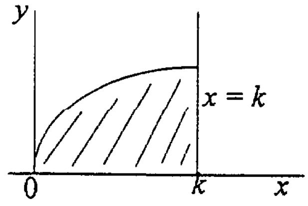
SECTION A

1. (i) (a) *(A1)*



(b) Area $=\int_0^k \sin x\,dx = [-\cos x]_0^k = 1-\cos k$ *(M1)(A1)*

(c) Volume required $=\pi\int_0^k \sin^2 x\,dx = \pi\int_0^k \frac{1-\cos 2x}{2}dx$ *(M1)(A1)*

$$=\frac{\pi}{2}\left[\left(x-\frac{\sin 2x}{2}\right)\right]_0^k = \frac{\pi}{4}(2k-\sin 2k)$$ *(M1)(A1)*

(ii) The random variable X has a hypergeometric distribution.

$$E(X)=\frac{(3)(4)}{10}=\frac{12}{10}=\frac{6}{5}=1.2$$ *(M1)(A1)*

$$V(X)=\frac{4(10-4)3(10-3)}{10^2(10-1)}=\frac{(24)(21)}{(100)(9)}=\frac{14}{25}=0.56$$ *(M1)(A1)*

Thus, $E(X)=\frac{6}{5}=1.2$

$V(X)=\frac{14}{25}=0.56$

several will do it differently

$E(X) = \Sigma x\,p(x) = \frac{1.60+2.36+3.4}{120} = 1.2$

$V(X) = \Sigma x^2 p(x) - \mu^2 = \frac{1.60+4.36+9.4}{120} - 1.2^2 = \boxed{.56}$

$\sigma(X) = .748$

$E(x) = \frac{nk}{N}$

$\sigma^2 = \frac{nk(N-k)(N-n)}{N^2(N-1)}$

(iii) Let X be the number of defective bulbs, p be the probability of finding a defective bulb.
X is a binomial random variable.
sample size $n = 200$
$p = 0.1$
$E(X) = np = 20$

Standard deviation of X $= \sqrt{(200)(0.1)(0.9)}$ $= \sqrt{18}$
~~= 4.24~~ *(M1)(A2)*

We want the probability that in a random sample of 200 bulbs more than 24, i.e. 25 or more, are defective.
Using continuity correction, we want to find $p(Y \geq 24.5)$ where Y is normally distributed with mean 2.0 and standard deviation 4.24. *(M1)*

Hence $p(X > 24) = p(Y \geq 24.5)$

$$= p\left(z \geq \frac{24.5 - 20}{4.24}\right) = p(z \geq 1.061)$$

$= 0.144$ (3 significant figures) *(M1)(A1)*

2. (i) The successive distance through which the ball falls form a geometric sequence with first term 81 and the common ratio $\frac{2}{3}$.

(a) The maximum height of the ball between the fifth and the sixth bounce is

$(81)\left(\frac{2}{3}\right)^5 = \frac{32}{3}$ metre. ***(M2)(A1)***

10.7

(b) The total distance traveled by the ball from the time it is dropped until it strikes the ground the sixth time is

$$\sum_{n=0}^{5} 81\left(\frac{2}{3}\right)^n + \sum_{n=0}^{4} 81\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^n$$

$$= \frac{81\left(1-\left(\frac{2}{3}\right)^6\right)}{1-\frac{2}{3}} + \frac{54\left(1-\left(\frac{2}{3}\right)^5\right)}{1-\frac{2}{3}}$$

$$= \frac{665}{3} + \frac{422}{3} = \frac{1087}{3} = 362\frac{1}{3} \text{ metres}$$ ***(M2)(A2)***

Note: Some candidates may calculate the total distance as follows:

$$\text{Total distance} = 81 + 2\times\left\{54 + 54\left(\frac{2}{3}\right) + 54\left(\frac{2}{3}\right)^2 + 54\left(\frac{2}{3}\right)^3 + 54\left(\frac{2}{3}\right)^4\right\}$$

$$= 81 + 108\left(\frac{1-\left(\frac{2}{3}\right)^5}{1-\frac{2}{3}}\right) = 81 + 324\left(\frac{243-42}{243}\right)$$

$$= 81 + 281\frac{1}{3} = 362\frac{1}{3} \text{ metres}$$ ***Award (M2)(A2)***

(c) If the ball continues to bounce indefinitely, then the distance traveled is

$$\sum_0^{\infty} 81\left(\frac{2}{3}\right)^n + \sum_0^{\infty} 81\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^n$$

$$= \frac{81}{1-\frac{2}{3}} + \frac{54}{1-\frac{2}{3}} = 243 + 162 = 405 \text{ metres}$$ *(M2)(A1)*

