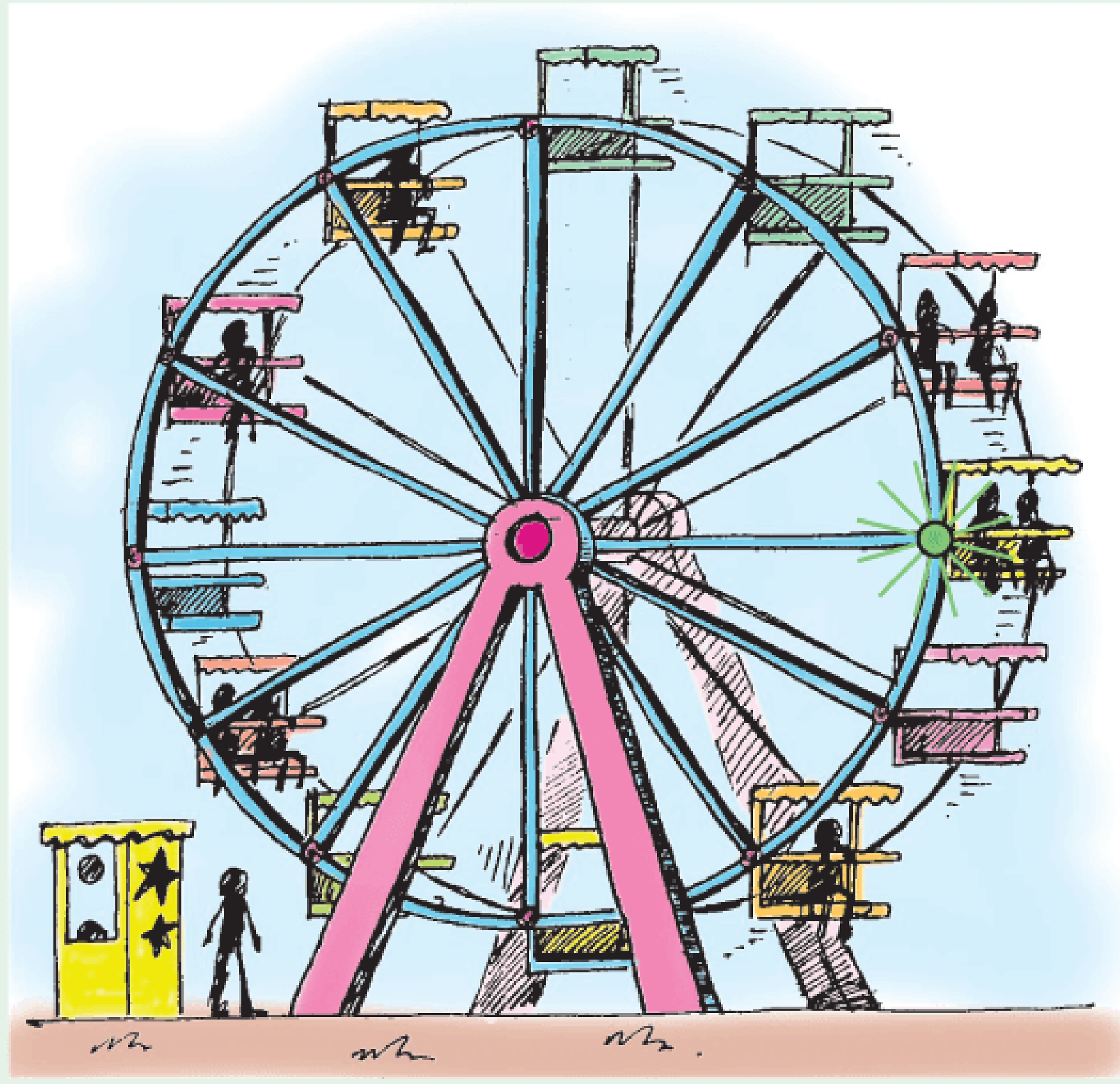
Chapter 17

Trigonometric functions

Contents:

OPENING PROBLEM

A Ferris wheel rotates anticlockwise at a constant speed. The wheel's radius is 10 m and the bottom of the wheel is 2 m above ground level. From his viewing point next to the ticket booth, Andrew is watching a green light on the perimeter of the wheel. He notices that the green light moves in a circle. It takes 100 seconds for a full revolution.



Click on the icon to visit a simulation of the Ferris wheel. You will be able to view the light from:

- in front of the wheel
- a side-on position
- above the wheel.

DEMO

You can then observe graphs of the green light's position as the wheel rotates at a constant rate.

Things to think about:

a Andrew estimates how high the light is above ground level at two second intervals. What will a graph of this data look like? Assume that the light is initially in the position shown.

b Andrew then estimates the horizontal position of the light at two second intervals. What will a graph of this data look like?

c What similarities and differences will there be between your two graphs?

d Can you write a function which will give the:

  i height of the light at any time t seconds

  ii horizontal displacement of the light at any time t seconds?

A PERIODIC BEHAVIOUR

Periodic phenomena occur all the time in the physical world. For example, in:

- seasonal variations in our climate
- variations in average maximum and minimum monthly temperatures
- the number of daylight hours at a particular location
- tidal variations in the depth of water in a harbour
- the phases of the moon
- animal populations.

These phenomena illustrate variable behaviour which is repeated over time. The repetition may be called **periodic**, **oscillatory**, or **cyclic** in different situations.

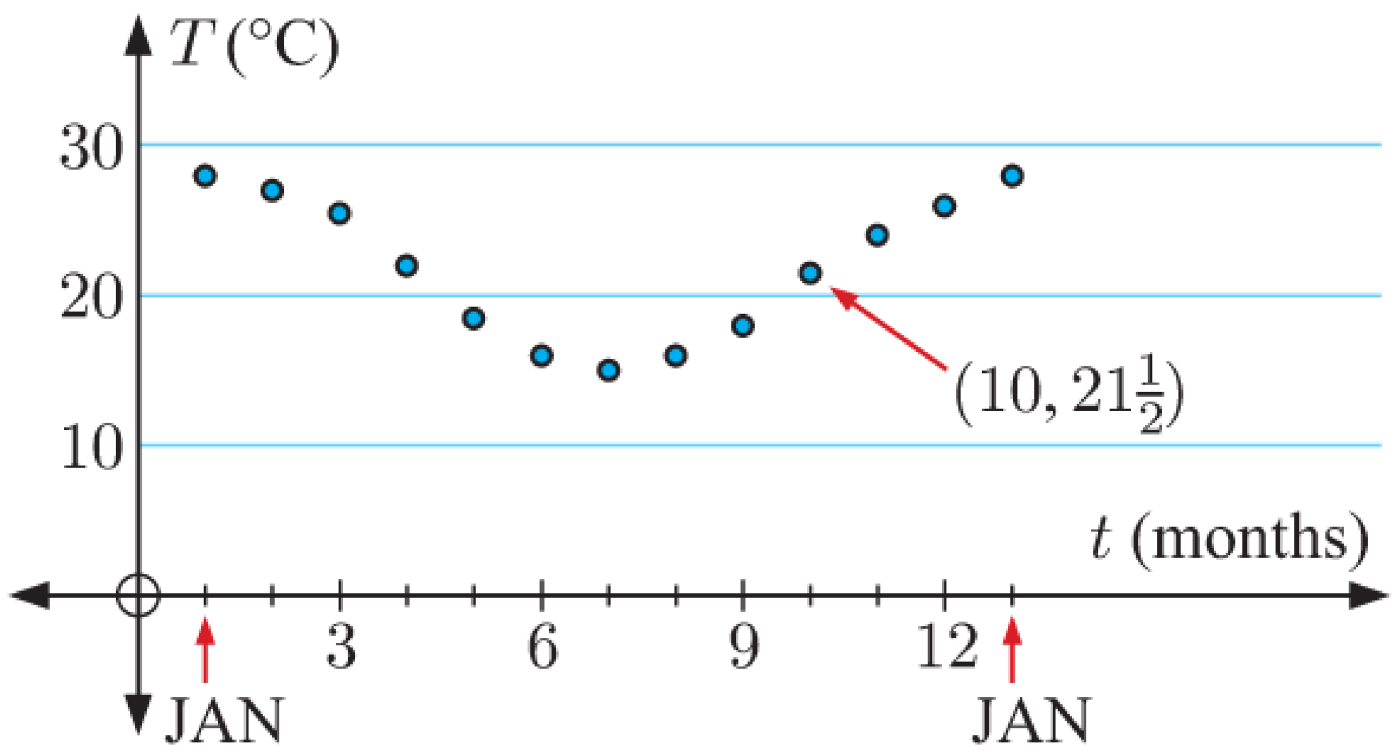
In this Chapter we will see how trigonometric functions can be used to model periodic phenomena.

OBSERVING PERIODIC BEHAVIOUR

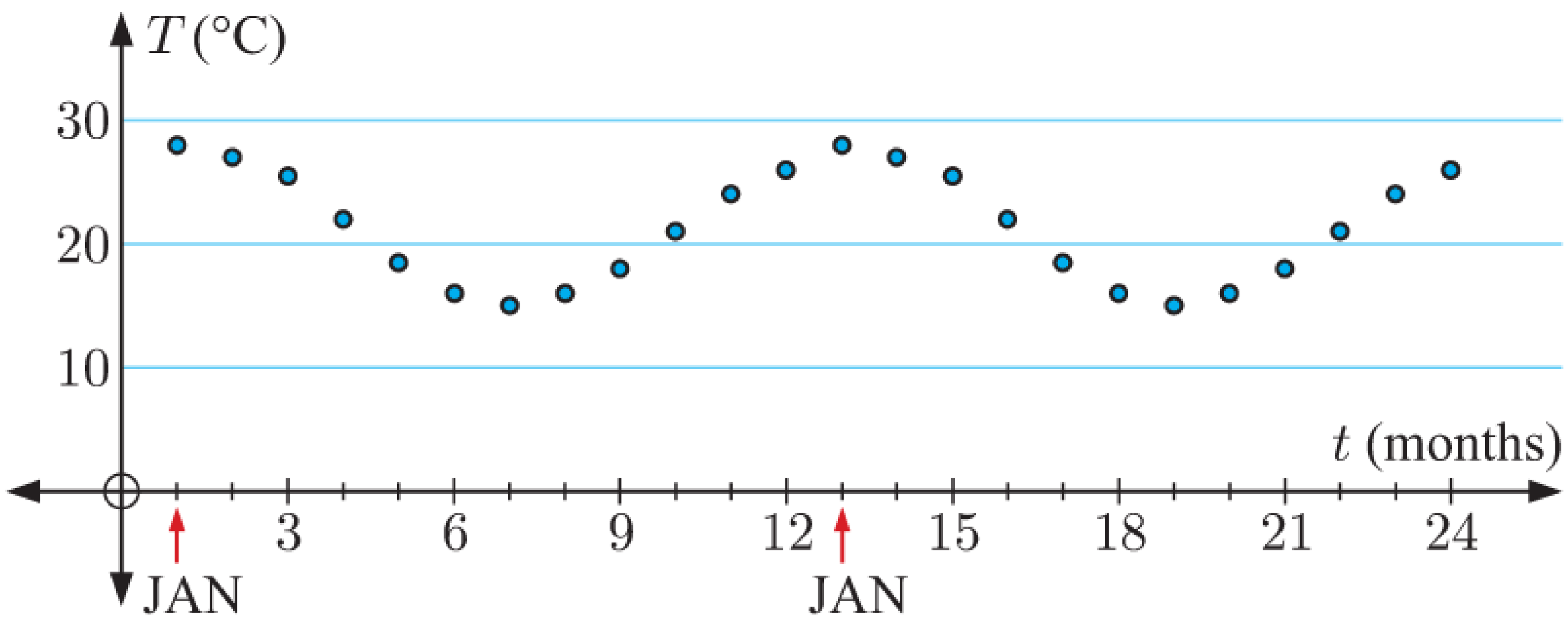
The table below shows the mean monthly maximum temperature for Cape Town, South Africa.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature T (°C)	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

