Sum and difference of angles

The goal is to learn the identities for the sine and cosine of a sum or a difference of two angles.

Identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$$
$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha$$
$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$
$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

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Easy consequences of the above identities:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$
$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

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If you don't think that the above are easy consequences, let me know and I will explain in class.

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They're easy consequences of the first four identities. For instance if we set $\alpha = \beta$ into the first identity, we get:

$$\sin(\alpha + \alpha) = \sin\alpha\cos\alpha + \sin\alpha\cos\alpha$$

Which gives:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha$$

Identities

We will prove:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

The other 3 follow easily by substituting certain angles.

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Proof

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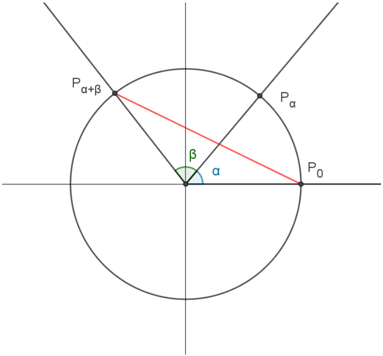

Think about the coordinates of the points P_0 and $P_{\alpha+\beta}$. What are these coordinates?

Proof

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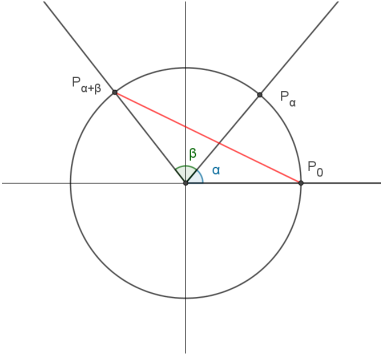

The coordinates of P_0 are $(1, 0)$ and the coordinates of $P_{\alpha+\beta}$ are simply $(\cos(\alpha + \beta), \sin(\alpha + \beta))$.

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Using Pythagorean Theorem:

$$|P_0 P_{\alpha+\beta}|^2 = (1 - \cos(\alpha + \beta))^2 + \sin^2(\alpha + \beta)$$

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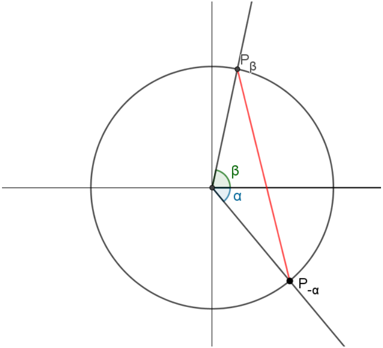
$$|P_0 P_{\alpha+\beta}|^2 = (1 - \cos(\alpha + \beta))^2 + \sin^2(\alpha + \beta)$$

We get:

$$|P_0 P_{\alpha+\beta}|^2 = 1 - 2\cos(\alpha + \beta) + \cos^2(\alpha + \beta) + \sin^2(\alpha + \beta) = 2 - 2\cos(\alpha + \beta)$$

Proof

Now if we rotate our triangle we get:



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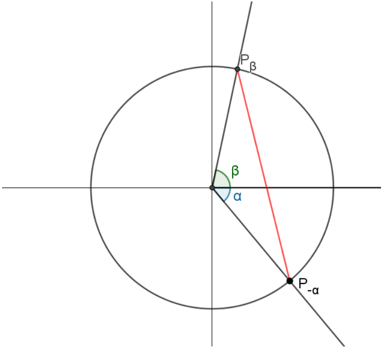

Of course the triangle did not change, so the length of the red segment is the same:

$$|P_0 P_{\alpha+\beta}| = |P_{-\alpha} P_\beta|$$

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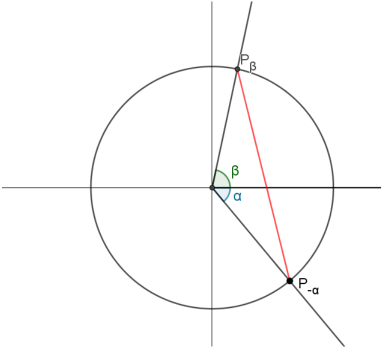

The coordinates of $P_{-\alpha}$ are of course $(\cos(-\alpha), \sin(\alpha))$.

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$$|P_{-\alpha}P_{\beta}|^2 = (\cos\beta - \cos(-\alpha))^2 + (\sin(-\alpha) - \sin\beta)^2$$

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We get:

$$\begin{aligned}|P_{-\alpha}P_{\beta}|^2 =& \cos^2\beta - 2\cos\beta\cos(-\alpha) + \cos^2(-\alpha) + \\ &+ \sin^2(-\alpha) - 2\sin(-\alpha)\sin\beta + \sin^2\beta = \\ =& 2 - 2\cos\beta\cos\alpha + 2\sin\alpha\sin\beta\end{aligned}$$

Proof

So finally we get:

$$2 - 2\cos(\alpha + \beta) = 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$$

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So:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Summary

Please try and understand this proof. If you think some details are missing, let me know.

Applications

Let's calculate $\sin 105^\circ$.

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$$\begin{aligned}\sin(105^\circ) = \sin(45^\circ + 60^\circ) =& \\ &= \sin 45^\circ \cos 60^\circ + \sin 60^\circ \cos 45^\circ = \\ &= \frac{\sqrt{2}}{2} \times \frac{1}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}\end{aligned}$$

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$$\begin{aligned}\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \\ = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \\ = \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \\ = \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

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We could've predicted the result, since:

$$\sin 105^\circ = \sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{2} + \frac{\pi}{12}\right) = \cos\frac{\pi}{12}$$

Na lekcji jeszcze raz przerobimy ten dowód, ale postarajcie sie go samodzielnie zrozumieć!