How CAN we make it better?

Designing the optimal can

IBDP Mathematics

Analysis and Approaches HL –Exploration

Introduction and Rationale:

Be it a tiring day at school, an exhausting session of sports, or simply when I want to enjoy in a party, my go to option is a soft drink. Soda drinks can refreshen a person with the very first sip, and since they are non-alcoholic, can be enjoyed by people of any ages. I have grown up drinking a few soda drinks that now seem to have established an emotional connect with me.

While drinking my favorite beverage- coca cola- from a can I noticed that the bottom is curved. It was something every can has, but I had never really thought why. I, being someone obsessed with understanding the working of things (the same reason I aim to pursue mechanical engineering in college), got interested, and saw the perfect opportunity for my mathematics exploration.

My tendency to observe and trying to unpack the reasons for why things are the way they are, had me question myself: ‘why are all these cans identical? Why a cylinder? Why the same opening lever?

But it did not stop here, and the creative part of my brain kicked in, making me wonder, how could the can be improved? Of course, this design of the can has been used for a long time, but I believe there must be some way to make it better: in terms of serving the product, its aesthetics, its usability, its packaging and its delivery.

And hence, cans became my area of focus for my mathematics internal assessment, as I saw an opportunity to explore real life scenarios using differential calculus, various types of geometric applications including derivations, along with applications of mathematical software and other aspects of mathematics. This huge array of applications of mathematics to a single topic is what constitutes my love for the subject, and hence I popped open my entry into the world of cans.

I started my process by creating a mind map relating to the topic. Doing so enabled me to think about my approach towards the research. I brainstormed and listed all the possible explorations I could investigate. (Attached Below)

A: This is too wordy. Not all of this is relevant. Although a rationale is no longer mandatory, when well written it helps to put the exploration into perspective.

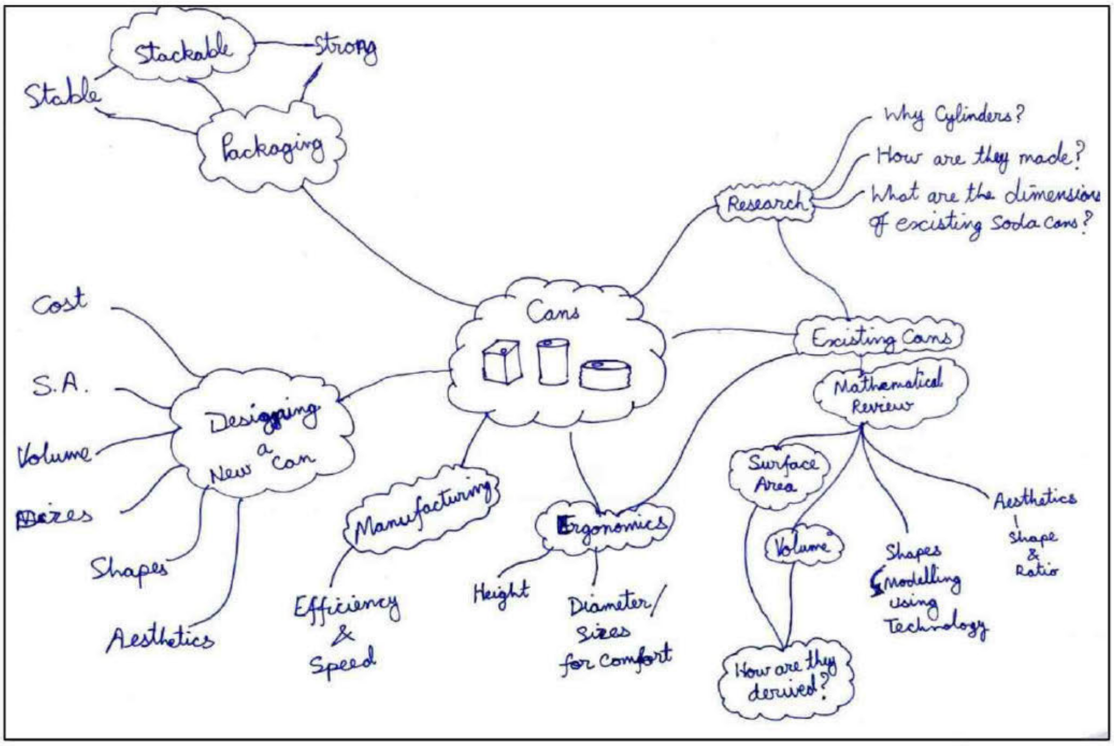

Figure 1: Self Made Mind Map

Aim and Approach:

The aim of this investigation is to understand the mathematics involved in currently produced cans and using it to design a better can. This will include learning the derivation of the formula of surface area for a figure formed by rotation, investigation of the volume, surface areas and other features of existing cans, and the designs will try to maximize the volume and minimize the surface are, while also taking account of better aesthetics and ergonomics.

A: The aim given seems to be too broad.

My action plan:

1) I will first research about cans- their material, the functions of different features, and the process of production.
2) I will then find data about common sizes of cans, which will be used to set a parameter for the developed design.

3) Thereafter, I will research and develop the proof for the surface area formula for a figure formed by rotation along the x axis.
4) Then I will search for specially shaped cans. The research would be specific to beverage cans, but other food cans with exceptional shapes may be considered as well, to see how the variations in design are used to balance the surface area, volume, and aesthetics. Using the software GeoGebra, I will model functions for the edges of these cans.
5) I will then integrate these functions according to the formulae and find the surface area and volume when the obtained function is rotated to form the can.
6) I will develop ideas for a new can, with regards to my understanding from the other cans, ergonomics, and aesthetics, and then use my knowledge to mathematically analyze them.
7) Based on all considerations, the final design will be chosen.

Background Research:

My instinctive move to start research was to get to know about cans, and hence I watched a few videos on YouTube regarding the engineering of a can and the production of a can.

Material: Soda cans are made of aluminum, a cheap, light weight, yet strong metal. They are produced at nearly 15000 cans per second! The production is huge: more than 100 billion cans of aluminum are produced every year, and therefore even the slightest amount of material used has a huge impact in terms of economy and saving the material.[1]

Bottom Dome: I also understood that the dome at the bottom of the can is to help reduce the required material, and to add strength; a dome is a rotated arch, and arches have greater tensile strength than straight beams.[2]

C: Research done by the candidate is acknowledged.