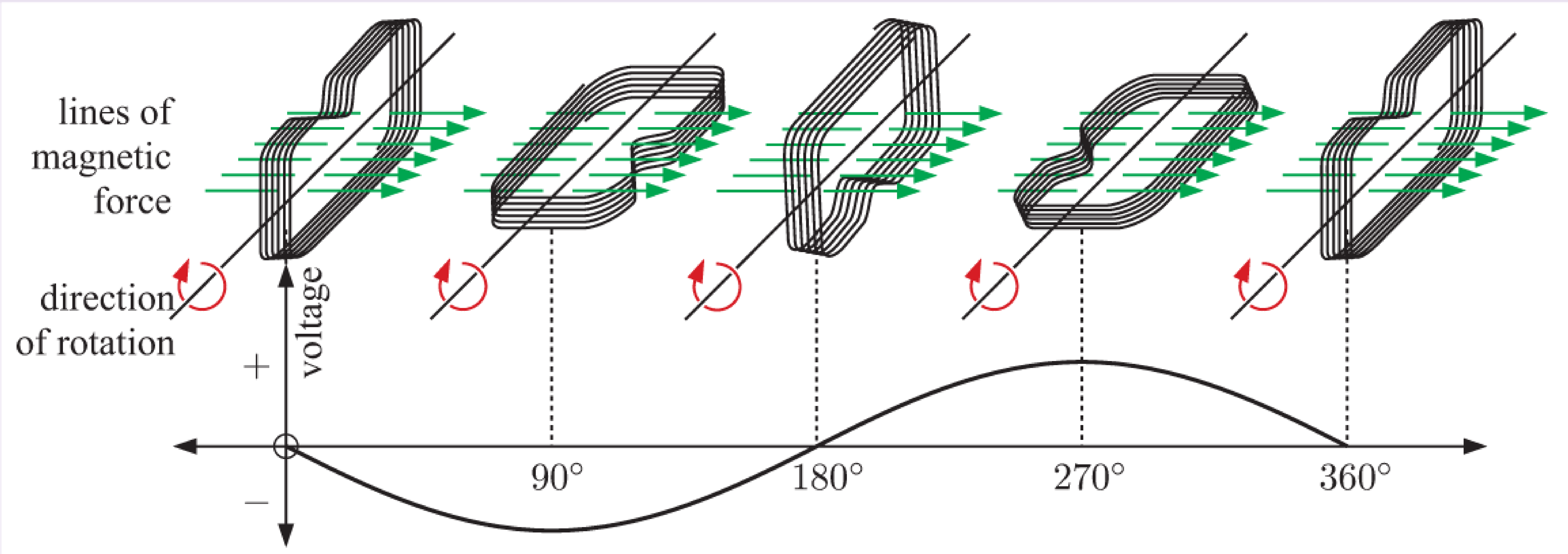
On the graph alongside we plot the temperature T on the vertical axis. We assign January as $t = 1$ month, February as $t = 2$ months, and so on for the 12 months of the year.



The temperature shows a variation from an average of 28°C in January through a range of values across the months. The cycle will approximately repeat itself for each subsequent 12 month period. By the end of the Chapter we will be able to establish a **periodic function** which approximately fits this set of points.



HISTORICAL NOTE



In 1831, **Michael Faraday** discovered that an electric current was generated by rotating a coil of wire at a constant speed through 360° in a magnetic field. The electric current produced showed a voltage which varied between positive and negative values in a periodic function called a **sine wave**.

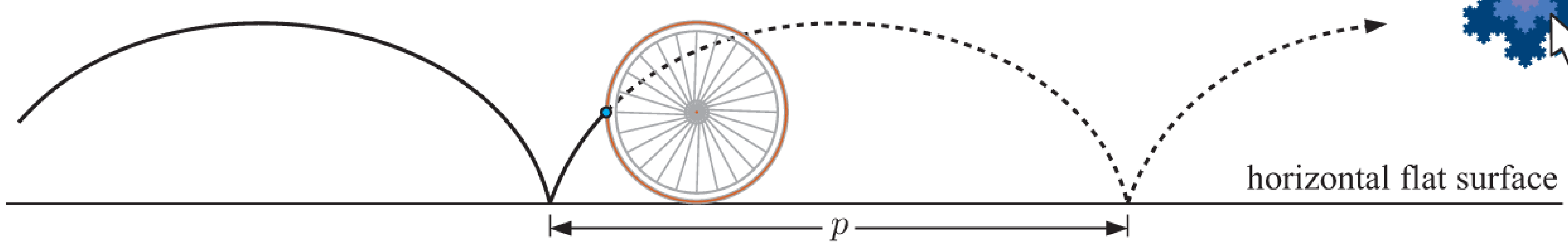
TERMINOLOGY USED TO DESCRIBE PERIODICITY

A **periodic function** is one which repeats itself over and over in a horizontal direction, in intervals of the same length. The **period** of a periodic function is the length of one repetition or cycle.

$f(x)$ is a periodic function with period p if $\boldsymbol{f(x+p) = f(x)}$ for all x, and p is the smallest positive value for this to be true.

A **cycloid** is an example of a periodic function. It is the curve traced out by a point on a circle as the circle rolls across a flat surface in a straight line.

DEMO



ACTIVITY 1 PERIODIC FUNCTIONS

GRAPHING PACKAGE

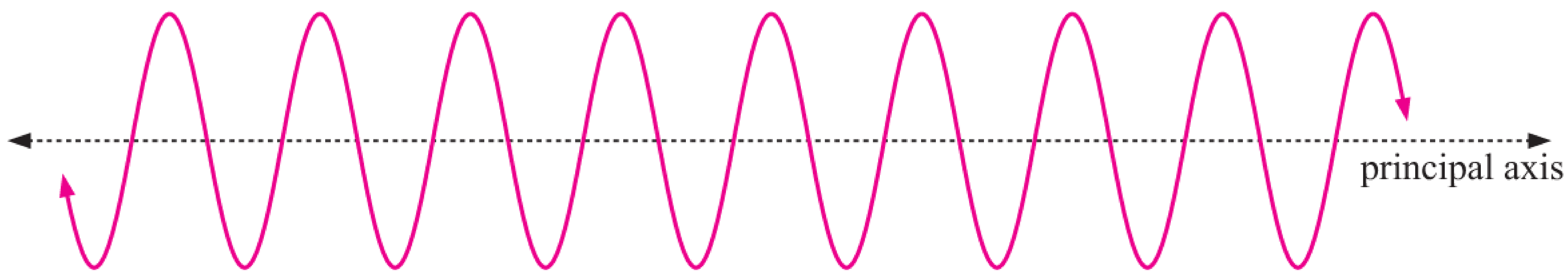
Use a **graphing package** to examine the function $f(x) = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is "the largest integer less than or equal to x".

In the graphing package, you type $\lfloor x \rfloor$ as floor(x).

Is $f(x)$ periodic? What is its period?

WAVES

In this course we are mainly concerned with periodic phenomena which show a wave pattern:



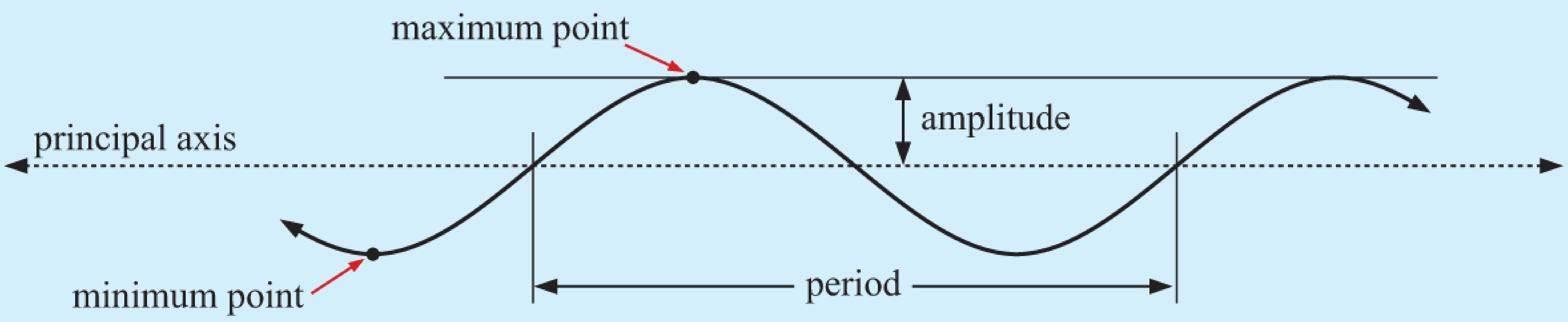
A **wave** oscillates about a horizontal line called the **principal axis** or **mean line**.

A **maximum point** occurs at the top of a crest, and a **minimum point** at the bottom of a trough.

If the maximum and minimum values of the wave are **max** and **min** respectively, then the principal axis has equation $\boldsymbol{y = \dfrac{\textbf{max} + \textbf{min}}{2}}$.

The **amplitude** is the distance between a maximum (or minimum) point and the principal axis.

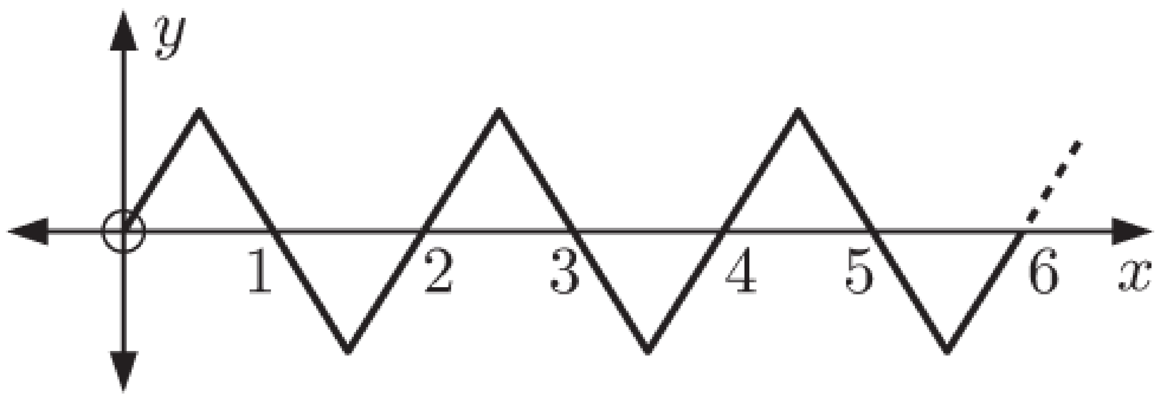
$$\textbf{amplitude} = \frac{\textbf{max} - \textbf{min}}{2}$$



EXERCISE 17A

1 Which of these graphs show periodic behaviour?

a



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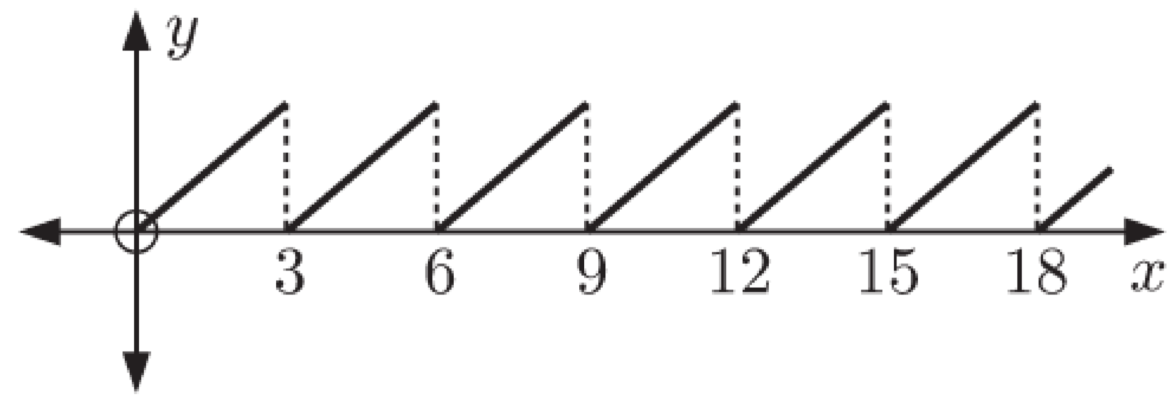

