The Optimum Solar Panel Tilt

Table of Contents:

A table of contents is not necessary for a piece of work of this length.

Introduction

Countries nowadays focus on clean energy alternatives to fossil fuel, to combat the climate changes caused by global warming. Solar panels are used to convert thermal radiation into electrical energy, and are seen as a promising source of clean energy. To capture as much sunlight as possible, solar panels are installed on many households. In a large solar panel plant with thousands of solar panels, there could exist a system to point all the solar panels to the sun, to maximize the energy produced. However, for most households, solar panels are fixed permanently to the roof. This paper will investigate the optimum tilt on a rooftop solar panel, so it can obtain the maximum amount of energy throughout one year.

A: Short and succinct introduction.

Aim

The aim of this investigation is to determine the optimum tilt angle of a solar panel depending on the position on Earth, i.e, depending on the latitude. The solar panel will be tilted in one direction only throughout the investigation. In Part 1, we will assume that Earth is a spherical object traveling around the sun with no axial tilt. In Part 2, we will expand the problem and add the axial tilt on Earth, which makes this investigation more realistic. Throughout the investigation, it is assumed that nothing is present between the solar panel and the sun at all times, and one day is 24 hours.

A: These subtitles are not necessary.

Rationale

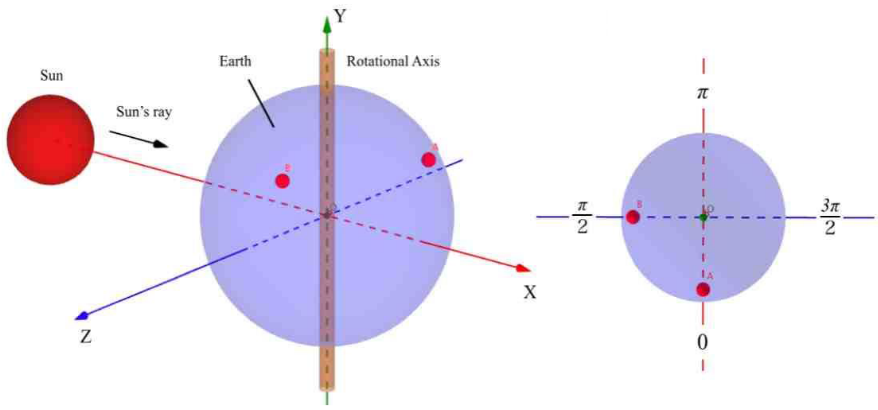
I chose to investigate this topic because of its realistic applications and its complexity. This investigation will produce a solution to a real world problem, which interested me. Knowing the optimum angle for a rooftop solar panel to maximize energy is relevant information for anyone installing solar panels. Simultaneously, I was amazed by the complexity of this investigation. Although the purpose of this investigation is to determine the tilt of a solar panel, a simple sounding problem, it is necessary to consider the movement of earth around the sun throughout one year, which involved complex mathematics involving vectors, trigonometry, and calculus.

Part 1: Optimum tilt angle for a spherical Earth with no axial tilt

Description of the situation

To begin the investigation, we will consider the simplest case where the Earth is a spherical object with no axial tilt, whereas in reality, the axial tilt of Earth is 23.5°. An axial tilt of a planet is the angle between its rotational axis and the perpendicular to its orbital plane(Nave).

Fig. 1



3

B: Good use of diagram and explanations to explain the situation being modelled.

Figure 1 illustrates this scenario. The center of Earth will be located on the origin of 3D xyz coordinate system. The center of the sun is located on the negative region of the x-axis. However, distance between the Earth and Sun is not to scale on this diagram. Since we have set the axial tilt of Earth to 0° for Part 1, the rotational axis of the Earth is equivalent to the y-axis.

When considering the position of an arbitrary point on Earth throughout one day, the day begins at midnight when the point is furthest away from the Sun. As time passes, the Earth will rotate clockwise when viewed directly above the axis of rotation. Throughout the investigation, we will consider the amount of time passed as the angular displacement of a point, denoted by θ_t. For example, in figure 1, point A is at the furthest it can be from the sun, hence it is the initial position for a day(which is midnight). At this point, the angular displacement is 0. After 6 hours, a quarter of a day has past, thus the Earth has rotated 90 degrees. Now, point A has moved to the position of point B, where its angular displacement is $\frac{\pi}{2}$, shown by the diagram on the right in figure 1.

The aim of this investigation is to calculate the optimum tilt angle of a solar panel which produces the maximum amount of energy throughout one year. For Part 1, since the axial tilt is 0, the amount of light any arbitrary point receives in 24 hours will be constant for any day in a year. Therefore, we will only consider the optimum angle for one day, i.e, one full rotation of Earth.

Another concept to define is the direction at which the solar panel will be tilted. For an arbitrary point on a spherical Earth, a plane tangent to that point can be created. Placing the solar panel on this plane is equivalent to placing the solar panel flat on the ground in real life. The axis of tilting the solar panel will be the component of this plane perpendicular to the Earth's axis of rotation.

Fig. 2

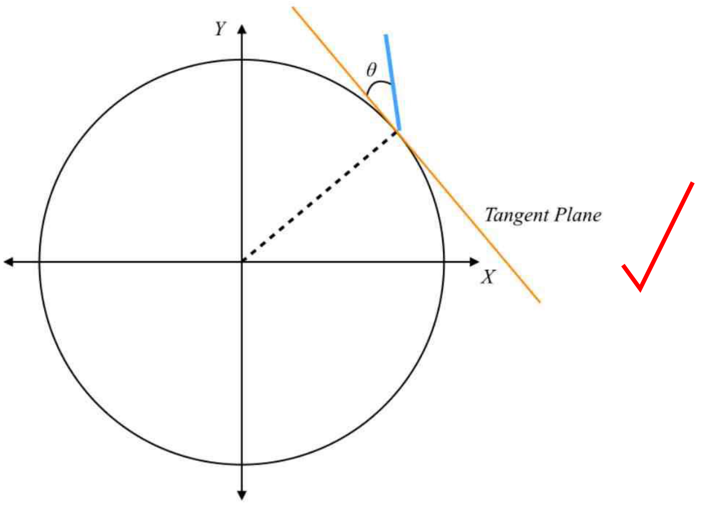


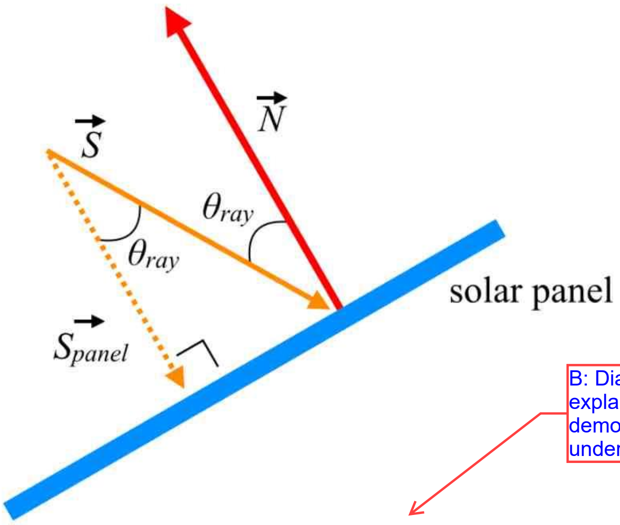
Figure 2 views the Earth directly above the xy-plane, exactly at midnight. The diagram shows how the solar panel, drawn in light blue, will be tilted by an angle θ to the tangent plane.