Mathematical reasoning behind cylindrical shape:

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[1] Hammack, Bill. "The Ingenious Design of the Aluminum Beverage Can." YouTube. Engineerguyvideo, 14 Apr. 2015. Web. 27 Sept. 2020. <https://www.youtube.com/watch?v=hUhisi2FBuw>

[2] IBID

1) Volume and packaging: Spheres are objects that have the maximum volume for the least surface area.[3] This can be proven analytically by comparing the surface area to volume ratio of a sphere to any other geometric solid, where the sphere yields the least ratio.
2) However, even when they are packed as closely as possible, they can only occupy 74% of the actual space. This is proven by the Hale's proof, a recently formed proof regarding the Kepler's Conjecture.[4]
3) While cuboids solve this problem, as they occupy all space in a cuboid box (100% space), they have a weakness of having an uncomfortable grip, and lack strength due to their sharp edges which cannot bear the pressure of an aerated drink.[5]
4) Therefore, the cylinder was chosen. A combination of spherical and rectangular properties. Being circular, there are no edges that become weak pressure points, and the rectangular shape ensures there is less waste of space and is better to hold.
5) Ease of production: A cylinder is easy to produce; the base requires a single stroke of a extruding machine on a single disk of aluminum.[6] The lesser the edges, the easier it is to extrude the metal without it breaking.

A: Although no penalty is applied here, all sources need to be cited at point of reference.

D: Some meaningful reflection to justify choice of cylinder is seen.

However, I also read facts regarding packaging that changed my perspective. Beer is a famous alcoholic beverage in the can industry, and I read that approximately 70% of the people that buy beer choose their brand at the aisle in a store.[7] This implies that it is of utmost importance to make sure the cans are stand out- and therefore aesthetics is a major driver of can design as well.

Along with this, another crucial factor is that of ergonomics. The openers of cans are engineered in a way that most people can open them easily.[8] However, improving the

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[3] Hammack, Bill. "The Ingenious Design of the Aluminum Beverage Can." YouTube. Engineerguyvideo, 14 Apr. 2015. Web. 27 Sept. 2020. <https://www.youtube.com/watch?v=hUhisi2FBuw>

[4] Weisstein, Eric W. "Kepler Conjecture." From Wolfram MathWorld. Web. 13 Oct. 2020. <https://mathworld.wolfram.com/KeplerConjecture.html>.

[5] Hammack, Bill. "The Ingenious Design of the Aluminum Beverage Can." YouTube. Engineerguyvideo, 14 Apr. 2015. Web. 27 Sept. 2020. <https://www.youtube.com/watch?v=hUhisi2FBuw>

[6] Rexam's Full Circle Film - the Lifecycle of an Aluminium Can. Dir. RexamPlc. YouTube. YouTube, 08 Nov. 2012. Web. 25 Oct. 2020. <https://www.youtube.com/watch?v=7dK1VVtja5c>.

[7] "DESIGN AUDIT SERIES." Nielsen. The Nielsen Company. Web. 30 Oct. 2020. <http://innovation.nielsen.com/craft-beer-audit-2016/craft-beer-audit-74KG-57426.html#iow93fJu6EKP3a8y3qLUhQ>.

[8] Hammack, Bill. "The Ingenious Design of the Aluminum Beverage Can." YouTube. Engineerguyvideo, 14 Apr. 2015. Web. 27 Sept. 2020. <https://www.youtube.com/watch?v=hUhisi2FBuw>

shape to assist a better grip, apart than just an ideal grip width, will make the experience much better for customers. This research gave me an understanding of cans in a better way and gave me ideas for the following steps.

Relevant Mathematical formulae:

Volume (V):

In my DP course, I have already studied the formula for finding the volume of a figure created by revolving a function along the x axis and the y axis. For a function $f(x)$, defined between $a < x < b$ revolved 2π along the x axis, the volume V is given by:

$$V = \pi \int_a^b \left(f(x)\right)^2 \times dx$$ [9]

A: There is no need to cite the source of this formula, as it is part of the syllabus content.

Surface Area (SA):

Note: The following proof is inspired from the source cited on page number 8.

This was a concept that was beyond my syllabus, and I studied this in depth to develop a profound understanding.

To prove this, consider the following diagram:

[9] Dawkins, Paul. "Calculus I - Volumes of Solids of Revolution / Method of Rings." Paul's Online Notes. Web. 1 Nov. 2020. <https://tutorial.math.lamar.edu/classes/calci/volumewithrings.aspx>.

