Mathematics HL: Analysis and Approaches

What is the expression of angles necessary for a robotic arm to reach a given position?

19 pages

1. Introduction

In my robotics club there was a robot with an arm of three segments and a motor at each joint so that the arm could fold and unfold. It was possible to set a target angle for the motors to turn to, thus controlling the position of the end of the arm. This made me wonder, how to express the needed angles for the motors in terms of the target position of the end of the arm and through the lengths of the arm segments. This is how I came to the research question: What is the expression of angles that are necessary for a robotic arm to reach a given position?

A: The aim of the exploration is given through a directed question.

The research of finding angles based on a target position has a wide application in robotics where it may be necessary to place objects precisely, for example, in filming, where cameras mounted on robotic arms need to reach a given position precisely and repeatedly. Another field of study where this research is useful are the CNC machines, where stepper motors determine the position of the machine's tool to mill out complex three-dimensional shapes (Thomas Publishing Company, 2020). Another application are computer-controlled welder arms that are used in car factories (6 axis CNC robot arm , 2017).

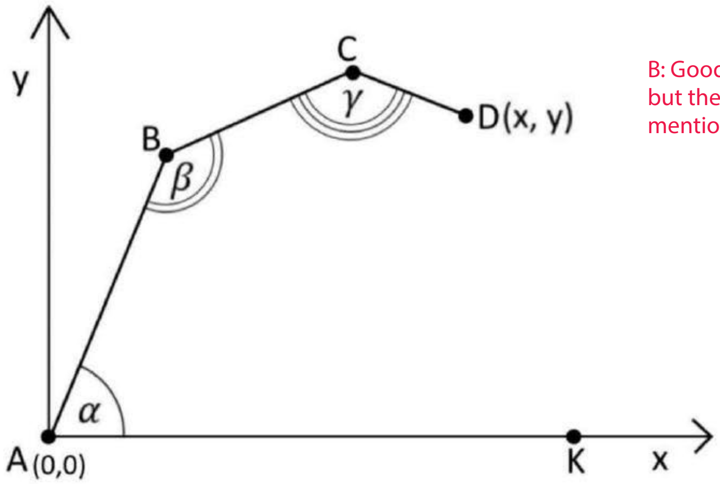
C/D: Some reflection is seen in this paragraph and the next, which justifies the aim and helps put the exploration in perspective.

Since simple geometry does not provide an easy solution to the research question, his paper will use linear algebra and trigonometry to answer it. The problem uses Cartesian coordinates; therefore, it will be imagined that the arm segments are vectors. All three segments of the arm move along one single plane, which means that we can use a 2D vectors for representation. Let point A be the origin of the coordinate plane and the base of the first segment. The target coordinates of the end of the arm, which is noted as point D, are (x,y). This means that $\overrightarrow{AD}$ is the vector over which the arm has to extend and the **sum of vector addition** of the vectors

$\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}$, as shown in Figure 1 below, where these three vectors represent the segments of the arm. The given parameters are:

- $|\overrightarrow{AB}|, |\overrightarrow{BC}|, |\overrightarrow{CD}|$, which are the **lengths of the segments** of the arm.
- x and y, which are the **target coordinates** for the end of the arm, point D.

The unknown parameters that need to be expressed through the given parameters are angles α, β, and γ, which are the target angles for the motors.



B: Good diagram, but there is still no mention of K.

Figure 1. Formalization of the problem. (self-made, 2020)

The general approach to answering the research question is to use dot product to solve the same problem, but for two arms only, and then use the resulting formula to investigate the two smaller segments.

A: This paragraph is difficult to follow without more explanation.

2. Answering the research question

C: Taking ownership of the problem by deciding on a simple approach.

Step 1: Simplify the problem by imagining that there are only two segments in the arm.

The intermediate step in solving this problem is to imagine that there are only two segments of the arm, represented by vectors $\overrightarrow{AB}$ and $\overrightarrow{BD}$, as shown in Figure 2 below. Vectors $\overrightarrow{BC}$ and $\overrightarrow{CD}$ are added to make one new vector $\overrightarrow{BD}$, where $|\overrightarrow{BD}|$ is unknown, but for now we imagine that we know it.