Note: Some candidates may also mention distance traveled

$$= 81 + 108\left(1 + \frac{2}{3} + \ldots\right)$$

$$= 81 + 108\left(\frac{1}{1-\frac{1}{3}}\right) = 81 + 324$$

$$= 405 \text{ metres}$$ ***Award (M2)(A1)***

(ii) FIRST METHOD

Let the three numbers in arithmetic progression be $x, x+r, x+2r$. Their sum is

$$x + (x+r) + (x+2r) = 3x + 3r = 24$$

Hence $x + r = 8$ or $r = 8 - x$ *(M1)(A1)*

We are also given that $x-1, x+r-2$ and $x+2r$ are in geometric progression. So

$$\frac{x+r-2}{x-1} = \frac{x+2r}{x+r-2}$$

or $(x+r-2)^2 = (x-1)(x+2r)$. *(M1)(A1)*

Substituting $x + r = 8$ and $r = 8 - x$, we get

$$(8-2)^2 = (x-1)\{x + 2(8-x)\}$$

or $(x-1)(16-x) = 36$

or $-x^2 + 17x - 16 = 36$

or $x^2 - 17x + 52 = 0$

or $(x-13)(x-4) = 0$

Hence, $x = 13$ or 4 *(M1)(A1)*

The solutions are obtained by taking $x = 13, r = 8 - 13 = -5$ and $x = 4, r = 4$.

So there are two sets of solutions

viz. 13, 8, 3 and 4, 8, 12 *(R1)(R1)*

SECOND METHOD

Since the three numbers are in arithmetic progression with sum equal to 24, let the numbers be $8 - x,\ 8,\ 8 + x$. *(M1)(A1)*

From these we form the new numbers $7 - x,\ 6,\ 8 + x$ which are in geometric progression.

Hence $(7 - x)(8 + x) = 6^2$ *(M1)(A1)*

We get $x^2 + x - 20 = 0$ i.e. $(x + 5)(x - 4) = 0$

Hence, $x = -5$ or $x = 4$ *(M1)(A1)*

When $x = 4$, the numbers are 4, 8, 12
and when $x = -5$, the numbers are 13, 8, 3 *(M1)(A1)*

3. (a) Line L_1 passes through (2, 3, 7) and is parallel to $\vec{v} = 3\vec{i} + \vec{j} + 3\vec{k}$.

Hence the parametric equation of the line is

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (2\vec{i} + 3\vec{j} + 7\vec{k}) + t(3\vec{i} + \vec{j} + 3\vec{k}), \quad -\infty < t < \infty$ *(M2)(A1)*

(b) $x = 2 + 3t, y = 3 + t, z = 7 + 3t$ is any point on the line. To find the point of intersection of the line and the plane $2x + 3y - 4z + 21 = 0$. Substitute $x = 2 + 3t, y = 3 + t, z = 7 + 3t$ in the equation of the plane and we get

$$2(2 + 3t) + 3(3 + t) - 4(7 + 3t) + 21 = 0$$

or $\quad 4 + 9 - 28 + (6 + 3 - 12)t + 21 = 0$

or $\quad -3t = -6$ or $t = 2$ ✓ *(M1)(A1)*

Hence the point of intersection is (8, 5, 13). *(A1)*

(c) Let E_1 be the plane which passes through the point (1, 2, 3) and parallel to the plane $2x + 3y - 4z + 21 = 0$. Normal to E_1 is $2\vec{i} + 3\vec{j} - 4\vec{k}$. *(M1)*

Since (1, 2, 3) lies on E_1, the equation of the plane E_1 is

$$2(x - 1) + 3(y - 2) - 4(z - 3) = 0$$ *(M1)*

or $\quad 2x + 3y - 4z + 4 = 0$ *(A1)*

(d) (i) L_2 has equation $x = t, y = t$ and $z = -t, \quad -\infty < t < \infty$.

Hence L_2 is parallel to the vector $\vec{i} + \vec{j} - \vec{k}$.

Since L_1 is parallel to the vector $3\vec{i} + \vec{j} + 3\vec{k}$, L_1 is not parallel to L_2. *(M1)(R1)*

(ii) A point on the line L_2 is given by $x = s, y = s$ and $z = -s, \quad -\infty < s < \infty$. If L_1 intersects L_2, then the equations

$$2 + 3t = s \qquad (1)$$
$$3 + t = s \qquad (2)$$
$$7 + 3t = -s \qquad (3)$$

will hold.

From (2) and (3) $10 + 4t = 0$ or $t = -\dfrac{5}{2}$. But from (1) and (2) $2t - 1 = 0$ or $t = \dfrac{1}{2}$. Thus the system of equations (1), (2) and (3) are inconsistent. Hence L_1 does not intersect L_2. *(M1)(R1)*

(e) (i) L_2 is parallel to the vector $w = \vec{i} + \vec{j} - \vec{k}$. *(A1)*

(ii) $\overrightarrow{PO} = \vec{i} - 2\vec{j} - 4\vec{k}$ *(A1)*

(iii) $$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 3 \\ 1 & 1 & -1 \end{vmatrix} = -4\vec{i} + 6\vec{j} + 2\vec{k}$$ *(M1)(A1)*

$$|\vec{v} \times \vec{w}| = |-4\vec{i} + 6\vec{j} + 2\vec{k}| = \sqrt{56}$$ *(A1)*