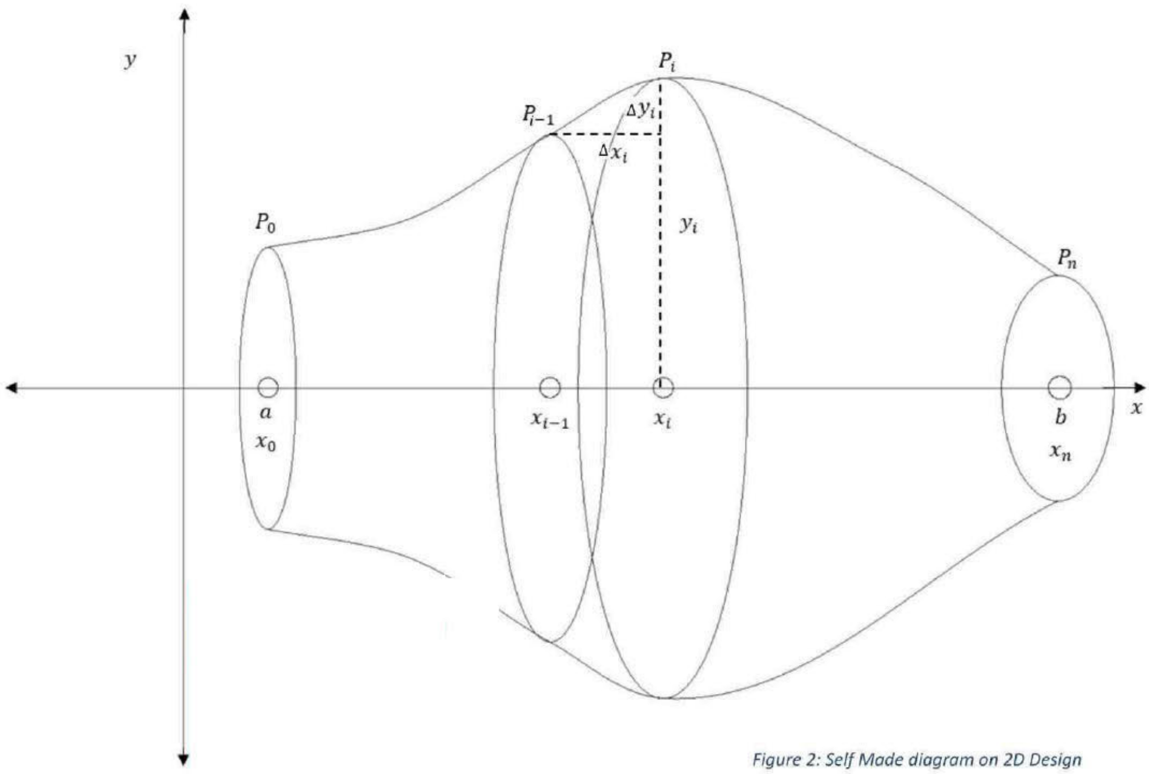

Figure 2: Self Made diagram on 2D Design

Consider the surface obtained by rotating the curve $y = f(x)$, for $a \leq x \leq b$, about the x axis, where f is positive and has a continuous derivative. The diagram above shows such a surface.

Divide the interval $[a, b]$ into n number of subintervals and let these have endpoints at
$x_0, x_1, \ldots, x_n$, each of an equal width Δx.

Let a point $P_i(x_i, y_i)$ lie on the curve; let $y_i = f(x_i)$. The part of the surface between x_{i-1} and x_i can then be approximated by taking the line segment from $\overline{P_{i-1}P_i}$ and rotating it 2π around the x axis.

This results in the formation of a frustrum, with a slant height $l = |\overline{P_{i-1}P_i}|$, and an average radius $r = \frac{1}{2}(y_{i-1} - y_i)$. Hence, the surface area (SA) is given by:

E: The candidate chose to derive the formula for SA generated by rotation. A full derivation is not necessary. Candidates need to demonstrate understanding of all the mathematics used.

$$SA = 2\pi r l = 2\pi \frac{(y_{i-1} + y_i)}{2} |P_{i-1}P_i| \qquad ---(1)$$

Using the Pythagoras Theorem,

$$|\overline{P_{i-1}P_i}| = \sqrt{\Delta x_i{}^2 + \Delta y_i{}^2} = \sqrt{\Delta x_i{}^2 + (f(x_i) - f(x_{i-1}))^2}$$

According to the mean value theorem, there is a point $x_i{}^*$ between x_i and x_{i-1} such that

$$f'(x_i{}^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$ [10]

$$\Rightarrow f(x_i) - f(x_{i-1}) = \Delta x_i f'(x_i{}^*)$$

$$\therefore |\overline{P_{i-1}P_i}| = \sqrt{(\Delta x_i)^2 + (\Delta x_i)^2 \left(f'(x_i{}^*)\right)^2}$$

$$= \Delta x_i \sqrt{1 + \left(f'(x_i{}^*)\right)^2}$$

From equation (1):

$$SA = \pi(y_{i-1} + y_i)\Delta x_i \sqrt{1 + \left(f'(x_i{}^*)\right)^2}$$

$$\therefore Total\ Surface\ Area = \sum_{i=1}^{n} \pi\left(f(x_i) + f(x_{i-1})\right)\left(\Delta x_i \sqrt{1 + \left(f'(x_i{}^*)\right)^2}\right)$$

$\because \Delta x_i$ is extremely small,

$\therefore y_i = f(x_i) \approx f(x_i{}^*)$, and $y_{i-1} = f(x_{i-1}) \approx f(x_i{}^*)$

[10] Dawkins, Paul. "Calculus I - The Mean Value Theorem." Calculus I - The Mean Value Theorem. Web. 5 Nov. 2020. <https://tutorial.math.lamar.edu/classes/calci/MeanValueTheorem.aspx>.

$$\therefore SA = \sum_{i=1}^{n} \pi\left(f(x_i^*) + f(x_i^*)\right)\left(\Delta x_i\sqrt{1 + \left(f'(x_i^*)\right)^2}\right)$$

$$SA = \sum_{i=1}^{n} 2\pi\left(f(x_i^*)\right)\left(\Delta x_i\sqrt{1 + \left(f'(x_i^*)\right)^2}\right)$$

$$\because \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi\left(f(x_i^*)\right)\left(\Delta x_i\sqrt{1 + \left(f'(x_i^*)\right)^2}\right) = \int_a^b 2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2} \times dx$$

Hence for a given function $y = f(x)$ between points a and b, rotated 2π about the x axis:

$$Surface\ Area = 2\pi \int_a^b f(x) \times \sqrt{1 + \left(f'(x)\right)^2} \times dx$$ [11]

E: The derivation is correct, but it is merely bookwork transcribed into the exploration. A full proof was not necessary—an explanation to justify the formula would have sufficed.

Data Collection

Standard Sizes: My next move, as planned, was to collect standard can sizes. I found the following data on the website of a can production firm:

UNIT SIZE	HEIGHT CLOSED	BODY DIAMETER
330 ml	115.2 mm	66.3 mm
440 ml	149.9 mm	66.2 mm
500 ml	168 mm	66.2 mm
568 ml	188.1 mm	66.2 mm

Table 1: Data Taken From: https://www.ball.com/eu/solutions/markets-capabilities/capabilities/beverage-cans/standard-range. For all citations refer to page no.21.

The data shows various sizes, but the company itself states that the standard or "iconic size" of drinks is the 330 ml can, the surface area of which is approximately 239.5 cm^2 (using the data above in the formula $SA = 2\pi rh$) I will take this into consideration for my design as well.

