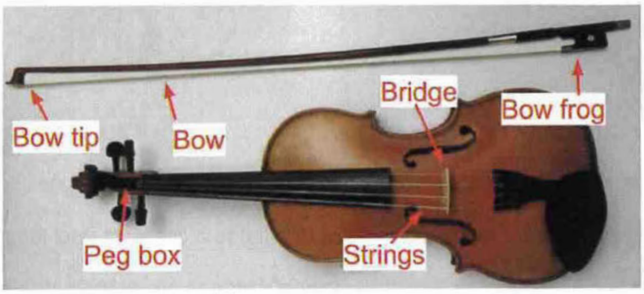
Mathematics Application and Approaches HL Internal Assessment

Path of a Violin Bow Frog

Introduction

The violin is a musical instrument with a wooden body, 4 strings and a bow (see figure 1). Usually, when violinists play music, they hold the bow frog, slide the bow on one string and rotate the bow about the string at the same time if they need to shift strings. If one traces the bow frog, one can see it gives a curved path. Different lengths of the section of bow used can give different musical effects. For example, the sound is more powerful if the frog is used. However, these changes can cause the path to vary. Professional violinists are physically trained to be familiar with those paths, so they can control the bow steadily to play good music. However, as an amatuer violinist, I always find it hard to understand intuitively, so I want to see the effect of bow length on its path in a more concrete way through mathematics.



A: The labelled picture is very helpful for the target audience.

Figure 1. Violin parts

The aim of this investigation is to model the path of the violin bow frog depending on the length of bow section used (L), when the bow plays all 4 strings and starts from the frog. I will find out the relationship between L and the radius curvature (R) of the path and use the appropriate arc from the corresponding circle to model the path. The model will be used to make a prediction when only L is given, which will be compared to the actual bow path for evaluation.

A: The aim is clearly stated.

Modelling the Bow

<u>Definition of variables</u>

Figure 2 and 3 illustrates the position of the strings and the bow when the violin is played.

It can be seen from figure 2 that the 4 strings are approximately equally spaced on the curved bridge. The inner pair and the outer pair of strings are at the same height respectively. The strings are stretched radially out from the peg box of the violin (see figure 1), but they are so long and the angle between them is so small that they are approximately parallel to each other near the bridge. When the violin is played, the bow only moves near the bridge, so I assume that the bow stays in a plane that is perpendicular to all the 4 strings. Thus, I simplify the 3-dimensional scenario into 2 dimensions.

D: The candidate explains and justifies assumptions made using the design of a violin.

Figure 2. Position of strings

Figure 3. Position of bow

Figure 4 shows the problem in 2 dimensions. I set up a Cartesian coordinate system in the plane where the bow moves. The bow is straight and it has a constant total length, so I represent it using a line segment of length L_0. The four strings are represented as points A, B, C, and D, whose positions satisfy AD || BC and the separation between neighboring strings is AB = BC = CD = s. s is a constant length on my violin. So the quadrilateral ABCD is an isosceles trapezoid and the two base angles are equal, as annotated on the graph as θ_s. Because AD || BC, angle 1 and 2 are also equal to those base angles. For convenience, I put A on the origin and AD on the x-axis.

B: Good mathematical communication with reference to the diagram.

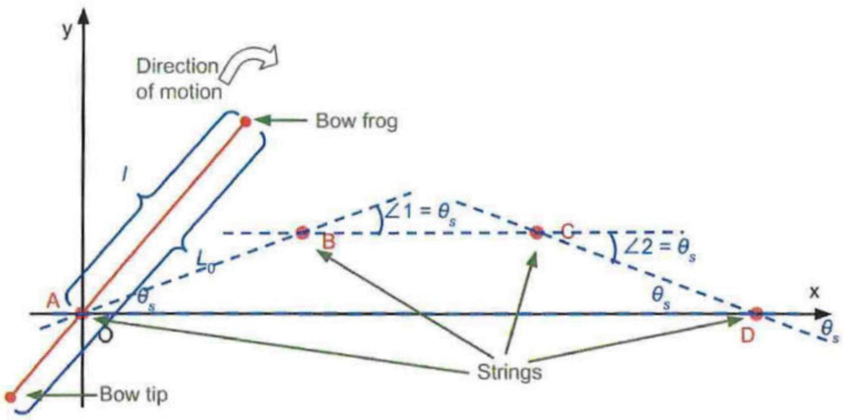


Figure 4. Bow and strings in Cartesian coordinate system

I consider the bow to move from string A to string D, starting from the frog. Its ending point depends on the length of bow section used (L), which is in the range of $s \leq L \leq L_0$, because the bow section to be used must be enough to reach the next string and it cannot be longer than the whole bow. The path of the frog should not change if the bow moves in the opposite direction and uses the same L, because the motion is reversible. As the bow moves translationally on a string, the distance between the bow frog and the string increases (see figure 4). I define this distance as l. Meanwhile, the bow rotates around the string to shift to the next string, so the angle between the bow and the x-axis changes. I define this angle as θ.