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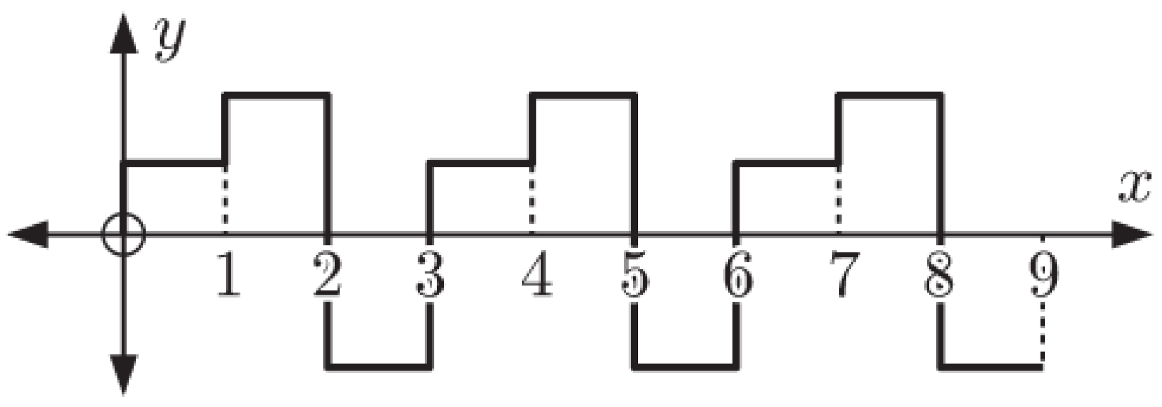

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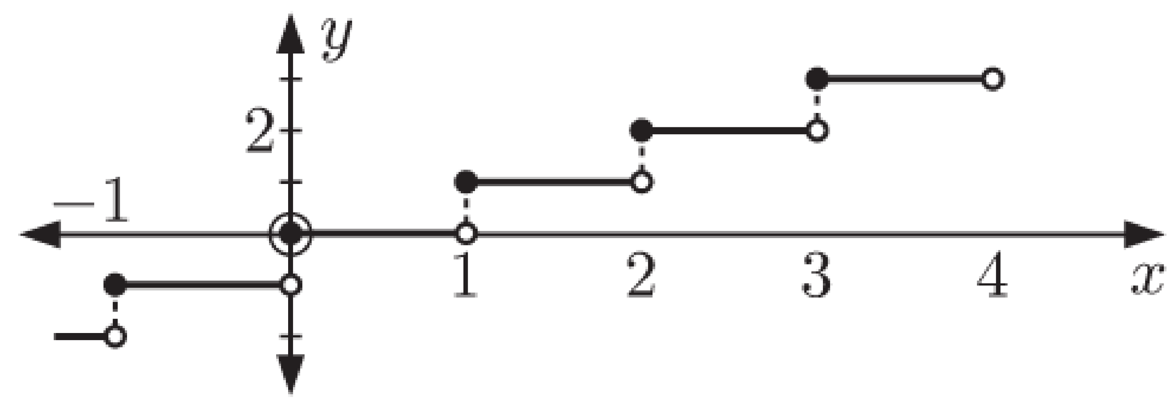

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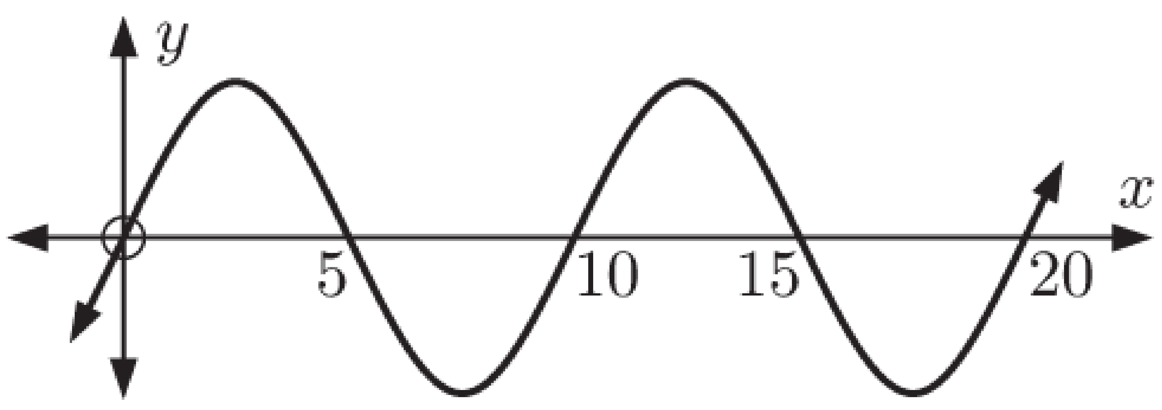
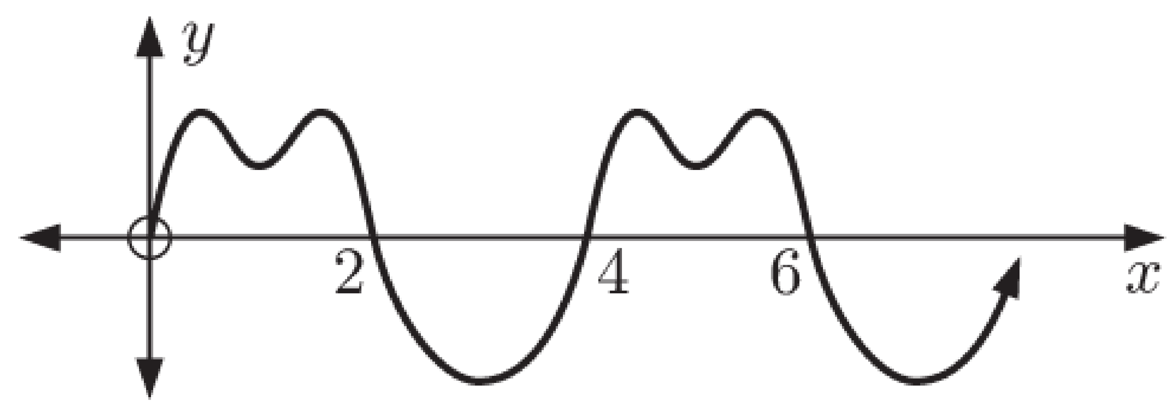


f



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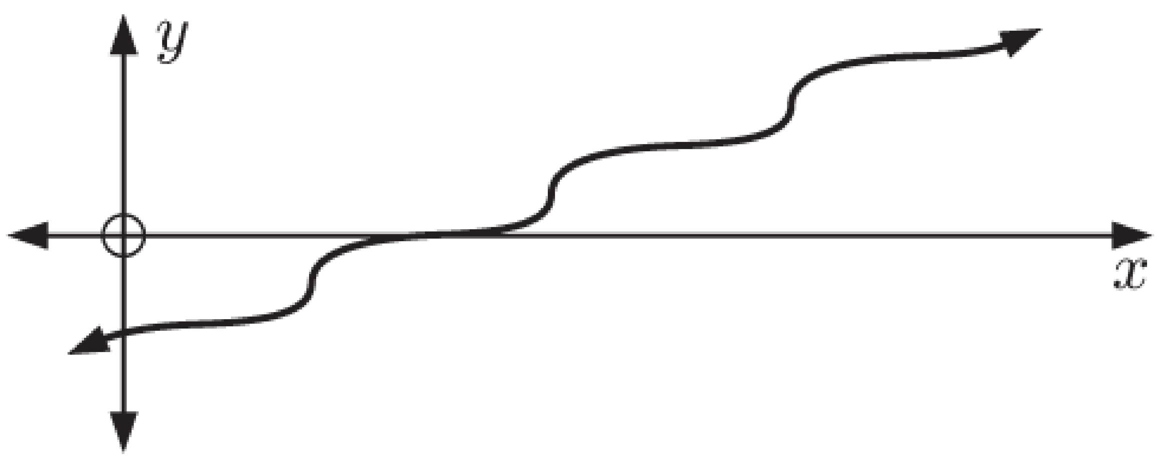
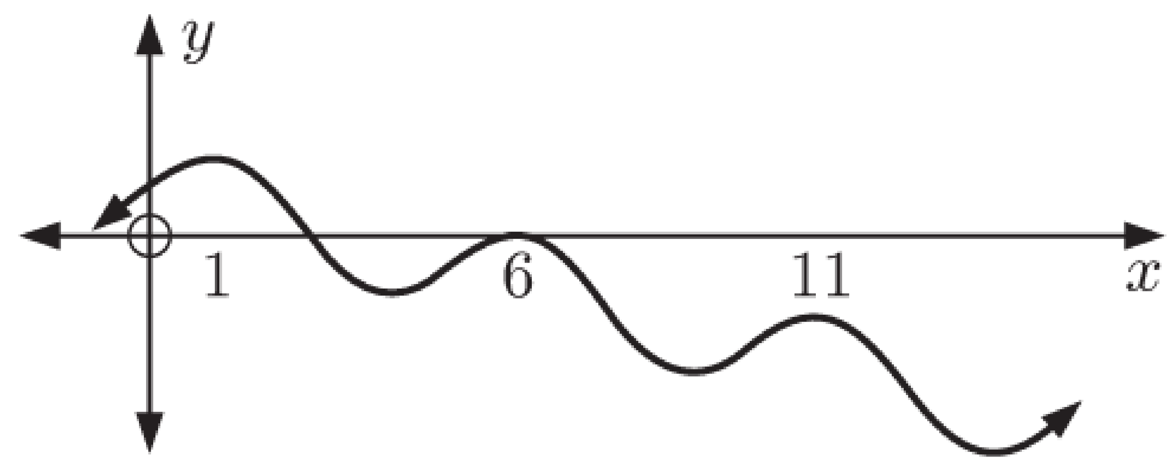


h



2 Paul spun the wheel of his bicycle. The following tabled values show the height above the ground of a point on the wheel at various times.

Time (seconds)	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1

Time (seconds)	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
Height above ground (cm)	5	27	40	55	63	60	44	24	9	3

a Plot the graph of height against time.

b Is it reasonable to fit a curve to this data, or should we leave it as discrete points?

c Is the data periodic? If so, estimate:

- **i** the equation of the principal axis
- **ii** the maximum value
- **iii** the period
- **iv** the amplitude.

3 Plot the points for each data set below. Is there any evidence to suggest the data is periodic?

a

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	−1	−1.4	−1	0	1	1.4	1	0

b

x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4

GRAPHICS CALCULATOR INSTRUCTIONS

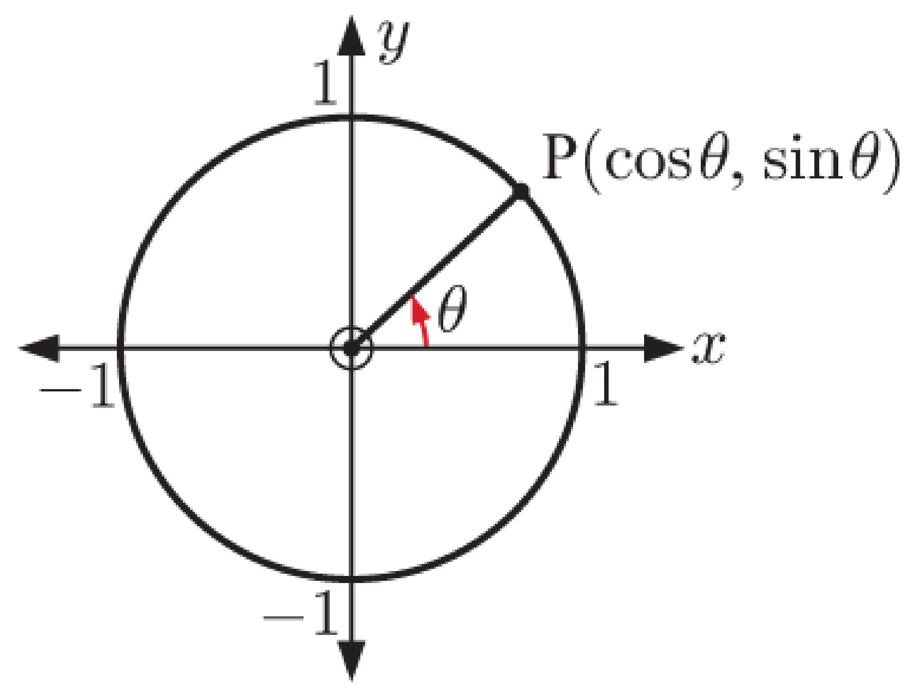
B THE SINE AND COSINE FUNCTIONS

A **trigonometric function** is a function which involves one of the trigonometric ratios.

