence find the exact value of $\tan 67.5°$. *[7 marks]*

3. A water wave has the profile shown in the graph, where y represents the height of the wave in metres, and x is the horizontal distance, also in metres.



(a) Given that the equation of the wave can be written as $y_1 = a\cos(px)$, find the values of a and p.

(b) A second wave has the profile given by the equation $y_2 = 0.9\sin\left(\dfrac{2\pi}{3}x\right)$.
Write down the amplitude and the period of the second wave.

When the two waves combine a new wave is formed, with the profile given by $y = y_1 + y_2$.

(c) Write the equation for y in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$

(d) State the amplitude and the period of the combined wave.

(e) Find the smallest positive value of x for which the height of the combined wave is zero.

(f) Find the first two positive values of x for which the height of the combined wave is 1.3 m.

4. (a) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $r\cos(\theta + \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\alpha}{2}$, giving r and α as exact values.

(b) Hence, or otherwise, for $0 \le \theta \le 2\pi$, find the range of values of $\sqrt{3}\cos\theta - \sin\theta$.

(c) Solve $\sqrt{3}\cos\theta - \sin\theta = -1$, for $0 \le \theta \le 2\pi$, giving your answers as **exact** values. *[10 marks]*

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5. (a) Write $t^3 - 3t^2 - 3t + 1$ as a product of a linear and a quadratic factor.

(b) Show that $\tan(3A) = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

(c) Write down the exact value of $\tan 45°$.

(d) Hence find the exact values of $\tan 15°$ and $\tan 75°$. *[10 marks]*