1. Given that $\frac{z}{z+2} = 2 - \mathrm{i}$, $z \in \mathbb{C}$, find z in the form $a + \mathrm{i}b$.

(Total 4 marks)

METHOD 1

$z = (2 - \mathrm{i})(z + 2)$ M1
$= 2z + 4 - \mathrm{i}z - 2\mathrm{i}$
$z(1 - \mathrm{i}) = -4 + 2\mathrm{i}$
$z = \frac{-4 + 2\mathrm{i}}{1 - \mathrm{i}}$ A1
$z = \frac{-4 + 2\mathrm{i}}{1 - \mathrm{i}} \times \frac{1 + \mathrm{i}}{1 + \mathrm{i}}$ M1
$= -3 - \mathrm{i}$ A1

METHOD 2

let $z = a + \mathrm{i}b$
$\frac{a + \mathrm{i}b}{a + \mathrm{i}b + 2} = 2 - \mathrm{i}$ M1
$a + \mathrm{i}b = (2 - \mathrm{i})((a + 2) + \mathrm{i}b)$
$a + \mathrm{i}b = 2(a + 2) + 2b\mathrm{i} - \mathrm{i}(a + 2) + b$
$a + \mathrm{i}b = 2a + b + 4 + (2b - a - 2)\mathrm{i}$
attempt to equate real and imaginary parts M1
$a = 2a + b + 4 (\Rightarrow a + b + 4 = 0)$
and $b = 2b - a - 2 (\Rightarrow -a + b - 2 = 0)$ A1

Note: Award A1 for two correct equations.

$b = -1; a = -3$
$z = -3 - \mathrm{i}$ A1

[4]

2. The complex numbers $z_1 = 2 - 2\text{i}$ and $z_2 = 1 - \text{i}\sqrt{3}$ are represented by the points A and B respectively on an Argand diagram. Given that O is the origin,

(a) find AB, giving your answer in the form $a\sqrt{b - \sqrt{3}}$, where $a, b \in \mathbb{Z}^+$;

(3)

(b) calculate $\text{A}\hat{\text{O}}\text{B}$ in terms of π.

(3)
(Total 6 marks)

(a) $\text{AB} = \sqrt{1^2 + (2 - \sqrt{3})^2}$ M1

$= \sqrt{8 - 4\sqrt{3}}$ A1

$= 2\sqrt{2 - \sqrt{3}}$ A1

(b) **METHOD 1**

$\arg z_1 = -\frac{\pi}{4}$, $\arg z_2 = -\frac{\pi}{3}$ A1A1

Note: Allow $\frac{\pi}{4}$ and $\frac{\pi}{3}$.

Note: Allow degrees at this stage.

$\text{A}\hat{\text{O}}\text{B} = \frac{\pi}{3} - \frac{\pi}{4}$

$= \frac{\pi}{12}$ (accept $-\frac{\pi}{12}$) A1

Note: Allow FT for final A1.

METHOD 2

attempt to use scalar product or cosine rule M1

$\cos \text{A}\hat{\text{O}}\text{B} = \frac{1 + \sqrt{3}}{2\sqrt{2}}$ A1

$\text{A}\hat{\text{O}}\text{B} = \frac{\pi}{12}$ A1

[6]

3. Given that $z = \cos\theta + \text{i}\sin\theta$ show that

(a) $\text{Im}\left(z^n + \dfrac{1}{z^n}\right) = 0, n \in \mathbb{Z}^+$;

(2)

(b) $\text{Re}\left(\dfrac{z-1}{z+1}\right) = 0, z \neq -1.$

(5)
(Total 7 marks)

(a) using de Moivre's theorem

$z^n + \dfrac{1}{z^n} = \cos n\theta + \text{i}\sin n\theta + \cos n\theta - \text{i}\sin n\theta$ $(= 2\cos n\theta)$, imaginary part of which is 0 M1A1

so $\text{Im}\left(z^n + \dfrac{1}{z^n}\right) = 0$ AG

(b) $\dfrac{z-1}{z+1} = \dfrac{\cos\theta + \text{i}\sin\theta - 1}{\cos\theta + \text{i}\sin\theta + 1}$

$= \dfrac{(\cos\theta - 1 + \text{i}\sin\theta)(\cos\theta + 1 - \text{i}\sin\theta)}{(\cos\theta + 1 + \text{i}\sin\theta)(\cos\theta + 1 - \text{i}\sin\theta)}$ M1A1

Note: Award M1 for an attempt to multiply numerator and denominator by the complex conjugate of their denominator.

