INTERNATIONAL BACCALAUREATE

MARKSCHEME

November 1997

MATHEMATICS

Higher Level

Paper 1

1. $x = k$ is a solution of the equation $x^3 + kx^2 - x - k = 0$ if

$k^3 + k^3 - k - k = 0$ (M1)

$k^3 - k = 0$

$k(k-1)(k+1) = 0$ (M1)

$\therefore k = 0, \pm 1.$ (A2) (C4)

2. C has the same order as B, and so C is $n \times p$. (R1)(A1) (C2)

D has the same order as AB, and so D is $m \times p$. (R1)(A1) (C2)

3. $p(A \cup B) = p(A) + p(B) - p(A \cap B)$ (M1)

$0.6 = 0.2 + p(B) - 0.2 \times p(B)$ since A, B are independent. (M1)(R1)

Therefore, $0.8 \times p(B) = 0.4$

and $p(B) = 0.5$ (A1) (C4)

4. (a) $\log_9 x^3 = \dfrac{\log_3 x^3}{\log_3 9} = \dfrac{3\log_3 x}{2} \Rightarrow k = \dfrac{3}{2}$ (A1)

$\log_{27} 512 = \dfrac{\log_3 512}{\log_3 27} = \dfrac{\log_3 8^3}{3} = \dfrac{3\log_3 8}{3} = \log_3 8 \Rightarrow m = 1$ (A1) (C2)

(b) $\log_9 x^3 + \log_3 x^{1/2} = \log_{27} 512$

$\Rightarrow \dfrac{3}{2}\log_3 x + \dfrac{1}{2}\log_3 x = \log_3 8$

$\Rightarrow 2\log_3 x = \log_3 8$

$\Rightarrow x^2 = 8$

$\Rightarrow x = \sqrt{8} = 2\sqrt{2}$ (since $x > 0$) (A1)(R1) (C2)

5. (a) $\dfrac{dy}{dt} = 1 + \cos t, \quad \dfrac{dx}{dt} = 2t + 2\cos 2t$

Hence, $\dfrac{dy}{dx} = \dfrac{1 + \cos t}{2t + 2\cos 2t} = 1$ at the point $t = 0$. (M2)

Therefore, the required gradient = 1. (A1) (C3)

(b) At the point $t = 0$, $x = y = 0$.

Therefore, the required equation is $y = x$. (A1) (C1)

6. $y = xe^{3x} + \ln x$

$\Rightarrow \dfrac{dy}{dx} = e^{3x} + 3xe^{3x} + \dfrac{1}{x}$ *(M1)(A1)* *(C2)*

$\Rightarrow \dfrac{d^2y}{dx^2} = 3e^{3x} + 3\left[e^{3x} + 3xe^{3x}\right] - \dfrac{1}{x^2} = \left[6 + 9x\right]e^{3x} - \dfrac{1}{x^2}$ *(M1)(A1)* *(C2)*

7. (a) $\dfrac{z}{\omega} = \dfrac{(3+ik)(k-7i)}{(k+7i)(k-7i)} = \dfrac{10k}{k^2+49} + i\left(\dfrac{k^2-21}{k^2+49}\right)$ *(M1)(A1)* *(C2)*

(b) $\dfrac{z}{\omega}$ is real if and only if $k^2 = 21$, i.e. $k = \pm\sqrt{21}$ *(M1)(A1)* *(C2)*

8. MATHEMATICS contains 11 letters with 2 M's, 2 A's and 2 T's.

(a) Number of arrangements $= \dfrac{11!}{2!2!2!}$ *(M1)*

$= 4\,989\,600$ *(A1)* *(C2)*

(b) Number of arrangements $= \dfrac{9!}{2!2!}$ *(M1)*

$= 90\,720$ *(A1)* *(C2)*

9. $5\sin x - 12\cos x = 6.5 \Rightarrow \dfrac{5}{13}\sin x - \dfrac{12}{13}\cos x = \dfrac{1}{2}$