I describe the motion of the bow again using the variables. I let the bow start on string A, at a position where $l = 0$ and $\theta_0 = 2\theta_s$, and end on string D, at $l = L$ and $\theta = -2\theta_s$. When the bow moves on one string and shifts to the next, θ decreases continuously and at the point of shifting, the total decrease is θ_s; l first increases continuously but at the point of shifting, it suddenly decreases by s. I assume the linear and angular speeds (v and ω) of the bow are constant. Although there is not another string after D for the bow to shift onto, I still let the bow rotate θ_s on D for consistency.

A: What is the reason behind the choice of angle here?

Now I find out v and ω. The linear speed v is the distance the bow travels per unit time. The distance is related to l because l increases as the bow moves, but l also decreases discontinuously by s every time when the bow shifts string. It is helpful to visualize the change in l over time. I let the time taken to play each string be t_1, so the total time is $4t_1$. Figure 5 is a l-t graph showing the discontinuity in l.

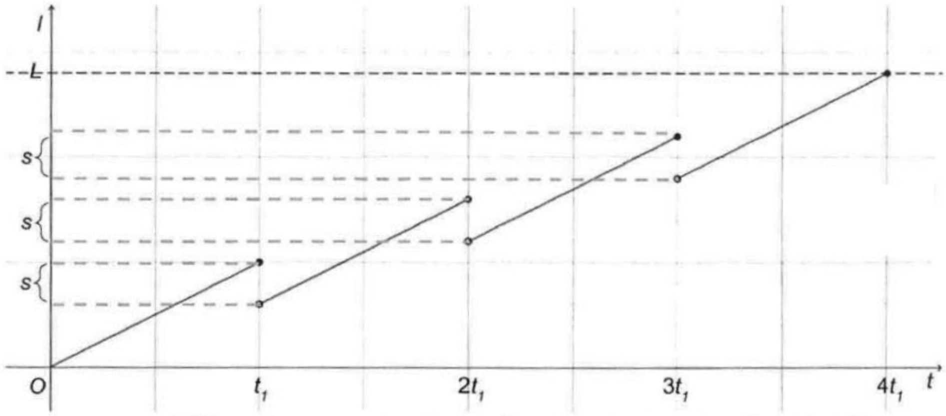


Figure 5. Distance from the string the bow is on and the bow frog (l) vs. time (t)

In figure 5, l reaches L when the motion ends. The total distance is L plus the 3 vertical gaps caused by the separation s between strings. So v is the total distance divided by the total time:

$$v = \frac{L + 3s}{4t_1}$$

The change in angle is continuous so ω is simpler to calculate. The angular speed ω is the total angle that the bow rotates divided by the total time. The bow rotates θ_s about each of the 4 strings, so the total angle of rotation is $4\theta_s$. The angular speed is:

$$\omega = \frac{4\theta_s}{4t_1} = \frac{\theta_s}{t_1}$$

It is not important how much time it takes the bow to move, because this exploration does not concern how fast the path is generated. To simplify the expressions, I take the time required to play on each string to be t_1 = 1 second. Substitute t_1 = 1 second into the equations for v and ω:

D: Some reflection seen here. Time could be left as a parameter, but as this investigation is tracing the path assuming t, a simple value is a good choice.

$$v = \frac{L + 3s}{4}$$

$$\omega = \theta_s$$

Because of the discontinuity in l, it is only reasonable to talk about the path on each string separately. I look at the path on string A first, for which I can formulate two functions $l(t)$ and $\theta(t)$:

$$l(t) = vt = \frac{L + 3s}{4}t$$

$$\theta(t) = 2\theta_s - \omega t = 2\theta_s - \theta_s t,$$

where the domain is $0 \le t \le 1$ second. At the beginning of motion, $l = 0$, so in $l(t)$, the constant term is 0. Because l increases at a constant speed v, there is a t term with a coefficient v. The angle θ starts with an initial value of $2\theta_s$, so in $\theta(t)$, there is a constant term $2\theta_s$. θ decreases as time increases because the rotation is clockwise, so the t term has a coefficient of $-\omega$.

A: It is not immediately clear that the candidate is explaining the equations above.

Using basic trigonometry, the path of the frog when the bow is on string A can be expressed parametrically:

$$x = \cos(\theta(t)) \times l(t) = \frac{L + 3s}{4}t\cos(2\theta_s - \theta_s t)$$

$$y = \sin(\theta(t)) \times l(t) = \frac{L + 3s}{4}t\sin(2\theta_s - \theta_s t)$$

C: Taking ownership of the problem. The candidate has resolved the path into vertical and horizontal components.

The domain is $0 \le t \le 1$ second. At an instant of time t, one can think of $l(t)$ as the radius of a circle with its center on the origin, and $\theta(t)$ as the angle of the radius with the x-axis. The projections of $l(t)$ onto the x-axis and y-axis are $l(t)\cos(\theta(t))$ and $l(t)\sin(\theta(t))$ respectively, which are the x- and y-coordinates of the end point of $l(t)$, representing the position of the bow frog. However, unlike the radius of a circle, $l(t)$ is not a constant but an increasing linear function of t, so the path of the bow end is not part of a circle with its center on the origin, even possibly not part of a circle at all.