Consider the point $P(\cos\theta, \sin\theta)$ on the unit circle.

As θ increases, the point P moves around the unit circle, and the values of $\cos\theta$ and $\sin\theta$ change.

We can draw the graphs of $y = \sin\theta$ and $y = \cos\theta$ by plotting the values of $\sin\theta$ and $\cos\theta$ against θ.

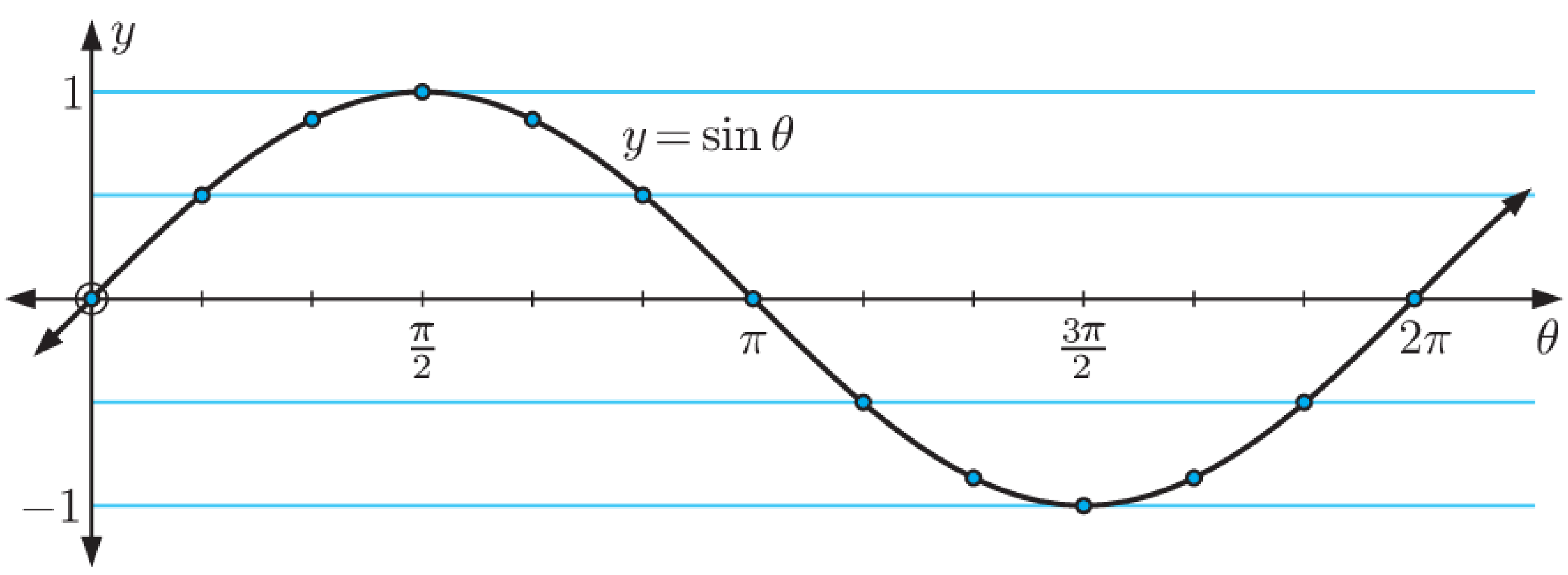


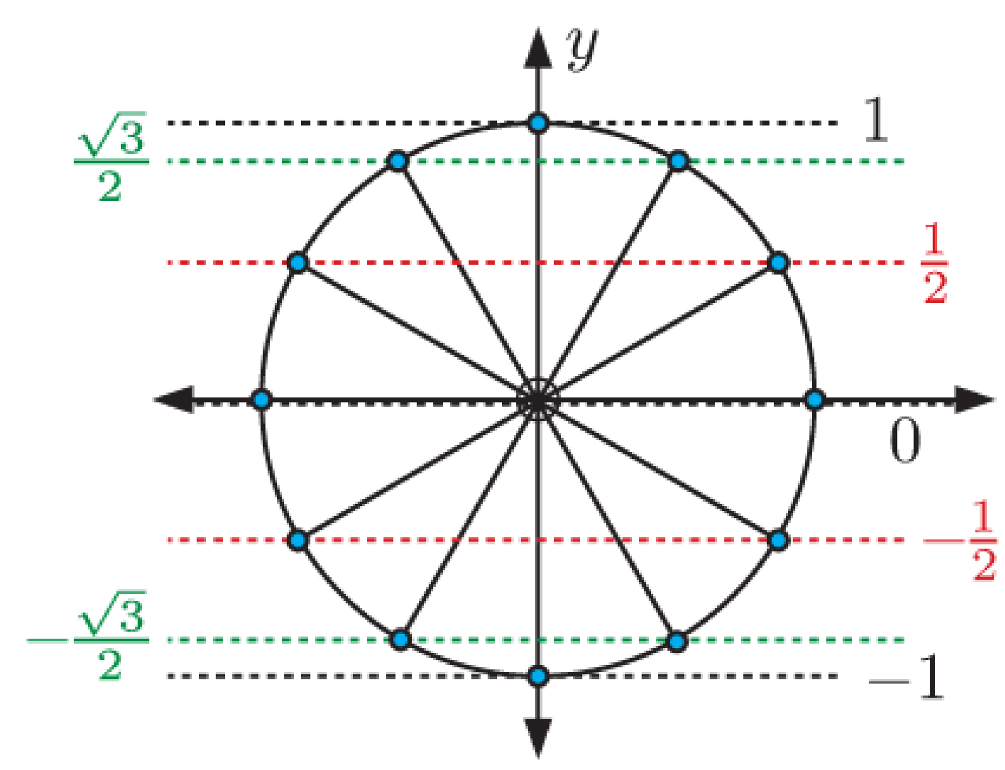
THE GRAPH OF $y = \sin\theta$

By considering the y-coordinates of the points on the unit circle at intervals of $\frac{\pi}{6}$, we can create a table of values for $\sin\theta$:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Plotting $\sin\theta$ against θ gives:





Once we reach 2π, P has completed a full revolution of the unit circle, and so this pattern repeats itself.

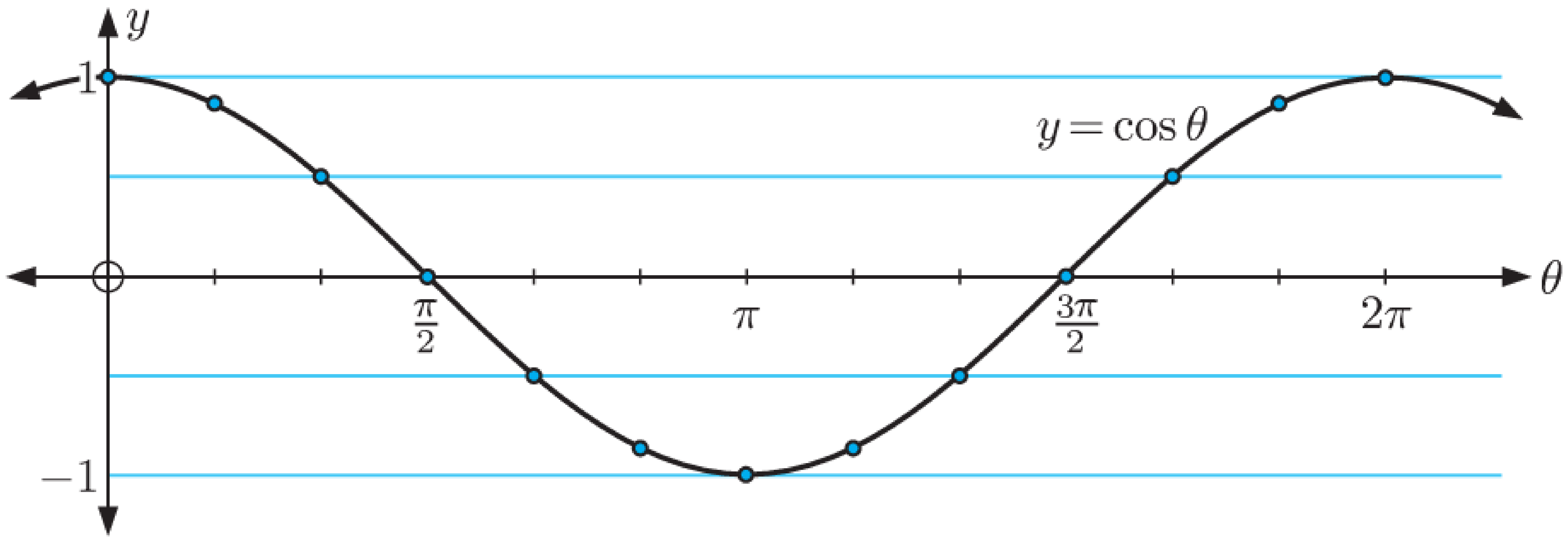
DEMO

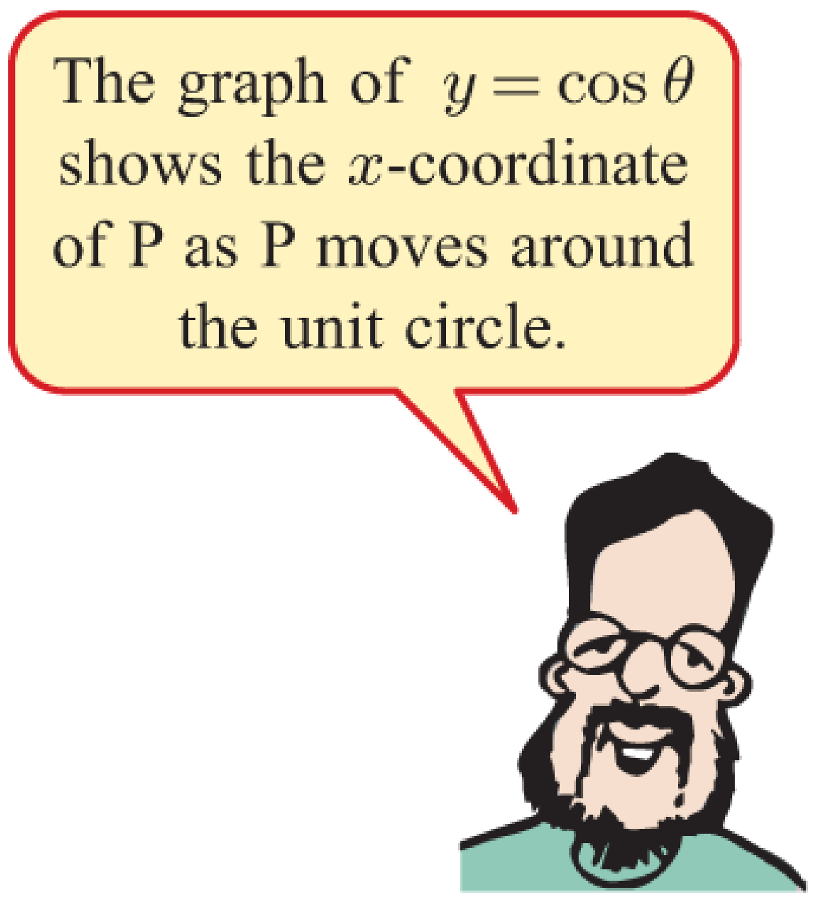
THE GRAPH OF $y = \cos\theta$

By considering the x-coordinates of the points on the unit circle at intervals of $\frac{\pi}{6}$, we can create a table of values for $\cos\theta$:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

