1.

METHOD 1

area = $\int_0^{\sqrt{3}} \arctan x \, dx$ A1

attempting to integrate by parts M1

$= [x \arctan x]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} x \frac{1}{1+x^2} dx$ A1A1

$= [x \arctan x]_0^{\sqrt{3}} - \left[\frac{1}{2}\ln(1+x^2)\right]_0^{\sqrt{3}}$ A1

Note: Award A1 even if limits are absent.

$= \frac{\pi}{\sqrt{3}} - \frac{1}{2}\ln 4$ A1

$\left(= \frac{\pi\sqrt{3}}{3} - \ln 2\right)$

METHOD 2

area = $\frac{\pi\sqrt{3}}{3} - \int_0^{\frac{\pi}{3}} \tan y \, dy$ M1A1A1

$= \frac{\pi\sqrt{3}}{3} + [\ln|\cos y|]_0^{\frac{\pi}{3}}$ M1A1

$= \frac{\pi\sqrt{3}}{3} + \ln\frac{1}{2} \left(= \frac{\pi\sqrt{3}}{3} - \ln 2\right)$ A1

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2.

(a) $f'(x) = \dfrac{2xe^x - x^2e^x}{e^{2x}} \left(= \dfrac{2x - x^2}{e^x}\right)$ M1A1

For a maximum $f'(x) = 0$ (M1)

$2x - x^2 = 0$

giving $x = 0$ or 2 A1A1

$f''(x) = \dfrac{(2-2x)e^x - e^x(2x - x^2)}{e^{2x}} \left(= \dfrac{x^2 - 4x + 2}{e^x}\right)$ M1A1

$f''(0) = 2 > 0 \Rightarrow$ minimum R1

$f''(2) = -\dfrac{2}{e^2} < 0 \Rightarrow$ maximum R1

Maximum value $= \dfrac{4}{e^2}$ A1

(b) For a point of inflexion,

$f''(x) = \dfrac{x^2 - 4x + 2}{e^x} = 0$ M1

giving $x = \dfrac{4 \pm \sqrt{16-8}}{2}$ (A1)

$= 2 \pm \sqrt{2}$ A1

(c) $\int_0^1 x^2e^{-x}dx = \left[-x^2e^{-x}\right]_0^1 + 2\int_0^1 xe^{-x}dx$ M1A1

$= -e^{-1} - 2\left[xe^{-x}\right]_0^1 + 2\int_0^1 e^{-x}dx$ A1M1A1

$= -e^{-1} - 2e^{-1} - 2\left[e^{-x}\right]_0^1$ A1A1

$= -3e^{-1} - 2e^{-1} + 2 \ (= 2 - 5e^{-1})$ A1

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3.

Recognition of integration by parts M1

$\int x^2 \ln x dx = \left[\dfrac{x^3}{3}\ln x\right] - \int \dfrac{x^3}{3} \times \dfrac{1}{x}dx$ A1A1

$= \left[\dfrac{x^3}{3}\ln x\right] - \int \dfrac{x^2}{3}dx$

$= \left[\dfrac{x^3}{3}\ln x - \dfrac{x^3}{9}\right]$ A1

$\Rightarrow \int_1^e x^2 \ln x dx = \left(\dfrac{e^3}{3} - \dfrac{e^3}{9}\right) - \left(0 - \dfrac{1}{9}\right) \quad \left(= \dfrac{2e^3 + 1}{9}\right)$ A1

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4.

(a) (i) $18(x-1)=0 \Rightarrow x=1$ A1

(ii) vertical asymptote: $x=0$ A1
horizontal asymptote: $y=0$ A1

(iii) $18(2-x)=0 \Rightarrow x=2$ M1A1

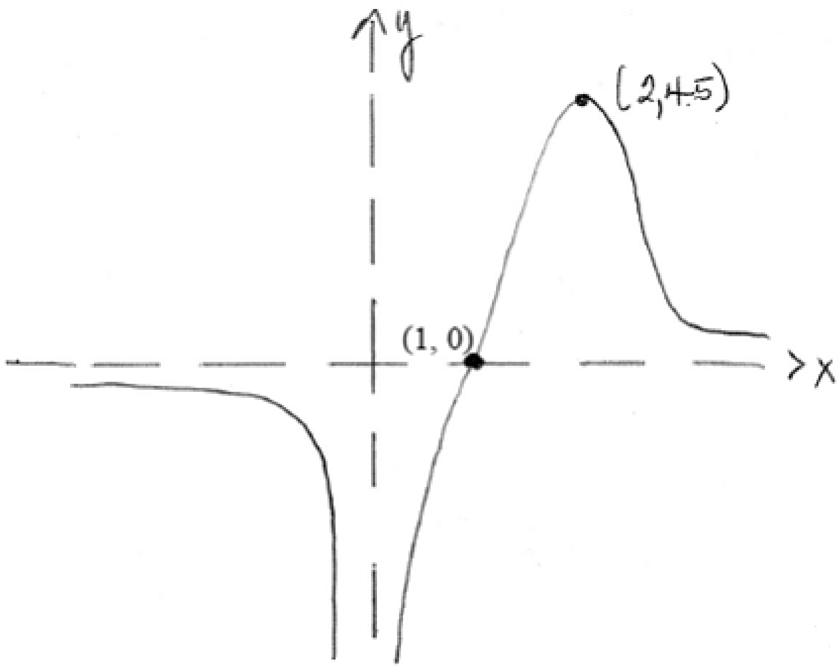
$f''(2)=\dfrac{36(2-3)}{2^3}=-\dfrac{9}{2}<0$ hence it is a maximum point R1