$\Rightarrow \sin(x-\alpha) = \dfrac{1}{2}$ where $\cos\alpha = \dfrac{5}{13}$ and $\sin\alpha = \dfrac{12}{13}$ *(M1)*

Thus, $\alpha = 67.4^\circ$ will do. *(A1)*

$\Rightarrow x - 67.4^\circ = 30^\circ, 150^\circ \; (+360^\circ k, k \in \mathbb{Z})$

which gives $x = 97.4^\circ, 217^\circ \; (0^\circ \le x \le 360^\circ)$ *(A2)* *(C4)*

10. The coefficient of x^2 in $(1+x)^{2n} = \dbinom{2n}{2}$ *(M1)*

The coefficient of x^2 in $(1+15x^2)^n = \dbinom{n}{1}15$ *(M1)*

Thus, $\dbinom{2n}{2} = 15\dbinom{n}{1} \Rightarrow \dfrac{2n(2n-1)}{2} = 15n$ *(M1)*

$\Rightarrow 2n - 1 = 15 \quad (n \ne 0)$

$\Rightarrow n = 8$ *(A1)* *(C4)*

11. $6x - x^2 - 5 = 4 - (x-3)^2$ *(M1)(A1)*

$$\Rightarrow \int \frac{dx}{\sqrt{6x - x^2 - 5}} = \int \frac{dx}{\sqrt{4-(x-3)^2}} = \arcsin\frac{x-3}{2} + c$$ *(M1)(A1)* *(C4)*

12. (a) $\det A \quad = k^2 + 1$ *(A1)*

$$A^{-1} \quad = \frac{1}{k^2+1}\begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix}$$ *(A1)* *(C2)*

(b) $$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{k^2+1}\begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix}\begin{pmatrix} 2k \\ 1-k^2 \end{pmatrix} = \begin{pmatrix} 1 \\ -k \end{pmatrix}$$

$\Rightarrow x = 1,\ y = -k$ *(M1)(A1)* *(C2)*

13. $$\frac{1}{x-\sqrt{x}} \geq \frac{4}{15}$$

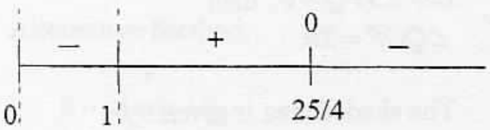
$$\Rightarrow \frac{1}{x-\sqrt{x}} - \frac{4}{15} \geq 0$$

$$\Rightarrow \frac{15 - 4x + 4\sqrt{x}}{15\left(x-\sqrt{x}\right)} \geq 0$$ *(M1)*

$$\Rightarrow \frac{\left(5-2\sqrt{x}\right)\left(3+2\sqrt{x}\right)}{15\sqrt{x}\left(\sqrt{x}-1\right)} \geq 0$$ *(M1)*

Now, $3 + 2\sqrt{x} > 0, x \neq 0, 1$ and $\sqrt{x} > 0$

The required sign diagram is:



(M1)

Therefore, $1 < x \leq \frac{25}{4}$. *(A1)* *(C4)*

14. $9x - y = 14$ has gradient 9

For $y = x^3 - 3x + a, \frac{dy}{dx} = 3x^2 - 3$ *(M1)*

$\Rightarrow 3a^2 - 3 = 9$

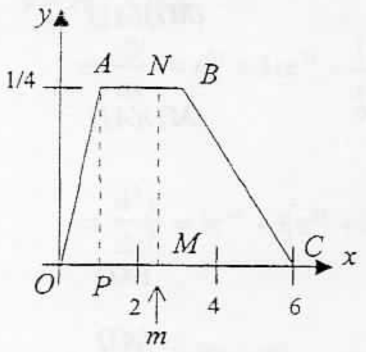
$\Rightarrow a = \pm 2.$ *(M1)(A1)*

When $a = 2$, $\quad a^3 - 3a + a = 4$ and $\quad 9 \times 2 - 4 = 14$

When $a = -2$, $\quad a^3 - 3a + a = -4$ and $\quad 9 \times (-2) - (-4) = -14$

Therefore, $\quad a = 2$, only. *(A1)* *(C4)*

15.