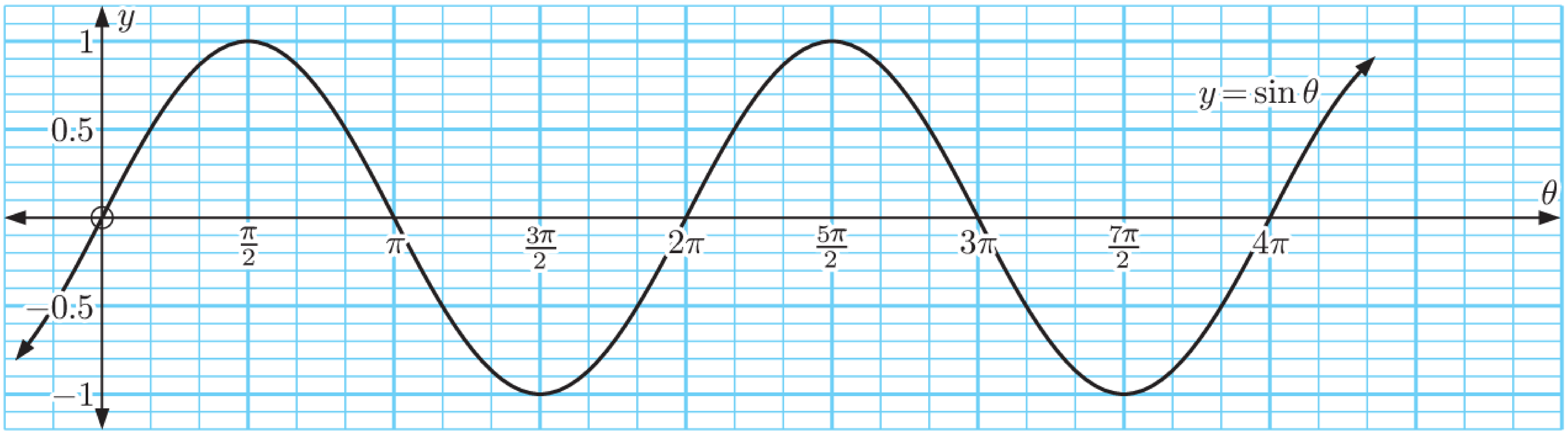
Plotting $\cos\theta$ against θ gives:





EXERCISE 17B

1 Below is an accurate graph of $y = \sin\theta$.



a Find the y-intercept of the graph.

b Find the values of θ on $0 \leqslant \theta \leqslant 4\pi$ for which:

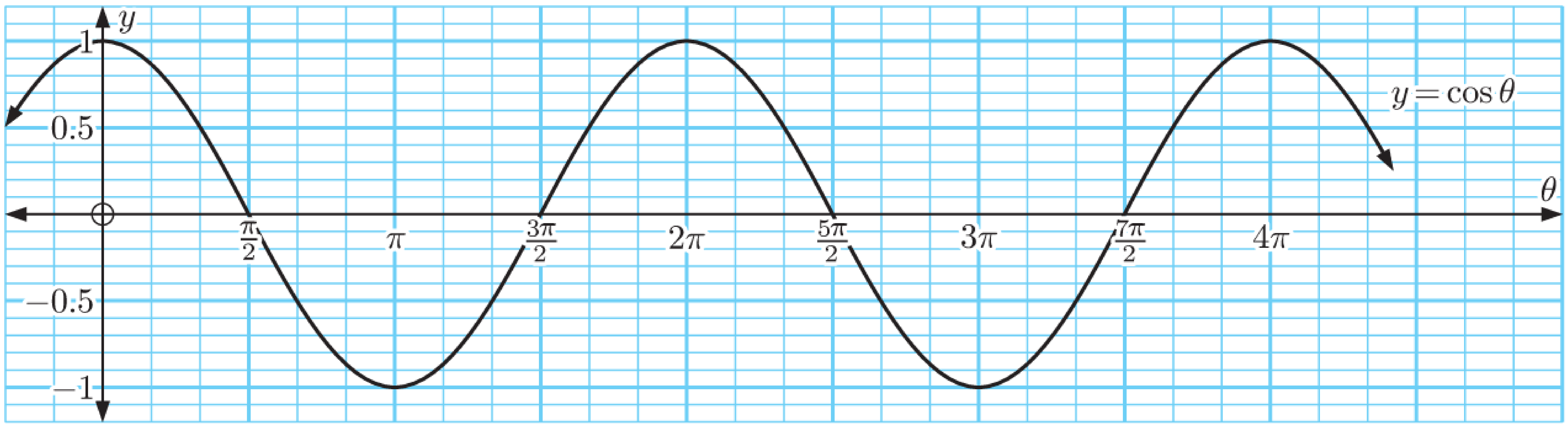
i $\sin\theta = 0$ **ii** $\sin\theta = -1$ **iii** $\sin\theta = \frac{1}{2}$ **iv** $\sin\theta = \frac{\sqrt{3}}{2}$

c Find the intervals on $0 \leqslant \theta \leqslant 4\pi$ where $\sin\theta$ is:

i positive **ii** negative.

d Find the range of the function.

2 Below is an accurate graph of $y = \cos\theta$.



a Find the y-intercept of the graph.

b Find the values of θ on $0 \leqslant \theta \leqslant 4\pi$ for which:

i $\cos\theta = 0$ **ii** $\cos\theta = 1$ **iii** $\cos\theta = -\frac{1}{2}$ **iv** $\cos\theta = -\frac{1}{\sqrt{2}}$

c Find the intervals on $0 \leqslant \theta \leqslant 4\pi$ where $\cos\theta$ is:

i positive **ii** negative.

d Find the range of the function.

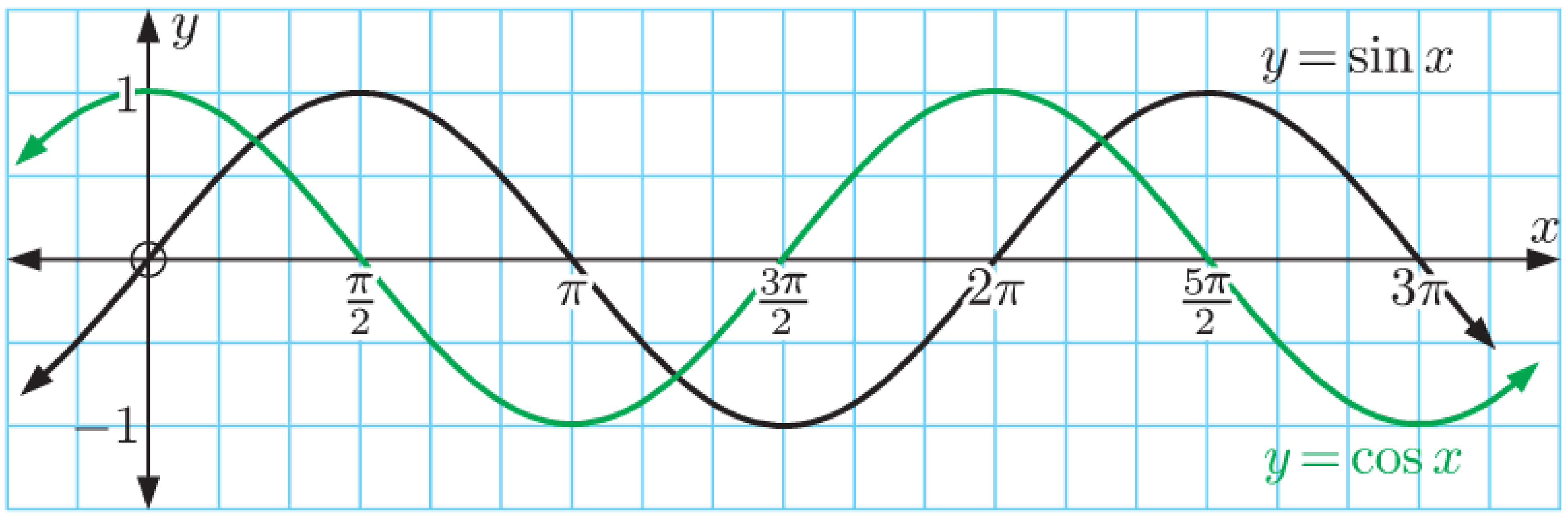
C GENERAL SINE AND COSINE FUNCTIONS

Now that we are familiar with the graphs of $y = \sin\theta$ and $y = \cos\theta$, we can use transformations to graph more complicated trigonometric functions.

Instead of using θ, we will now use x to represent the angle variable. This is just for convenience, so we are dealing with the familiar function form $y = f(x)$.

For the graphs of $y = \sin x$ and $y = \cos x$:

- the **period** is 2π
- the **amplitude** is 1
- the **principal axis** is the line $y = 0$.



We immediately notice that $y = \sin x$ is a horizontal translation of $y = \cos x$ by $\frac{\pi}{2}$ units to the right.

> For all x, $\quad \boldsymbol{\sin x = \cos\left(x - \frac{\pi}{2}\right)}$
>
> and $\quad \boldsymbol{\cos x = \sin\left(x + \frac{\pi}{2}\right)}$

INVESTIGATION FAMILIES OF TRIGONOMETRIC FUNCTIONS

GRAPHING PACKAGE

What to do:

1 **a** Use the graphing package to graph on the same set of axes:

i $y = \sin x$ **ii** $y = 2\sin x$ **iii** $y = \frac{1}{2}\sin x$

iv $y = -\sin x$ **v** $y = -\frac{1}{3}\sin x$ **vi** $y = -\frac{3}{2}\sin x$

b For graphs of the form $y = a\sin x$, comment on the significance of:

i the sign of a **ii** the size of a, or $|a|$.

2 **a** Use the graphing package to graph on the same set of axes:

i $y = \sin x$ **ii** $y = \sin 2x$ **iii** $y = \sin\left(\frac{1}{2}x\right)$ **iv** $y = \sin 3x$

b For graphs of the form $y = \sin bx$, $b > 0$, what is the period?

3 **a** Graph on the same set of axes:

i $y = \sin x$ **ii** $y = \sin\left(x - \frac{\pi}{3}\right)$ **iii** $y = \sin\left(x + \frac{\pi}{6}\right)$

b What translation moves $y = \sin x$ to $y = \sin(x - c)$?

4 **a** Graph on the same set of axes:

i $y = \sin x$ **ii** $y = \sin x + 2$ **iii** $y = \sin x - 2$

b What translation moves $y = \sin x$ to $y = \sin x + d$?

c What is the principal axis of $y = \sin x + d$?

5 What sequence of transformations maps $y = \sin x$ onto $y = a\sin b(x - c) + d$?

From the **Investigation** you should have observed the following properties of the general sine function:

For the **general sine function**

$$\boldsymbol{y = a\sin(b(x-c)) + d}$$

- a affects **amplitude**
- b affects **period**
- c affects **horizontal translation**
- d affects **vertical translation**

- the amplitude is $|\,a\,|$
- the period is $\dfrac{2\pi}{b}$ for $b > 0$
- the principal axis is $y = d$
- $y = a\sin(b(x-c)) + d$ is obtained from $y = \sin x$ by a vertical stretch with scale factor a and a horizontal stretch with scale factor $\dfrac{1}{b}$, followed by a horizontal translation of c units and a vertical translation of d units.

The properties of the **general cosine function** $y = a\cos(b(x-c)) + d$ are the same as those of the general sine function.