A: Confusing here. Aren't the lengths of both arms fixed constants?

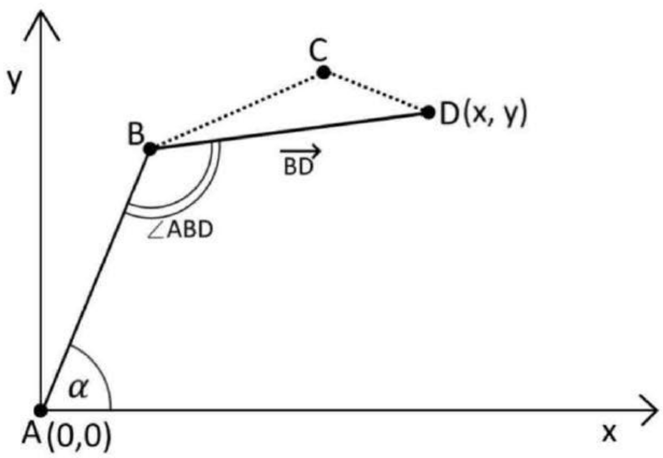


Figure 2. Simplification of the problem with two segments (self-made, 2020)

Step 2: Express the dot products of $\overrightarrow{AB}$ and $\overrightarrow{BD}$ in terms of $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$, and the angles $m\angle ABD$ and α. Re-arrange the expressions to find two different expressions for $m\angle ABD$ in terms of $\overrightarrow{AB}$, $\overrightarrow{BD}$, and angle α. Later on, these two expressions will be used to solve for angle α.

Step 2.1: Expressing $m\angle ABD$ in the first way through the givens and angle α.

A: The presentation of work renders it a little incoherent. Rather than first explaining the process and then executing the work, explanations should be alongside the actual work.

First, the general relation of the vectors and angles, as seen in the simplified problem, must be written as this dot product:

$$\overrightarrow{AB} \cdot \overrightarrow{BD} = |\overrightarrow{AB}||\overrightarrow{BD}|\cos(\pi - m\angle ABD)$$

Where π - $m\angle ABD$ is the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{BD}$.

In order to use dot product, we need to express vector $\overrightarrow{AB}$ and $\overrightarrow{BD}$ in cartesian form. This is done in the following Step:

Step 2.1.1: Find cartesian form of vector $\overrightarrow{AB}$, i.e., the coordinates of point B.

Let the cartesian coordinates of point B be noted as x_B and y_B. Then,

B: Notation should be (x_B, y_B) .

$$x_B = |\overrightarrow{AB}|cos\alpha$$

$$y_B = |\overrightarrow{AB}|sin\alpha$$

because x_B and y_B are equal to the two catheti (the two sides of a right triangle that form the right angle) of the right triangle shown in the Figure below by segments AP and BP that correspond to x_B and y_B respectively:

A: Attention to detail when addressing target audience.

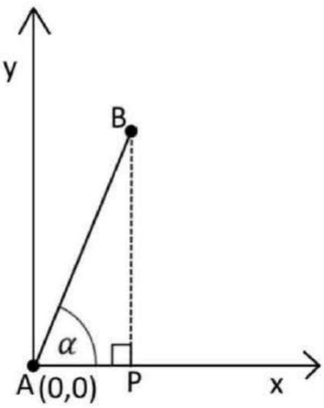


Figure 3. Cartesian coordinates of point B (self-made, 2020).

Since $\overrightarrow{AD} = \overrightarrow{BD} + \overrightarrow{AB}$, $\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$, and $\overrightarrow{AD} = (x, y)$, we can write the cartesian form of $\overrightarrow{BD}$ as

B: Notation should be $\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\overrightarrow{BD} = \begin{bmatrix} x - |\overrightarrow{AB}|cos\alpha \\ y - |\overrightarrow{AB}|sin\alpha \end{bmatrix}$$

B: Candidate chose to use square brackets for vectors; this is condoned.