Hence, $$d = \left|\frac{\overrightarrow{PO} \cdot (\vec{v} \times \vec{w})}{|\vec{v} \times \vec{w}|}\right| = \left|\frac{(\vec{i} - 2\vec{j} - 4\vec{k}) \cdot (-4\vec{i} + 6\vec{j} + 2\vec{k})}{\sqrt{56}}\right|$$

$$= \left|\frac{-24}{\sqrt{56}}\right| = \frac{12}{\sqrt{14}}$$ *(M1)(A1)*

4. (i) (a) $$A^2 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$$ *(A1)*

$$A^3 = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$$ *(A1)*

(b) Conjective: $A^n = \begin{bmatrix} n+1 & -n \\ n & -(n-1) \end{bmatrix}$ for all $n \in \mathbb{N}^*$ ***(A3) If all the 4 entries are correct, –1 for each error.***

(c) Let $P(n)$ be the statement that

$$A^n = \begin{bmatrix} n+1 & -n \\ n & -(n-1) \end{bmatrix}$$

$P(1)$ is true because

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$ *(C1)*

Suppose $P(k)$ is true for some $k \in \mathbb{N}^*$. *(M1)*

Then

$$A^{k+1} = A^k A = \begin{bmatrix} k+1 & -k \\ k & -(k-1) \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2k+2-k & -(k+1) \\ 2k-(k-1) & -k \end{bmatrix}$$

$$= \begin{bmatrix} (k+1)+1 & -(k+1) \\ k+1 & -((k+1)-1) \end{bmatrix}$$ *(M1)(A1)*

Hence $P(k+1)$ is true.

By mathematical induction $P(n)$ is true for all $n \in \mathbb{N}^*$. *(R1)*

(ii) (a) Given $f(x)=\dfrac{ax+b}{cx^2+dx+e}$

$f\left(-\dfrac{5}{2}\right)=0$ implies $\dfrac{-\frac{5}{2}a+b}{\frac{25}{4}c+\frac{5}{2}d+e}=0$

Hence

$$-\frac{5}{2}a+b=0\text{.................}(1)$$

Since $x=-1$ and $x=-4$ are asymptotes, $cx^2+dx+e=(x+1)(x+4)=x^2+5x+4$.

Hence

$$c=1,\ d=5 \text{ and } e=4.$$

Also $f(0)=\dfrac{5}{4}$ implies

$$\frac{b}{e}=\frac{5}{4} \text{ or } b=\frac{5e}{4}$$

Since $e=4, b=5$.

Using $b=5$ in (1), we get

$$-\frac{5}{2}a+5=0 \text{ or } a=2$$

Hence,

$$a=2, b=5, c=1, d=5 \text{ and } e=4$$

(M3)(A1)

$f(-5/2)=0 \Rightarrow$ $\boxed{5a-2b=0}$

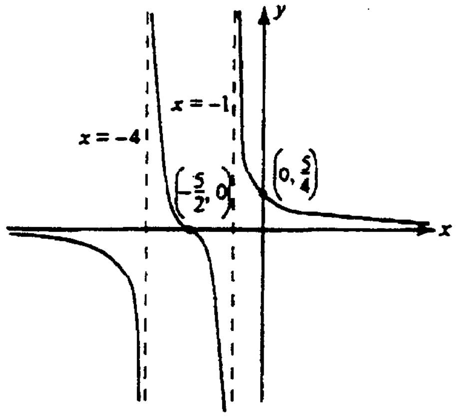
$f(0)=\frac{5}{4} \Rightarrow$ $\boxed{\frac{b}{e}=\frac{5}{4}}$ M1

$x=-4$ Asym. $\Rightarrow$ $\boxed{16c-4d+e=0}$ M1

$x=-1 \Rightarrow$ $\boxed{c-d+e=0}$

$\Rightarrow$ $\boxed{\begin{array}{l} a=2t \\ b=5t \\ c=t \\ d=5t \\ e=4t \end{array}}$ M1 A1

(b) Since $f'(x) < 0$ when $f(x)$ is decreasing, we see from the given graph of $f(x)$ that $f(x)$ is decreasing when $x < -4, -4 < x < -1$ and $x > -1$. Hence $f'(x) < 0$ when $x < -4, -4 < x < -1$ and $x > -1$. *(A1)(A1)(A1)*



Note: Some candidates may calculate $f'(x)$ and conclude

$$f'(x) = -\frac{2x^2 + 10x + 17}{(x^2 + 5x + 4)^2}$$ *(M1)*

Since $2x^2 + 10x + 17 > 0$ for all x, *(M1)*

$f'(x) < 0$ for all values of x for which $f(x)$ is defined viz.

$x < -4,\ -4 < x < -1,\ -1 < x$. *(R1)*

(c) $$f(x) = \frac{2x+5}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1} = \frac{A(x+1) + B(x+4)}{(x+4)(x+1)}$$

$(A+B)x + (A+4B) = 2x + 5$ M 1

Thus, on equating coefficients of like powers of x,

$A + B = 2$ and $A + 4B = 5$ M 1

From these two equations, we get, $B = 1$ and $A = 1$. Hence

$$f(x) = \frac{1}{x+4} + \frac{1}{x+1} \quad \text{.............(2)}$$ A 1 *(M2)(A1)*

(d) $f'(x) = -(x+4)^{-2} - (x+1)^{-2}$

and

$f''(x) = 2(x+4)^{-3} + 2(x+1)^{-3}$................(3)