DEMO

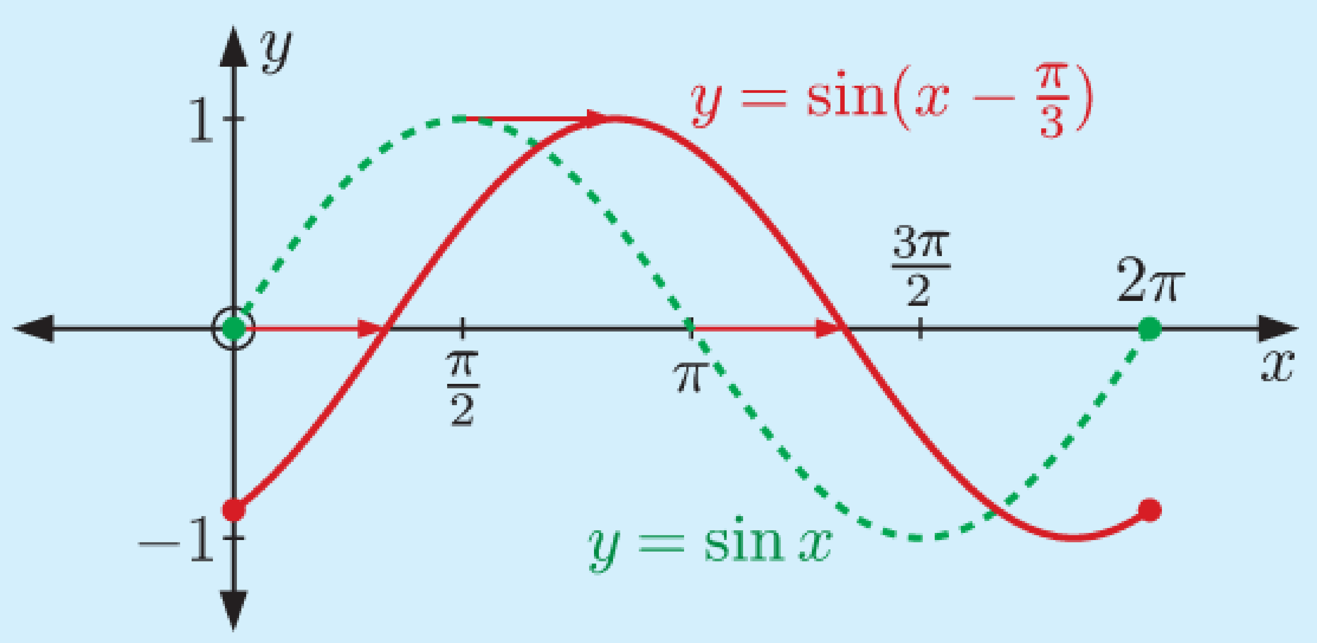
Example 1

Self Tutor

Sketch the graphs of the following on $0 \leqslant x \leqslant 2\pi$:

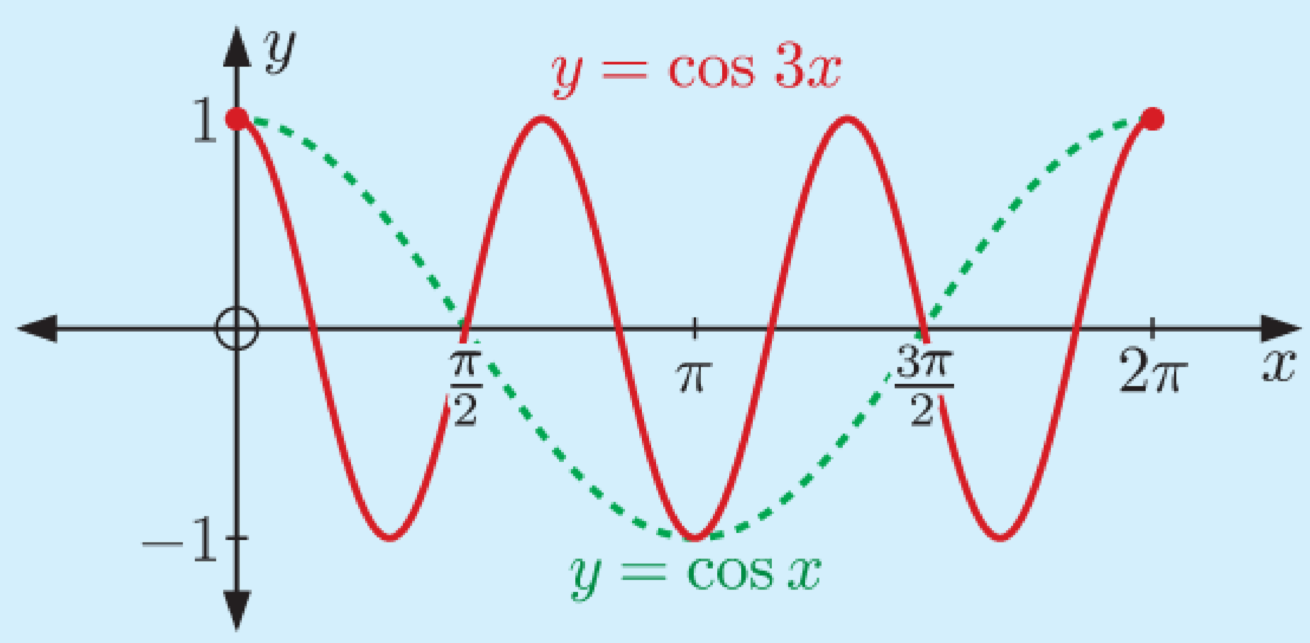
a $y = \sin\left(x - \frac{\pi}{3}\right)$ **b** $y = \cos 3x$ **c** $y = \cos\left(x + \frac{\pi}{6}\right) + 1$ **d** $y = -\sin x$

a We translate $y = \sin x$ horizontally $\frac{\pi}{3}$ units to the right.

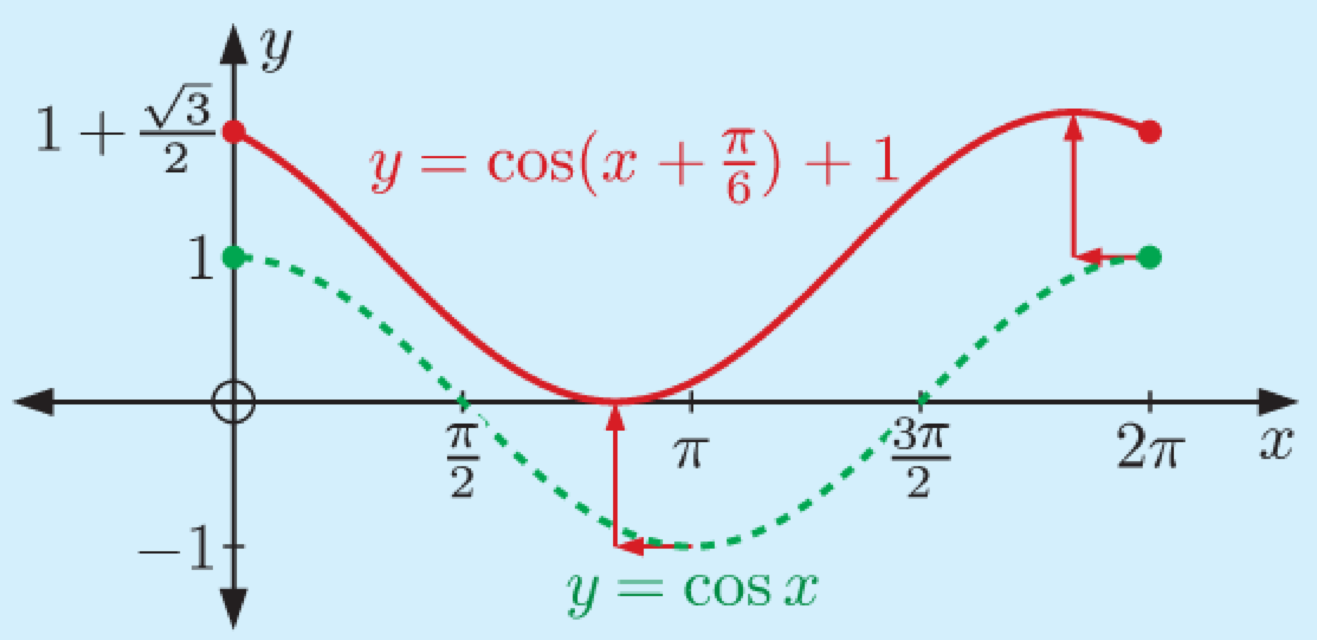


b We stretch $y = \cos x$ horizontally with scale factor $\frac{1}{3}$.

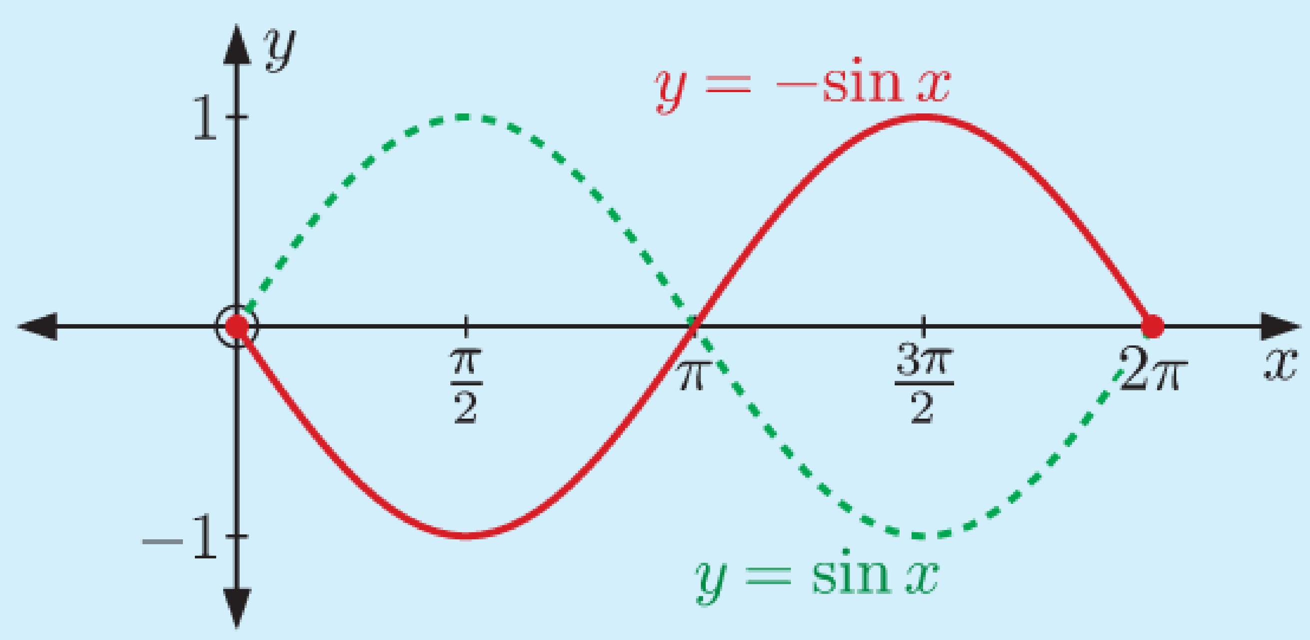
$\therefore$ $y = \cos 3x$ has period $\frac{2\pi}{3}$.



c We translate $y = \cos x$ horizontally $\frac{\pi}{6}$ units to the left, and 1 unit upwards.



d We reflect $y = \sin x$ in the x-axis.



EXERCISE 17C

1 State the transformation which maps $y = \sin x$ onto:

a $y = \sin x - 1$ **b** $y = \sin\left(x - \frac{\pi}{4}\right)$ **c** $y = 2\sin x$

d $y = \sin 4x$ **e** $y = \sin \frac{x}{4}$ **f** $y = \sin\left(x - \frac{\pi}{3}\right) + 2$

2 State the transformation which maps $y = \cos x$ onto:

a $y = \frac{1}{2}\cos x$ **b** $y = -\cos x$ **c** $y = \cos\left(x + \frac{\pi}{6}\right) - 2$

3 State the period of:

a $y = \sin 5x$ **b** $y = \sin(0.6x)$ **c** $y = \sin \pi x$

d $y = \cos 3x$ **e** $y = \cos \frac{x}{3}$ **f** $y = \cos \frac{\pi x}{50}$

4 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a 5π **b** $\frac{2\pi}{3}$ **c** 12π **d** 4 **e** 100

5 State the maximum and minimum value of:

a $y = 4\cos 2x$ **b** $y = 3\cos x + 5$ **c** $y = -2\cos(x - 3) - 4$

6 For the function $y = 4\sin 3x + 2$, state the:

a amplitude **b** period **c** range.