$\Rightarrow \text{Re}\left(\dfrac{z-1}{z+1}\right) = \dfrac{(\cos\theta - 1)(\cos\theta + 1) + \sin^2\theta}{\text{real denominator}}$ M1A1

Note: Award M1 for multiplying out the numerator.

$\dfrac{\cos^2\theta + \sin^2\theta - 1}{\text{real denominator}}$ A1

$= 0$ AG

[7]

4. Consider the complex number $\omega = \dfrac{z+\mathrm{i}}{z+2}$, where $z = x + \mathrm{i}y$ and $\mathrm{i} = \sqrt{-1}$.

(a) If $\omega = \mathrm{i}$, determine z in the form $z = r \operatorname{cis} \theta$.

(6)

(b) Prove that $\omega = \dfrac{(x^2+2x+y^2+y)+\mathrm{i}(x+2y+2)}{(x+2)^2+y^2}$.

(3)

(c) **Hence** show that when $\mathrm{Re}(\omega) = 1$ the points (x, y) lie on a straight line, l_1, and write down its gradient.

(4)

(d) Given $\arg(z) = \arg(\omega) = \dfrac{\pi}{4}$, find $|z|$.

(6)
(Total 19 marks)

(a) **METHOD 1**

$\dfrac{z+\mathrm{i}}{z+2} = \mathrm{i}$

$z+\mathrm{i} = \mathrm{i}z + 2\mathrm{i}$ M1

$(1-\mathrm{i})z = \mathrm{i}$ A1

$z = \dfrac{\mathrm{i}}{1-\mathrm{i}}$ A1

EITHER

$z = \dfrac{\operatorname{cis}\left(\frac{\pi}{2}\right)}{\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)}$ M1

$z = \dfrac{\sqrt{2}}{2}\operatorname{cis}\left(\dfrac{3\pi}{4}\right)\left(\text{or } \dfrac{1}{\sqrt{2}}\operatorname{cis}\left(\dfrac{3\pi}{4}\right)\right)$ A1A1

OR

$z = \dfrac{-1+\mathrm{i}}{2}\left(= -\dfrac{1}{2}+\dfrac{1}{2}\mathrm{i}\right)$ M1

$z = \dfrac{\sqrt{2}}{2}\operatorname{cis}\left(\dfrac{3\pi}{4}\right)\left(\text{or } \dfrac{1}{\sqrt{2}}\operatorname{cis}\left(\dfrac{3\pi}{4}\right)\right)$ A1A1

METHOD 2

$\mathrm{i} = \dfrac{x+\mathrm{i}(y+1)}{x+2+\mathrm{i}y}$ M1

$x + \mathrm{i}(y + 1) = -y + \mathrm{i}(x + 2)$ A1

$x = -y;\ x + 2 = y + 1$ A1

solving, $x = -\dfrac{1}{2};\ y = \dfrac{1}{2}$ A1

$z = -\dfrac{1}{2} + \dfrac{1}{2}\mathrm{i}$

$z = \dfrac{\sqrt{2}}{2}\operatorname{cis}\left(\dfrac{3\pi}{4}\right)\left(\text{or } \dfrac{1}{\sqrt{2}}\operatorname{cis}\left(\dfrac{3\pi}{4}\right)\right)$ A1A1

Note: Award A1 fort the correct modulus and A1 for the correct argument, but the final answer must be in the form $r \operatorname{cis}\theta$.
Accept 135° for the argument.