Having these cartesian expressions at hand, we can proceed with the main direction of Step 2.1 by writing the equation with the cartesian coordinates:

$$\begin{bmatrix} |\overrightarrow{AB}|cos\alpha \\ |\overrightarrow{AB}|sin\alpha \end{bmatrix} \cdot \begin{bmatrix} x - |\overrightarrow{AB}|cos\alpha \\ y - |\overrightarrow{AB}|sin\alpha \end{bmatrix} = |\overrightarrow{AB}||\overrightarrow{BD}| \cos(\pi - m\angle ABD)$$

After which we expand the multiplication and addition on the left hands side and keep solving for $m\angle ABD$.

$$|\overrightarrow{AB}|(xcos\alpha - |\overrightarrow{AB}|\cos^2\alpha) + |\overrightarrow{AB}|(ysin\alpha - |\overrightarrow{AB}|\sin^2\alpha) = |\overrightarrow{AB}||\overrightarrow{BD}|\cos(\pi - m\angle ABD),$$

where $|\overrightarrow{AB}|$ cancels out on both sides.

$$x cos\alpha + \ y sin\alpha - |\overrightarrow{AB}|\cos^2\alpha - |\overrightarrow{AB}|\sin^2\alpha = |\overrightarrow{BD}|\cos\left(\pi - m\angle ABD\right)$$

$$x cos\alpha + \ y sin\alpha - |\overrightarrow{AB}|(\cos^2\alpha + \sin^2\alpha) = |\overrightarrow{BD}|\cos(\pi - m\angle ABD)$$

$$x cos\alpha + y sin\alpha - |\overrightarrow{AB}| = \ |\overrightarrow{BD}|\cos\left(\pi - m\angle ABD\right)$$

By dividing both sides by $|\overrightarrow{BD}|$ and then by taking arccosine of both sides we are left with the following:

$$\pi - m\angle ABD = \arccos\left(\frac{x cos\alpha + \ y sin\alpha - |\overrightarrow{AB}|}{|\overrightarrow{BD}|}\right)$$

E: The mathematics on this page and on the previous page is clearly laid out to demonstrate understanding and address the target audience.

Hence, this is the **first** expression for $m\angle ABD$:

$$m\angle ABD = \pi - \ \arccos\left(\frac{x cos\alpha + \ y sin\alpha - |\overrightarrow{AB}|}{|\overrightarrow{BD}|}\right)$$

Step 2.2: Find the second way to express $m\angle ABD$.

Step 2.2.1: To find the second expression for $m\angle ABD$ we need to express the angle between vector $\overrightarrow{BD}$ and x – axis, **noted as** $\angle X$, in terms of the givens and angle α with the use of dot product, as in this case this is most straightforward:

$$\hat{\imath} \cdot \overrightarrow{BD} = |\hat{\imath}||\overrightarrow{BD}|\cos(m\angle X),$$

where $\hat{\imath}$ is the unit vector of the x – axis.

As established previously, $\overrightarrow{BD}$ can be re-written as a difference of $\overrightarrow{AD}$ and $\overrightarrow{AB}$:

$$\begin{bmatrix}1\\0\end{bmatrix} \cdot (\overrightarrow{AD} - \ \overrightarrow{AB}) = |\overrightarrow{BD}|\cos(m\angle X)$$

To use dot product, we will use the cartesian form of vector $\overrightarrow{AB}$ that was established earlier:

$$\begin{bmatrix}1\\0\end{bmatrix} \cdot \begin{bmatrix}x - |\overrightarrow{AB}|cos\alpha\\ y - |\overrightarrow{AB}|sin\alpha\end{bmatrix} = |\overrightarrow{BD}|\cos(m\angle X)$$

$$x - |\overrightarrow{AB}|cos\alpha = |\overrightarrow{BD}|\cos\,(m\angle X)$$

$$\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|} = \cos(m\angle X)$$

$$m\angle X = arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)$$

Now we can proceed to the main task of Step 2.2, which is finding the second expression for angle $m\angle ABD$ in terms of angle α and the givens. This expression is evident form the following diagram.

