8 Differentiation

8.1 Differentiation of polynomials, trigonometric, exponential and logarithmic functions; product and quotient rules; composite functions

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

f(x)	**f′(x)**	**f(x)**	**f′(x)**
$(ax+b)^n$	$an(ax+b)^{n-1}$	$\sin^{-1} x$	$\dfrac{1}{\sqrt{(1-x^2)}}$
$\sin ax$	$a \cos ax$		
$\cos ax$	$-a \sin ax$	$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
$\tan ax$	$a \sec^2 ax$		
$\sec ax$	$a \sec ax \tan ax$	$\ln ax$	$\dfrac{1}{x}$
$\operatorname{cosec} ax$	$-a \operatorname{cosec} ax \cot ax$	e^{ax}	ae^{ax}
$\cot ax$	$-a \operatorname{cosec}^2 ax$		

Product rule: if $y = uv$, then $\dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}$

Quotient rule: if $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Chain rule: if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Basic

1 Find $\dfrac{dy}{dx}$ if y is:

(a) x^5 (b) $x^{1/2}$ (c) $\sqrt[3]{x}$ (d) x^{-1}

(e) $\dfrac{1}{x^2}$ (f) $\dfrac{1}{\sqrt{x}}$ (g) $\dfrac{1}{x^n}$

2 Differentiate the following with respect to x.

(a) $3x^7$ (b) $6\sqrt{x}$
(c) $2x^4 - x^3 + 2x$ (d) $5x + 3$
(e) 3 (f) $3x^4 - \frac{x^3}{3} + 4x^2 - \frac{x}{2} + 10$
(g) $x + \frac{1}{x}$ (h) $x^2 - \frac{1}{x^2}$
(i) $x^4 - \frac{8}{x^2}$ (j) $3x + \frac{4}{x}$
(k) $6x^4 - 4x^6$ (l) $ax^2 + bx + c$

3 Using the chain rule, find $f'(x)$ if $f(x)$ is:

(a) $(x+1)^4$ (b) $(2x+3)^5$ (c) $4(x-2)^2$ (d) $3(2-3x)^3$
(e) $(x^2+1)^2$ (f) $(3x^2+1)^4$ (g) $(1-x^2)^6$ (h) $(x^2-x)^5$
(i) $(\sqrt{x}+1)^{10}$ *(j) $\sqrt{1-x}$ (k) $\sqrt{x^2+2}$ (l) $(ax+b)^n$

4 Using the chain rule, differentiate the following with respect to t.

(a) $(t+3)^{-3}$ (b) $(2t+3)^{-1}$ *(c) $\frac{1}{3t+2}$ (d) $\frac{1}{\sqrt{(3t+1)}}$
(e) $\frac{1}{(3-t^2)^5}$ (f) $\frac{1}{(1+\sqrt{t})^2}$ (g) $\frac{2}{\sqrt{2+t^2}}$ (h) $\frac{1}{(at+b)^n}$

5 Using the product rule, find the derivative with respect to x of the following, simplifying your answers.

*(a) $2x(x+1)^3$ (b) $x^2(2x^3+1)^5$ (c) $(x+1)^2(x+2)^3$
(d) $(x^2+2)\sqrt{x}$ (e) $x\sqrt{x^2+2}$ (f) $x^n(x+1)^m$

6 Using the quotient rule, find $\frac{dy}{dx}$ for the following, simplifying your answers.

(a) $\frac{x}{1+x^2}$ (b) $\frac{x+1}{x-1}$ (c) $\frac{x}{2x^2+1}$ (d) $\frac{1-x^2}{1+x^2}$
*(e) $\frac{x}{(3x+1)^2}$ (f) $\frac{1+\sqrt{x}}{1-\sqrt{x}}$ (g) $\frac{x^2+1}{(x+1)^2}$ (h) $\frac{x^2}{\sqrt{1+x^2}}$

7 For the following functions, find $f'(x)$ if $f(x)$ is:

(a) $2\sin x$ (b) $-4\cos x$ (c) $0.5\tan x$
(d) $\sin 3x$ (e) $\cos 0.5x$ (f) $\tan 6x$