4

Method to determine the amount of sunlight the solar panel receives

The amount of energy the solar panel will produce, will depend on the amount of sunlight it receives. Therefore, in the investigation we will calculate the optimum angle where the solar panel can collect the maximum amount of sunlight. To calculate the amount of sunlight the solar panel will receive at any instant, we must represent the Sun's rays as vectors. We will assume the magnitude and direction of all the Sun's rays to be constant, which is a reasonable assumption. For convenience, we will set the magnitude of the Sun's rays to be 1 unit. When the Sun's ray will hit the solar panel, the sunlight the solar panel will receive is the component of the Sun's ray which is perpendicular to the solar panel's surface.

Fig. 3



B: Diagram and explanation demonstrate understanding.

Figure 3 illustrates this situation. Vector S is the Sun's rays, vector N is the normal vector to the solar panel, and vector S_{panel} is the vector component perpendicular to the solar panel. The amount of light the solar panel will receive is the magnitude of vector S_{panel}. From the figure 3, we can deduce,

$$cos(\theta_{ray}) = \frac{|S_{panel}^{\rightarrow}|}{|\vec{S}|} \Leftrightarrow |S_{panel}^{\rightarrow}| = |\vec{S}| cos(\theta_{ray})$$

$$\text{Since} \quad |\vec{S}| = 1 \qquad |S_{panel}^{\rightarrow}| = |\vec{S}||cos(\theta_{ray})| = cos(\theta_{ray})$$

$$cos(\theta_{ray}) \geq 0 \quad since, \ \theta_{ray} \in [0, \frac{\pi}{2}]$$

To calculate $cos(\theta_{ray})$, we can take the dot product of the vector N and vector S.

B: Although we usually use small letters for single letter vectors this is condoned as it is defined by the student who uses this notation consistently.

$$if\ \vec{N} = \begin{pmatrix} x_N \\ y_N \\ z_N \end{pmatrix} \quad \text{and} \quad \vec{S} = \begin{pmatrix} x_S \\ y_S \\ z_s \end{pmatrix}$$

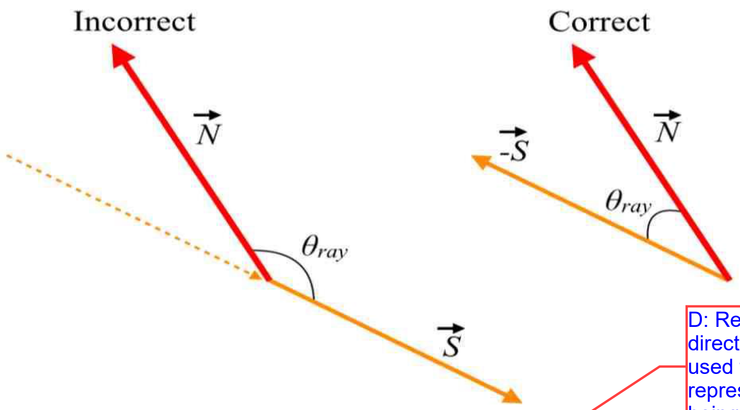
$$\vec{N} \cdot \vec{S} = x_N x_S + y_N y_S + z_N z_S = |\vec{N}||\vec{S}| cos(\theta_{ray})$$

$$cos(\theta_{ray}) = \frac{x_N x_S + y_N y_S + z_N z_S}{|\vec{N}||\vec{S}|}$$

5

One aspect to keep in mind is the direction of vector S, which is pointing towards the solar panel. However, to calculate the correct value of θ_{ray}, we must take the dot product of vector N and vector S pointing to the opposite direction. This concept is shown in figure 4.

Fig. 4



D: Reflects on vector directions that should be used that realistically represent the situation being modelled.

Therefore, the correct calculation for $cos(\theta_{ray})$ will be,

$$\vec{N}\cdot(-\vec{S})=-(x_N x_S+y_N y_S+z_N z_S)=|\vec{N}||\vec{S}|cos(\theta_{ray})$$

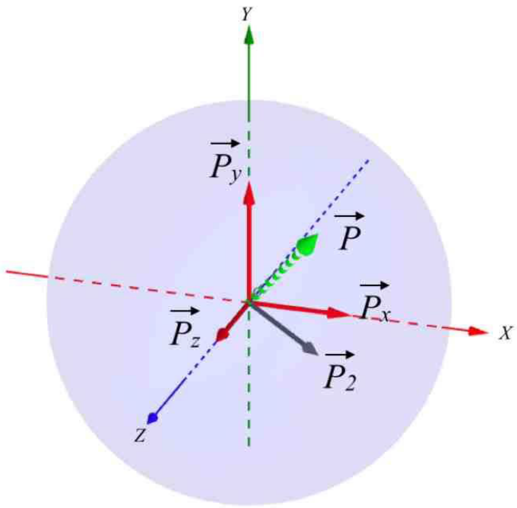
$$\text{As} \quad |\vec{S}|=1$$

$$cos(\theta_{ray})=-(\frac{x_N x_S+y_N y_S+z_N z_S}{|\vec{N}|})$$

<u>Determine the amount of sunlight the solar panel will receive</u>

To calculate the amount of sunlight received, given by $cos(\theta_{ray})$, we must determine the normal vector of the solar panel. To do so, first we must determine a general equation for a position vector for any point on Earth.

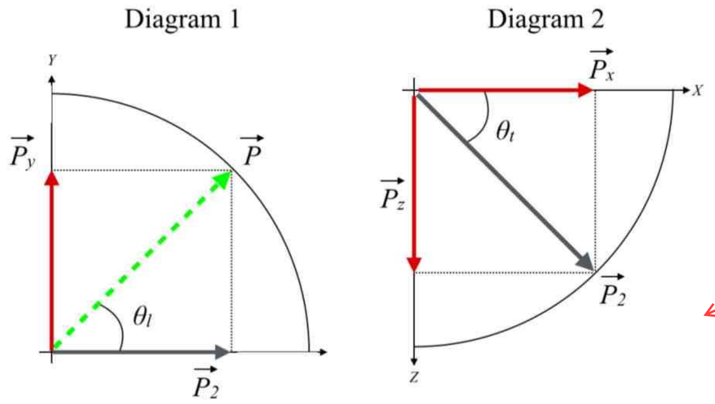
Fig. 5



6

Figure 5 illustrates the components which make up the position vector P. The aim is to determine the xyz components of the position vector in terms of the θ_l, the latitude, and θ_t, the angular displacement. We can use two 2D diagrams to calculate each component vector.

Fig. 6



E: The candidate is breaking up the model into smaller parts making it easier to follow.

We can use the diagrams in figure 6 to calculate the component vectors. In diagram 1, we are viewing directly above the plane that contains vector P_y and P_2. In diagram 2, we are viewing directly above the xz-plane. Vector P_2 is contained in the xz-plane, and is the sum of vector P_x and P_z.