When $x=2$, $f(x)=\dfrac{9}{2}$ A1

$\left(f(x)\text{ has a maximum at }\left(2,\dfrac{9}{2}\right)\right)$

(iv) $f(x)$ is concave up when $f''(x)>0$ M1
$36(x-3)>0 \Rightarrow x>3$ A1

(b)



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Note: Award A1 for shape, A1 for maximum, A1 for x-intercept, A1 for horizontal asymptote and A1 for vertical asymptote.

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5.

Attempting to differentiate implicitly (M1)

$3x^2y + 2xy^2 = 2 \Rightarrow 6xy + 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^2 + 4xy\frac{\mathrm{d}y}{\mathrm{d}y} = 0$ A1

Substituting $x = 1$ and $y = -2$ (M1)

$-12 + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 8 - 8\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ A1

$\Rightarrow -5\frac{\mathrm{d}y}{\mathrm{d}x} = 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{5}$ A1

Gradient of normal is $\frac{5}{4}$ A1 N3

[6]

6.

$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan(x-1)) = \frac{1}{1+(x-1)^2}$ (or equivalent) A1

$m_N = -2$ and so $m_T = \frac{1}{2}$ (R1)

Attempting to solve $\frac{1}{1+(x-1)^2} = \frac{1}{2}$ (or equivalent) for x M1

$x = 2$ (as $x > 0$) A1

Substituting $x = 2$ **and** $y = \frac{\pi}{4}$ to find c M1

$c = 4 + \frac{\pi}{4}$ A1 N1

[6]

7.

(a) $f'(x) = (1 + 2x)\,e^{2x}$ A1

$f'(x) = 0$ M1

$\Rightarrow (1 + 2x)e^{2x} = 0 \Rightarrow x = -\dfrac{1}{2}$ A1

$f''(x) = (2^2x + 2 \times 2^{2-1})e^{2x} = (4x + 4)e^{2x}$ A1

$f''\left(-\dfrac{1}{2}\right) = \dfrac{2}{e}$ A1

$\dfrac{2}{e} > 0 \Rightarrow$ at $x = -\dfrac{1}{2}$, $f(x)$ has a minimum. R1

$P\left(-\dfrac{1}{2}, -\dfrac{1}{2e}\right)$ A1

(b) $f''(x) = 0 \Rightarrow 4x + 4 = 0 \Rightarrow x = -1$ M1A1

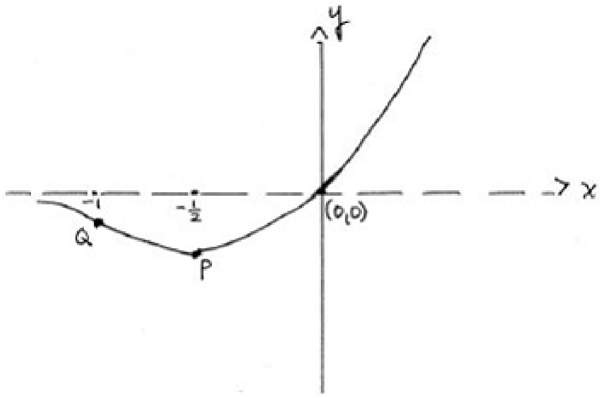
Using the 2nd derivative $f''\left(-\dfrac{1}{2}\right) = \dfrac{2}{e}$ and $f''(-2) = -\dfrac{4}{e^4}$, M1A1

the sign change indicates a point of inflexion. R1

(c) (i) $f(x)$ is concave up for $x > -1$. A1

(ii) $f(x)$ is concave down for $x < -1$. A1

(d)



A1A1A1A1

Note: Award A1 for P and Q, with Q above P,
A1 for asymptote at y = 0,
A1 for (0, 0),
A1 for shape.