(g) $-3\sin 2x$ (h) $2\cos\frac{1}{2}x$ (i) $\frac{1}{6}\tan\frac{1}{6}x$
(j) $3\sin\pi x - 2\cos\frac{1}{2}\pi x$ (k) $3\tan\frac{1}{12}\pi x - 0.5\sin 2\pi x$ (l) $\sec 7x$

8 Differentiate the following with respect to x.

*(a) $\sin^2 x$ (b) $\sin x^2$ (c) $\sin 2x$
(d) $\cos^2 x - 3\sin^2 x$ (e) $4\tan^2 x$ (f) $6\sin^3 2x$
(g) $\sqrt{\sin x}$ (h) $4\cos 3x^2$ (i) $(\sin x + \cos x)^3$
(j) $\sin^2 \pi x$ (k) $\operatorname{cosec} x^2$

9 Find $\dfrac{dy}{dx}$ if y is equal to the following.

(a) $2e^x$ (b) e^{-x} (c) e^{4x} (d) $3e^{2x}$ *(e) e^{x^2}
(f) $e^{\sin x}$ (g) $e^{1/x}$ (h) $\exp e^x$ (i) $(e^{2x}+1)^2$ (j) $(e^{2x+1})^2$

10 For the following functions f(x), find f$'(x)$, simplifying your answers.

(a) $2\ln x$ (b) $\ln 2x$ (c) $3\ln 4x$ *(d) $\ln x^2$
(e) $\ln\sin x$ (f) $\ln(x^2+1)$ (g) $(\ln x)^2$

11 Using the product or quotient rule, differentiate the following with respect to x.

(a) $x\sin x$ (b) $x^2\cos 4x$ (c) $\dfrac{e^x}{x}$

(d) $x\ln x$ (e) $\dfrac{\ln x}{x^2}$ (f) x^2e^{3x}

(g) $\dfrac{\sin x}{x}$ (h) $\dfrac{\tan x}{x}$ (i) $\dfrac{x^2}{\sin x}$

(j) $e^{2x}\cos\pi x$ (k) $e^{x\sin x}$ (l) $\sin 3x\cos x + \cos 3x\sin x$

(m) $x^2\sec x$ (n) $\sqrt{x}\cot x$

12 (a) (i) By rewriting $y = \sin^{-1}x$ as $x = \sin y$ find $\dfrac{dx}{dy}$.

(ii) Using the fact that $\dfrac{dy}{dx} = \dfrac{1}{dx/dy}$, find $\dfrac{dy}{dx}$ in terms of x.

(b) Using a similar method, find $\dfrac{dy}{dx}$ if $y = \tan^{-1}x$.

Intermediate

1 Find the derivative of $y = 2x^3 + 3x^2 - 36x$. Factorise your answer and hence find the values of x for which the derivative is equal to zero.

2 The derivative of the curve $y = ax^3 + 2bx^2 + 3cx$ is $\frac{dy}{dx} = 6x^2 + 6x - 6$.

Find the values of the constants a, b and c, simplifying your answers as much as possible.

3 Given that $f(x) = x\ln(x^3 - 4)$, find the value of $f'(2)$. Leave your answer in terms of natural logarithms.

4 If $y = \frac{\ln x}{x}$ show that $\frac{dy}{dx} = 0$ when $x = e$.

5 (a) Find $f'\left(\frac{\pi}{3}\right)$ if $f(x) = 2\sin x - 3\cos 2x$.

(b) Find $f'(1)$ if $f(x) = 4x\ln x$.
(c) Find $f'(0)$ and $f'(1)$ if $f(x) = \exp(x^2 - 2x)$.

6 If $y = 64\cos x - \frac{8\cos x}{\sin x}$ find the value of $\sin x$ for which $\frac{dy}{dx} = 0$.

7 (a) If $x^4 y = 3$, find $\frac{dy}{dx}$ when $x = 1$.