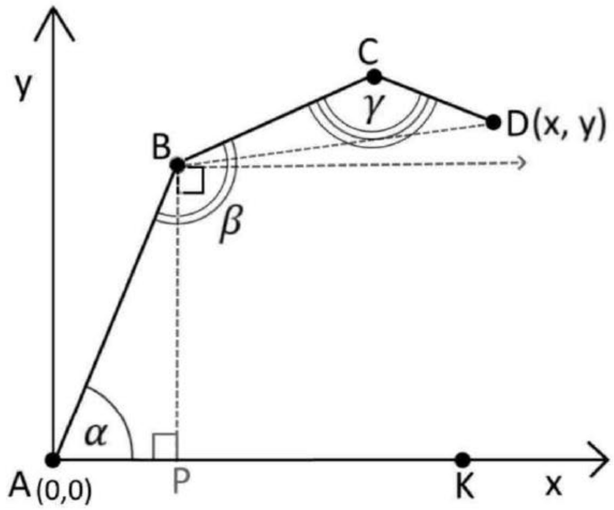


Figure 4. Finding angle m∠ABD (self-made, 2020).

It is evident from this diagram that $m\angle ABD$ is equal to the sum of $m\angle ABP$, $\frac{\pi}{2}$, and the angle between vector $\overrightarrow{BD}$ and the x-axis, conveniently expressed earlier as $m\angle X$.

$$m\angle ABD = m\angle ABP + \frac{\pi}{2} + m\angle X$$

Where $m\angle ABP = \pi - \left(\alpha + \frac{\pi}{2}\right)$, and $m\angle X = arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)$. Hence,

$$m\angle ABD = \pi - \left(\alpha + \frac{\pi}{2}\right) + \frac{\pi}{2} + arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)$$

A/B: Clear explanations in words and notation address target audience. The inconsistency in italicising rather than arccos is condoned, as it could have been produced by software.

Therefore, this is the **second** way to express $m\angle ABD$:

$$m\angle ABD = \pi - \alpha + arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)$$

Step 3: Equating the two different expressions of angle $m\angle ABD$ and solving for angle α in terms of the knowns:

$$\pi - \arccos\left(\frac{xcos\alpha + ysin\alpha - |\overrightarrow{AB}|}{|\overrightarrow{BD}|}\right) = \pi - \alpha + arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)$$

$$\arccos\left(\frac{xcos\alpha + ysin\alpha - |\overrightarrow{AB}|}{|\overrightarrow{BD}|}\right) = \alpha - arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)$$

$$\frac{xcos\alpha + ysin\alpha - |\overrightarrow{AB}|}{|\overrightarrow{BD}|} = \cos\left(\alpha - arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)\right)$$

Due to the angle subtraction formula $cos(A - B) = cosAcosB + sinAsinB$, we can re-write $\cos\left(\alpha - arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)\right)$ as shown in the following expression:

B: It would have been better to have written this as $\frac{\left(x-\left|\overrightarrow{AB}\right|\right)\cos\alpha}{\left|\overrightarrow{BD}\right|}$.

$$\frac{xcos\alpha + ysin\alpha - |\overrightarrow{AB}|}{|\overrightarrow{BD}|} = cos\alpha \frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|} + sin\alpha sin\left(arccos\left(\frac{x-|\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)\right)$$

Where by multiplying both sides by $|\overrightarrow{BD}|$ we simplify the fractions. Since $sin(A) = cos(\pi - A)$, we can re-write $sin\left(arccos\left(\frac{x-|\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)\right)$ and substitute as shown in the following equation, where we also multiply both sides by $|\overrightarrow{BD}|$:

E: Error in mathematics

$\sin(A) = \cos\left(\frac{\pi}{2} - A\right)$.