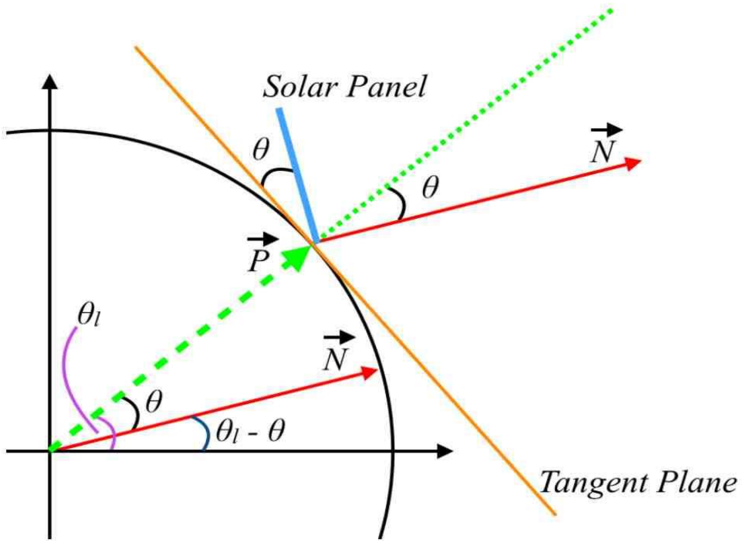
$$\text{Let the radius of Earth} = r = |\vec{P}|$$

$$\vec{P}_2 = r\cos(\theta_l) \qquad \vec{P}_y = r\sin(\theta_l)$$

$$\vec{P}_x = \vec{P}_2\cos(\theta_t) = r\cos(\theta_l)\cos(\theta_t) \qquad \vec{P}_z = \vec{P}_2\sin(\theta_t) = r\cos(\theta_l)\sin(\theta_t)$$

$$\text{Thus,} \quad \vec{P} = \begin{pmatrix} r\cos(\theta_l)\cos(\theta_t) \\ r\sin(\theta_l) \\ r\cos(\theta_l)\sin(\theta_t) \end{pmatrix}$$

Fig. 7



7

Figure 7 is a view directly above the plane which contains the position vector P and its component vector P_y. The red vector N is the normal vector to the solar panel. If we set the magnitude of the normal vector to r, the radius of Earth, and translate it to the origin, we end up with this diagram. From the diagram, we can deduce the angle between vector N and the horizontal is $\theta_l - \theta$. Knowing this, we can express the normal vector in the same way as the position vector, but with latitude $\theta_l - \theta$.

Therefore, the normal vector N of the solar panel tilted by an angle θ, for a point on Earth with latitude θ_l, and angular displacement θ_t is,

$$\vec{N} = \begin{pmatrix} r\cos(\theta_l - \theta)\cos(\theta_t) \\ r\sin(\theta_l - \theta) \\ r\cos(\theta_l - \theta)\sin(\theta_t) \end{pmatrix}$$

E: Very clearly developed.

To determine $\cos(\theta_{ray})$, we need to know the components of vector S, that represents the Sun's rays. Recall that the center of the Sun is located on the negative side of the x-axis, and vector S has a magnitude of 1 unit. Since the center of the Earth is also located on the x-axis(the origin), vector S must be,

$$\vec{S} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore, $$\cos(\theta_{ray}) = -\left(\frac{x_N x_S + y_N y_S + z_N z_S}{|\vec{N}|}\right) = -\left(\frac{r\cos(\theta_l - \theta)\cos(\theta_t)}{r}\right) = -\cos(\theta_l - \theta)\cos(\theta_t)$$

<u>Determine a function to calculate the total amount of sunlight</u>

We have found an expression for the amount of light received for a specific angular displacement value. Thus we can express and amount of light received as,

$$f(\theta_t) = -\cos(\theta_l - \theta)\cos(\theta_t)$$

The total amount of sunlight the solar panel receives in a day can be determined using the limit definition of the integral. To do this, we must first determine the interval when the solar panel receives sunlight throughout a day, in terms of θ_t. This interval should be between sunrise and sunset. Since the axial tilt of Earth is 0, 6AM is sunrise and 6PM is sunset. In terms of θ_t, $\theta_t = \frac{\pi}{2}$ is sunrise and $\theta_t = \frac{3\pi}{2}$ is sunset. The amount of sunlight received over a period of time is determined by the amount of sunlight multiplied with the time interval. Since we are using the angular displacement to represent time, the amount of sunlight received for a small amount of time is, $f(\theta_t)\,\Delta\theta_t$, where $\Delta\theta_t$ corresponds to the small amount of time. If we add all the values of $f(\theta_t)\Delta\theta_t$, from $\theta_t = \frac{\pi}{2}$ to $\theta_t = \frac{3\pi}{2}$, we can approximate the total amount of sunlight received; given by,

$$f\left(\frac{\pi}{2}\right)\Delta\theta_t + f\left(\frac{\pi}{2} + \Delta\theta_t\right)\Delta\theta_t + f\left(\frac{\pi}{2} + 2\Delta\theta_t\right)\Delta\theta_t + \ldots + f\left(\frac{3\pi}{2}\right)\Delta\theta_t = \sum_{i=0}^{n} f\left(\frac{\pi}{2} + \Delta\theta_t i\right)\Delta\theta_t$$

In this expression, $\Delta\theta_t$ is the time interval(sunrise to sunset) divided into n intervals.

8

$$\text{Thus,} \quad \Delta\theta_t = \frac{\frac{3\pi}{2} - \frac{\pi}{2}}{n}$$

Substituting this back into the original expression, we get,

$$\textit{Total amount of sunlight} \approx \sum_{i=0}^{n} f\left(\frac{\pi}{2} + \frac{\frac{3\pi}{2} - \frac{\pi}{2}}{n} i\right)\left(\frac{\frac{3\pi}{2} - \frac{\pi}{2}}{n}\right)$$

If we take the limit as n goes to infinity, we can determine the exact value of the total amount of sunlight

$$\textit{Total amount of sunlight} = \lim_{n \to \infty} \sum_{i=0}^{n} f\left(\frac{\pi}{2} + \frac{\frac{3\pi}{2} - \frac{\pi}{2}}{n} i\right)\left(\frac{\frac{3\pi}{2} - \frac{\pi}{2}}{n}\right)$$

We can compare sections of this infinite sum with the limit definition of the integral, so we can represent the total amount of sunlight received in a day through and integral, rather than an infinite sum.

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{i=0}^{n} f(x_i) \frac{b-a}{n}$$

E: Good development to develop the integral from the limit of sums.

where $x_i = a + \Delta x$ $\quad \Delta x = \dfrac{b-a}{n}$ $\quad a$ = initial value of x $\quad b$ = final value of x

By comparing the two equations and the terms inside them, we can represent the function as such,

$$\textit{Total amount of sunlight} = f(\theta) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} f(\theta_t)\,d\theta_t = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos(\theta_l - \theta)\cos(\theta_t)\,d\theta_t$$

This function given by this integral calculates the total amount of sunlight as a function of the tilt angle θ.