(b) If $pq = 9$, find the value of $\frac{dp}{dq}$ when $q = 3$.

8 If $y = a\cos x - b\sin x$ and $\frac{dy}{dx} = 0$ when $x = \frac{\pi}{3}$, show that $a = \frac{-b}{\sqrt{3}}$.

9 Find the derivatives of the following, simplifying your answers where possible.

(a) $x^4 + 3x^2$ (b) $\frac{1}{(3 - 2x)^3}$ (c) $x^2 \cos x$ (d) $\frac{2x + 1}{3x - 1}$

(e) $\frac{x^2 + 1}{x^2 - 1}$ (f) $\sqrt{x^2 - 1}$ (g) $\sec^2 x$ (h) $\cot 5x$

10 Differentiate the following with respect to x.

*(a) $\sin^{-1} 2x$ (b) $\tan^{-1} x^2$ (c) $\sin^{-1} \sqrt{x}$

11 Find and simplify the derivative of $\frac{x}{1 + x^2} + \tan^{-1} x$.

12 Differentiate the following with respect to t.

(a) $5\sin^3 3t$ (b) $\frac{\sin t + 1}{\cos t + 1}$ (c) $2\sin 2t \cos 3t$ (d) $\tan^{-1} 3t$

(e) $\frac{\sin t}{e^t}$ (f) $\ln(1 + \tan^2 t)$

13 (a) Use the product rule to differentiate:

(i) $e^{-2x}\cos 2x$ (ii) $3x\ln 2x$ (iii) $e^x \sin x$

(b) Use the quotient rule to differentiate:

(i) $\dfrac{x}{1+x}$ (ii) $\dfrac{\tan^{-1} x}{1+x^2}$ (iii) $\dfrac{1-e^{-x}}{1+e^x}$

14 Differentiate:

(a) $\sqrt{\dfrac{1+4x}{2x-1}}$ (b) $\tan^{-1}\left(\dfrac{1-4x}{4x+1}\right)$ (c) $\dfrac{x^2 \sin x}{\ln x}$

(d) $\dfrac{\exp(-x^2)}{\sqrt{x}}$ (e) $\cos^{-1} x - x\sqrt{1-x^2}$ (f) $\ln(x+\sqrt{1+x^2})$

15 (a) Differentiate:

*(i) $\ln\left(\dfrac{x^2}{1+x^2}\right)$ (ii) $\ln\sqrt{1+x}$ (iii) $\ln\left(\dfrac{x^2+1}{2x+1}\right)$

(b) Given that $f(x) = \ln\left(\dfrac{1+\sin x}{1-\sin x}\right)$ show that $f'(x) = \dfrac{2}{\cos x}$.

16 (a) (i) Differentiate $y = (x^2+1)(x+3)^{-2}$ as a product.

(ii) Now differentiate $y = \dfrac{x^2+1}{(x+3)^2}$ as a quotient.

(b) Why should your answers to part (a) be the same? Show that this is the case.

Advanced

1 Differentiate with respect to x:

(a) $\dfrac{x}{\sqrt{ax^2+b}}$ (b) $\sin(\cos(x^3))$ (c) $e^{\sin^2 x}$

2 Differentiate $f(x) = x^{-2}$ from first principles.

3 Differentiate with respect to x, simplifying your answers,

(a) $\ln\left(\dfrac{a-x}{a+x}\right)^{1/3}$ (b) $e^{2\ln 3x}$ (c) $\cos^4 x - \sin^4 x$

4 Given that $f(x) = \dfrac{3x+5a}{x^2-a^2}$, find the values of a for which $f'(12) = 0$.

5 If $y = e^{-x}(ax+b)$, and y satisfies the differential equation

$\dfrac{d^2y}{dx^2} + p\dfrac{dy}{dx} + qy = 0$ for all a and b, find the values for p and q.