Follow-through used from here onwards.

$$xcos\alpha + ysin\alpha - |\overrightarrow{AB}| = \ldots$$

$$\ldots = cos\alpha(x - |\overrightarrow{AB}|cos\alpha) + |\overrightarrow{BD}|sin\alpha cos\left(\pi - arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)\right)$$

Since $cos(A - B) = cosAcosB + sinAsinB$, we can re-write $cos\left(\pi - arccos\left(\frac{x-|\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)\right)$ as

$$\cos(\pi)\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|} + sin\pi\sin\left(arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)\right)$$

Which is substituted into the main equation:

$$xcos\alpha + ysin\alpha - |\overrightarrow{AB}| = xcos\alpha - |\overrightarrow{AB}|cos^2\alpha + |\overrightarrow{BD}|sin\alpha\left(\cos(\pi)\frac{x-|\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right) + \ldots$$

$$\ldots |\overrightarrow{BD}|sin\alpha\left(sin\pi\sin\left(arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)\right)\right)$$

Where the term $|\overrightarrow{BD}|sin\alpha\left(sin\pi\sin\left(arccos\left(\frac{x-|\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)\right)\right)$ cancels out because $sin\pi = 0$.

After this, we are left with the following:

$$ysin\alpha - |\overrightarrow{AB}| = -|\overrightarrow{AB}|\cos^2\alpha + |\overrightarrow{BD}|sin\alpha(-1)\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}$$

Notice, $|\overrightarrow{BD}|$ cancels out, which means that we do not need to know $|\overrightarrow{BD}|$, i.e., the length of the second segment, when applying the final expression. Simplification continues as the following:

B: As in previous comment, this should have been written as a coefficient of $\sin\alpha$.

$$ysin\alpha - |\overrightarrow{AB}| = -|\overrightarrow{AB}|\cos^2\alpha - sin\alpha(x - |\overrightarrow{AB}|cos\alpha)$$

$$ysin\alpha - |\overrightarrow{AB}| = -|\overrightarrow{AB}|\cos^2\alpha - xsin\alpha + |\overrightarrow{AB}|sin\alpha cos\alpha$$

$$ysin\alpha + xsin\alpha - |\overrightarrow{AB}| = -|\overrightarrow{AB}|\cos^2\alpha + |\overrightarrow{AB}|sin\alpha cos\alpha$$

$$(x + y)sin\alpha - |\overrightarrow{AB}| = -|\overrightarrow{AB}|\cos^2\alpha + |\overrightarrow{AB}|sin\alpha cos\alpha$$

$$(x + y)sin\alpha = |\overrightarrow{AB}| - |\overrightarrow{AB}|\cos^2\alpha + |\overrightarrow{AB}|sin\alpha cos\alpha$$

A/E: Although the process here could have been reduced by a couple of lines, the written work helps peers to consistently follow the process.

$$\frac{x+y}{|\overrightarrow{AB}|}sin\alpha = 1 - \cos^2\alpha + sin\alpha cos\alpha$$

Since $1 = cos^2\alpha + sin^2\alpha$ and $1 - cos^2\alpha = sin^2\alpha$, $1 - cos^2\alpha$ can be re-written in the equation:

$$\frac{x+y}{|\overrightarrow{AB}|}sin\alpha = sin^2\alpha + sin\alpha cos\alpha$$

$$\frac{x + y}{|\overrightarrow{AB}|} = sin\alpha + cos\alpha$$

$$\left(\frac{x + y}{|\overrightarrow{AB}|}\right)^2 = sin^2\alpha + 2sin\alpha cos\alpha + cos^2\alpha$$

Since the double angle formula is $sin(2A) = 2sinAcosA$, we can re-write $2sin\alpha cos\alpha$ as $sin2\alpha$ in the following expression:

$$\left(\frac{x+y}{|\overrightarrow{AB}|}\right)^2 = 1 + sin2\alpha$$

$$sin2\alpha = \left(\frac{x+y}{|\overrightarrow{AB}|}\right)^2 - 1$$

This is the expression for angle α:

$$\alpha = \frac{arcsin\left(\left(\frac{x+y}{|\overrightarrow{AB}|}\right)^2 - 1\right)}{2}$$