This will be explored further later, but now let us consider the path when the bow shifts to string B. The speeds *v* and ω are constant, so they are as calculated before.

The initial *l* between the bow frog and the string it is on (string B) is now $\frac{L+3s}{4} - s = \frac{L-s}{4}$.

This is because $\frac{L+3s}{4}$ is the maximum *l* when the bow is on string A. When the bow shifts to string B, the bow skips the separation *s* between string A and B, so the distance decreases by *s* (see figure 5). The initial θ between the bow and the x-axis is θ_s, which is the angle at which the bow ended on string A. The position of the path also differs. When the bow is on string B, the bow rotates about point B with coordinates ($-x_B$, $-y_B$), instead of the origin, like when the bow is on string A. ✓

However, the path on string B can be found similarly as I did for string A. I can first find the path of the bow on string A when the initial *l* is $\frac{L-s}{4}$ and the initial θ is θ_s. Then I can translate this path through the vector (x_B, y_B). In this case, he functions *l*(*t*) and θ(*t*) are:

B: Wrong vector notation, should be $\begin{pmatrix} x_r \\ y_r \end{pmatrix}$.

$$l(t) = \frac{L-s}{4} + vt = \frac{L-s}{4} + \frac{L+3s}{4}t$$

$$\theta(t) = 2\theta_s - \omega t = 2\theta_s - \theta_s t$$

For the path on string B, the parametric equations before the translation are:

$$x = \cos(\theta(t)) \times l(t) = \left(\frac{L-s}{4} + \frac{L+3s}{4}t\right)\cos(\theta_s - \theta_s t)$$

$$y = \sin(\theta(t)) \times l(t) = \left(\frac{L-s}{4} + \frac{L+3s}{4}t\right)\sin(\theta_s - \theta_s t)$$

Using the geometric relationship shown in figure 5, (x_B, y_B) can be found to be ($\cos(\theta_s)$, $\sin(\theta_s)$), so after the translation, the parametric equations are:

E: Should be $(s\cos\theta_s, s\sin\theta_s)$.

$$x = \left(\frac{L-s}{4} + \frac{L+3s}{4}t\right)\cos(\theta_s - \theta_s t) + \cos(\theta_s)$$

$$y = \left(\frac{L-s}{4} + \frac{L+3s}{4}t\right)\sin(\theta_s - \theta_s t) + s\sin(\theta_s)$$

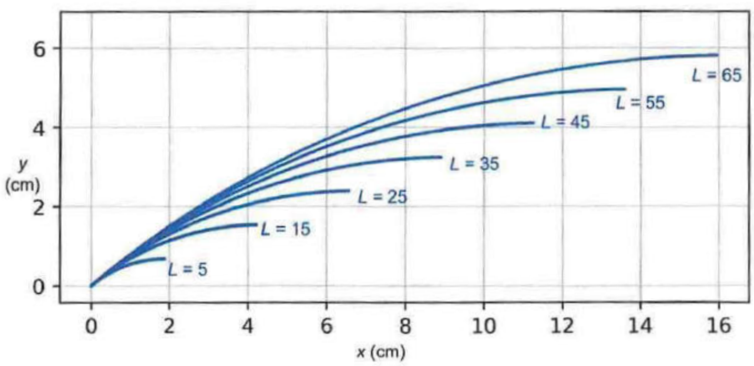
The domain of t is $0 \leq t \leq 1$ second. Similarly, using the corresponding initial l and initial θ values, and then applying the translation using the corresponding vector, one can find the paths for string C and D.

B/D: Should be approximations. How were measurements taken? What about accuracy?

<u>Plotting and fitting to the paths</u>

D: Taking $s=1$ eliminates the error noted in the previous comment.

To plot the path, I need numeric values of θ_s and s. I measured on my violin that $\theta_s = 20° = \frac{\pi}{9}$ rad, s = 1cm and L_0 = 65cm. That constrains L to the domain of $1 \leq L \leq 65$ cm. Using these values, I can apply the parametric equations for the path when the bow is on string A to plot the actual bow path. I change the length of the bow section used (L) to be L = 5, 15, 25, 35, 45, 55, and 65cm, in order to find the relationship between L and the shape of the path. These paths are drawn on the same coordinate system in figure 6.



A: It would have helped the reader had the student used L to represent the length of bow used to travel from string A to string B.

Figure 6. Paths of bow frog (string A)

The paths are all curves that slightly concave down, starting from a relatively greater gradient and ending at a very shallow gradient. This is reasonable because the bow rotates towards the horizontal position. The length of path increases as L increases, which should be true because the length of the path must be longer when L is longer.

It is hard to tell by eye the exact shape of the curves in figure 5, but they look very similar. To study the paths more closely, my first thought is to use the gradient of the path. But it leads to some difficulties. The gradient of a path consists of a range of values and is expressed as a function of x. It is hard to quantitatively compare functions, which consists of a range of values. It is even harder to predict a function using a single value *L*.