When $x = -\frac{5}{2}, f''(x) = 2\left(-\frac{5}{2}+4\right)^{-3} + 2\left(-\frac{5}{2}+1\right)^{-3}$

$$= 2\left(\frac{3}{2}\right)^{-3} + 2\left(-\frac{3}{2}\right)^{-3} = 0$$

Since $f'(x)$ is negative throughout $(-4, -1)$, $f''(x) = 0$ when $x = -\frac{5}{2}$, $f''(x)$ changes sign at $x = -\frac{5}{2}$. Hence $x = -\frac{5}{2}$ is a point of inflection. *(M2)(R1)*

(e) $f''(x) > 0$ when $-4 < x < -\frac{5}{2}$ and $x > -1$ *(A1)(A1)*

Note: Some candidates may write $f''(x) > 0$ if
$(x+1)(x+4)^4 + (x+1)^4(x+4) > 0$
i.e. $(x+1)(x+4)\{(x+4)^3 + (x+1)^3\} > 0$
i.e. $(x+1)(x+4)(2x+5)(x^2+5x+13) > 0$

Since $x^2 + 5x + 13 > 0$ for all x,
$f''(x) > 0$ when $-4 < x < -\frac{5}{2}$ or $x > -1$ *(M1)(A1)*

SECTION B

Abstract Algebra

5. (i) $\mathbb{R}^* = \mathbb{R} - \{0\}$ and $a \# b = b|a|$

(a) Yes. $\mathbb{R}^*$ is closed under the binary operation # since $a \# b = b|a| \in \mathbb{R}$ and when $a \neq 0$ $|a| \neq 0$, ~~then~~ and $b \neq 0$, ~~$a \neq 0$~~ $b|a| \neq 0$ imply $a \# b \neq 0$. Thus $a \# b \in \mathbb{R}^*$. ? *(C1)(R1)*

(b) Let $a,b,c \in \mathbb{R}^*$. Then $(a \# b) \# c = (b|a|) \# c = c|b|a|| = c|b||a| = a \# (c|b|)$ $= a \# (b \# c)$. Hence # is an associative binary operation on $\mathbb{R}^*$. *(M1)(R1)*

(c) If $k \in \mathbb{R}^*$ such that $k \# a = a$, then $a|k| = a$. Hence $|k| = 1$. Thus $k = -1$ or 1. *(M1)(R1).*

(d) We want m so that $a \# m = 1$ or $a \# m = -1$. $a \# m = m|a|$ implies $m = \frac{1}{|a|}$ or $-\frac{1}{|a|}$. *(M1)(R1)*

(e) $(\mathbb{R}^*, \#)$ can not be a group because in that case there is an element $e \in \mathbb{R}^*$ so that $a \# e = e \# a = a$ for every $a \in \mathbb{R}^*$. But $a \# e = a$ implies $e|a| = a$ or $e = \frac{a}{|a|}$ which is not a constant. So we do not have an identity in $\mathbb{R}^*$ and hence $(\mathbb{R}^*, \#)$ is not a group. [or no unique identity] *(M1)(A1)*

(f) $S = \{x \in \mathbb{R} \mid x < 0\}$ and $a \# b = b|a| < 0$ for all $a, b \in S$.
So # is a closed binary operation. Also for all $a,b,c \in S$.

$$(a \# b) \# c = (b|a|) \# c = c|b|a|| = c|b||a| = a \# (c|b|) = a \# (b \# c).$$

Thus # is an associative binary operation on S. *(R1)*

-1 is the identity, since for any $a \in S$, $a \# e = e|a| = a$ and $e \# a = a|e| = a$. *(R1)*

Corresponding to each $a \in s$ there is $-\frac{1}{|a|} \in S$, so that $a \# \left(-\frac{1}{|a|}\right) = \left(-\frac{1}{|a|}\right) \# a = -1$.

Hence $-\frac{1}{|a|}$ is the inverse of a. *(R1)*

(ii) (a) $a \bullet b = a \bullet c$ implies $a^{-1} \bullet (a \bullet b) = a^{-1} \bullet (a \bullet c)$.

By associativity, we have

$(a^{-1} \bullet a) \bullet b = (a^{-1} \bullet a) \bullet c$

or $e \bullet b = e \bullet c$

or $b = c$, where e is the identity element of $(G, \bullet)$. *(M2)(A2)*

(b) Let e, e' be two identities (if possible) in $(G, \bullet)$.

From $a \bullet e = a = a \bullet e'$, we get $e = e'$, *(M2)(R1)*

so identity is unique.

Remark: Some candidates may attempt the problem as follows:

$e = e \bullet e' = e'$ implies $e = e'$ *Award (M2)(R1)*

(c) Suppose, for some $a \in G$, there are two inverses viz. a^{-1} and b. Then $a \bullet a^{-1} = e = a \bullet b$. By cancellation law $a^{-1} = b$. Hence each element of the group G has exactly one inverse. *(M2)(R1)*

(iii) (a) A group $(G, \bullet)$ is said to be cyclic if there exists an element $a \in (G, \bullet)$ such that $G = \{a^n \mid n \in \mathbb{Z}\}$. The element a is called a generator. *(C2)(C2)*

(b) By the structure of the Cayley table given for $(H, *)$, $*$ is a closed binary operation on H. a is the identity. Each element of H has an inverse as mentioned below:

Element of H	Inverse
a	a
b	d
c	c
d	b

Since $*$ is given to be associative, $(H, *)$ is a group. *(M2)(A1)*

$b^0 = a, b^1 = b, b^2 = c, b^3 = d$ and $b^4 = a$

Thus $(H, *)$ is a cyclic group with a generator b. *(M1)(A1)*

(c) One can, in a similar manner, show that d is a generator for the cyclic group $(H, *)$, since

$d^1 = d, d^2 = c, d^3 = b$ and $d^4 = a$.