Step 4: Shift vectors $\overrightarrow{BC}$ and $\overrightarrow{CD}$ to the origin by subtracting $\overrightarrow{AB}$ from $\overrightarrow{AD}$ to make it easier to solve for angle γ. The resulting transformation is shown in the following Figure:

E: Although vectors are essentially free, the fact that the candidate uses a translation of points B, C and D eases understanding for the reader. It would have been good to see the original diagram alongside.

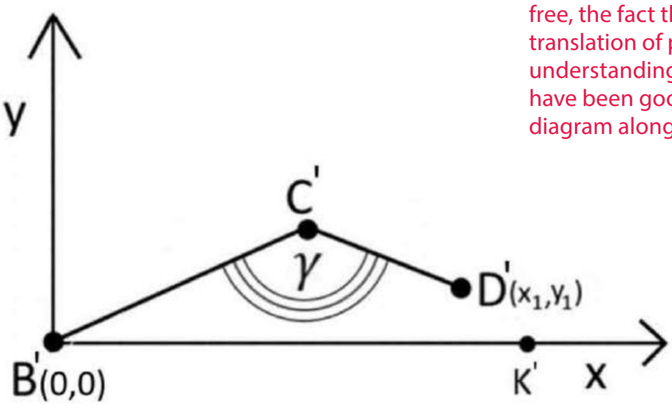


Figure 5. Looking at the other two segments. (self-made, 2020)

Where x_1 and y_1 are the coordinates of the point D', to which the original point D is shifted.

$|\overrightarrow{C'D'}| = |\overrightarrow{CD}|$ and $|\overrightarrow{B'C'}| = |\overrightarrow{BC}|$ because the two original vectors got shifted linearly without distortion of the angles and without changing the magnitudes.

E: It is clear that the candidate understands the issue addressed in the previous comment.

This is expressed mathematically as follows:

$$\overrightarrow{B'D'} = \overrightarrow{AD} - \overrightarrow{AB} = \overrightarrow{BC} + \overrightarrow{CD}$$

$$\overrightarrow{B'D'} = \begin{bmatrix} x - x_B \\ y - y_B \end{bmatrix} = \begin{bmatrix} x - |\overrightarrow{AB}|cos\alpha \\ y - |\overrightarrow{AB}|sin\alpha \end{bmatrix}$$

Notice that this setup is analogous to the simplified two-segment version in Step 1. This means that angle γ is analogous to $m\angle ABD$ and the same formulas are applicable here as well.

D: The candidate is drawing connections between methods used.

Step 5: Express angle γ with the same formula as $m\angle ABD$.

First, express $m\angle K'B'C'$ from Figure 5 with the same formula as used for α because the two angles are analogous.

$$m\angle K'B'C' = \frac{arcsin\left(\left(\frac{x_1 + y_1}{|\overrightarrow{B'C'}|}\right)^2 - 1\right)}{2}$$

E: Elegant and concise application of a formula found earlier.

Where x_1 and y_1 are the coordinates of the point D' from Figure 5.

Then express γ with the same formula as for $m\angle ABD$, which is the following:

$$m\angle ABD = \pi - \alpha + arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)$$

This is the expression of angle γ:

$$\gamma = \pi - m\angle K'B'C' + arccos\left(\frac{x_1 - |\overrightarrow{B'C'}|\cos(m\angle K'B'C')}{|\overrightarrow{C'D'}|}\right)$$

Where $x_1 = x - |\overrightarrow{AB}|cos\alpha$, $y_1 = y - |\overrightarrow{AB}|sin\alpha$, $m\angle K'B'C' = \frac{arcsin\left(\left(\frac{x_1+y_1}{|\overrightarrow{B'C'}|}\right)^2 - 1\right)}{2}$ and

$|\overrightarrow{C'D'}| = |\overrightarrow{CD}|$, $|\overrightarrow{B'C'}| = |\overrightarrow{BC}|$.