(b) substituting $z = x + \mathrm{i}y$ to obtain $w = \dfrac{x+(y+1)\mathrm{i}}{(x+2)+y\mathrm{i}}$ (A1)

use of $(x + 2) - y\mathrm{i}$ to rationalize the denominator M1

$\omega = \dfrac{x(x+2)+y(y+1)+\mathrm{i}(-xy+(y+1)(x+2))}{(x+2)^2+y^2}$ A1

$= \dfrac{(x^2+2x+y^2+y)+\mathrm{i}(x+2y+2)}{(x+2)^2+y^2}$ AG

(c) $\operatorname{Re}\omega = \dfrac{x^2+2x+y^2+y}{(x+2)^2+y^2} = 1$ M1

$\Rightarrow x^2 + 2x + y^2 + y = x^2 + 4x + 4 + y^2$ A1

$\Rightarrow y = 2x + 4$ A1

which has gradient $m = 2$ A1

(d) **EITHER**

$\arg(z) = \frac{\pi}{4} \Rightarrow x = y$ (and $x, y > 0$) (A1)

$\omega = \frac{2x^2 + 3x}{(x+2)^2 + x^2} + \frac{i(3x+2)}{(x+2)^2 + x^2}$

if $\arg(\omega) = \theta \Rightarrow \tan\theta = \frac{3x+2}{2x^2+3x}$ (M1)

$\frac{3x+2}{2x^2+3x} = 1$ M1A1

OR

$\arg(z) = \frac{\pi}{4} \Rightarrow x = y$ (and $x, y > 0$) A1

$\arg(w) = \frac{\pi}{4} \Rightarrow x^2 + 2x + y^2 + y = x + 2y + 2$ M1

solve simultaneously M1

$x^2 + 2x + x^2 + x = x + 2x + 2$ (or equivalent) A1

THEN

$x^2 = 1$

$x = 1$ (as $x > 0$) A1

Note: Award A0 for $x = \pm 1$.

$|z| = \sqrt{2}$ A1

Note: Allow FT from incorrect values of x.

[19]

5. Consider the complex numbers $z = 1 + 2\text{i}$ and $w = 2 + a\text{i}$, where $a \in \mathbb{R}$.

Find a when

(a) $|w| = 2|z|$;

(3)

(b) $\text{Re}(zw) = 2\,\text{Im}(zw)$.

(3)

(Total 6 marks)

(a) $|z| = \sqrt{5}$ and $|w| = \sqrt{4 + a^2}$

$|w| = 2|z|$

$\sqrt{4 + a^2} = 2\sqrt{5}$

attempt to solve equation M1

Note: Award M0 if modulus is not used.

$a = \pm 4$ A1A1 N0

(b) $zw = (2 - 2a) + (4 + a)\text{i}$ A1

forming equation $2 - 2a = 2(4 + a)$ M1

$a = -\dfrac{3}{2}$ A1 N0

[6]

6. If z is a non-zero complex number, we define $L(z)$ by the equation

$$L(z) = \ln|z| + \mathrm{i}\arg(z),\ 0 \le \arg(z) < 2\pi.$$

(a) Show that when z is a positive real number, $L(z) = \ln z$.

(2)

(b) Use the equation to calculate

(i) $L(-1)$;

(ii) $L(1 - \mathrm{i})$;

(iii) $L(-1 + \mathrm{i})$.

(5)

(c) Hence show that the property $L(z_1z_2) = L(z_1) + L(z_2)$ does not hold for all values of z_1 and z_2.