Another problem is that gradient only gives steepness, which can be different from curvature. Take a circle as an example: An arc from the top of a circle has a much smaller overall gradient compared to an arc from the side of the same circle, while they have exactly the same curvature. It is also harder to intuitively imagine how curved a path is based on gradient, especially when compared to using the radius of curvature of the paths. The radius of curvature (*R*) is the radius of the circle that best approximates a curve at a point (Weisstein). This circle is analogous to the tangent to a curve at a point, which is the line that best approximates the curve at the point (Weisstein). The center of this circle is the center of curvature. This involves a geometric shape that is easier to imagine its size and curvature compared to gradient. *R* for a point on a curve expressed by function *y=f*(*x*) can be found using:

D: Meaningful reflection on the work and models created.

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1.5}}{\left|\frac{d^2y}{dx^2}\right|}$$ (Weisstein).

C/D: The candidate explains the formula for radius of curvature qualitatively.

The derivation of this formula is not the focus of this exploration, so it is not elaborated here, but this formula can be understood qualitatively. The sign of the second derivative $\frac{d^2y}{dx^2}$ indicates whether the curve is concave up (positive) or concave down (negative), which has no effect on *R*, so an absolute value is used. As the second derivative of *y=f*(*x*) at a point increases, the graph is more curved around the point, so the radius of curvature will be smaller. This matches the relationship shown in the formula—as the denominator increases, *R* decreases.

A possible approach is to calculate the mean radius of curvature based on a range of points on the string-A path. Once I have the result, I can draw a circle of that radius and draw the circle on the graph to fit the path. From the path of the bow on string A, I selected 10 equally spaced *t* (0.1, 0.2, 0.3, …, 0.9, 1 seconds) to calculate the *R* at each point and found the mean *R*. However, the circle with this mean *R* does not always fit the corresponding path perfectly. For example, when L = 65cm, the mean R = 25.681… cm ≈ 25.7 cm, but as shown in figure 7, the radius can be adjusted manually to fit better.

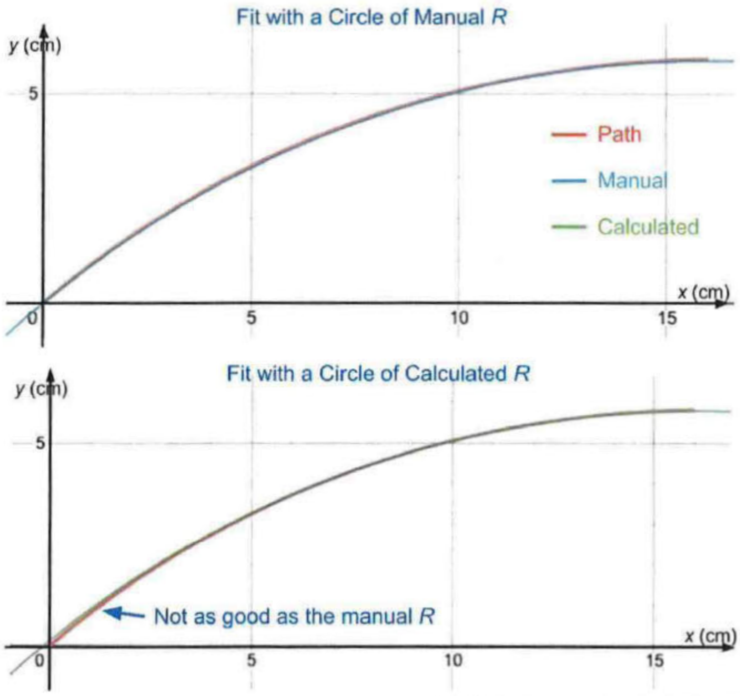


Figure 7. Comparing the R found manually (upper) and using numerical calculation (lower)

D: Good reflection and discussion about GOF. Failed to explain how manual fit was done and the other curves when $L < 65$.

Therefore, I decide to find R manually by drawing a circle to fit each path and measuring the radius of each circle. The blue arc in figure 7 is part of the circle I drew for L = 65cm and it fits well to the path because they almost completely coincide. The circle is too large so I cannot show it completely. When fitting circles to paths, I also record the x- and y-coordinates (x_c and y_c) of the center of each circle.

D: Candidate only explains best fit based on visual interpretation. No empirical analysis done.

Building the model for string A

I record the measurements for all string-A paths in table 1. I can adjust the radius finely (the smallest adjustment is 0.01 cm), so the circles and paths generally fit very well like in figure 7. This means I had good measurements for the overall radius of curvature. I plot R against L, which shows a relationship that is strongly positive linear, so I add a regression (best-fit) line of y onto x (see figure 8). I choose the best fit of y onto x because L is entered into my parametric equations without an uncertainty, while R is found manually which means the uncertainty is greater.

D: How is this achieved?