So the two generators are b and d. *(M1)(A1)*

(d) The subgroups of $(H, *)$ are $\{a\}$, $\{a, c\}$ and H. *(M2)*

The proper subgroups of ~~$(H, *)$ are $\{a\}$~~ and $\{a, c\}$. L 3 *(R1)*

(e) $(H, *)$ can not have any subgroup of order 3 because Lagrange's theorem requires that the order of a subgroup divides the order of a group and $(H, *)$ is a group of order four. *(M2)(R1)*

Remark: Some candidates may mention that by Lagrange's theorem $(H, *)$ can only have subgroups of order 1, 2 or 4. Hence, $(H, *)$ can not have any subgroup of order 3. *(M2)(R1)*

Graph Theory

6. (i) (a) Adjacency matrix is given by

$$A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix}$$

(M1)(A2)

–1 for two mistakes.

(b) Incidence matrix is given by

$$B = \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix}$$

(M1)(A2)

–1 for two errors.

(c) To determine the number of ways to go from v_1 to v_2 traversing exactly four edges, we compute A^4 and select the (1, 2) entry.

$$A^4 = \begin{bmatrix} 44 & 46 & 40 & 12 \\ 46 & 62 & 50 & 14 \\ 40 & 50 & 42 & 12 \\ 12 & 14 & 12 & 4 \end{bmatrix}$$

(M2)(A2)

Since (1, 2) entry is 46, there are 46 different ways to go from v_1 to v_2 in the required manner.

(d) Let $G = (V, E)$ be an undirected graph or multigraph with no isolated points. G is said to have an Eulerian circuit if there is a circuit in G that traverses every edge of the graph exactly once. *(A2)*

The above graph does not contain an Eulerian circuit because the vertex v_1 has degree 3. *(R1)*

Note that if $G = (V, E)$ is an undirected connected graph with an Eulerian circuit then every vertex has an even degree. *(R1)* ?

Note: Some candidates may write the following:
The graph does not contain an Eulerian circuit since not all vertices have even degree. Some may say that e_7 makes it impossible to have an Eulerian circuit.
Award (R2)

(e) If in a graph G there exists a closed circuit which passes exactly once through each vertex of G, then such a circuit is called a Hamiltonian circuit. *(A2)*

Since no closed circuit contains v_4 the graph does not contain a Hamiltonian circuit. *(R1)*

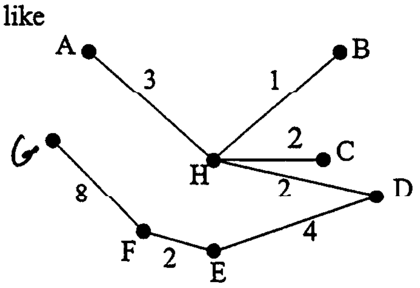
(ii) Prim's algorithm requires that we start at the vertex A and consider it as a tree and then look for the shortest path that joins a vertex on this tree to any of the remaining vertices to obtain a minimal spanning tree. We make choices, choose corresponding edges to be added and keep track of the weights.

Choice	Edge	Weight
1	AH	3
2	HB	1
3	HC	2
4	HD	2
5	DE	4
6	EF	2
7	FG	8
Total weight		22

(M3)(A3)
–1 for each error.

Note: Some candidates may only mention:
Starting at A and using Prim's algorithm $AH + HB + HC + HD + DE + EF + FG$ yields $3+1+2+2+4+2+8=22$. ***Award (M3)(A3).***
–1 for each error.

The network looks like

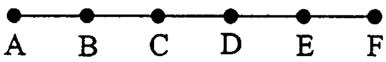


(A2)

(iii) (a) Each edge of a graph is incident on two vertices and thereby contributes two to the sum of the degree of the vertices.
If a graph has n edges then the sum of the degrees of the vertices is $2n$. *(M2)(A1)*

(b) The sum of the degrees in the degree sequence $\{3, 3, 2, 2, 2, 2, 2, 2, 1\}$ is 19. Since it is an odd number there can not be any graph with the given degree sequence as the degree of the vertices. *(M1)(A1)*

On the other hand the graph



(A1)

corresponds to the degree sequence $\{2, 2, 2, 2, 1, 1\}$. Note that degree of A = degree of $F = 1$ and degree of B = degree of C = degree of D = degree of E. *(A1)*

(iv) (a) Two graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are isomorphic if there is a one to one and onto map $\varphi : V_1 \to V_2$ such that $a, b \in V_1$ are adjacent if and only if $\varphi(a)$ and $\varphi(b)$ are adjacent. *(A3)*

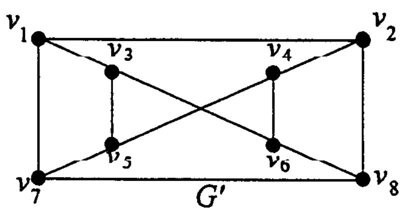
Remark: Some candidates may write an isomorphism between two graphs is a one to one and onto mapping between vertices so that it preserves adjacency and incidence. or 1-1 correspondence.