Determine the optimum tilt angle

To determine the optimum tilt angle, we must differentiate $f(\theta)$ and determine the angle for the local maximum point. To do so, we must first simplify the integral in the function.

$$f(\theta) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos(\theta_l - \theta)\cos(\theta_t)\,d\theta_t = -\cos(\theta_l - \theta)\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(\theta_t)\,d\theta_t$$

$$= -\cos(\theta_l - \theta)\left[\sin(\theta_t)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = -\cos(\theta_l - \theta)\left(\sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)\right)$$

$$= -\cos(\theta_l - \theta)(-1-1) = 2\cos(\theta_l - \theta)$$

9

We can determine the maximum point by solving $f'(\theta)=0$

$$f'(\theta)=-2\sin(\theta_l-\theta)(-1)=2\sin(\theta_l-\theta)$$

$$f'(\theta)=0 \Leftrightarrow 2\sin(\theta_l-\theta)=0 \Leftrightarrow \sin(\theta_l-\theta)=0$$

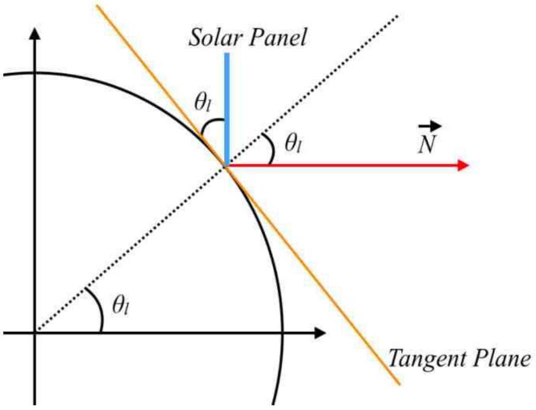
$$\theta_l-\theta=k\pi \Leftrightarrow \theta=\theta_l+k\pi \quad k\in\mathbb{Z}$$

$$\text{Since} \quad \theta\in[0,\frac{\pi}{2}] \ ,\theta=\theta_l$$

Therefore, the maximum amount of energy can be obtained when the tile angle is equal to the latitude. This result makes sense because the solar panel will face the sun directly when the tilt angle is equal to the latitude, shown in figure 8.

D: Meaningful reflection.

Fig. 8

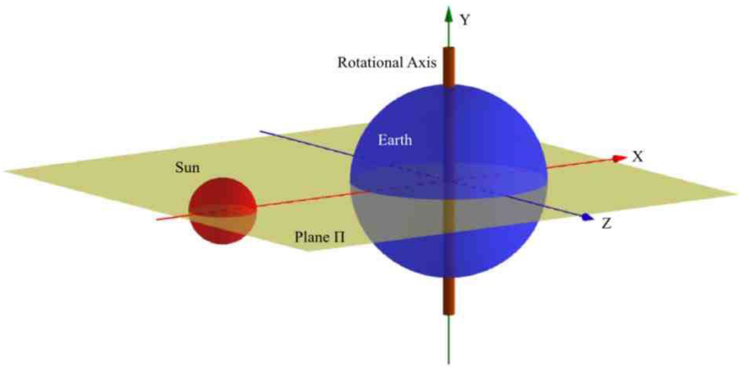


Part 2: Optimum tilt angle for a spherical Earth with an axial tilt

Description of the situation

In this section, we will take the same approach as Part 1 to determine the optimum angle for a spherical Earth with an axial tilt of 23.5°.

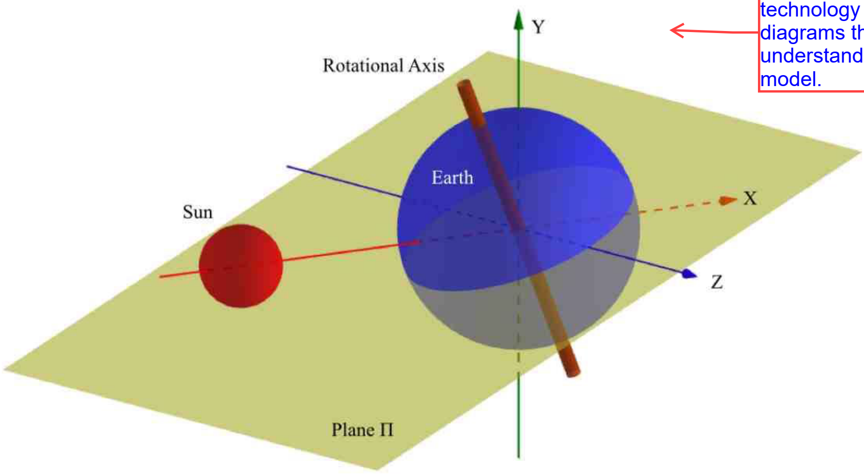
Fig. 9



10

Figure 9 illustrates the scenario for Part 1 with the addition of Plane Π, which is a plane containing the equator of Earth(Plane Π is shown only to visualize the difference between figure 9 and 10).

Fig. 10



B: Excellent use of technology to produce diagrams that enhance understanding the model.

Figure 10 views the Sun and the Earth from the same direction as figure 9, but the Earth has rotated 23.5° on the z-axis. This rotation causes the rotational axis of the Earth to deviate from the y-axis.

The method to calculate the optimum tilt will be the same as Part 1 of this investigation. First, a general expression for the normal vector of the solar panel will be determined. Then, we will take the dot product of the normal vector and the vector to represent the Sun's ray. Finally, we will use integration to create a function of θ, the tilt angle, and determine the optimum tilt angle. Also, we will keep axis at which the solar panel is tilted, and the system of using angular displacement to represent time the same as Part 1.

However, unlike Part 1, due to the axial tilt, the optimum angle will differ depending on the day in an year. A method to comprehend to this change will be discussed later in this investigation.

Determine the amount of sunlight the solar panel will receive

To determine the general expression for the normal vector, we will begin with the general expression for the normal vector when the axial tilt was 0. From Part 1, we know this expression is,

$$\vec{N}=\begin{pmatrix} r\cos(\theta_l-\theta)\cos(\theta_t) \\ r\sin(\theta_l-\theta) \\ r\cos(\theta_l-\theta)\sin(\theta_t) \end{pmatrix}$$

Recall the method in which the normal vector was calculated in Part 1. If the normal vector is translated to the origin, it is equivalent to a position vector created from latitude $\theta_l-\theta$, and angular displacement θ_t. The point on the surface of the sphere, created by the translated normal vector, will have coordinates,

$$(x,y,z)=(\ r\cos(\theta_l-\theta)\cos(\theta_t), r\sin(\theta_l-\theta), r\cos(\theta_l-\theta)\sin(\theta)\ )$$

Let this point be point *N*.

11

Shown by figure 9 and 10, when the axial tilt on the Earth is applied, the whole sphere is rotated by an angle θ_r, about the z-axis. This means, all the points on the sphere will have rotated about the z-axis as well. Therefore, point N must be rotated about the z-axis by θ_r radians.
Since we are rotating the point about the z-axis, the z-coordinate will remain constant. Thus, this problem is equivalent to rotating a point with two x,y coordinates about the origin, when viewed directly above the xy-plane.

Fig. 11

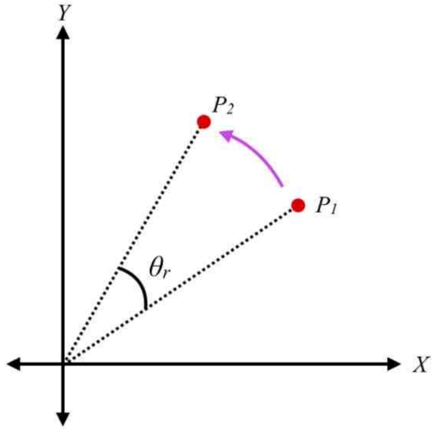


Figure 11 illustrates the situation. In this case, point P_1 has has coordinates,

$$P_1(x, y) = (\ r\cos(\theta_l - \theta)\cos(\theta_t), r\sin(\theta_l - \theta)\ )$$

And the aim is to determine the coordinate of point P_2. To do so, we can use the rotation matrix. The rotation matrix is a tool to determine the coordinate of a point when it is rotated counterclockwise about the origin in a cartesian plane(Weisstein). The coordinate of a point after the rotation is calculated by multiplying the rotation matrix and the coordinates expressed in a 2×1 matrix(Weisstein).