Table 1: R, x_c and y_c depending on L (string A)

L (cm)	5	15	25	35	45	55	65
R (cm)	2.94	6.75	10.5	14	18	21.5	25.3
x_c (cm)	1.88	4.29	6.67	8.98	11.42	13.70	16.10
y_c (cm)	-2.26	-5.20	-8.05	-10.76	-13.88	-16.56	-19.52

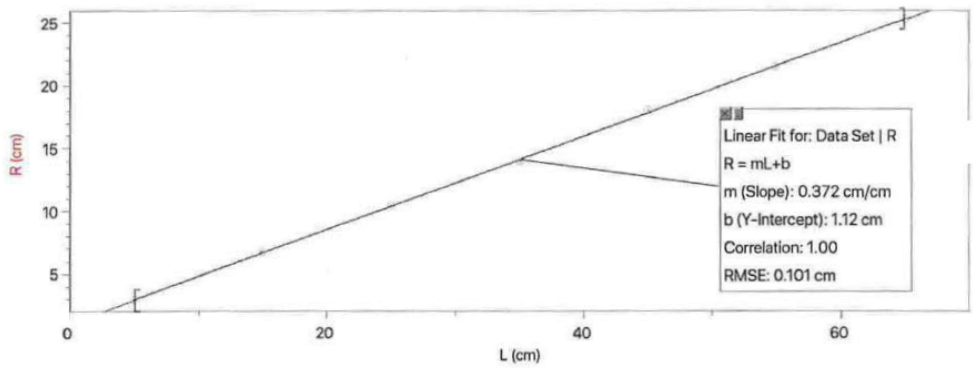
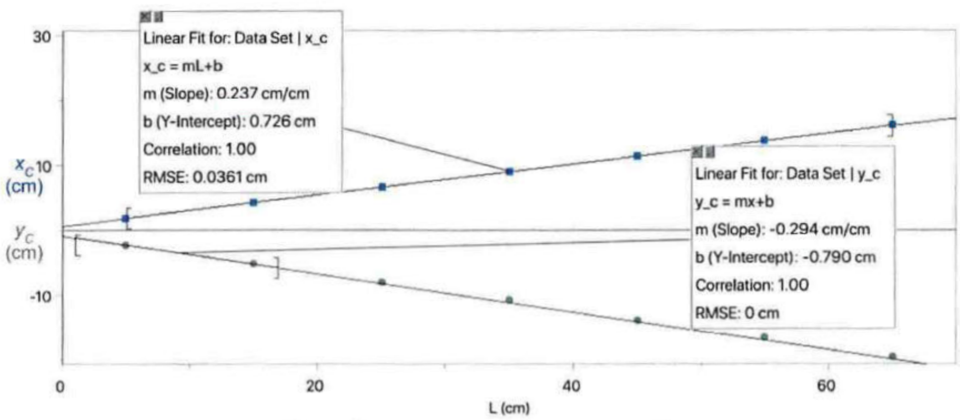


Figure 8. Radius of curvature (R) depending on length of bow section used (L) (string A)

The line of best fit is:

$$R = 0.372L + 1.12$$

To model the bow path on string A, it is not enough to predict the radius of the curvature of the paths. I still need to find out the position of the center of curvature and exactly which arc section of the circle can be used to model the path. To do that, I plot x_c and y_c against L respectively to find the relationship between them.



D: The two lines are not symmetrical, as indicated by the slope values. Candidate could have reflected on this.

Figure 9. x_c and y_c against L (string A)

For similar reasons as explained before, I fit y onto x regressions to the points in each data set in figure 9:

$$x_c = 0.237L + 0.726$$

$$y_c = -0.294L - 0.790$$

Now my model for the string-A path includes the radius of the circle and the position of its center. Next, I need to pick out the exact arc on the circle that is used to model the path, because not the whole circle coincided with the path when fitting (see figure 7). When the bow is on string A, it starts with $\theta = \frac{2\pi}{9}$ and ends with $\theta = \frac{\pi}{9}$, so the starting point of the path is always on the line $y = \tan(\frac{2\pi}{9})x$, and the ending point is always on $y = \tan(\frac{\pi}{9})x$. I can find the intersections between the circle and these two lines to find the arc. This is illustrated in figure 10, where an example of L = 65 cm is used.

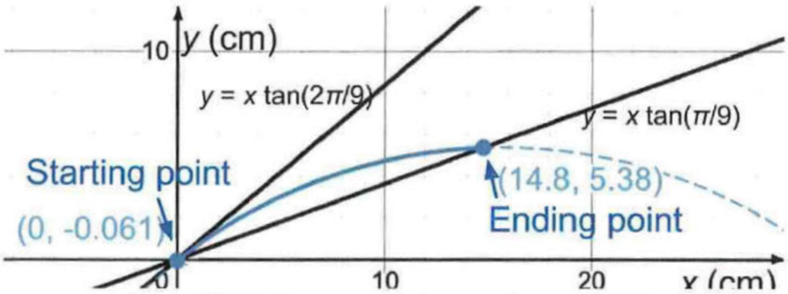


Figure 10. Finding the arc using angular constraints

Using the relationships between R-L, x_c-L, and y_c-L, I found R = 25.3 cm, x_c = 15.7 cm and y_c = -19.9 cm. The equation of the circle in figure 10 is $(x-15.7)^2 + (y+19.9)^2 = 25.3^2$. The starting point and ending point of the arc are annotated on the graph.