(b) Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isomorphic with an isomorphism φ. Let $v_1, v_2, \ldots, v_k, v_1$ be a cycle of length k in G_1 with $v_i \in V_1$, $1 \le i \le k$. Then, by the isomorphism,

$\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_k), \varphi(v_1)$ *(M2)(R1)*

is a cycle of length k since $\varphi(v_{i-1})$ is adjacent to $\varphi(v_i)$, $2 \le i \le k$ and $\varphi(v_k)$ is adjacent to $\varphi(v_1)$.

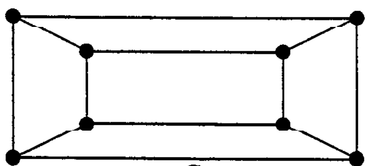
(c) If G and G' were isomorphic and one of them has a cycle of length k then the other must have a cycle of length k.



G'

G' has a cycle of length 5 which is given by $v_1, v_3, v_6, v_8, v_2, v_1$ *(M2)*

But G has no cycle of length 5. So G and G' are not isomorphic. *(R1)*



G

Statistics

7. (i) One unit of time is one minute. On a weekday morning a switch board receives 25 calls during a five minute period so λ, the number of calls per minute, is 5.

(a) The probability that a switch board receives zero telephone calls between 10.31 and 10.32 next Thursday morning is

$$e^{-5}\frac{5^0}{0!} = e^{-5} = 0.00674$$

So the probability that the switch board receives at least one telephone call is $1 - e^{-5} = 0.993$. *(M2)(A1)*

Remark: The answer may be given as $1 - e^{-5}$

(b) The probability that the switch board receives at least two or three telephone calls between 10.31 and 10.32 next Thursday morning is

$$e^{-5}\frac{5^2}{2!} + e^{-5}\frac{5^3}{3!} = e^{-5}\left[\frac{25}{2} + \frac{125}{6}\right]$$

$$= e^{-5}\left[\frac{200}{6}\right] = \frac{100}{3}e^{-5} \approx 0.225$$ *(M2)(A1)*

(ii) Let us set up the following hypothesis:

H_0: Proportion of students failed by X, Y and Z are equal

H_1: Proportion of students failed by X, Y and Z are not equal *(C1)(C1)*

If H_0 were true, then the teachers would have failed $\frac{27}{180} = 15\%$ of students and would have passed 85% of students.

Hence the expected frequencies are given by the following table:

EXPECTED FREQUENCIES

	X	Y	Z	Total
Passed	46.75	51.85	54.4	153
Failed	8.25	9.15	9.6	27
Total	55	61	64	180

(M2)(A2)

ν, the number of degrees of freedom is given by $\nu = (2-1)(3-1) = 2$.

$$\chi^2 = \frac{(50-46.75)^2}{46.75} + \frac{(47-51.85)^2}{51.85} + \frac{(56-54.40)^2}{54.40} + \frac{(5-8.25)^2}{8.25} + \frac{(14-9.15)^2}{9.15} + \frac{(8-9.60)^2}{9.60}$$
$$= 4.84$$ *(M2)(A2)*

At 10% level of significance $\chi^2 = 4.61$.

Since $4.84 > 4.61$, the critical value corresponding to a probability of 0.1, we reject the null hypothesis. *(M1)(R1)*

At 5% level of significance $\chi^2 = 5.99$. Since $4.84 < 5.99$, we can not reject the null hypothesis. *(M1)(R1)*

(iii) Let μ be the mean thickness of the washers.

$H_0: \mu = 0.50$ and the machine is in proper working order.
$H_1: \mu \neq 0.50$ and the machine is not in proper working order. *(C1)(C1)*

We need a two tailed small sample test. Under H_0,

$t = 3.1622$

$$t = \frac{\bar{x}-\mu}{s/\sqrt{N-1}} = \left(\frac{0.53-0.50}{0.03}\right)\sqrt{10-1} = 3$$ also $\sqrt{n}$ *(M2)(A2)*

We accept H_0 if t is between $-t_{.975}$ to $t_{.975}$ with $10-1=9$ degrees of freedom. Thus, we accept H_0 if t is between -2.26 and 2.26. *(M1A1)*

Since calculated t value is 3, we reject H_0. *(M1)(R1)*

SECOND METHOD

$H_0: \mu = 0.50$, machine is in working order. *(C1)*

$H_1: \mu \neq 0.50$, machine is faulty. *(C1)*

Sample size is 10. So estimate for population standard deviation is $\left(\sqrt{\frac{10}{9}}\right)0.03$.