$$\text{Rotation matrix: } \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$Coordinates\ of\ P_2 = \begin{bmatrix} \cos(\theta_r) & -\sin(\theta_r) \\ \sin(\theta_r) & \cos(\theta_r) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\theta_r) - y\sin(\theta_r) \\ x\sin(\theta_r) + y\cos(\theta_r) \end{bmatrix}$$

E: Very well explained.

$$= \begin{bmatrix} r\cos(\theta_l - \theta)\cos(\theta_t)\cos(\theta_r) - r\sin(\theta_l - \theta)\sin(\theta_r) \\ r\cos(\theta_l - \theta)\cos(\theta_t)\sin(\theta_r) + r\sin(\theta_l - \theta)\cos(\theta_r) \end{bmatrix}$$

The coordinates of point P_2 is the x and y coordinates of point N after it has been rotated by θ_r radians about the z-axis. Therefore, the new point N denoted as N_{new} will be,

$$N_{new} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\theta_l - \theta)\cos(\theta_t)\cos(\theta_r) - r\sin(\theta_l - \theta)\sin(\theta_r) \\ r\cos(\theta_l - \theta)\cos(\theta_t)\sin(\theta_r) + r\sin(\theta_l - \theta)\cos(\theta_r) \\ r\cos(\theta_l - \theta)\sin(\theta_t) \end{pmatrix}$$

12

Recall that point N before the rotation is point created on the surface of the sphere by the normal vector of the solar panel. Therefore, point N_{new} is the normal vector to the solar panel tilted θ radians, which is located on a point on Earth with latitude θ_l, angular displacement θ_t, and an axial tilt of θ_r radians.
Now that we know a general expression for the normal vector, we can use the formula derived in Part 1 to calculate the amount of sunlight received at one instance, given by,

$$Amount\ of\ sunlight = cos(\theta_{ray}) = -(\frac{x_N x_S + y_N y_S + z_N z_S}{|\vec{N}|})$$

where, $\vec{N} = \begin{pmatrix} rcos(\theta_l - \theta)cos(\theta_t)cos(\theta_r) - rsin(\theta_l - \theta)sin(\theta_r) \\ rcos(\theta_l - \theta)cos(\theta_t)sin(\theta_r) + rsin(\theta_l - \theta)cos(\theta_r) \\ rcos(\theta_l - \theta)sin(\theta_t) \end{pmatrix}$ and $\vec{S} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Therefore, $Amount\ of\ sunlight = -(\frac{rcos(\theta_l - \theta)cos(\theta_t)cos(\theta_r) - rsin(\theta_l - \theta)sin(\theta_r)}{r})$

$$= sin(\theta_l - \theta)sin(\theta_r) - cos(\theta_l - \theta)cos(\theta_t)cos(\theta_r)$$

Since this is a function of angular displacement, θ_t,

$$f(\theta_t) = sin(\theta_l - \theta)sin(\theta_r) - cos(\theta_l - \theta)cos(\theta_t)cos(\theta_r)$$

Represent the movement of Earth around the Sun

In reality, the Earth will orbit around the Sun. However, throughout the investigation, the position of Earth has been fixed to the origin. Also, considering the orbit of a rotating body makes this problem too complicated and difficult to imagine. To simplify the problem, we can focus on the apparent change on the axial tilt of Earth.

Fig. 12

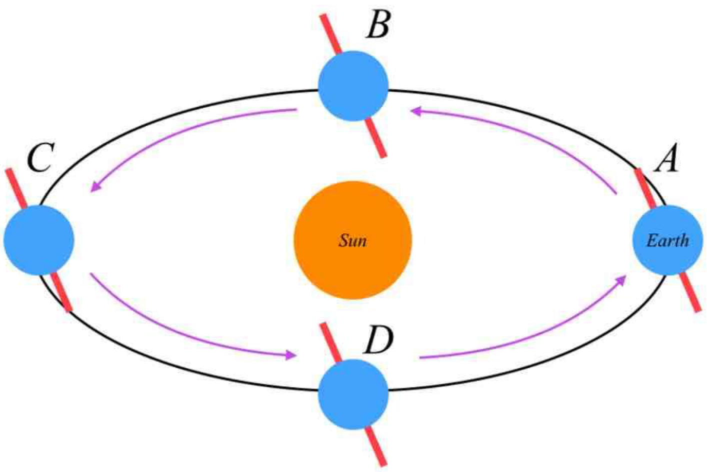


Figure 12 illustrates how the Earth travels around the Sun. If we view the Earth from the sun, at point A, the apparent axial tilt of Earth is 23.5°. Similarly, at point C, the apparent axial tilt of is -23.5°. This means, at some point, the apparent axial tilt was equal to 0. The two points at which this will happen is point B and D.

Fig. 13

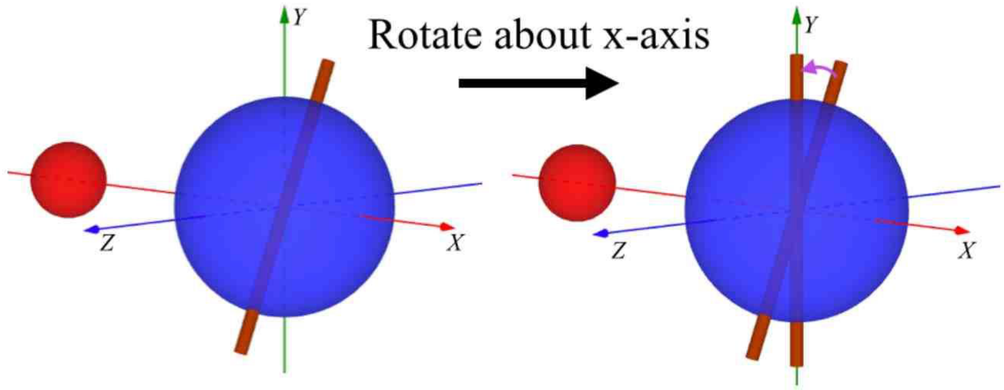


Figure 13 shows why the apparent axial tilt of the Earth is 0° at points B and D. For both points, the axis of rotation of the Earth appears to be rotated about the x-axis by 23.5°. Although this might not be obvious, if we rotate both the Earth and the Sun 23.5° about the x-axis, we return to the scenario examined in Part 1, where the axial tilt = 0. Rotating both celestial bodies about the x-axis only changes the viewer's perspective. The relationship between the Earth and the Sun will be unchanged.

Fig. 14

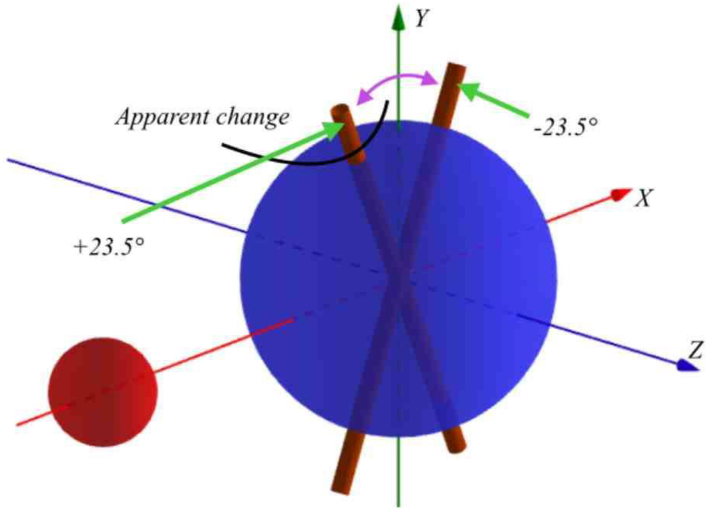


Figure 14 illustrates the apparent change in the axial tilt of Earth throughout the year. If we assume the Earth orbits around the sun in a circular path, we can express the apparent axial tilt of Earth as a function of a specific day of a year.

$$\theta_r = \frac{23.5}{180}\pi \cos\left(\frac{2\pi}{365}n\right) \qquad n \in \mathbb{Z}, n \in [1, 365]$$

Here, n represents the number of days in a year. In this formula, we are assuming the apparent axial tilt will stay constant for a day. While this is not true, because the change in axial tilt for one day is very small, it is a reasonable assumption to simplify the investigation. ✓

Determine a function to calculate the total amount of sunlight

Now, we know the amount of light the solar panel tilted by an angle θ, will receive at an instant depending the position of Earth, given by θ_l, θ_t, and a specific day of the year given by n.
The total amount of sunlight the solar panel will receive in a particular day can be determined the same way as explained in Part 1, using integration.