To evaluate the accuracy of this model for string A, it is not possible to compare the vertical distance between the modelled path and the actual path by subtracting one from the other, because one is expressed using the cartesian form and one is expressed parametrically. Even if I convert the equation of the circle into parametric form, it is still not possible because at the same t, the x values of the point on the modelled path and the actual path do not match. I can only find the vertical distance (difference in y) as t changes, not the vertical distance (difference in y) as x changes, while the latter one is related to my aim of modelling the path. Therefore, I decide to use the position of the ending points and the gradient at those points to compare the model and the actual path (see figure 11).

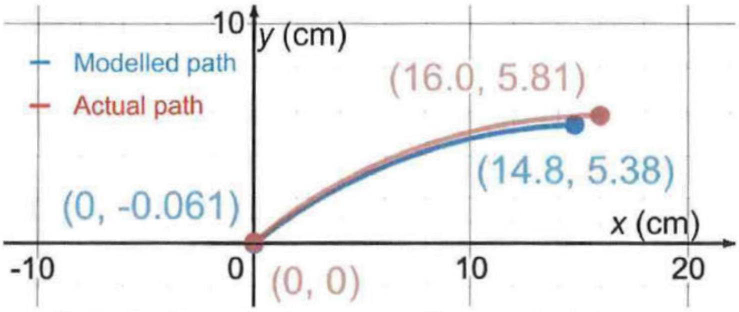


Figure 11. Comparing the model and the actual path (string A)

The distance between the ending points can be calculated using the formula for distance between two points: $\sqrt{(16.0-14.8)^2+(5.81-5.38)^2} \approx 1.27 \text{ cm}$. To see how much the model differs from the actual scenario, this distance can be compared to the distance between the starting and ending point of the actual path: $\sqrt{16.0^2+5.81^2} \approx 17.0 \text{ cm}$. The deviation 1.27cm is only a small percentage of about 7.50% of the actual distance 17.0 cm, so this shows the model is valid.

D: How can validity be justified?
What measure is used for validity?

It is simple to calculate the gradient of the modelled path using the cartesian equation of a circle using rise over run, but to calculate the gradient $\frac{dy}{dx}$ of the actual path the using a pair of parametric equations, I need to use the chain rule:

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{dy}{dt}\times\frac{1}{\frac{dx}{dt}}$$

The gradient can be calculated numerically and the results are in table 2:

Table 2: Comparing gradient at starting and ending points

	Starting point	Ending point
Modelled path gradient	≈ 0.791	≈ 0.0356
Actual path gradient	≈ 0.839	≈ 0.0132

It is evident from table 2 that the starting point gradients are very close to each other. For the ending point, even though the gradients differ relatively greatly, they are both very small and very close to 0. That means both paths are almost flat at the ending point. Overall, the model is still valid. Therefore, I can model the path by finding a best-fit circle to the path using the relationships between R-L, x_C-L, and y_C-L, and then finding the appropriate arc on the circle using the starting and ending angles.

D: The candidate validates decisions that are being made at each stage to help develop the exploration.

Because the relationships between R-L is linear, it seems like as L increases, the circle is stretched. This can be seen clearly from figure 6, which includes multiple string-A paths. This is evident from the parametric equation describing the path on string A:

$$x = \frac{L+3s}{4} t \cos(2\theta_s - \theta_s t)$$

$$y = \frac{L+3s}{4} t \sin(2\theta_s - \theta_s t),$$

where s and θ_s are both constants. As L changes, $\frac{L+3s}{4}$ changes, which is equivalent to multiplying both parametric equations by a factor. Therefore, change in L results in a dilation of the path when the bow is on string A.

This can be confirmed with a graph of x_C against y_C, which indeed shows features of a dilation as L increases, because the centers of the circle line up in a straight line (see figure 12).

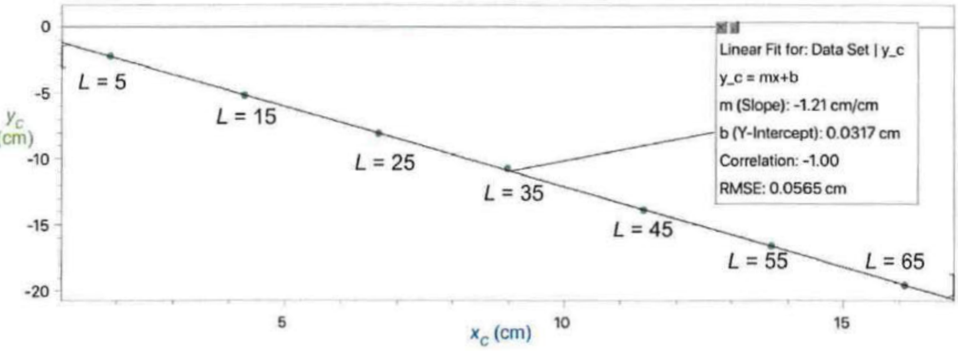


Figure 12. Centers of circle (string A)

Because L causes only a dilation of the string-A path, once the circle fits the path well for one value of L, it should fit the path well for other values of L. This means using linear relationships between R-L, x_C-L, and y_C-L to model the path is valid.