(2)
(Total 9 marks)

(a) $|z| = z$, $\arg(z) = 0$ A1A1
so $L(z) = \ln z$ AG N0

(b) (i) $L(-1) = \ln 1 + \mathrm{i}\pi = \mathrm{i}\pi$ A1A1 N2

(ii) $L(1 - \mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\dfrac{7\pi}{4}$ A1A1 N2

(iii) $L(-1 + \mathrm{i}) = \ln\sqrt{2} + \mathrm{i}\dfrac{3\pi}{4}$ A1 N1

(c) for comparing the product of two of the above results with the third M1
for stating the result $-1 + \mathrm{i} = -1 \times (1 - \mathrm{i})$ and $L(-1 + \mathrm{i}) \ne L(-1) + L(1 - \mathrm{i})$ R1
hence, the property $L(z_1z_2) = L(z_1) + L(z_2)$
does not hold for all values of z_1 and z_2 AG N0

[9]

7. Find, in its simplest form, the argument of $(\sin\theta + \text{i}\,(1 - \cos\theta))^2$ where θ is an acute angle.

(Total 7 marks)

$(\sin\theta + \text{i}\,(1 - \cos\theta))^2 = \sin^2\theta - (1 - \cos\theta)^2 + \text{i}\,2\sin\theta(1 - \cos\theta)$ M1A1

Let α be the required argument.

$\tan\alpha = \dfrac{2\sin\theta(1-\cos\theta)}{\sin^2\theta - (1-\cos\theta)^2}$ M1

$= \dfrac{2\sin\theta(1-\cos\theta)}{(1-\cos^2\theta)-(1-2\cos\theta+\cos^2\theta)}$ (M1)

$= \dfrac{2\sin\theta(1-\cos\theta)}{2\cos\theta(1-\cos\theta)}$ A1

$= \tan\theta$ A1

$\alpha = \theta$ A1

[7]

8. (a) Use de Moivre's theorem to find the roots of the equation $z^4 = 1 - \text{i}$.

(6)

(b) Draw these roots on an Argand diagram.

(2)

(c) If z_1 is the root in the first quadrant and z_2 is the root in the second quadrant, find $\dfrac{z_2}{z_1}$ in the form $a + \text{i}b$.

(4)

(Total 12 marks)

(a) $z = (1-\text{i})^{\frac{1}{4}}$

Let $1 - \text{i} = r(\cos\theta + \text{i}\sin\theta)$

$\Rightarrow r = \sqrt{2}$ A1

$\theta = -\dfrac{\pi}{4}$ A1

$z = \left(\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + \text{i}\sin\left(-\dfrac{\pi}{4}\right)\right)\right)^{\frac{1}{4}}$ M1

$= \left(\sqrt{2}\left(\cos\left(-\dfrac{\pi}{4} + 2n\pi\right) + \text{i}\sin\left(-\dfrac{\pi}{4} + 2n\pi\right)\right)\right)^{\frac{1}{4}}$

$= 2^{\frac{1}{8}}\left(\cos\left(-\dfrac{\pi}{16} + \dfrac{n\pi}{2}\right) + \text{i}\sin\left(-\dfrac{\pi}{16} + \dfrac{n\pi}{2}\right)\right)$ M1

$= 2^{\frac{1}{8}}\left(\cos\left(-\dfrac{\pi}{16}\right) + \text{i}\sin\left(-\dfrac{\pi}{16}\right)\right)$

Note: Award M1 above for this line if the candidate has forgotten to add 2π and no other solution given.

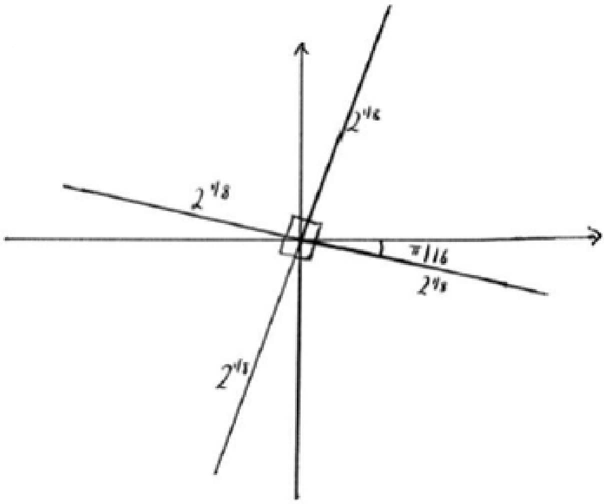
$$= 2^{\frac{1}{8}}\left(\cos\left(\frac{7\pi}{16}\right) + \mathrm{i}\sin\left(\frac{7\pi}{16}\right)\right)$$

$$= 2^{\frac{1}{8}}\left(\cos\left(\frac{15\pi}{16}\right) + \mathrm{i}\sin\left(\frac{15\pi}{16}\right)\right)$$

$$= 2^{\frac{1}{8}}\left(\cos\left(-\frac{9\pi}{16}\right) + \mathrm{i}\sin\left(-\frac{9\pi}{16}\right)\right)$$ A2

Note: Award A1 for 2 correct answers. Accept any equivalent form.