[11] Dawkins, Paul. "Calculus II - Surface Area." Paul's Online Notes. 30 May 2018. Web. 5 Nov. 2020. <https://tutorial.math.lamar.edu/classes/calcii/surfacearea.aspx>.

Existing Can Shape Analysis:

Designing a newer version of any product requires a detailed study of the original products, and since cans are a type of classic product; they haven't changed a lot over a large period, it is absolutely essential to understand their mathematics. This would help me with details including the sizes for hand grip, reasons for certain features such as the curved bottom surface, how the surface area is minimized, volume maximized, etc. that would help me create a new product that is more efficient.

To study pre-existing cans, I initially thought of personally exploring supermarkets and picking different types of cans and measuring them. However, this plan was not possible due to the covid-19 pandemic. Instead, I decided to surf the internet for pictures of interesting cans and cans already at home. This was safe and more importantly provided me with a wider range of different cans.

A point to consider here was that the sources of the images of these cans did not reveal their actual volumes, or their dimensions. Therefore, the best option for me was to take the height of the common 330 ml can, which is equal to 11.5 cm, for all the cans. Another limitation because of the same reason was that the dome beneath the can would be ignored and assumed to be flat as different cans have different depths of domes, and these were not visible in the images.

A: Too wordy on this page.

I started by putting one of the cans on a software called "GeoGebra." I put a line on the x axis of length 11.5, rotated the image and set the height of the can equal to the line. I then started plotting the points on the curve of the can. I copied the coordinates and made a table on the software. This table graphed the points and then enabled me to try different functions for the points. I carefully went through all possible modelling options and chose the seemingly best fit.

I wanted to limit the number of significant figures, however, while using the software I realized that very minute changes also caused great variation in the shape of the graph because the powers were great. I hence fixed the software accuracy to 10 decimal places, which helped me make precise and accurate calculations. However, I was able limited my answers to 6 significant figures.

D: Some reflection is noted on the specification and size of cans before starting the modelling process.

The process can be seen here:

Can 1:

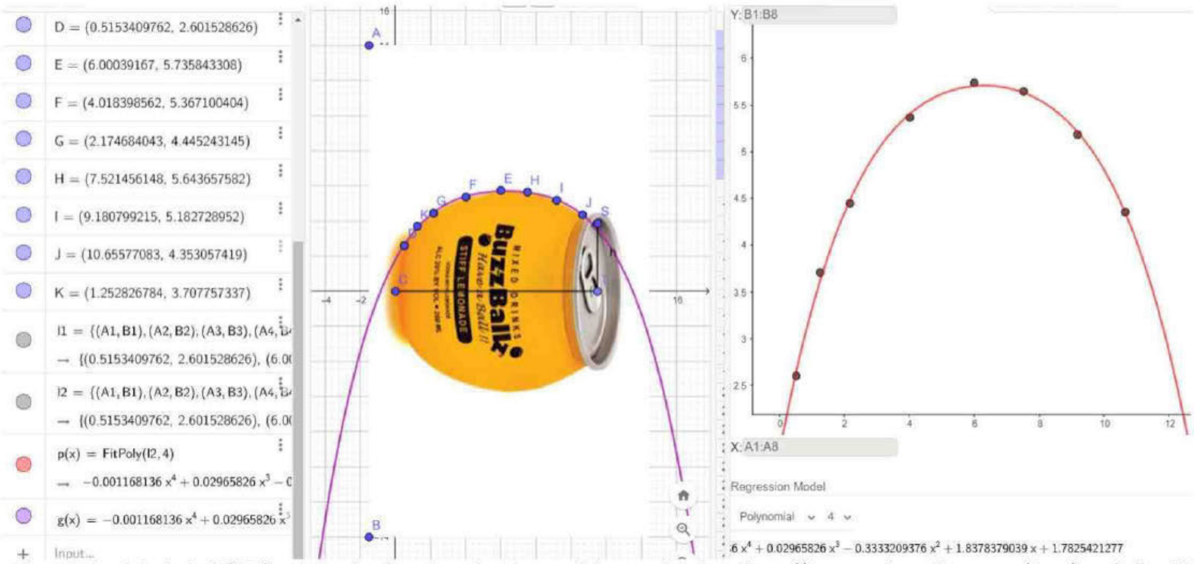

Figure 3: Can 1 Analysis. Self-Taken screenshot from Geogebra. Image of the can taken from: https://bremerswineandliquor.com/shop/buzz-ballz-stiff-lemonade/

B: Good use of technology.

A quartic function was derived for this can. I saw the single curvature and tried the first polynomial fit with degree 2, however, the curve fit the points best when I increased the power to 4 as seen in the graph towards the right. I took the first point a little further from $x = 0$, as that part does not account for the volume (the image is slightly angled, and at $x = 0$ extra metal at the base is shown.

D: Candidate chose best fit visually rather than by a quantitative GOF analysis.

$$f(x) = -0.0011681360\,x^4 + 0.0296582600\,x^3 - 0.3333209376\,x^2 + 1.8378379039\,x + 1.7825421277$$

I hence applied the formula the Volume:

$$V = \pi \int_{0.5153409762}^{11.5} (-0.0011681360x^4 + 0.0296582600x^3 - 0.3333209376x^2 + 1.8378379039x + 1.7825421277)^2\,dx$$

$$\therefore V = 872.528\ cm^3$$

B: This should be an approximation sign.

$SA\ of\ body$

$$= 2\pi \int_{0.5153409762}^{11.5} ((-0.0011681360x^4 + 0.0296582600x^3 - 0.3333209376x^2$$
$$+ 1.8378379039x$$
$$+ 1.7825421277)(\sqrt{1 + (-0.0046725440x^3 + 0.0889747800x^2 - 0.666642x + 1.8378379039)^2}))dx$$

$$= 400.879\ cm^2$$

$$SA\ of\ the\ top\ and\ bottom\ circles = 2\pi\big(f(0.5153409762) + f(11.5)\big)$$
$$= 2\pi(2.645110018 + 3.511792826) = 38.6849cm^2$$

B: Consistent misuse of equality sign when it should be approximately equal to.

$$Total\ SA = 439.564cm^2$$

E: This gives the circumference of the two circles, not the area.