Let the median be m.

Then, the area of $OANM = \frac{1}{2}$ *(M1)*

Area of $\Delta OAP = \frac{1}{8}$, and *(A1)*

area of $ANMP = \frac{1}{4}(m-1)$ *(A1)*

Thus, $\frac{1}{8} + \frac{1}{4}(m-1) = \frac{1}{2}$, giving $m = 2\frac{1}{2}$ *(A1)*

(C4)

Alternatively, since the area of $\Delta OAP = \frac{1}{8}$, then $\int_1^m \frac{1}{4}\,dx = \frac{3}{8}$ *(A1)(M2)*

giving $\frac{1}{4}(m-1) = \frac{3}{8}$ and finally $m = 2\frac{1}{2}$ *(A1)*

(C4)

16. $\alpha + \beta = -5$ and $\alpha\beta = k$ *(A1)*

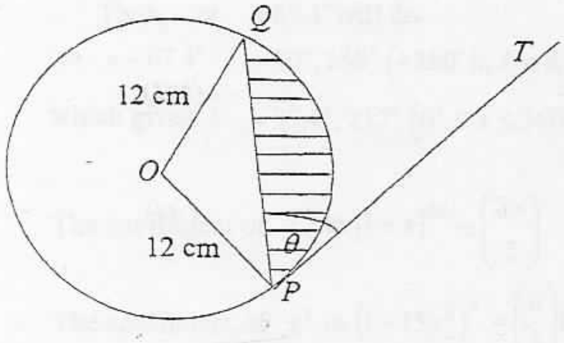
The sum of the roots is $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = 25 - 2k$,

and the product of the roots is $\alpha^2\beta^2 = (\alpha\beta)^2 = k^2$. *(M1)(A1)*

A suitable quadratic equation is $x^2 + (2k-25)x + k^2 = 0$. *(A1)*

(C4)

17.



Let $\angle TPQ = \theta$, then $\angle QOP = 2\theta$.

The shaded area is given by

$A = \frac{1}{2} \times 12^2 \times (2\theta - \sin 2\theta)$

i.e. $A = 72(2\theta - \sin 2\theta)$ *(A1)*

$\frac{dA}{dt} = 72(2 - 2\cos 2\theta)\frac{d\theta}{dt}$

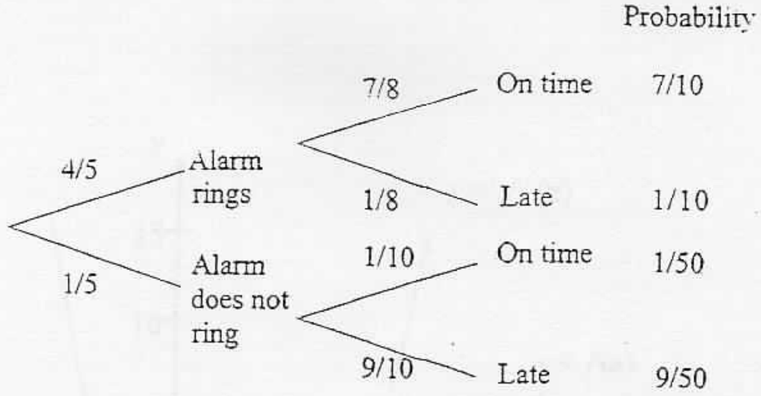
(M1)

When $\theta = 30^\circ$ and $\frac{d\theta}{dt} = \frac{\pi}{60}$ rad s^{-1}, $\frac{dA}{dt} = 72\left(2 - 2 \times \frac{1}{2}\right) \times \frac{\pi}{60} = 1.2\pi$ *(M1)*

Therefore, the area is *increasing* at the rate of 1.2π cm^2s^{-1}. *(A1)*

(C4)

18.