Hence, means of sample size 10 have a t-distribution with standard deviation

$\left(\sqrt{\frac{10}{9}} \times 0.03\right)\frac{1}{\sqrt{10}} = 0.01$. *(M2)(A2)*

Critical values at 5% level under H_0 are

$0.50 \pm (2.262)(0.01) \quad (\nu = 9)$

i.e. 0.50 ± 0.0226 *(M1)(A1)*

The observed value is outside this interval, so we reject the claim. *(M1)(R1)*

(iv) (a) The 95% confidence limits are

$$\hat{p} \pm z\sqrt{\frac{\hat{p}(1-\hat{p})}{N}} \qquad (N = \text{sample size})$$

we have $\hat{p} = 0.55$, $z = 1.96$ (for 95% confidence level) and $N = 100$.

So the confidence interval is given by

$$0.55 \pm 1.96\sqrt{\frac{(0.55)(0.45)}{100}}$$
$$= 0.55 \pm 0.098 \qquad \textit{(M2)(A1)}$$

So the confidence interval is [0.452, 0.648] *(A1)*

(b) Let N be the required sample size. We want N to be such that

$$0.50 < \hat{p} - z\sqrt{\frac{\hat{p}(1-\hat{p})}{N}} \qquad \textit{(M1)(A1)}$$

where $\hat{p} = 0.55$ and z (for 95% confidence) = 1.96.

So we want

$$\frac{50 - \hat{p}}{-z} > \sqrt{\frac{\hat{p}(1-\hat{p})}{N}}.$$

Thus $\sqrt{N} > \dfrac{z}{\hat{p} - 0.50}\sqrt{\hat{p}(1-\hat{p})}$. *(M1)(A1)*

Substitute $\hat{p} = 0.55, z = 1.96$ to get

$$N > \left(\frac{1.96}{0.55 - 0.50}\right)^2 \cancel{\sqrt{}}(0.55)(0.45)$$
$$= 380.3184. \qquad \textit{(M1)}$$

$\therefore$ sample size required is at least 381. *(R1)*

Analysis and Approximation

8. (i) (a) The interval [0, 1] is divided into four sub-intervals. The trapezium rule approximation of $\int_0^1 e^{x^2}\,dx$ is given by

$$\frac{1-0}{(2)(4)}\left\{f(0)+2f\left(\frac{1}{4}\right)+2f\left(\frac{2}{4}\right)+2f\left(\frac{3}{4}\right)+f(1)\right\}$$

$$=\frac{1}{8}\left\{1+2e^{1/16}+2e^{1/4}+2e^{9/16}+e\right\}=1.49$$ ***(M2)(A2)***

(b) The error E_n in the trapezium rule approximation is given by

$E_n=-\frac{(b-a)^3}{12n^2}f''(c)$ where c is a point in $[a, b]$ and n is the number of sub intervals. In our case, $n=4, a=0, b=1$.

$$E_4=-\frac{1}{(12)(16)}\left(\left(\frac{d}{dx}\right)^2 e^{x^2}\right)_{x=c}$$ ***(A1)***

Where c is such that $0<c<1$.

If $f(x)=e^{x^2}, f'(x)=2xe^{x^2}$ and $f''(x)=2e^{x^2}+(2x)^2e^{x^2}=(2+4x^2)e^{x^2}$

Hence, $f''(c)=(2+4c^2)e^{c^2}$

Since $f''(c)$ is positive and increasing over [0, 1], $0<f''(c)\le(2+4)e=6e$.

Hence $|E_4|\le\frac{6e}{(12)(16)}=\frac{e}{32}=0.085$ ***(M2)(A1)***

(ii) (a) Set $u_k=k\left(\frac{1}{2}\right)^k$. Then $u_{k+1}=\frac{k+1}{2^{k+1}}$

$$\lim_{k\to\infty}\frac{u_{k+1}}{u_k}=\lim_{k\to\infty}\frac{1}{2}\left(\frac{k+1}{k}\right)=\frac{1}{2}$$ ***(M2)(A1)***

Since $0<\frac{1}{2}<1$, the series $\sum_{k=1}^{\infty}k\left(\frac{1}{2}\right)^k$ converges by ratio test. ***(R1)***

Remark: Some candidates may use root test.

(b) Set $f(x) = \dfrac{10}{x\ln x}$, $x \geq 2$.

$f(x)$ is positive and continuous on $[2, \infty)$. Also $f(x)$ decreasing with the fact that

$f(k) = a_k = \dfrac{10}{k\ln k}$, $k = 2, 3, \ldots$ *(M1)*

By integral test the series $\sum_{k=2}^{\infty} \dfrac{10}{k\ln k}$ and the integral $\int_2^{\infty} \dfrac{10}{x\ln x}\,dx$ converge or diverge together.

Since,

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{R\to\infty} (\ln\ln x)_2^R = \infty$$ *(M1)(A1)*

the series diverges. *(R1)*

(c) $\sum_{k=1}^{\infty} (-1)^k \dfrac{k}{k^2+1}$ is an alternating series.

Let us write it as $\sum_{k=1}^{\infty} (-1)^k u_k$ where $u_k = \dfrac{k}{k^2+1}$, $k = 1, 2, \ldots$.

If we wrote $u(x) = \dfrac{x}{x^2+1}$, then $u'(x) = \dfrac{(x^2+1) - 2x^2}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} < 0$ for $x \geq 2$.