7 The general cosine function is $y = a\cos(b(x - c)) + d$.
State the geometrical significance of a, b, c, and d.

8 Sketch the graphs of the following for $0 \leqslant x \leqslant 4\pi$:

a $y = \sin x - 2$ **b** $y = \sin x + 3$ **c** $y = \sin x - 0.5$

d $y = \sin(x - 2)$ **e** $y = \sin(x + 2)$ **f** $y = \sin\left(x - \frac{\pi}{4}\right)$

g $y = \sin\left(x - \frac{\pi}{6}\right) + 1$ **h** $y = \sin(x - 1) - 2$ **i** $y = \sin\left(x + \frac{\pi}{4}\right) + 2$

j $y = 3\sin x$ **k** $y = \frac{1}{2}\sin x$ **l** $y = \frac{3}{2}\sin x$

m $y = \sin 3x$ **n** $y = \sin \frac{x}{2}$ **o** $y = \sin 4x$

9 Sketch the graphs of the following for $-2\pi \leqslant x \leqslant 2\pi$:

a $y = \cos x + 2$ **b** $y = \cos\left(x - \frac{\pi}{4}\right)$ **c** $y = \cos\left(x + \frac{\pi}{6}\right)$

d $y = \frac{3}{2}\cos x$ **e** $y = -\cos x$ **f** $y = \cos\left(x - \frac{\pi}{6}\right) + 1$

g $y = \cos\left(x + \frac{\pi}{4}\right) - 1$ **h** $y = \cos 2x$ **i** $y = \cos \frac{x}{2}$

10 **a** Sketch the curve $y = 4\sin x$ for $0 \leqslant x \leqslant 2\pi$.

b Find the value of y when: **i** $x = \frac{5\pi}{6}$ **ii** $x = \frac{7\pi}{4}$
Mark these points on your graph in **a**.

11 For what values of d does the graph of $y = 3\cos x + d$ lie:

a entirely above the x-axis

b entirely below the x-axis

c partially above and partially below the x-axis?

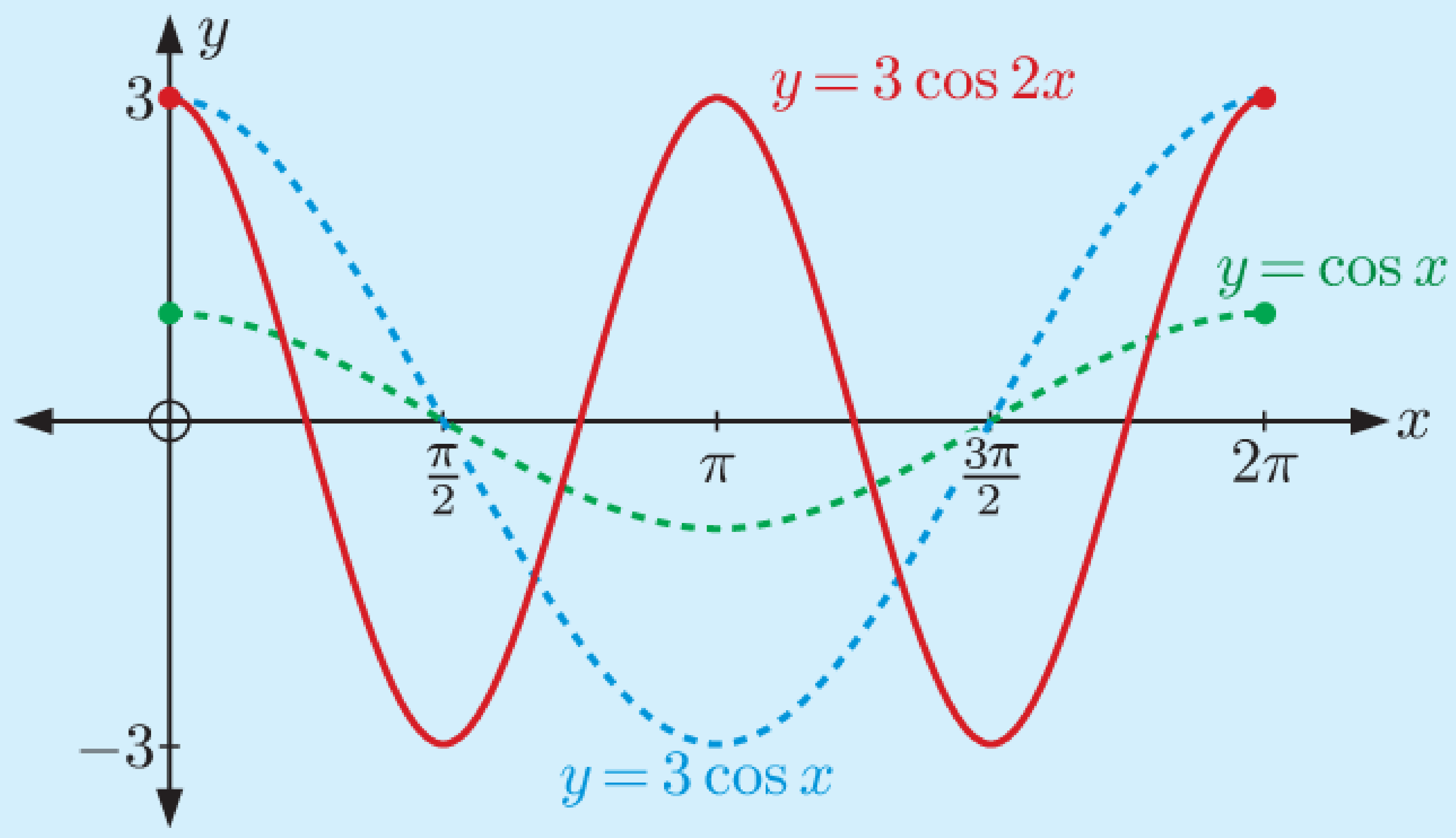
Example 2 **Self Tutor**

Sketch the graph of $y = 3\cos 2x$ for $0 \leqslant x \leqslant 2\pi$.

$a = 3$, so the amplitude is $|3| = 3$.

$b = 2$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

We stretch $y = \cos x$ vertically with scale factor 3 to give $y = 3\cos x$, then stretch $y = 3\cos x$ horizontally with scale factor $\frac{1}{2}$ to give $y = 3\cos 2x$.



12 State the transformations which map:

a $y = \sin x$ onto $y = 2\sin 3x$

b $y = \cos x$ onto $y = -2\cos x$

c $y = \sin x$ onto $y = 3\sin x - 5$

d $y = \cos x$ onto $y = \cos\left(2\left(x + \frac{\pi}{6}\right)\right)$

13 Sketch the graphs of the following for $-2\pi \leqslant x \leqslant 2\pi$:

a $y = -3\sin x$

b $y = \cos 2x + 1$

c $y = \frac{1}{2}\sin\left(x + \frac{\pi}{6}\right) - \frac{1}{3}$

d $y = \frac{1}{3}\cos\left(x + \frac{\pi}{4}\right) + 1$

e $y = 3\sin\left(x - \frac{\pi}{3}\right) - 1$

f $y = -\cos\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right)$

14 Consider the general sine function $y = a\sin(b(x - c)) + d$. State which of the variables a, b, c, and d can be changed to always produce a change in:

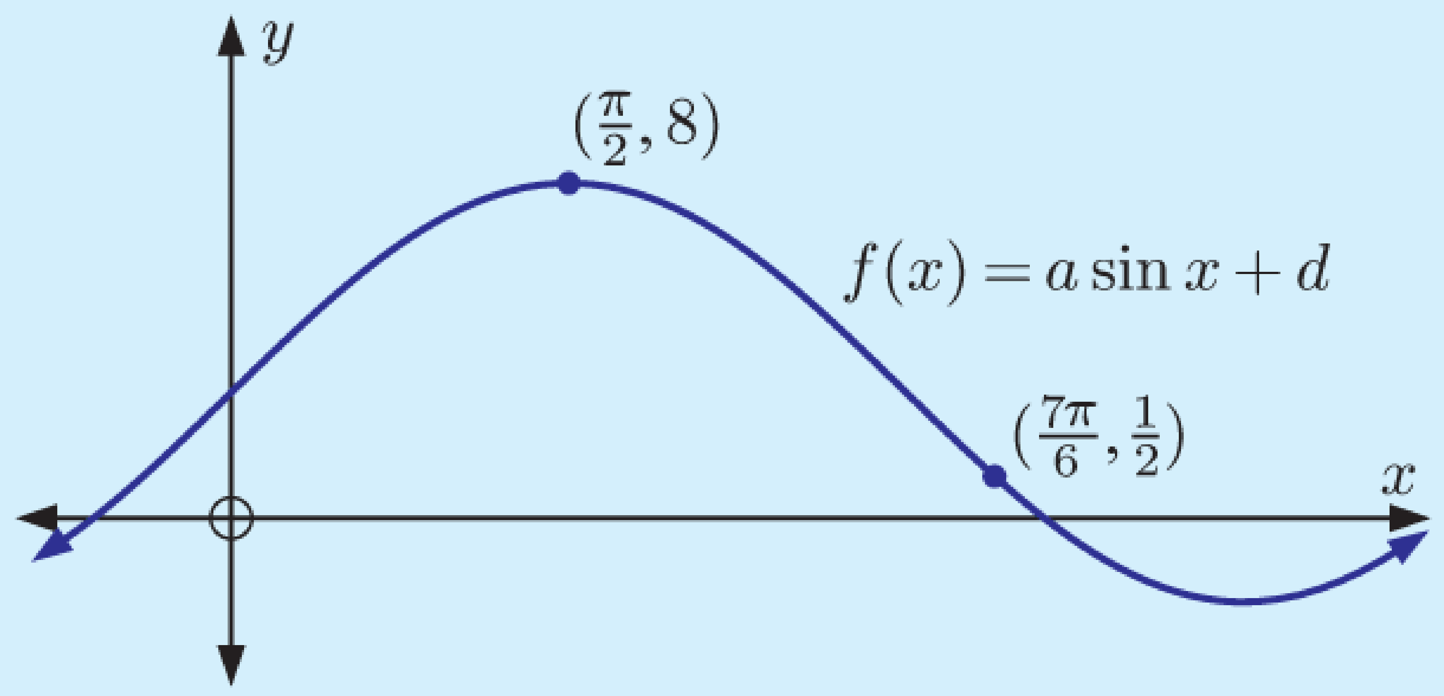
a the x-intercepts of the function

b the y-intercept of the function

c the range of the function.