6 Differentiate the following functions with respect to x.

(a) $\cos(\sin(ax))$ (b) $\dfrac{x^3}{e^x - e^{-x}}$ (c) $\tan^{-1}(e^{-x^3})$

(d) $\int_0^\infty ye^{-xy^2}\,dy$, where x and y are independent variables and $x > 0$

7 Let $f(x) = x - \ln(1+x) - \dfrac{x^2}{2(1+x)}$, where $x > 0$. Find $f'(x)$ and hence, or otherwise, show that, when $x > 0$, $x > \ln(1+x) + \dfrac{x^2}{2(1+x)}$.
[OCR (Cambridge), 1987]

8 Differentiate with respect to x:

(a) $1 + \operatorname{cosec} 2x - \dfrac{2(\cos^2 x - \sin^2 x)}{\sin 4x}$

(b) $\cos\{\tan^{-1}(x^2)\}$
(c) $\sin(e^x - |x|)$

9 (i) By considering $(1 + x + x^2 + \cdots + x^n)(1 - x)$ show that, if $x \neq 1$,

$$1 + x + x^2 + \cdots + x^n = \frac{(1 - x^{n+1})}{1 - x}$$

(ii) By differentiating both sides and setting $x = -1$ show that

$$1 - 2 + 3 - 4 + \cdots + (-1)^{n-1}n$$

takes the value $-\dfrac{n}{2}$ if n is even and the value $\dfrac{(n+1)}{2}$ if n is odd.

(iii) Show that

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}(An^2 + Bn)$$

where the constants A and B are to be determined.
[OCR (STEP), 1995]

10 Differentiate from first principles:

(a) $|x^3|$ (b) $x\sin^2 x$

Revision

1 Given that $y = 3x^4 - 4x^3 - 12x^2$ find $\dfrac{dy}{dx}$ in factorised form.

2 If $f(x) = x^{1/3} + \dfrac{1}{2x - 1}$ find $f'(x)$.

3 Find $\dfrac{dv}{dt}$ if:

(a) $v = 4t(t^2 - 1)^3$ (b) $v = t\sqrt{2 - t}$

4 A curve is given by $y = \dfrac{x^2 - 5x + 4}{x}$ for $x > 0$. Show that $\dfrac{dy}{dx} = 1 - \dfrac{4}{x^2}$.

5 Differentiate the following with respect to t.

(a) $e^{2t}\cos t$ (b) $\sin(t^3 + 4)$ (c) $\dfrac{\ln t}{t}$ (d) $\ln(1 - \sin^2 t)$

6 For $y = x(1 + 2x)^3$ use the product rule to find an expression for $\dfrac{dy}{dx}$, simplifying your answer as far as possible.

7 Given that $y = \ln\sqrt{1 + x^2}$ show that $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$.

8 Using the quotient rule, differentiate $y = \dfrac{e^x}{1 + e^{-x}}$.

9 Given that $y = \sin^{-1} 4x$, show that $\dfrac{dy}{dx} = \dfrac{4}{\sqrt{1 - 16x^2}}$.

10 Show that the result of differentiating $\sqrt{x} + \dfrac{1}{\sqrt{x}}$ with respect to x may be written in the form $\dfrac{x - 1}{2x\sqrt{x}}$.

11 Find the derivatives of the following functions.

(a) $\sin 5x$ (b) $\tan 0.5x$ (c) e^{3x} (d) $(1 + x^2)^4$
(e) $\ln(1 + x^3)$ (f) $\cos^{-1}(2x)$

12 Use the quotient rule to differentiate the function $f(x) = \dfrac{x^2}{1 - x^2}$.

8.2 Increasing and decreasing functions; rates of change; tangents and normals; maxima, minima and stationary points; points of inflexion; optimisation problems

Velocity $v = \dfrac{dx}{dt}$, acceleration $a = \dfrac{dv}{dt}$

At a stationary point $\dfrac{dy}{dx} = 0$.

For a maximum, $\dfrac{d^2y}{dx^2} < 0$; for a minimum, $\dfrac{d^2y}{dx^2} > 0$.

At a point of inflexion $\dfrac{d^2y}{dx^2} = 0$ and changes sign as x moves across the point.