Analysis: I chose this can because it is spherical, which gave the opportunity to see how effectively the sphere maximizes volume. The values obtained are clearly not realistic for a drink can, but this is because it is extended to fit the general height. The actual volume is 200 cm^2 as given on the image. This shows how drastically the volume increased. The surface area to volume ratio is almost half (1:1.985).

For Can 2, I had to break the model into pieces:

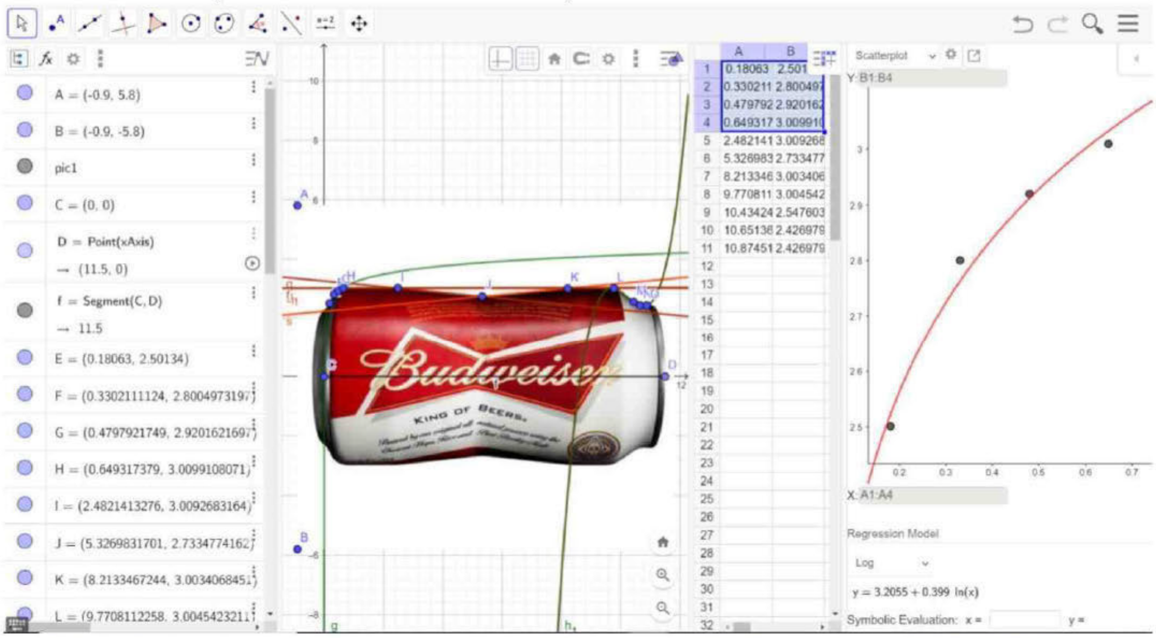

Figure 4: Can 2 Analysis. Self made on geogebra Image of can taken from: https://www.grubstreet.com/2013/04/budweiser-bowtie-can.html

I carefully divided the curve wherever I saw a change in the nature of the curvature, and individually selected the best fit models. Again, I took the first point a slightly further from $x = 0$, and the last point before $x = 11.5$, as it was an extra part because of the curvature in the image. For $x = 0.18$ to $x = 0.65$, a natural logarithmic curve was the best fit. After this, there were 4 linear segments, which I divided according to the changes in curvature. From $x = 8.21$ to $x = 9.77$, a segment of a cubic function fit best. Using the following, I created the following piece-wise function:

D: Candidate does not justify why this level of accuracy is being used.

$$f(x) = \begin{cases} y = 0.3989877366\ln(x) + 3.2055253166, & 0.1806300000 \le x < 0.6493173790 \\ y = -0.0003505469x + 3.0101384233, & 0.6493173790 \le x < 2.4821413276 \\ y = -0.0969441943x + 3.2498975074, & 2.4821413276 \le x < 5.3269831701 \\ y = 0.0935188599x + 2.2353040233, & 5.3269831701 \le x < 8.2133467244 \\ y = 0.0022244642x + 3.0216771407, & 8.2133467244 \le x < 9.7708112258 \\ y = 1.0133520968x^3 - 31.1250243741x^2 + 317.8181328971x - 1076.1358903522, & 9.7708112258 \le x < 10.8745178331 \end{cases}$$

I individually applied the formula on these functions and added them to find total volume (individual integrals not shown as they are extremely lengthy):

$$V = 11.8302 + 52.1537 + 73.743 + 74.6644 + 44.0855 + 24.7091 = 281.186\ cm^3$$

Similarly, I applied the SA formula as well:

$$SA\ for\ body = 12.394 + 34.6584 + 51.5655 + 52.2478 + 29.7655 + 21.574$$

$$= 202.205\ cm^2$$

$$SA\ for\ top\ and\ bottom\ circles: 2\pi\big(f(0.18063) + f(10.8745178331)\big)$$

$$= 2\pi(2.522735792 + 2.426979856) = 31.1000\ cm^2$$

$$Total\ Surface\ Area = 233.305\ cm^2$$

Analysis: This can is a very clever design, because of multiple reasons. This can uses the dimensions of the generally produced cans, tweaking it a little bit to make it bent in a bow shape. By doing such a minor change they have made the can extremely attractive (the indent gives it a good look, while reducing the volume and the surface area of the original can by a little margin. The surface area to volume ratio for this is 1:1.205

D: The candidate gives the SA to volume ratio but does not discuss its significance.