Building the model for string B, C, D

For the other 3 strings, there is not a dilation. As an example, the parametric equation for the string-B path before transformation is (as discussed on page 6 and 7):

$$x = \left(\frac{L-s}{4} + \frac{L+3s}{4}t\right)\cos(\theta_s - \theta_s t) + \cos(\theta_s)$$

$$y = \left(\frac{L-s}{4} + \frac{L+3s}{4}t\right)\sin(\theta_s - \theta_s t) + \sin(\theta_s)$$

Where changing L no longer means multiplying the same factor to the whole equation, because there are two different terms of L, but I can still manually find the relationships between R-L, x_C-L, and y_C-L for string B, as I did for string A (see table 3, figure 13 and figure 14).

Table 3: R, x_c and y_c depending on L (string B)

L (cm)	5	15	25	35	45	55	65
R (cm)	3.19	7.68	12.25	16.50	21.20	26.00	30.75
x_c (cm)	2.42	4.79	7.13	9.53	11.87	14.70	17.16
y_c (cm)	-2.46	-6.12	-9.85	-13.24	-17.11	-21.66	-25.51

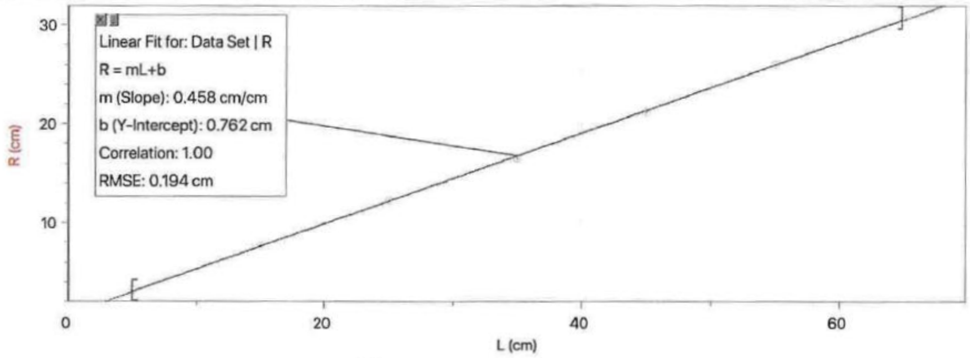


Figure 13. R-L (string B)

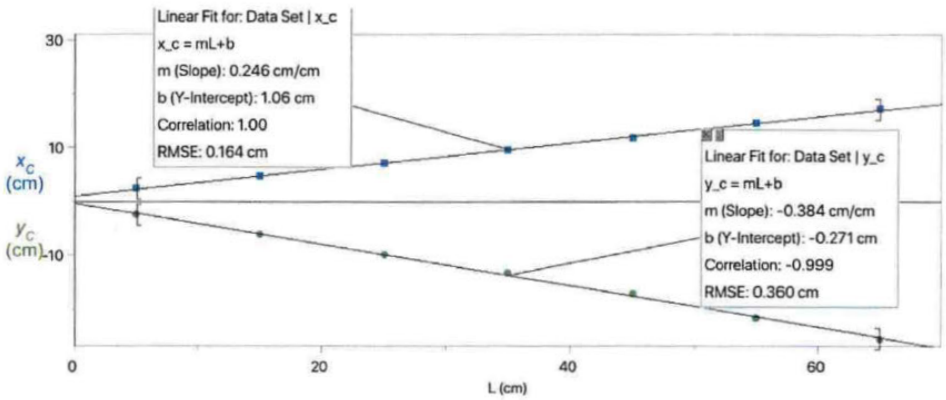


Figure 14. x_c-L, and y_c-L (string B)

Although change in L no longer causes a dilation of the path for string, it is clear in figure 13 and 14 that the relationships between R-L, x_c-L, and y_c-L can still be approximately modelled as linear relationships. To examine this approximation further, I plot the residuals. An example residual plot is shown in figure 15.

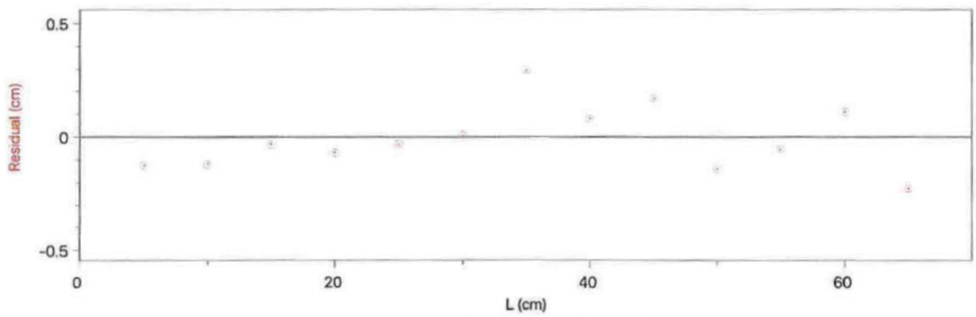


Figure 15. Residuals of R compared to the linear best-fit in figure 13

More situations when L = 10, 20, 30, 40, 50, and 60 cm were included to show the residuals better in figure 15. These new situations do not affect the best fit lines in figure 13 and 14 in any significant way. It can be seen in figure 15 that overall there is no clear trend in the residuals. Although figure 15 seems to suggest that there is a moderate increasing trend and then a weak decreasing trend in the residuals as L increases, I have attempted a quadratic or a cubic fit, which do not make much improvements. Moreover, the residuals are very small (between -0.5cm and 0.5cm) and this indicates the best-fit line is sufficient for my aim of modelling the bow path.