From Part 1 we know that,

$$Total\ amount\ of\ sunlight\ for\ one\ day = \int_{Sunrise}^{Sunset} f(\theta_t)\,d\theta_t$$

$$= \int_{Sunrise}^{Sunset} sin(\theta_l-\theta)\,sin(\theta_r) - cos(\theta_l-\theta)\,cos(\theta_t)\,cos(\theta_r)\,d\theta_t$$

$$= \int_{Sunrise}^{Sunset} sin(\theta_l-\theta)\,sin(\frac{23.5}{180}\pi\,cos(\frac{2\pi}{365}n)) - cos(\theta_l-\theta)\,cos(\theta_t)\,cos(\frac{23.5}{180}\pi\,cos(\frac{2\pi}{365}n))\,d\theta_t$$

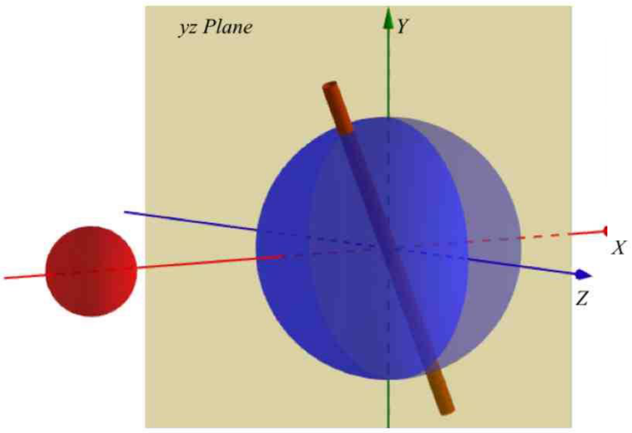
The lower and upper bounds for the integral is the angular displacement value which represent the time of sunrise and sunset, respectively.

To calculate the total amount of sunlight received for a year, we have to add this function 365 times for every value of n from 1 to 365. Therefore,

$$f(\theta) = \sum_{n=1}^{365} \int_{\theta_{sunrise}}^{\theta_{sunset}} sin(\theta_l-\theta)\,sin(\frac{23.5}{180}\pi\,cos(\frac{2\pi}{365}n)) - cos(\theta_l-\theta)\,cos(\theta_t)\,cos(\frac{23.5}{180}\pi\,cos(\frac{2\pi}{365}n))\,d\theta_t$$

The final step to complete this function is to determine an expression for the $\theta_{sunrise}$ and θ_{sunset}.

Fig. 15



Regardless of the apparent axial tilt, the borderline between day and night is constant throughout the year on this diagram, and is visualized in figure 15. Since both the centers of the Sun and the Earth are fixed on the x-axis, this borderline is always represented by the yz-plane given by the equation $x = 0$. In other words, the angular displacement between midnight and sunrise can determined, by solving the equation,

$$x\ component\ of\ \vec{N}_{new} = 0 \quad \text{for } \theta_t$$

15

$$r\cos(\theta_l-\theta)\cos(\theta_t)\cos(\theta_r)-r\sin(\theta_l-\theta)\sin(\theta_r)=0$$

$$\cos(\theta_l-\theta)\cos(\theta_t)\cos(\theta_r)=\sin(\theta_l-\theta)\sin(\theta_r)$$

$$\cos(\theta_t)=\frac{\sin(\theta_l-\theta)\sin(\theta_r)}{\cos(\theta_l-\theta)\cos(\theta_r)}$$

$$\cos(\theta_t)=\tan(\theta_l-\theta)\tan(\theta_r)$$

$$\theta_t=\cos^{-1}(\tan(\theta_l-\theta)\tan(\theta_r))=\cos^{-1}(\tan(\theta_l-\theta)\tan(\frac{23.5}{180}\pi\cos(\frac{2\pi}{365}n)))$$

This is the angle between midnight and sunrise,

Thus, $\theta_{sunrise}=\cos^{-1}(\tan(\theta_l-\theta)\tan(\frac{23.5}{180}\pi\cos(\frac{2\pi}{365}n)))$

Since the time it takes from midnight to sunrise is the same as sunset to midnight,

$$\theta_{sunset}=2\pi-\cos^{-1}(\tan(\theta_l-\theta)\tan(\frac{23.5}{180}\pi\cos(\frac{2\pi}{365}n)))$$

Therefore, the total amount of sunlight received throughout a year is given by the function,

$$f(\theta)=\sum_{n=1}^{365}\int_{\cos^{-1}(\tan(\theta_l-\theta)\tan(\theta_r))}^{2\pi-\cos^{-1}(\tan(\theta_l-\theta)\tan(\theta_r))}\sin(\theta_l-\theta)\sin(\theta_r)-\cos(\theta_l-\theta)\cos(\theta_t)\cos(\theta_r)\,d\theta_t$$

where $\theta_r=\frac{23.5}{180}\pi\cos(\frac{2\pi}{365}n)$

(θ_r is used to tidy the function)

<u>Determine the optimum tilt angle</u>

Before we can determine the optimum tilt angle, we can simplify the function by simplifying the integral.

$$f(\theta)=\sum_{n=1}^{365}\int_a^b\sin(\theta_l-\theta)\sin(\theta_r)-\cos(\theta_l-\theta)\cos(\theta_t)\cos(\theta_r)\,d\theta_t$$

$$f(\theta)=\sum_{n=1}^{365}\left[\sin(\theta_l-\theta)\sin(\theta_r)\theta_t-\cos(\theta_l-\theta)\cos(\theta_r)\sin(\theta_t)\right]_a^b$$

$$f(\theta)=\sum_{n=1}^{365}\sin(\theta_l-\theta)\sin(\theta_r)(b-a)-\cos(\theta_l-\theta)\cos(\theta_r)(\sin(a)-\sin(b))$$

where, $a=\cos^{-1}(\tan(\theta_l-\theta)\tan(\theta_r))$, $b=2\pi-\cos^{-1}(\tan(\theta_l-\theta)\tan(\theta_r))$

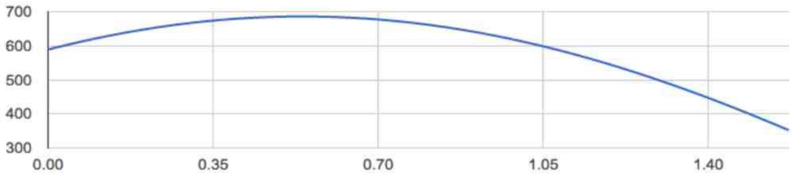
and $\theta_r=\frac{23.5}{180}\pi\cos(\frac{2\pi}{365}n)$

Since differentiating this function and calculating the maximum is impossible by hand, I created a computer program written in javascript(see appendix) which will graph this function for the domain, $\theta\in[0,\frac{\pi}{2}]$, and calculate the value of θ which produces the maximum amount of sunlight the solar panel will receive in a year.