(a) Probability student is on time $= \dfrac{7}{10} + \dfrac{1}{50} = \dfrac{18}{25}$ *(M1)(A1)* *(C2)*

(b) $p(\text{alarm did not ring} \mid \text{student is late for school})$

$$= \frac{p(\text{alarm did not ring and student is late for school})}{p(\text{student is late})}$$ *(M1)*

$$= \frac{9/50}{1/10 + 9/50}$$

$$= \frac{9}{14}$$ *(A1)* *(C2)*

19. $|1 - \mathrm{i}z| = |z + 1|$

$\Rightarrow |1 + y - \mathrm{i}x|^2 = |1 + x + \mathrm{i}y|^2$ *(M1)*

$\Rightarrow (1 + y)^2 + x^2 = (1 + x)^2 + y^2$ *(M1)(A1)*

$\Rightarrow y = x$, which is the required locus. *(A1)* *(C4)*

Alternative Method:

$|1 - \mathrm{i}z| = |z + 1|$

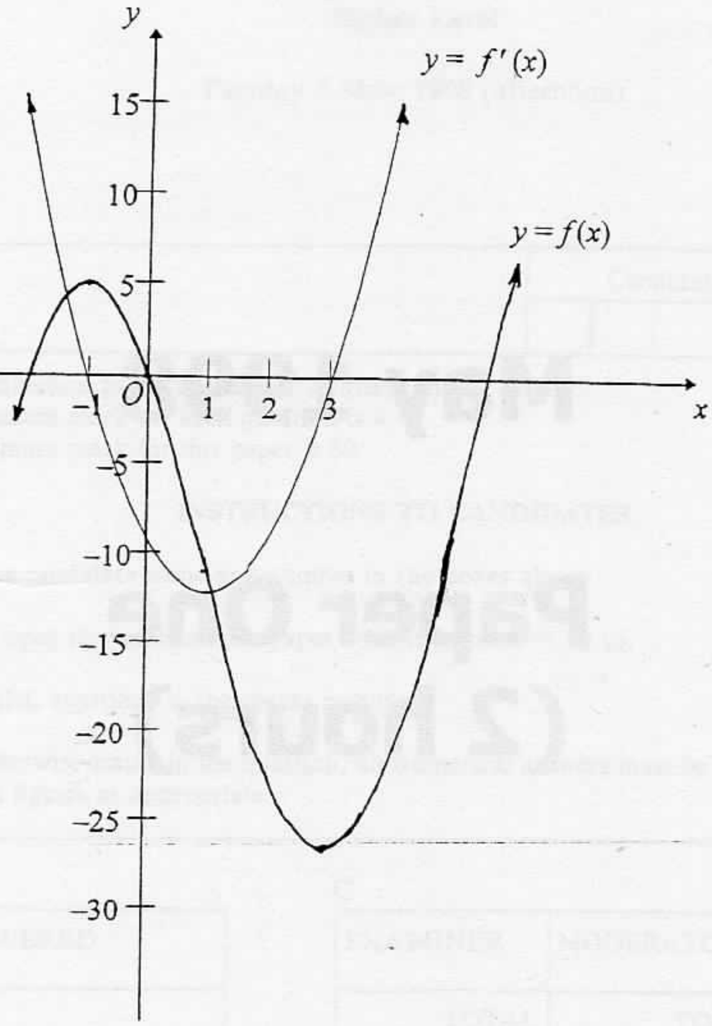
$\Rightarrow |-\mathrm{i}||z + \mathrm{i}| = |z + 1|$ *(M1)*

$\Rightarrow |z + \mathrm{i}| = |z + 1|.$ *(A1)*

Therefore, P is equidistant from $-\mathrm{i}$ and -1. *(R1)*

Thus, the locus of P is the straight line $y = x$. *(A1)* *(C4)*

20.



(A1) (shape)

(A1) (max at $(-1, 5)$)

(A1) (for $(0, 0)$)

(A1) (min at $(3, -27)$)

(C4)