Step 6: Solve for angle β.

C: Evidence of authentic and outstanding personal engagement is evident in the perseverance of the candidate to add a third arm.

The expression for angle β is derived from the following Figure:

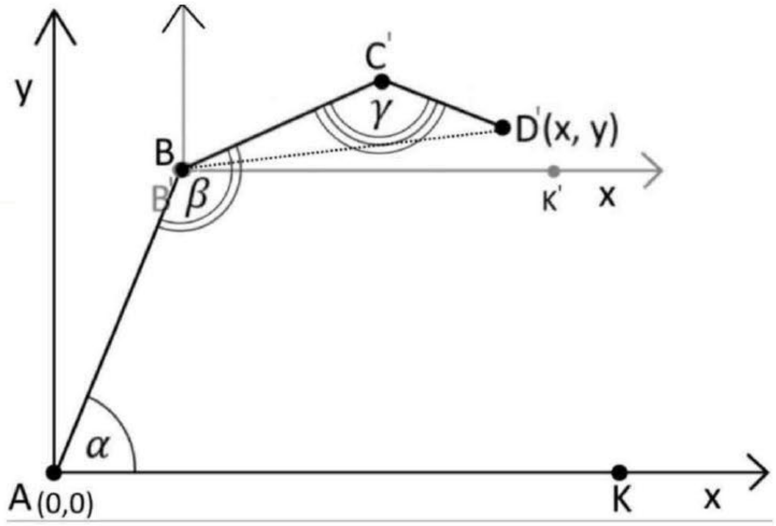


Figure 6. How to solve for angle β (self-made, 2020)

From this Figure we can see that angle β is the difference of the sum of angles $m\angle ABD$ and $m\angle K'B'C'$, and $m\angle K'B'D'$ (noted and expressed earlier as $m\angle X$). This is expressed as follows:

$$\beta = m\angle ABD + m\angle K'B'C' - m\angle X$$

B: The diagram above supports the explanation.

Where $m\angle ABD = \pi - \alpha + arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)$, $m\angle K'B'C' = \frac{arcsin\left(\left(\frac{x_1+y_1}{|\overrightarrow{B'C'}|}\right)^2 - 1\right)}{2}$, and

$m\angle X = arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)$; the angle between vector $\overrightarrow{BD}$ and the x-axis.

Summing the angles is shown in the following:

$$\beta = \pi - \alpha + arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right) + \frac{arcsin\left(\left(\frac{x_1+y_1}{|\overrightarrow{B'C'}|}\right)^2 - 1\right)}{2} - arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right),$$

Where $arccos\left(\frac{x - |\overrightarrow{AB}|cos\alpha}{|\overrightarrow{BD}|}\right)$ cancels out.

Hence, the **expression for angle β is this:**

$$\beta = \pi - \alpha + \frac{arcsin\left(\left(\frac{x_1 + y_1}{|\overrightarrow{B'C'}|}\right)^2 - 1\right)}{2}$$

Where $x_1 = x - |\overrightarrow{AB}|cos\alpha$, $y_1 = y - |\overrightarrow{AB}|sin\alpha$, and $|\overrightarrow{C'D'}| = |\overrightarrow{CD}|$, $|\overrightarrow{B'C'}| = |\overrightarrow{BC}|$.

Step 7: The answer to the research question.