(b)



A2

Note: Award A1 for roots being shown equidistant from the origin and one in each quadrant.
A1 for correct angular positions. It is not necessary to see written evidence of angle, but must agree with the diagram.

(c) $$\frac{z_2}{z_1} = \frac{2^{\frac{1}{8}}\left(\left(\cos\frac{15\pi}{16}\right) + \mathrm{i}\sin\left(\frac{15\pi}{16}\right)\right)}{2^{\frac{1}{8}}\left(\left(\cos\frac{7\pi}{16}\right) + \mathrm{i}\sin\left(\frac{7\pi}{16}\right)\right)}$$ M1A1

$$= \cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}$$ (A1)

$= \mathrm{i}$ A1 N2

$(\Rightarrow a = 0, b = 1)$

[12]

9. Given that $(a + bi)^2 = 3 + 4i$ obtain a pair of simultaneous equations involving a and b. Hence find the two square roots of $3 + 4i$.

(Total 7 marks)

$a^2 + 2iab - b^2 = 3 + 4i$	
Equate real and imaginary parts	(M1)
$a^2 - b^2 = 3,\ 2ab = 4$	A1
Since $b = \frac{2}{a}$	
$\Rightarrow a^2 - \frac{4}{a^2} = 3$	(M1)
$\Rightarrow a^4 - 3a^2 - 4 = 0$	A1
Using factorisation or the quadratic formula	(M1)
$\Rightarrow a = \pm 2$	
$\Rightarrow b = \pm 1$	
$\Rightarrow \sqrt{3+4i} = 2 + i,\ -2 - i$	A1A1

[7]

10. (a) Factorize $z^3 + 1$ into a linear and quadratic factor.

(2)

Let $\gamma = \frac{1+i\sqrt{3}}{2}$.

(b) (i) Show that γ is one of the cube roots of -1.

(ii) Show that $\gamma^2 = \gamma - 1$.

(iii) Hence find the value of $(1 - \gamma)^6$.

(9)

(Total 11 marks)

(a) using the factor theorem $z + 1$ is a factor	(M1)
$z^3 + 1 = (z + 1)(z^2 - z + 1)$	A1
(b) (i) **METHOD 1**	
$z^3 = -1 \Rightarrow z^3 + 1 = (z + 1)(z^2 - z + 1) = 0$	(M1)
solving $z^2 - z + 1 = 0$	M1
$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$	A1
therefore one cube root of -1 is γ	AG
METHOD 2	
$\gamma^2 = \left(\frac{1+i\sqrt{3}}{2}\right)^2 = \frac{-1+i\sqrt{3}}{2}$	M1A1
$\gamma^3 = \frac{-1+i\sqrt{3}}{2} \times \frac{-1+i\sqrt{3}}{2} = \frac{-1-3}{4}$	A1
$= -1$	AG

(ii) **METHOD 1**

as γ is a root of $z^2 - z + 1 = 0$ then $\gamma^2 - \gamma + 1 = 0$ M1R1

$\therefore \gamma^2 = \gamma - 1$ AG

Note: Award M1 for the use of $z^2 - z + 1 = 0$ in any way.
Award R1 for a correct reasoned approach.