Therefore, I model the path on string B as I did for string A. For string C and D, the model can still approximate the path well enough for those two strings, so I modelled the paths similarly. It is worth noting that due to the translation (see page 6 and 7), for strings B, C, and D, the lines defining the arc on the circle fits should depend on the angle θ (like for string A, on page12), but should also depend on the translation. Figure 16 shows all the lines that define the starting and ending points of each section of the bow path. The path is also drawn with each section in different colors to clarify the idea.

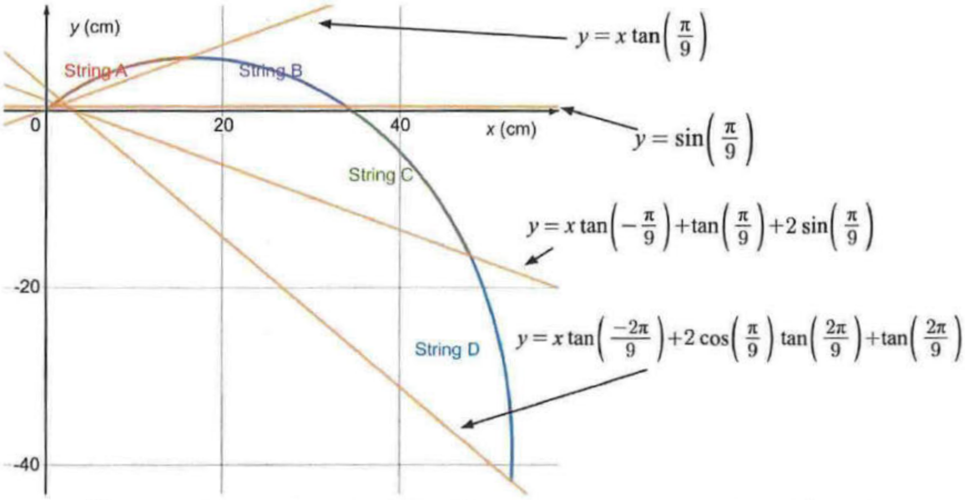
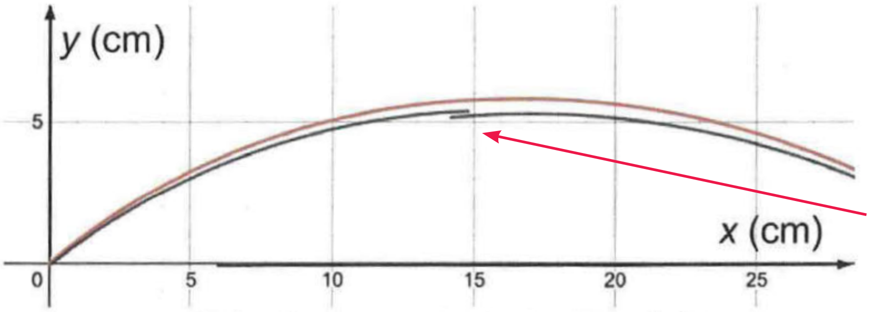


Figure 16. Lines defining starting and ending points of each section of the path

Prediction using the complete model

The complete model consists of equations describing relationships between R-L, x_C-L, and y_C-L. For conciseness, the equations are not repeated here. Using the complete model, I make a prediction of the path when L = 65 cm (continuing with the prediction discussed for string A). A section of the predicted path (in black) and the actual path (in red) are shown in figure 17.



D: What about the discontinuity here?

Figure 17. A section of the predicted path and the actual path

As shown in figure 17, a limitation of this model is at the connection points. When the bow path on different strings should connect to each other. In other words, the path should be continuous. However this is not the case because the starting and ending points of each section of the path are decided by the angle of the bow. The use of circles to model the bow path also becomes a bit more problematic as the change in L does not lead to a dilation of the path for string B, C and D, as discussed previously. Overall, the modelled path has a similar shape as the actual path, but the modelled path tends to be on the "inner side", leading to an underestimation of the length of the path.

D: Meaningful reflection.

Conclusion

In this exploration, I found that the path of a bow can be described using parametric equations similar to the following two, which are the equations for the path when the bow is on string A:

$$x = \frac{L+3s}{4} t\cos(2\theta_s - \theta_s t)$$

$$y = \frac{L+3s}{4} t\sin(2\theta_s - \theta_s t)$$

To allow for better intuitive understanding of the path, which is helpful when I play the violin, I used circles to fit to the paths and used the appropriate arc from the circle to model the path. For the aim of this exploration, it is valid to approximate the relationships between R-L, x_c-L, and y_c-L, and the modelled path is generally consistent with the actual path, as shown in the last section.

Works Cited

Weisstein, Eric W. "Osculating Circle." *Wolfram MathWorld*, mathworld.wolfram.com/OsculatingCircle.html. Accessed 8 Jan. 2021.

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