I repeated the process for another different can that I found aesthetically attractive:

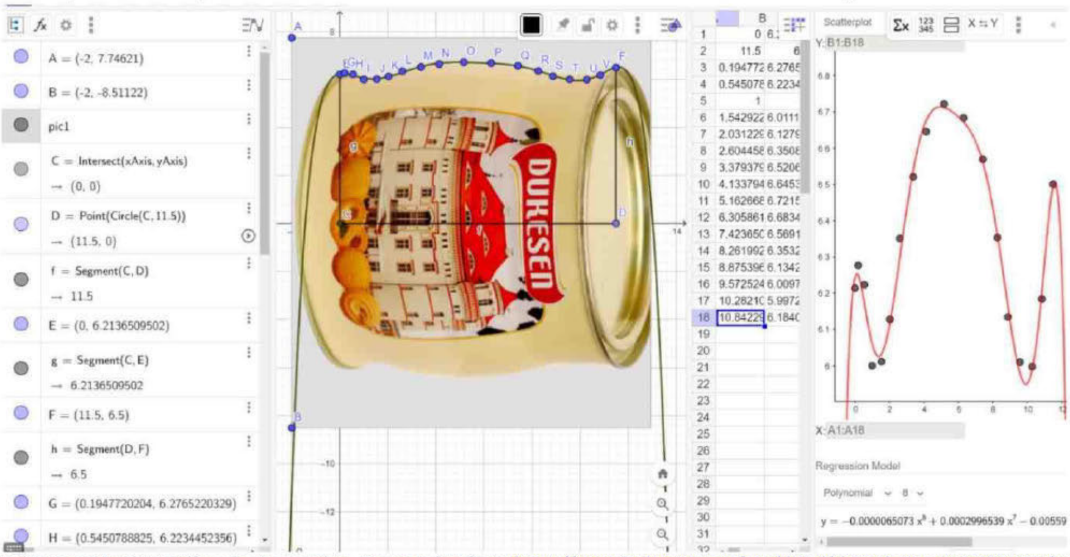

Figure 5: Can 3 Analysis.Self made on geogebra. Picture taken from: https://www.indiamart.com/proddetail/shaped-cans-19369991855.html

For this can, I first tried trigonometric functions, as I saw a periodic nature. But since the correlation between the points and the function was not strong, I tried polynomial functions as well and saw that following degree 8 function was fitting the best (as can be seen above). Also, I fit the first point in this to $x = 0$ by shifting the extra part behind the y axis. The last point was also fit at $x = 11.5$ as shown.

D: No evidence is seen to justify this claim.

$$f(x) = -0.0000065073x^8 + 0.0002996539x^7 - 0.0055995535x^6 + 0.0544098938x^5 - 0.2904533131x^4 + 0.80899426x^3 - 0.9388126885x^2 + 0.1888793286x + 6.2425972526$$

$$V = \pi \int_0^{11.5} (-0.0000065073x^8 + 0.0002996539x^7 - 0.0055995535x^6 + 0.0544098938x^5 - 0.2904533131x^4 + 0.80899426x^3 - 0.9388126885x^2 + 0.1888793286x + 6.2425972526)^2 \times dx$$

$$V = 1467.69\ cm^3$$

$SA\ of\ body$

$$= 2\pi \int_0^{11.5} \begin{pmatrix} -0.0000065073x^8 + 0.0002996539x^7 - 0.0055995535x^6 + 0.0544098938x^5 \\ -0.2904533131x^4 + 0.80899426x^3 - 0.9388126885x^2 + 0.1888793286x + 6.2425972526 \end{pmatrix}$$

$$\times \sqrt{\left(1 + \begin{pmatrix} -0.0000520584\,x^7 + 0.00209752773x^6 - 0.0335973x^5 + \\ 0.272049x^4 - 1.16181x^3 + 2.42698x^2 - 1.87763x + 0.188879 \end{pmatrix}^2\right)} \times dx$$

$$= 471.862\ cm^2$$

$$SA\ for\ top\ and\ bottom\ circles = 2\pi\big(f(0) + f(10.8422937393)\big) \times$$

$$= 2\pi(6.24260 + 6.51023) = 80.1284\ cm^2$$

$$Total\ Surface\ Area = 551.987\ cm^2$$

Analysis: This can is a food container, chosen because of its curvy body which makes it extremely attractive. While this is not to scale, it can again be seen that the curvature in the middle, affected the volume much more than the surface area, as seen with the other cans. The surface area to volume ratio is 1: 2.66, which is the best out of the three cans.

E: In spite of the ratios found being incorrect due to the repeated error, no mention is ever made of their significance.

From this research I reflected over my learning and made decisions for my final can :

- Aesthetics: Indents in the shape and curvy body of a can helps in making the can look very attractive. Hence I decided to add both of these elements in my cans.
- Ergonomics: The indents make gripping the can easy, hence are a must.
- Sphere: The spherical nature of the cans showed their effectiveness in increasing the volume, hence these were critical to add.
- Since the best Surface Area to Volume ratio was found for the curvy can, a curvy element must be included in the design.

With these specifications in place, I started developing initial ideas for my can.

My Designs:

Figure 6: Design 1. Self Made.

C: Personal engagement is seen on this page.

The first design is simple looking, but a much more efficient design for a can in terms of ergonomics. The added indent increases the holding ability, as it would feel like it has been squeezed and fit more comfortably in the hand.

However, this design is thin at the center, but does not incorporate a spherical feature, which makes it inefficient in terms of area to volume ratio. I thus made my next idea:

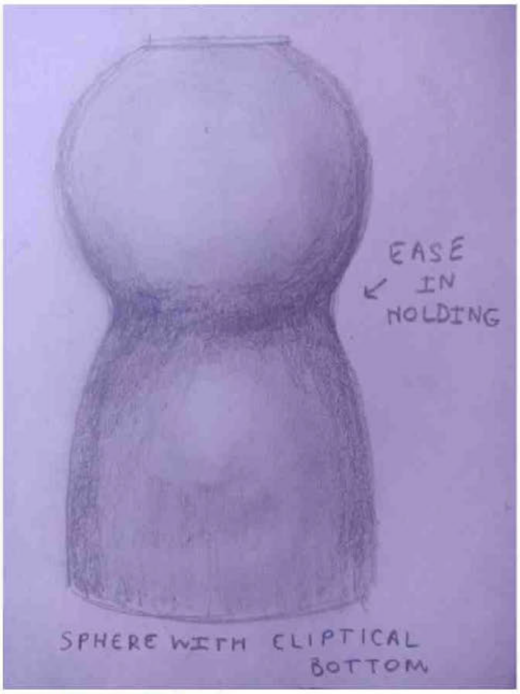

Figure 7: Design 2 Self Made

I tried to incorporate all the elements in this can. The sphere at the top, the indent at the center, and an overall curvy shape.