METHOD 2

$\gamma^2 = \frac{-1 + i\sqrt{3}}{2}$ M1

$\gamma - 1 = \frac{1 + i\sqrt{3}}{2} - 1 = \frac{-1 + i\sqrt{3}}{2}$ A1

(iii) **METHOD 1**

$(1 - \gamma)^6 = (-\gamma^2)^6$ (M1)

$= (\gamma)^{12}$ A1

$= (\gamma^3)^4$ (M1)

$= (-1)^4$

$= 1$ A1

METHOD 2

$(1 - \gamma)^6$

$= 1 - 6\gamma + 15\gamma^2 - 20\gamma^3 + 15\gamma^4 - 6\gamma^5 + \gamma^6$ M1A1

Note: Award M1 for attempt at binomial expansion.

use of any previous result e.g. $= 1 - 6\gamma + 15\gamma^2 + 20 - 15\gamma + 6\gamma^2 + 1$ M1

$= 1$ A1

Note: As the question uses the word 'hence', other methods that do not use previous results are awarded no marks.

11. Given that $|z| = \sqrt{10}$, solve the equation $5z + \frac{10}{z^*} = 6 - 18i$, where z^* is the conjugate of z.

(Total 7 marks)

$5zz^* + 10 = (6 - 18i)z^*$ M1

Let $z = a + ib$

$5 \times 10 + 10 = (6 - 18i)(a - bi)$ $(= 6a - 6bi - 18ai - 18b)$ M1A1

Equate real and imaginary parts (M1)

$\Rightarrow 6a - 18b = 60$ and $6b + 18a = 0$

$\Rightarrow a = 1$ and $b = -3$ A1A1

$z = 1 - 3i$ A1

[7]

12. Solve the simultaneous equations

$$iz_1 + 2z_2 = 3$$
$$z_1 + (1 - i)z_2 = 4$$

giving z_1 and z_2 in the form $x + iy$, where x and y are real.

(Total 9 marks)

$iz_1 + 2z_2 = 3 \Rightarrow z_2 = -\frac{1}{2}iz_1 + \frac{3}{2}$

$z_1 + (1 - i)z_2 = 4$

$\Rightarrow z_1 + (1 - i)\left(-\frac{1}{2}iz_1 + \frac{3}{2}\right) = 4$ M1A1

$\Rightarrow z_1 - \frac{1}{2}iz_1 + \frac{3}{2} + \frac{1}{2}i^2z_1 - \frac{3}{2}i = 4$

$\Rightarrow \frac{1}{2}z_1 - \frac{1}{2}iz_1 = \frac{5}{2} + \frac{3}{2}i$

$\Rightarrow z_1 - iz_1 = 5 + 3i$ A1

EITHER

Let $z_1 = x + iy$ (M1)

$\Rightarrow x + iy - ix - i^2y = 5 + 3i$

Equate real and imaginary parts M1

$\Rightarrow x + y = 5$

$-x + y = 3$

$2y = 8$

$y = 4 \Rightarrow x = 1$ i.e. $z_1 = 1 + 4i$ A1A1

$z_2 = -\frac{1}{2}i(1 + 4i) + \frac{3}{2}$ M1

$z_2 = -\frac{1}{2}i - 2i^2 + \frac{3}{2}$

$z_2 = \frac{7}{2} - \frac{1}{2}i$ A1

OR

$z_1 = \frac{5 + 3i}{1 - i}$ M1

$z_1 = \frac{(5 + 3i)(1 + i)}{(1 - i)(1 + i)}$ $\left(= \frac{5 + 8i - 3}{2}\right)$ M1A1

$z_1 = 1 + 4i$ A1

$z_2 = -\frac{1}{2}i(1 + 4i) + \frac{3}{2}$ M1

$z_2 = -\frac{1}{2}i - 2i^2 + \frac{3}{2}$

$z_2 = \frac{7}{2} - \frac{1}{2}i$ A1

[9]

13. (a) Write down the expansion of $(\cos\theta + \mathrm{i}\sin\theta)^3$ in the form $a + \mathrm{i}b$, where a and b are in terms of $\sin\theta$ and $\cos\theta$.

