Name:

Mathematics IB HL Test 4

January 20, 2022

1 hour 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Calculators are **not allowed** for this examination paper.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is [**72 marks**].
- Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to **show all working**.
- Write your solutions in the space provided.

1. [Maximum mark: 5]

Complex number z satisfies the equation:

$$(2 - i)z = 9 + 8i$$

Find z and calculate z^3, give your answers in the form $x + yi$, where $x, y \in \mathbb{R}$.

2. [Maximum mark: 4]

Solve the inequality:

$$|x + 4| > |2x - 3|$$

3. [Maximum mark: 8]

Consider the polynomial

$$P(x) = x^3 + ax^2 + bx + c$$

The remainders when this polynomial is divided by $(x - 1)$ and $(x + 1)$ are both equal to 3.
The remainder when it is divided by $(x - 2)$ is 15.

a) Find the values of a, b and c. [5 *points*]

b) The remainder when $P(x)$ is divided by $(x - p)$ is 1. Find the value of p given that $p > 0$. [3 *points*]

4. [Maximum mark: 7]

Prove that

$$\sum_{j=1}^{n} j2^j = (n-1)2^{n+1} + 2$$

5. [Maximum mark: 7]

Prove that $3^{2n+2} - 8n - 9$ is divisible by 64 for all positive integers n.

6. [Maximum mark: 9]

a) Show that when n^2 is divided by 3, the only possible remainders are 0 and 1, for all $n \in \mathbb{Z}$. [3 *points*]

b) Show that if n^2 is divisible by 3, then n is divisible by 3, for all $n \in \mathbb{Z}$. [2 *points*]

c) Prove that $\sqrt{3}$ is irrational. [4 *points*]

7. [Maximum mark: 5]

Solve the equation:

$$2(\log_9 x)^2 + \log_{\frac{1}{9}} x = \log_{\sqrt{2}} 2\sqrt{2}$$

8. [Maximum mark: 5]

Find all possible values of k for which the graph of the function

$$f(x) = (k+1)x^2 - 2(k-1)x + 3k - 3$$

lies entirely below the x-axis.

9. [Maximum mark: 6]

Consider the system of equations:

$$\begin{cases} 8x + 3y + az = 12 \\ x + 2z = 3 \\ 2x + y - z = b \end{cases}$$

a) Find the values of a and b for which the system has infinitely many solutions. [4 *points*]

b) For the values of a and b found in part (a) find the solutions to the system in terms of parameter λ. [2 *points*]

10. [Maximum mark: 5]

A small classroom has six one-person desks arranged in 3 rows of 2. Six students are to be seated in the room for Mathematics examination. Find the number of possible arrangements of students if 2 of the 6 students: Tomasz and Maria cannot sit next to each other in the same row and one cannot sit directly behind the other.

11. [Maximum mark: 11]

A sequence is given by $u_1 = 7$ and $u_{n+1} = 2u_n + 3$ for $n \geqslant 1$.

a) Using the definition above find u_2 and u_3. [2 *points*]

b) Show that a sum of an even number and an odd number is odd and hence show that every term of this sequence is odd. [3 *points*]

c) Show that the formula for the n-th term of the sequence is $u_n = 5 \cdot 2^n - 3$. [6 *points*]