However, this can was not extremely pleasing in terms of looks. I then decided to build further on it, and came up with a design with a helix:

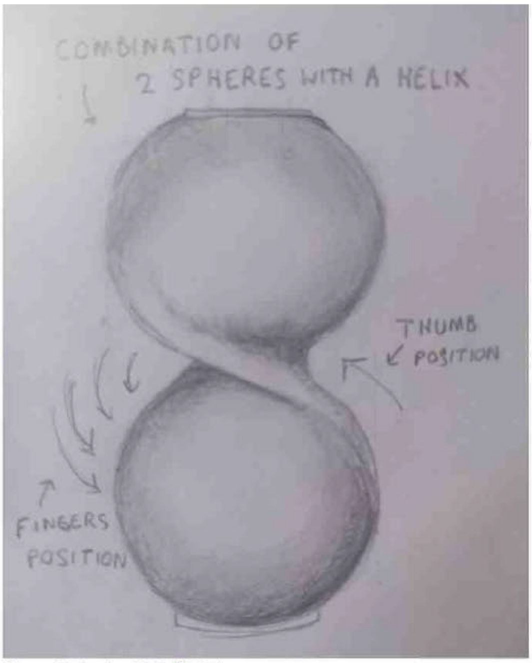

Figure 8: Design 3, Self Made

Not only does this design have 2 spheres, but the helical connection makes it look very attractive. It also acts as an effective ergonomic design, as the thumb can fit in the slot while fingers can wrap around the sphere.

However, I realized that the addition of the helical strips will add unnecessary surface area without any significant volume addition.

This fulfilled every other criteria, so I came up with the final design:

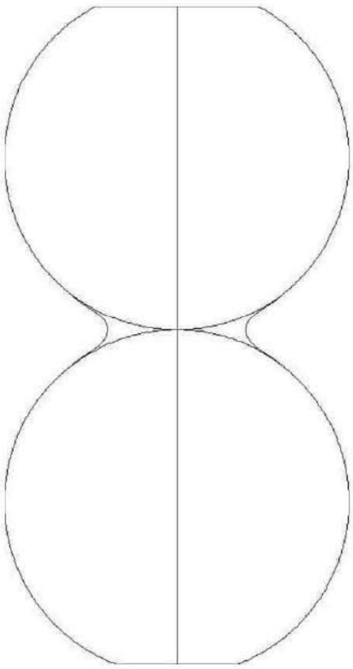
Figure 9:Final Can Self Made on 2D design

I was designing this can in competition to the ones available on market, and thus kept the same height as the general cans: 11.5 cm. The design was made initially using perfect circles, and is in scale, thus it was adjusted according to the height and the points were plotted:

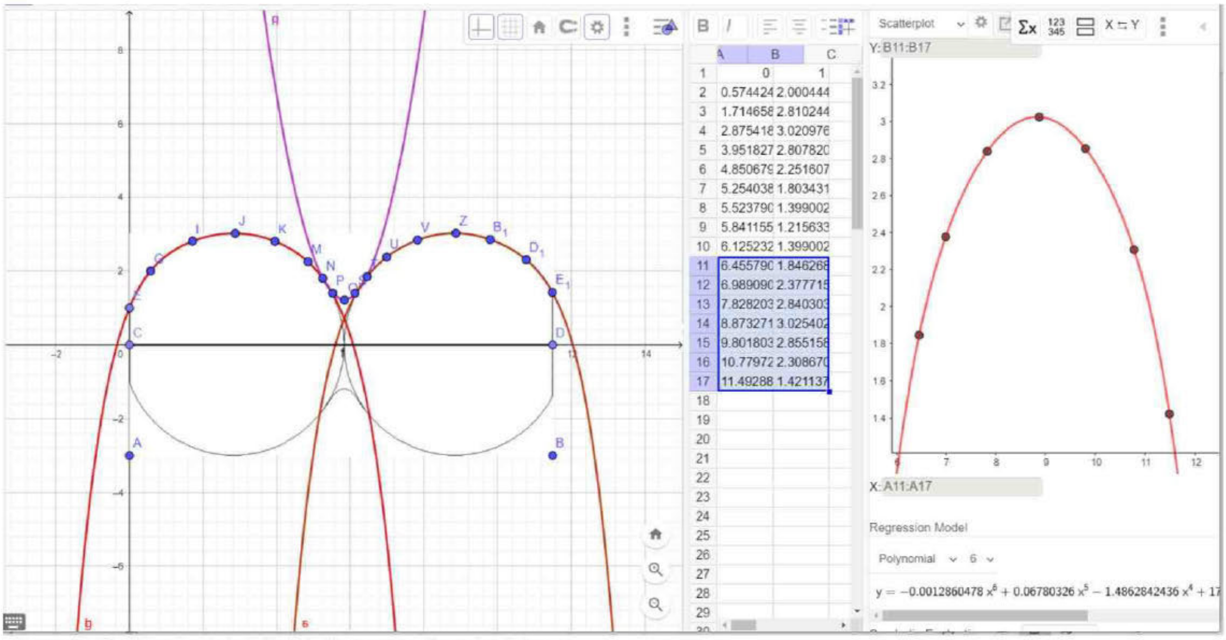

Figure 10: Final Can Analysis. Self-clicked screenshot from GeoGebra.