Fig. 16

latitude = 34.5

Maximum point, theta = 0.5387831400906495 = 30.87 degrees



This is the graph which shows the total amount of light received by the solar panel depending on the tilt angle. Osaka, Japan, has a latitude of approximately 34.5 degrees. According to this graph, the maximum amount of energy can be produced if the solar panel is tilted 30.87 degrees. This optimum tilt angle is calculated in the program by taking 1000 points within the domain of the function and compares which value or θ produces the largest value.

When creating the program, one aspect which had to be considered was the upper and lower bounds of the integral, $\theta_{sunrise}$ and θ_{sunset}, which was originally part of the function. The element which exists in both of the bounds, is

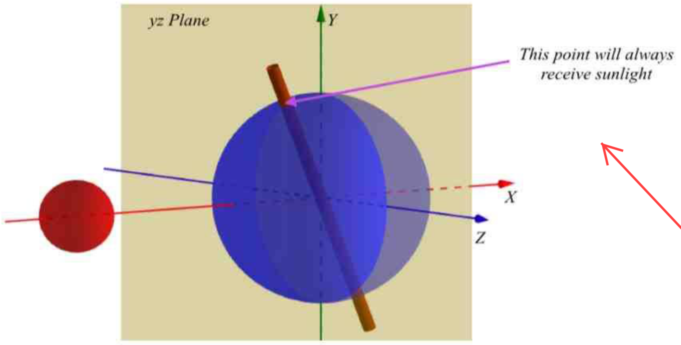
$$\cos^{-1}(\tan(\theta_l-\theta)\tan(\theta_r))$$

Since the domain of $f(x)=\cos^{-1}(x)$ is $x\in[-1,1]$, $-1<\tan(\theta_l-\theta)\tan(\theta_r)<1$
However, for some specific values of θ_l, θ, and θ_r, $\tan(\theta_l-\theta)\tan(\theta_r)$ becomes greater than 1 or smaller than -1. This caused the program to halt the calculations.

To overcome this problem, I examined the conditions which lead to this problem. As a result, I found out that $\tan(\theta_l-\theta)\tan(\theta_r)>1$ resulted when a specific point on Earth received sunlight for the whole day, and $\tan(\theta_l-\theta)\tan(\theta_r)<-1$ resulted when that specific point on Earth received no sunlight for the whole day.

For example, a point located on the north pole $\left(\theta_l=\frac{\pi}{2}\right)$ will receive sunlight for the whole day when the apparent axial tilt is 23.5°. Figure 17 illustrates this situation.

Fig. 17



D: Good reflection showing how model is realistic.

17

On the other hand, if $tan(\theta_l - \theta)tan(\theta_r) = 1$, $\theta_{sunrise} = 0$, $\theta_{sunset} = 2\pi$, which is equivalent to the solar panel receiving sunlight for the whole day. And, if $tan(\theta_l - \theta)tan(\theta_r) = -1$, $\theta_{sunrise} = \pi$, and $\theta_{sunset} = \pi$, which is equivalent to the solar panel receiving no sunlight for the whole day

From this observation, to satisfy the domain of the inverse cosine function, two conditions were set in the program,

$$if \quad tan(\theta_l - \theta)tan(\theta_r) > 1 \quad then, \quad tan(\theta_l - \theta)tan(\theta_r) = 1$$
$$if \quad tan(\theta_l - \theta)tan(\theta_r) < -1 \quad then, \quad tan(\theta_l - \theta)tan(\theta_r) = -1$$

By introducing these conditions, the program was able to graph the functions for all values of θ_l, θ, and θ_r while satisfying the domain of each variable.

While this program will calculate the optimum tilt angle for a given latitude, the aim of this investigation is to determine the optimum tilt angle for every latitude.

A: The exploration is now going over to programming which makes it long and also move on to a different course of study, computer science, which cannot be checked by a mathematics teacher with no knowledge in this area.

To do so, I created a second program, which will calculate the optimum tilt angle for every integer latitude value in degrees, and plot the result on a graph.

Fig. 18

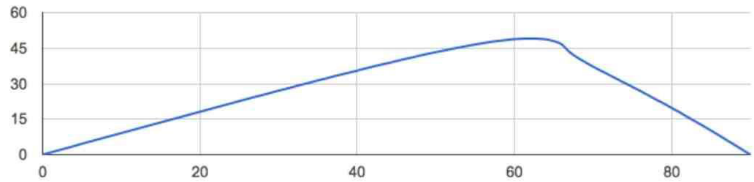


Figure 18 is a graph where the x-axis is the latitude of a specific point on Earth, and the y-axis is the optimum tilt angle, both in degrees. From 0° latitude to around 60° latitude, the optimum tilt angle is close to the latitude. This result is similar to the result for Part 1, and makes intuitive sense. However, for higher latitudes, the optimum angle decreases, and eventually reaches to 0°. While this result is difficult to judge intuitively, as the latitude approaches 90°, there are more days when the whole day is dark or light, which may be one of the reasons behind this result. Another interpretation of this result could relate to the method I used to overcome the domain problem of,

$$cos^{-1}(tan(\theta_l - \theta)tan(\theta_r))$$

A latitude around 66° is when some days in a year will produce a value of $tan(\theta_l - \theta)tan(\theta_r)$ small than -1 or greater than 1. This latitude of 66° also the point in which the graph(figure 18) begins to slope downwards almost in a straight line. Meaning, there could be a connection between these two aspects.

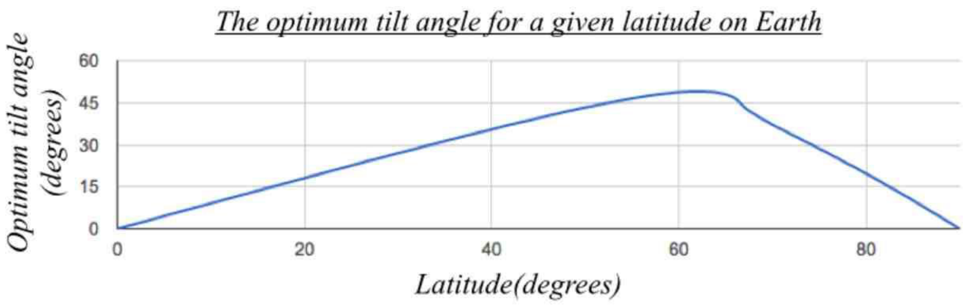
18

Conclusion

Results

From Part 1 and Part 2, we have determined the optimum tilt angle for a solar panel, for every latitude, to receive the most sunlight throughout the year, which corresponds to producing the maximum amount of energy.

To conclude, in Part 1, when the Earth did not have an axial tilt, the optimum tilt angle was equal to the latitude of that specific point on Earth. In Part 2, when the Earth had an axial tilt of 23.5°, the optimum tilt angle, dependent on the latitude, can be determined from the graph below.



While this result only show the optimum tilt angles for one hemisphere, for the other hemisphere the magnitude of the optimum angles will be equal but opposite in direction.