(e) Show true for $n = 1$ (M1)

$f'(x) = e^{2x} + 2xe^{2x}$ A1

$= e^{2x}(1 + 2x) = (2x + 2^0)e^{2x}$

Assume true for $n = k$, ie $f^{(k)}x = (2^k x + k \times 2^{k-1})e^{2x}$, $k \geq 1$ M1A1

Consider $n = k + 1$, ie an attempt to find $\frac{d}{dx}\left(f^k(x)\right)$ M1

$f^{(k+1)}(x) = 2^k e^{2x} + 2e^{2x}(2^k x + k \times 2^{k-1})$ A1

$= (2^k + 2(2^k x + k \times 2^{k-1}))e^{2x}$

$= (2 \times 2^k x + 2^k + k \times 2 \times 2^{k-1})e^{2x}$

$= (2^{k+1}x + 2^k + k \times 2^k)e^{2x}$ A1

$= (2^{k+1}x + (k + 1)2^k)e^{2x}$ A1

$P(n)$ is true for $k \Rightarrow P(n)$ is true for $k + 1$, and since true for $n = 1$, result proved by mathematical induction $\forall n \in \mathbb{Z}^+$ R1

Note: Only award R1 if a reasonable attempt is made to prove the $(k + 1)^{\text{th}}$ step.

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8.

(a) AQ = $\sqrt{x^2+4}$ (km) (A1)

QY = $(2 - x)$ (km) (A1)

$T = 5\sqrt{5}\,\text{AQ} + 5\text{QY}$ (M1)

$= 5\sqrt{5}\sqrt{(x^2+4)} + 5(2-x)$ (mins) A1

(b) Attempting to use the chain rule on $5\sqrt{5}\sqrt{(x^2+4)}$ (M1)

$\frac{d}{dx}\left(5\sqrt{5}\sqrt{(x^2+4)}\right) = 5\sqrt{5} \times \frac{1}{2}(x^2+4)^{-\frac{1}{2}} \times 2x$ A1

$\left(= \frac{5\sqrt{5}x}{\sqrt{x^2+4}}\right)$

$\frac{d}{dx}(5(2-x)) = -5$ A1

$\frac{dT}{dx} = \frac{5\sqrt{5}x}{\sqrt{x^2+4}} - 5$ AG N0

(c) (i) $\sqrt{5}\,x = \sqrt{x^2+4}$ or equivalent A1

Squaring both sides and rearranging to obtain $5x^2 = x^2 + 4$ M1

$x = 1$ A1 N1

Note: Do not award the final A1 for stating a negative solution in final answer.

(ii) $T = 5\sqrt{5}\sqrt{1+4} + 5(2-1)$ M1

$= 30$ (mins) A1 N1

Note: Allow FT on incorrect x value.

(iii) **METHOD 1**

Attempting to use the quotient rule M1

$u=x, v=\sqrt{x^2+4}, \frac{du}{dx}=1$ and $\frac{dv}{dx}=x(x^2+4)^{-\frac{1}{2}}$ (A1)

$$\frac{d^2T}{dx^2}=5\sqrt{5}\left[\frac{\sqrt{x^2+4}-\frac{1}{2}(x^2+4)^{-\frac{1}{2}}\times 2x^2}{(x^2+4)}\right]$$ A1

Attempt to simplify (M1)

$$=\frac{5\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}\left[x^2+4-x^2\right]$$ or equivalent A1

$$=\frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}$$ AG

When $x = 1$, $\frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}} > 0$ and hence $T = 30$ is a minimum R1 N0

Note: Allow FT on incorrect x value, $0 \le x \le 2$.

METHOD 2

Attempting to use the product rule M1

$u=x, v=\sqrt{x^2+4}, \frac{du}{dx}=1$ and $\frac{dv}{dx}=x(x^2+4)^{-\frac{1}{2}}$ (A1)

$$\frac{d^2T}{dx^2}=5\sqrt{5}\,(x^2+4)^{-\frac{1}{2}}-\frac{5\sqrt{5}x}{2}(x^2+4)^{-\frac{3}{2}}\times 2x$$ A1

$$\left(=\frac{5\sqrt{5}}{(x^2+4)^{\frac{1}{2}}}-\frac{5\sqrt{5}\,x^2}{(x^2+4)^{\frac{3}{2}}}\right)$$

Attempt to simplify (M1)

$$=\frac{5\sqrt{5}(x^2+4)-5\sqrt{5}\,x^2}{(x^2+4)^{\frac{3}{2}}} \quad \left(=\frac{5\sqrt{5}\,(x^2+4-x^2)}{(x^2+4)^{\frac{3}{2}}}\right)$$ A1

$$=\frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}}$$ AG

When $x = 1$, $\frac{20\sqrt{5}}{(x^2+4)^{\frac{3}{2}}} > 0$ and hence $T = 30$ is a minimum R1 N0

Note: Allow FT on incorrect x value, $0 \le x \le 2$.

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