The formulas for the angles α, β, and γ, as expressed through coordinates *(x,y)* and the three lengths $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$, and $|\overrightarrow{CD}|$ are the following:

$$\alpha = \frac{arcsin\left(\left(\frac{x + y}{|\overrightarrow{AB}|}\right)^2 - 1\right)}{2}$$

$$\beta = \pi - \alpha + \frac{arcsin\left(\left(\frac{x - |\overrightarrow{AB}|cos\alpha + y - |\overrightarrow{AB}|sin\alpha}{|\overrightarrow{BC}|}\right)^2 - 1\right)}{2}$$

$\gamma = \ldots$

$$= \pi - \frac{\arcsin\left(\left(\frac{x - |\overrightarrow{AB}|\cos\alpha + y - |\overrightarrow{AB}|\sin\alpha}{|\overrightarrow{BC}|}\right)^2 - 1\right)}{2} \ldots$$

$$+ \arccos\left(\frac{x - |\overrightarrow{AB}|\cos\alpha - |\overrightarrow{BC}|\cos\left(\frac{\arcsin\left(\left(\frac{x - |\overrightarrow{AB}|\cos\alpha + y - |\overrightarrow{AB}|\sin\alpha}{|\overrightarrow{BC}|}\right)^2 - 1\right)}{2}\right)}{|\overrightarrow{CD}|}\right)$$

Limitations of the formulas:

The overall restriction for the target *x* and *y* coordinates can be expressed as follows:

$$|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CD}| \geq \sqrt{x^2 + y^2}$$

Where $\sqrt{x^2 + y^2} = |\overrightarrow{AD}|$. This means that it is impossible to reach point *D* if the lengths of the given arm segments are in sum shorter than the distance between the target position and the origin, i.e., if the robotic arm is too short.

D: Valid consideration.

Restrictions on the formula of angle α is restricted by the maximum possible sine, namely, 1. The formula is the following:

$$\alpha = \frac{arcsin\left(\left(\frac{x+y}{|\overrightarrow{AB}|}\right)^2 - 1\right)}{2}$$

Where the element $\left(\frac{x+y}{|\overrightarrow{AB}|}\right)^2$ cannot be larger than two, otherwise the sine of an angle would be larger than one, which is outside the domain of sine. This means that the following is true for the relationship between x, y, and $|\overrightarrow{AB}|$:

D: Another valid limitation.

$$0 \leq \left(\frac{x+y}{|\overrightarrow{AB}|}\right)^2 \leq 2$$

$$0 \leq \frac{x+y}{|\overrightarrow{AB}|} \leq \sqrt{2}$$

$$0 \leq x + y \leq |\overrightarrow{AB}|\sqrt{2}$$

From this, all other restrictions on every parameter expressed through angle α and the givens follow. This restriction also implies that in order to be calculable by a computer, the real-world lengths of the segments, as well as the target coordinates will need to be scaled down to fit into the sine and cosine domains.

There are restrictions created by the practical application of this study in robotics. Since angle α is calculated with arcsine, there will be two solutions for this angle. The computer would choose the smaller angle by default. This means that the program must contain an "*if*" statement that would check if the small solution for α works with the given coordinates. If it does not, the program chooses the angle that is formed by the line that goes through x and y of point B and the x-axis.

C/D: A good discussion of implications drawn from the limitations described, put in the perspective of a real-life application.

Conclusion and further research:

In conclusion, answer to the research question "What is the expression of angles that are necessary for a robotic arm to reach a given position?" is expressed through the formulas given at the end of Step 7. The restrictions on the given variables are stated in the previous section.

An advantage of the approach of finding the expressions through dot product of vectors is that the resulting formulas do not require to convert the vectors from polar to Cartesian coordinates and the other way around. The drawback of this approach is that it results in an expression of an angle that has two solutions and therefore requires a brute force step during computation.

D: Weighing up the pros and cons of the chosen approach.

A major conclusion of this research is that if a problem consists of alike parameters, in this case, arm segments, it can be simplified into the elementary problem, in this case, of only two segments, and then solved step-by-step for more complex cases. This general method is similar to proof by induction and infinite series, where calculations are based on the previous step.

This research could be further enhanced by investigating what is the minimal distance the arm must move to reach the given position, how to express the angles for an arm with n segments, and how to express a function of angles that would describe a path drawn by the tip of the arm.

D: The candidate concludes by generalizing what was learned through the process and how it might apply to future projects. Connections are also made with other parts of the syllabus.

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