Evaluation

The method used to determine the optimum tilt angle represented the physical conditions of Earth to a reasonable accuracy. In Part 2, I was able to incorporate all three significant pieces of information, the axial tilt of Earth, the Earth's orbit around the Sun, and the sunrise/sunset times, into one equation. Hence, the results obtained in Part 2 are reasonably accurate compared to reality. However, the results are not perfect due to some assumptions made throughout the investigation. For example, we assume the Earth's orbit around the Sun is circular, whereas in reality, it is elliptical.

This investigation can be further extended in different directions. One extension is to compare the difference between the amount of energy produced by a solar panel which is fixed throughout the year, a solar panel which is adjusted n times a year, and a solar panel which can be electronically adjusted to always face the sun. Another extension is to add a second axis of tilt for the solar panel. By adding another axis of rotation, a tilt angle which produces more energy than this investigation may be found.

Work Cited

Nave, Carl. R "Axis Tilt is Critical for Life." *HyperPhysics*, http://hyperphysics.phy-astr.gsu.edu/hbase/Astro/orbtilt.html. Accessed 3 January 2017

Weisstein, Eric W. "Rotation Matrix." *From MathWorld--A Wolfram Web Resource*, http://mathworld.wolfram.com/RotationMatrix.html. Accessed 14 November 2016.

19

Appendix

This is the first javascript program I wrote to graph the amount of energy gained for every solar panel tilt angle for a given latitude(used in figure 16).

```
<html>

    <body>

    <style>
    div.position {
    position: relative;
    left: 0px;
    top: 0px;
    }
    </style>

    <p>Set Latitude: <input type="search" id="set_latitude" onsearch="set_latitude()"> (hit
enter)</p>
    <p id="data1"></p>
    <p id="max_theta"></p>

    <div class="position" id="chart_div" style="float:top; width: 800px;"></div>

    <script type="text/javascript" src="https://www.gstatic.com/charts/loader.js"></script>

    <script type="text/javascript">

        var energy = [];
        var maximum = [0,0];

        var latitude = 34.5;

        function calculate(){

        energy = [];
        maximum = [0,0];

        var l = (latitude/180)*Math.PI; //latitude
        var p = (23.4371/180)*Math.PI; //tilt of earth


        var points = 1000; //number of calculations
        var min = 0;
        var max = (Math.PI)/2;

        var incrament = (Math.abs(max)-Math.abs(min))/points;
        var d = (2*Math.PI)/365;

        for(j=0;j<=points;j++)
            {
                var x = min+incrament*j;

                var y = 0;

                for(i=1;i<=365;i++)
                    {
```

20

```
                        var inside = Math.tan(l)*Math.tan(p*Math.cos(d*i)); // inside the acos
function
                        if(inside > 1)
                            {
                                inside = 1;
                            }
                        if(inside < -1)
                            {
                                inside = -1;
                            }
                        var a = Math.acos(inside);
                        var b = 2*Math.PI-Math.acos(inside);

                        var calc = Math.cos(l-x)*Math.cos(p*Math.cos(d*i))*(Math.sin(a)-Math.sin(b))
+Math.sin(l-x)*Math.sin(p*Math.cos(d*i))*(b-a);

                        y = y + calc;
                    }

                energy.push([x,y]);
            }


        for(k=0;k<=points;k++)
            {
                if(energy[k][1] > maximum[1])
                    {
                        maximum[1] = energy[k][1];
                        maximum[0] = energy[k][0];
                    }
            }

        document.getElementById("max_theta").innerHTML = "Maximum point, theta = "+ maximum[0] + " =
" + (maximum[0]/Math.PI)*180+" degrees"; //  amount of energy = "+maximum[1];

            activate();
        }

        function set_latitude()
        {
            latitude = document.getElementById("set_latitude").value;
            calculate();
        }

        // GRAPHING STUFF FROM HERE:

        google.charts.load('current', {packages: ['corechart', 'line']});

        function drawChart() {
            var data = new google.visualization.DataTable();
                data.addColumn('number', 'theta');
                data.addColumn('number', 'f(θ)');

                data.addRows(energy);

                var options = {
                               curveType: 'function',
                              };
```

21

```
            var chart = new google.visualization.LineChart(document.getElementById('chart_div'));
            chart.draw(data,options);
        }


        function activate()
        {
            document.getElementById("data1").innerHTML = "latitude = " + latitude;
            google.charts.setOnLoadCallback(drawChart); //activate graph
        }

        calculate();
    </script>
    </body>
</html>
```

This is the second javascript program I wrote to graph the optimum angle which yields the maximum amount of energy gained by the solar panel for every latitude(used in figure 18).

```
<html>

    <body>

    <style>
    div.position {
    position: relative;
    left: 0px;
    top: 0px;
    }
    </style>

    <p>Set Latitude: <input type="search" id="set_latitude" onsearch="set_latitude()"> (hit
enter)</p>
    <p id="data1"></p>
    <p id="max_theta"></p>

    <div class="position" id="chart_div" style="float:top; width: 800px;"></div>

    <script type="text/javascript" src="https://www.gstatic.com/charts/loader.js"></script>

    <script type="text/javascript">

        var energy = [];
        var maximum = [0,0];
        var final = [];
        var latitude = 34.5;

        function calculate(){

        energy = [];
        maximum = [0,0];

        var l = (latitude/180)*Math.PI; //latitude
        var p = (23.4371/180)*Math.PI; //tilt of earth


        var points = 1000; //number of calculations
        var min = 0;
```

22

```
        var max = (Math.PI)/2;

        var incrament = (Math.abs(max)-Math.abs(min))/points;
        var d = (2*Math.PI)/365;

        for(j=0;j<=points;j++)
            {
                var x = min+incrament*j;

                var y = 0;

                for(i=1;i<=365;i++)
                    {
                        var inside = Math.tan(l)*Math.tan(p*Math.cos(d*i)); // inside the acos
function
                        if(inside > 1)
                            {
                                inside = 1;
                            }
                        if(inside < -1)
                            {
                                inside = -1;
                            }
                        var a = Math.acos(inside);
                        var b = 2*Math.PI-Math.acos(inside);

                        var calc = Math.cos(l-x)*Math.cos(p*Math.cos(d*i))*(Math.sin(a)-Math.sin(b))
+Math.sin(l-x)*Math.sin(p*Math.cos(d*i))*(b-a);

                        y = y + calc;
                    }

                energy.push([x,y]);
            }


        for(k=0;k<=points;k++)
            {
                if(energy[k][1] > maximum[1])
                    {
                        maximum[1] = energy[k][1];
                        maximum[0] = energy[k][0];
                    }
            }
        }

            for(m=0;m<=90;m++)
            {
                latitude = m;
                maximum = [];
                energy = [];
                calculate();

                var optimum = (maximum[0]/Math.PI)*180;

                final.push([latitude,optimum]);
            }
        activate();
```

```
        // GRAPHING STUFF FROM HERE:

        google.charts.load('current', {packages: ['corechart', 'line']});

        function drawChart() {
            var data = new google.visualization.DataTable();
                data.addColumn('number', 'theta');
                data.addColumn('number', 'f(θ)');

                data.addRows(final);

                var options = {
                               curveType: 'function',
                              };


            var chart = new google.visualization.LineChart(document.getElementById('chart_div'));
            chart.draw(data,options);
        }


        function activate()
        {
            document.getElementById("data1").innerHTML = "latitude = " + latitude;
            google.charts.setOnLoadCallback(drawChart); //activate graph
        }


    </script>
    </body>
</html>
```

24