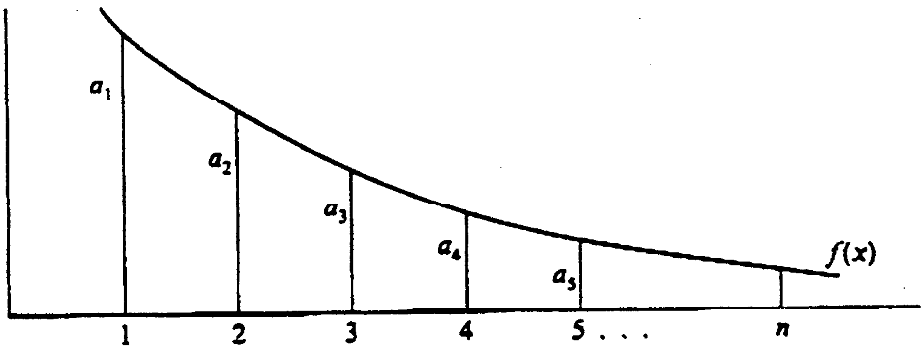
Hence $u(x)$ is a decreasing function and consequently u_k is a decreasing sequence for $k = 2, 3, \ldots$.

Also $\lim_{k\to\infty} u_k = \lim_{k\to\infty} \dfrac{k}{k^2+1} = 0$ *(M2)(A1)*

Hence, by alternating series test, the series $\sum_{k=1}^{\infty} (-1)^k \dfrac{k}{k^2+1}$ is a convergent series.

(R1)

(iii)(a)



$f(x)$ is a decreasing function over $[1, \infty)$. Further, we have $f(n) = a_n, n \geq 1$. Hence, $a_{i-1} \geq f(x) \geq a_i$ for $x \in [i-1, i]$; $i \geq 2$.

Thus $\int_{i-1}^{i} a_{i-1} \mathrm{d}x \geq \int_{i-1}^{i} f(x) \mathrm{d}x \geq \int_{i-1}^{i} a_i \mathrm{d}x, \quad i \geq 2$

or $\quad a_{i-1} \geq \int_{i-1}^{i} f(x) \mathrm{d}x \geq a_i, \quad i \geq 2$ *(M1)(A1)*

$$\sum_{i=2}^{n} a_{i-1} \geq \sum_{i=2}^{n} \int_{i=1}^{i} f(x) \mathrm{d}x \geq \sum_{i=2}^{n} a_i$$

or $\quad a_1 + a_2 + \ldots + a_{n-1} \geq \int_1^n f(x) \mathrm{d}x \geq a_2 + a_3 + \ldots + a_n$

or $\quad a_2 + \ldots + a_n \leq \int_1^n f(x) \mathrm{d}x \leq a_1 + a_2 + \ldots + a_{n-1}$ *(M1)(A1)*

(b) From (a)

$$a_1 + a_2 + \ldots + a_n \le a_1 + \int_1^n f(x)\mathrm{d}x$$

and writing $s_n = a_1 + a_2 + \ldots + a_n$, we have

$$s_n \le \int_1^n f(x)\mathrm{d}x + a_1$$

Also, from part (a),

$$\int_1^n f(x)\mathrm{d}x \le a_1 + a_2 + \ldots + a_{n-1} = s_{n-1}$$

Hence,

$$\int_1^n f(x)\mathrm{d}x \le s_{n-1} + a_n = s_n$$

Since $a_n \ge 0$.

Thus

$$\int_1^n f(x)\mathrm{d}x \le s_n \le \int_1^n f(x)\mathrm{d}x + a_1 \qquad \textit{(M2)(R1)}$$

If we take $f(x) = \dfrac{1}{x}$

$$\int_1^n \frac{\mathrm{d}x}{x} \le \sum_1^n \frac{1}{n} \le \int_1^n \frac{\mathrm{d}x}{x} + 1 \qquad \textit{(M1)}$$

But

$$\int_1^n \frac{\mathrm{d}x}{x} = \ln n - \ln 1 = \ln n \qquad \textit{(A1)}$$

Hence

$$\ln n \le \sum_1^n \frac{1}{n} < \ln n + 1$$

When $n = 10\,000$, $\ln n = 9.2103$ and we get

$$9.21 < \sum_{n=1}^{10000} \frac{1}{n} < 10.21$$

Thus the sum of the series $\sum_{n=1}^{10000} \frac{1}{n}$ is in the interval [9.21, 10.21] *(M2)(A1)*

(iv) By the mean value theorem for any $x \in (3, 7)$ there is some c, $3 < c < x$, such that

$$\frac{f(x) - f(3)}{x - 3} = f'(c).$$ *(M1)*

But $|f'(x)| \leq 4$. Hence,

$$\left| \frac{f(x) - f(3)}{x - 3} \right| = |f'(c)| \leq 4$$ *(A1)*

Thus,

$$|f(x) - f(3)| \leq 4|x - 3| \leq 16$$ *(M1)(A1)*

From this, we conclude that

$$f(3) - 16 \leq f(x) \leq f(3) + 16$$

Substituting $f(3) = -16$, we get

$$-32 \leq f(x) \leq 0 \text{ for } 3 \leq x \leq 7$$ *(M2)(A2)*