✓B

Function derived:

$$f(x) = \begin{cases} -0.0011547607x^6 + 0.0204214599x^5 - 0.1507224376x^4 + 0.589633358x^3 - 1.4478601068x^2 \\ \quad +2.40519241x + 1, & 0 \le x < 5.2540381835 \\ 1.6334026291x^2 - 19.0790436377x + 56.9508203419, & 5.2540381835 \le x < 6.4557909536 \\ \quad -0.0012860478x^6 + 0.0678032600x^5 - 1.4862842436x^4 + 17.3406661060x^3 \\ -113.7659238265x^2 + 399.3001008385x - 585.7315131452, & 6.4557909536 \le x \le 11.5 \end{cases}$$

B: This is not the best mathematical presentation.

$$Volume:$$

$$\pi \int_0^{5.2540381835} (-0.0011547607x^6 + 0.0204214599x^5 - 0.1507224376x^4 + 0.589633358x^3 - 1.4478601068x^2 + 2.40519241x + 1)^2 \times dx = 111.999$$

$$\pi\int_{5.2540381835}^{6.4557909536}(1.6334026291x^2-19.0790436377x+56.9508203419)^2\times dx=7.88475$$

$$\pi\int_{6.4557909536}^{11.5}\begin{pmatrix}-0.0012860478x^6+0.0678032600x^5-1.4862842436x^4+17.3406661060x^3\\-113.7659238265x^2+399.3001008385x-585.7315131452\end{pmatrix}^2$$
$$\times dx=110.247$$

$$V=111.999+7.88475+110.247=230.131\ cm^3$$

A/E: All these definite integrals could have been evaluated using technology. No penalty for manual evaluation.

$$Surface\ Area:$$

$$2\pi\int_{0}^{5.2540381835}\begin{pmatrix}-0.0011547607x^6+0.0204214599x^5-0.1507224376x^4+0.589633358x^3\\-1.4478601068x^2+2.40519241x+1\end{pmatrix}$$
$$\times\sqrt{1+\begin{pmatrix}-0.00692856x^5+\ 0.102107\ x^4-0.60289x^3+1.7689x^2\\-2.8957202136x+2.40519\end{pmatrix}^2}\times dx=100.089\ \ cm^2$$

$$2\pi\int_{5.2540381835}^{6.4557909536}(1.6334026291x^2-19.0790436377x+56.9508203419)$$
$$\times\sqrt{1+(3.2668052582x-19.0790436377)^2}\times dx=16.3663\ \ cm^2$$

$$2\pi\int_{6.4557909536}^{11.5}\begin{pmatrix}-0.0012860478x^6+0.0678032600x^5-1.4862842436x^4+17.3406661060x^3\\-113.7659238265x^2+399.3001008385x-585.7315131452\end{pmatrix}$$
$$\times\sqrt{1+\begin{pmatrix}-0.00771629x^5+0.339016x^4-5.9451369744x^3+52.0219983180x^2\\-227.5318476530x+399.3001008385\end{pmatrix}^2}\times dx$$
$$=95.7625$$

$$Surface\ Area\ of\ body=100.089+16.3663+95.7625=212.218\ cm^2$$

$$Surface\ area\ of\ top\ and\ bottom\ circles=2\pi\big(f(0)+f(11.5)\big)=2\pi(1+1.40820)$$
$$=15.1312\ cm^2$$

$$Total\ Surface\ Area=227.349\ cm^2$$

Can Dimensions, as obtained from the above figure:

Feature	Dimension (cm)
Vertical Height	11.5
Bottom circle (base) radius	1
Top circle radius	1.41
Maximum radius	3.03
Minimum radius	1.24

Table 2: Can Dimensions. Self Made on MS Word

Analysis and implications of the new can: This can incorporates all the desired elements, which makes it very aesthetically pleasing. My analysis shows that it is very close to can number 2 in terms of volume and surface area. My can has a volume of $230.131\ cm^3$, and surface area of $227.349\ cm^2$, which are close to can 2. This is evidence that this comparable to similar products available on the market in terms of size. My can has a surface area to volume ratio of 1:1.012, which is lower than that of can 2 (1:1.205), which means for the same surface area, the volume is slightly lesser. However, there are many other benefits of this shape when compared to the other cans.

D: Comparison of ratios is acknowledged but still no mention of its significance.

As mentioned in the background research, the aesthetics of a can are a very significant factor as many customers choose their beverage brand based on what they see in the aisles in a store. This means that the can must stand out. The indent is much greater than the other cans available in market. Such a combination of two spheres merging into each other to form a can is also a unique feature. Thus, this design, because of its unique shape, will catch the eye of the target market and thus increase the sales of beverage. To increase sales, another important feature is the user experience. As the can is made in accordance with the dimensions or regular cans in the market, the maximum radius is 3.03 cm, which is easy to grab by a wide range of people. The spherical shape allows a more comfortable grip, as if holding a ball. The indent at the center, with the minimum radius 1.24 cm, acts as a slot to fit the thumb and the first finger, and the overall grip becomes more stable and comfortable. Unlike a proper sphere, the can has been given a flat bottom, which makes the base stable to keep the can. These good ergonomic considerations make the user experience better, which will further push sales.

D: Some reflection is seen here.

Another factor to consider, however is the packaging and transportation. Since the height and maximum radius of the can is comparable to the generally produced can, the same number of cans will be able to fit in a standard box. There will be slightly less volume percentage covered than the other cans, but as mentioned, the other factors will cover up economically.

D: Meaningful reflection, which includes the requirement for packaging and transportation.

Reflection: Through the process of this mathematics IA, I understood the mathematics behind the derivation of the surface area formula, used technology effectively to analyze properties of different cans, and used my knowledge to design an analyze a new, effective can. I used the concepts I previously knew, such as differentiation, limits, and application of geometric theorems, along with newer concepts, such as the mean value theorem, which helped me increase my mathematical knowledge. A major learning was the use of software in mathematics- I learnt various skills in the software 'GeoGebra'- where I devised ways to scale the cans, add points, and use 2 variable regression to derive the functions. I also explored '2D Design' to draw diagrams and design my own can. It was fascinating to experience how mathematics and software was used to study a real-life context; existing cans in the market, and then also to design a completely new can.

Limitations: Firstly, I was not able to physically look for shaped cans which could have given me more inputs for my own cans. As I mentioned, the online sources of these cans did not give many details, such as the actual dimensions of height, width, dome height, etc. These would have helped me analyze the actual can sizes instead of the scaled sizes. The software gave me very precise results, however, the correlations were not perfect, meaning there would have been certain errors too. Apart from these, the overall analysis of cans was done in a strong manner.

Extensions: Statistics can be used to study the can market and how the design affect sales and to develop business models. This can further help in assessment of production processes. Packaging can also be explored with the use of further geometrical analysis. Further, more cans can be developed in terms of aesthetics and analyzed using all these concepts.

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