1. Prove by mathematical induction that, for $n \in \mathbb{Z}^+$,

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)^3 + \ldots + n\left(\frac{1}{2}\right)^{n-1} = 4 - \frac{n+2}{2^{n-1}}.$$

(Total 8 marks)

2. Use mathematical induction to prove that $5^n + 9^n + 2$ is divisible by 4, for $n \in \mathbb{Z}^+$.

(Total 9 marks)

3. (a) Find the sum of the infinite geometric sequence 27, −9, 3, −1, ...

(3)

(b) Use mathematical induction to prove that for $n \in \mathbb{Z}^+$,

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

(7)

(Total 10 marks)

4. (a) The sum of the first six terms of an arithmetic series is 81. The sum of its first eleven terms is 231. Find the first term and the common difference.

(6)

(b) The sum of the first two terms of a geometric series is 1 and the sum of its first four terms is 5. If all of its terms are positive, find the first term and the common ratio.

(5)

(c) The r^{th} term of a new series is defined as the product of the r^{th} term of the arithmetic series and the r^{th} term of the geometric series above. Show that the r^{th} term of this new series is $(r + 1)2^{r-1}$.

(3)

(d) Using mathematical induction, prove that

$$\sum_{r=1}^{n} (r+1)2^{r-1} = n2^n, n \in \mathbb{Z}^+.$$

(7)

(Total 21 marks)

5. (a) Consider the following sequence of equations.

$$1 \times 2 = \frac{1}{3}(1 \times 2 \times 3),$$

$$1 \times 2 + 2 \times 3 = \frac{1}{3}(2 \times 3 \times 4),$$

$$1 \times 2 + 2 \times 3 + 3 \times 4 = \frac{1}{3}(3 \times 4 \times 5),$$

$$.... .$$

(i) Formulate a conjecture for the n^{th} equation in the sequence.

(ii) Verify your conjecture for $n = 4$.

(2)

(b) A sequence of numbers has the n^{th} term given by $u_n = 2^n + 3$, $n \in \mathbb{Z}^+$. Bill conjectures that all members of the sequence are prime numbers. Show that Bill's conjecture is false.

(2)

(c) Use mathematical induction to prove that $5 \times 7^n + 1$ is divisible by 6 for all $n \in \mathbb{Z}^+$.

(6)
(Total 10 marks)

6. (a) Show that $\sin 2nx = \sin((2n + 1)x) \cos x - \cos((2n + 1)x) \sin x$.

(2)

(b) **Hence** prove, by induction, that

$$\cos x + \cos 3x + \cos 5x + ... + \cos((2n - 1)x) = \frac{\sin 2nx}{2\sin x},$$

for all $n \in \mathbb{Z}^+$, $\sin x \neq 0$.

(12)

(c) Solve the equation $\cos x + \cos 3x = \frac{1}{2}$, $0 < x < \pi$.

(6)
(Total 20 marks)