(2)

(b) Hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(3)

(c) Similarly show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(3)

(d) **Hence** solve the equation $\cos 5\theta + \cos 3\theta + \cos\theta = 0$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(6)

(e) By considering the solutions of the equation $\cos 5\theta = 0$, show that

$\cos\frac{\pi}{10} = \sqrt{\frac{5+\sqrt{5}}{8}}$ and state the value of $\cos\frac{7\pi}{10}$.

(8)
(Total 22 marks)

(a) $(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(\mathrm{i}\sin\theta) + 3\cos\theta(\mathrm{i}\sin\theta)^2 + (\mathrm{i}\sin\theta)^3$ (M1)

$= \cos^3\theta - 3\cos\theta\sin^2\theta + \mathrm{i}(3\cos^2\theta\sin\theta - \sin^3\theta)$ A1

(b) from De Moivre's theorem

$(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$ (M1)

$\cos 3\theta + \mathrm{i}\sin 3\theta = (\cos^3\theta - 3\cos\theta\sin^2\theta) + \mathrm{i}(3\cos^2\theta\sin\theta - \sin^3\theta)$

equating real parts

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ M1

$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$ A1

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$ AG

Note: Do not award marks if part (a) is not used.

(c) $(\cos\theta + \mathrm{i}\sin\theta)^5 =$

$\cos^5\theta + 5\cos^4\theta(\mathrm{i}\sin\theta) + 10\cos^3\theta(\mathrm{i}\sin\theta)^2 + 10\cos^2\theta(\mathrm{i}\sin\theta)^3$

$+ 5\cos\theta(\mathrm{i}\sin\theta)^4 + (\mathrm{i}\sin\theta)^5$ (A1)

from De Moivre's theorem

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ M1

$= \cos^5\theta - 10\cos^3\theta(1 - \cos^2\theta) + 5\cos\theta(1 - \cos^2\theta)^2$ A1

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$\therefore \cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ AG

Note: If compound angles used in (b) and (c), then marks can be allocated in (c) only.

(d) $\cos 5\theta + \cos 3\theta + \cos\theta$

$= (16\cos^5\theta - 20\cos^3\theta + 5\cos\theta) + (4\cos^3\theta - 3\cos\theta) + \cos\theta = 0$ M1

$16\cos^5\theta - 16\cos^3\theta + 3\cos\theta = 0$ A1

$\cos\theta\,(16\cos^4\theta - 16\cos^2\theta + 3) = 0$

$\cos\theta\,(4\cos^2\theta - 3)(4\cos^2\theta - 1) = 0$ A1

$\therefore \cos\theta = 0; \pm\frac{\sqrt{3}}{2}; \pm\frac{1}{2}$ A1

$\therefore \theta = \pm\frac{\pi}{6}; \pm\frac{\pi}{3}; \pm\frac{\pi}{2}$ A2

(e) $\cos 5\theta = 0$

$5\theta = \ldots\frac{\pi}{2}; \left(\frac{3\pi}{2}; \frac{5\pi}{2}\right); \frac{7\pi}{2}; \ldots$ (M1)

$\theta = \ldots\frac{\pi}{10}; \left(\frac{3\pi}{10}; \frac{5\pi}{10}\right); \frac{7\pi}{10}; \ldots$ (M1)

Note: These marks can be awarded for verifications later in the question.

now consider $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$ M1

$\cos\theta\,(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

$\cos^2\theta = \frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}$; $\cos\theta = 0$ A1

$\cos\theta = \pm\sqrt{\frac{20 \pm \sqrt{400 - 4(16)(5)}}{32}}$

$\cos\frac{\pi}{10} = \pm\sqrt{\frac{20 + \sqrt{400 - 4(16)(5)}}{32}}$ since max value of cosine $\Rightarrow$ angle closest to zero R1

$\cos\frac{\pi}{10} = \sqrt{\frac{4.5 + 4\sqrt{25 - 4(5)}}{4.8}} = \sqrt{\frac{5 + \sqrt{5}}{8}}$ A1

$\cos\frac{7\pi}{10} = -\sqrt{\frac{5 - \sqrt{5}}{8}}$ A1A1

[22]