Example 3 **Self Tutor**

Find the unknowns in this function:



$f\left(\frac{\pi}{2}\right) = 8$, so $a\sin\frac{\pi}{2} + d = 8$

$\therefore\ a + d = 8$ .... (1)

$f\left(\frac{7\pi}{6}\right) = \frac{1}{2}$, so $a\sin\frac{7\pi}{6} + d = \frac{1}{2}$

$\therefore\ -\frac{1}{2}a + d = \frac{1}{2}$ .... (2)

So, we have
$a + d = 8$ {(1)}
$-a + 2d = 1$ {2 × (2)}

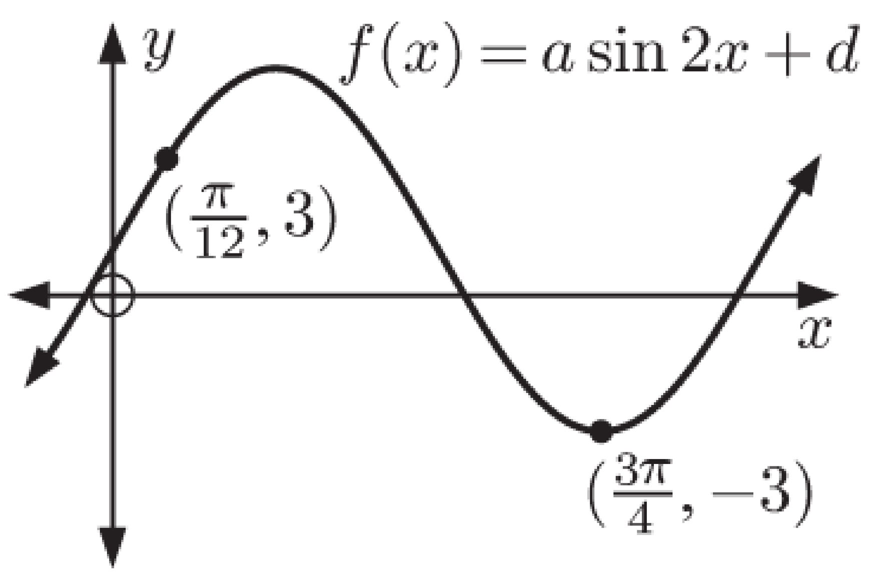
Adding, $3d = 9$ and so $d = 3$

Substituting $d = 3$ into (1) gives $a + 3 = 8$

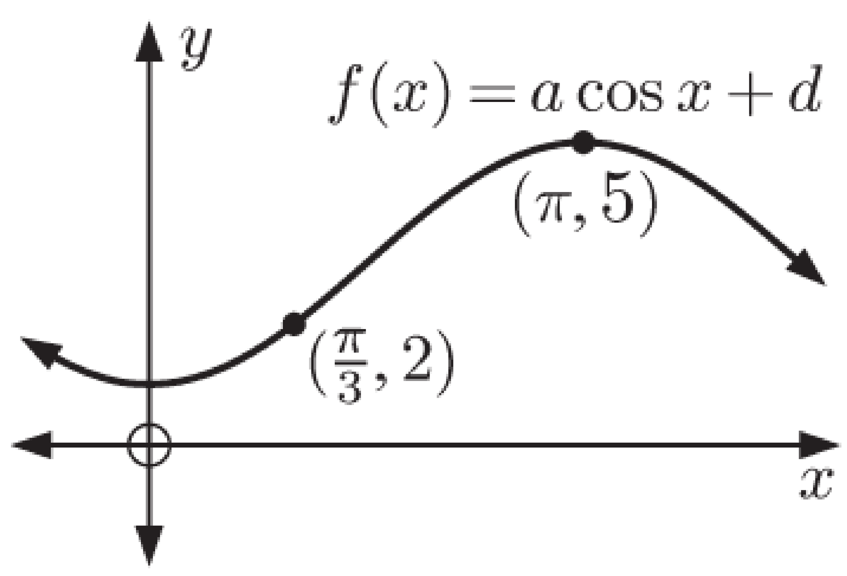
$\therefore\ a = 5$

15 Find the unknowns in each function:

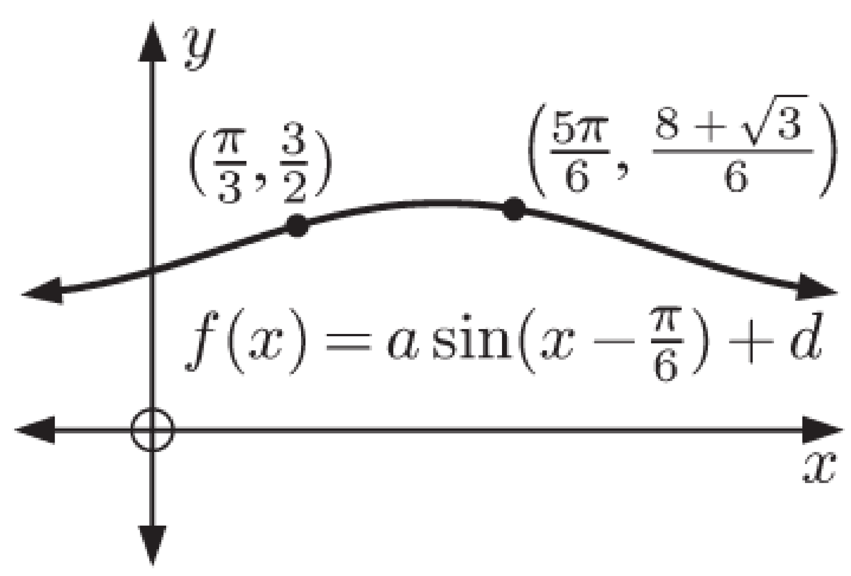
a



b

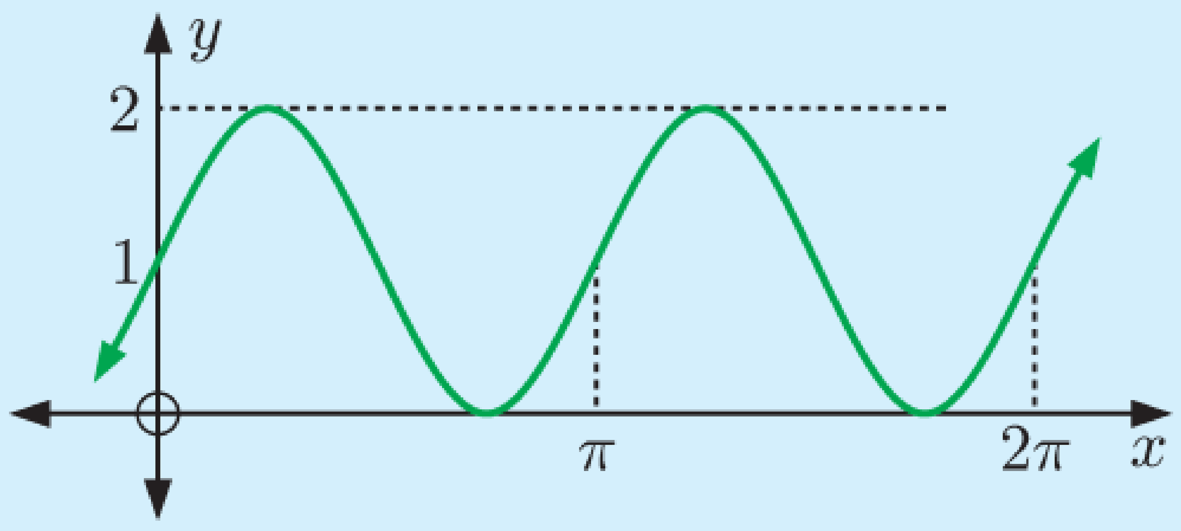


c



Example 4 Self Tutor

Find the equation of this sine function.



The amplitude is 1, so $a = 1$.

The period is π, so $\frac{2\pi}{b} = \pi$ and $\therefore$ $b = 2$.

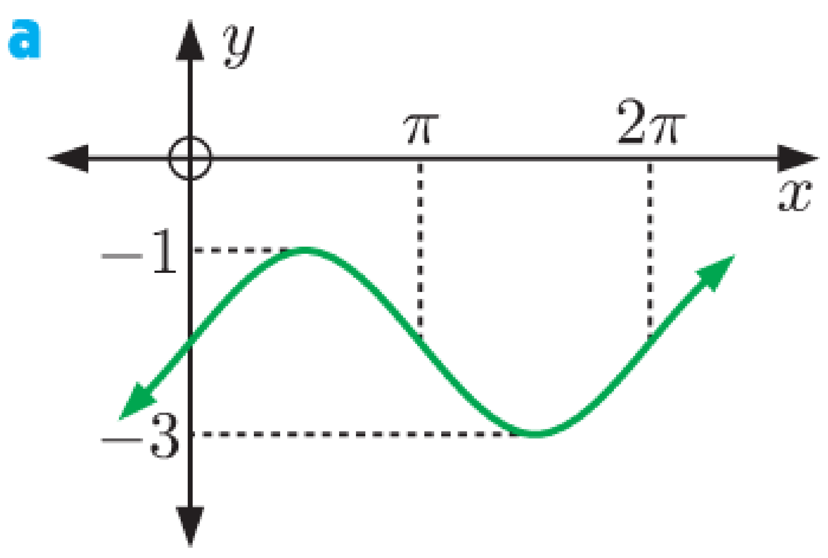
There is no horizontal translation, so $c = 0$.

The principal axis is $y = 1$, so $d = 1$.

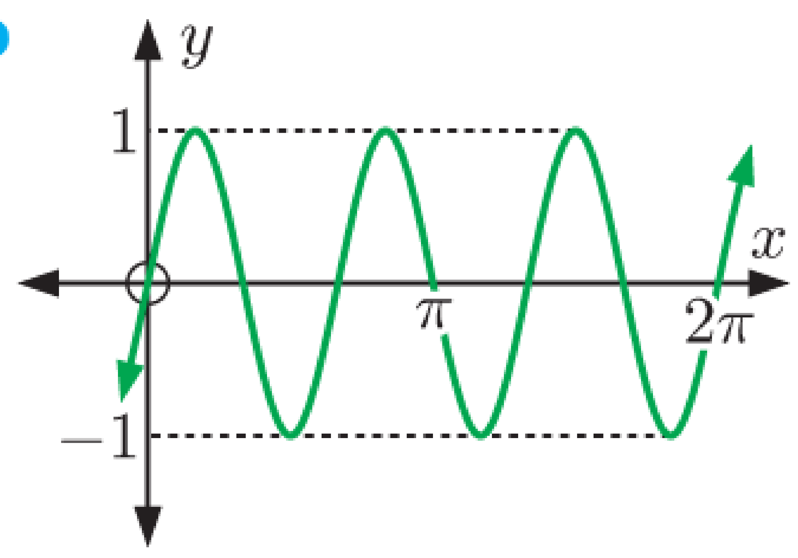
The equation of the function is $y = \sin 2x + 1$.

16 Find the equation of each sine function:

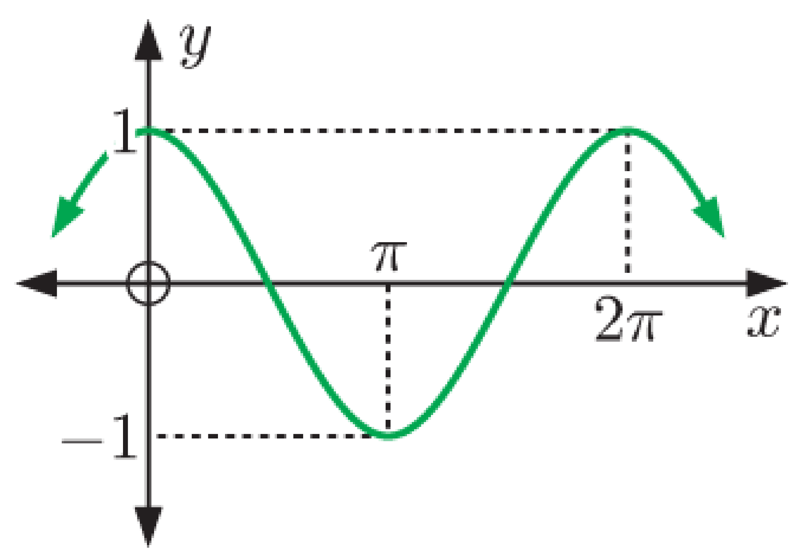
a



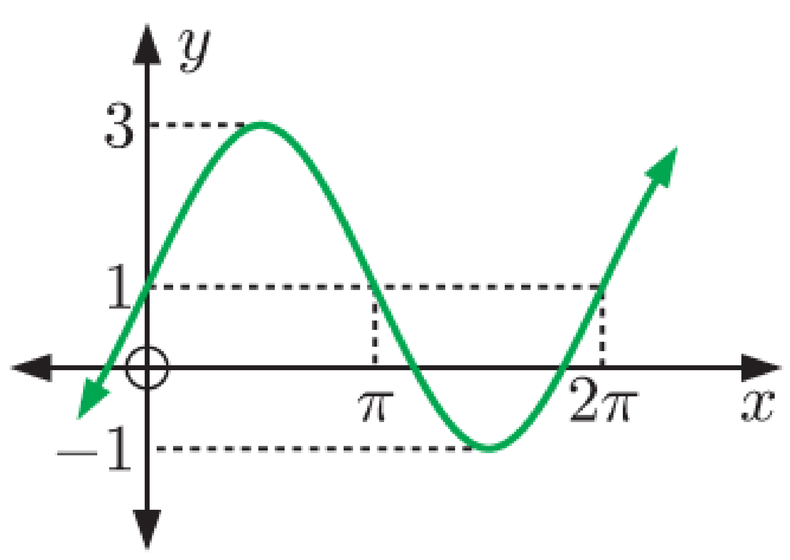
b



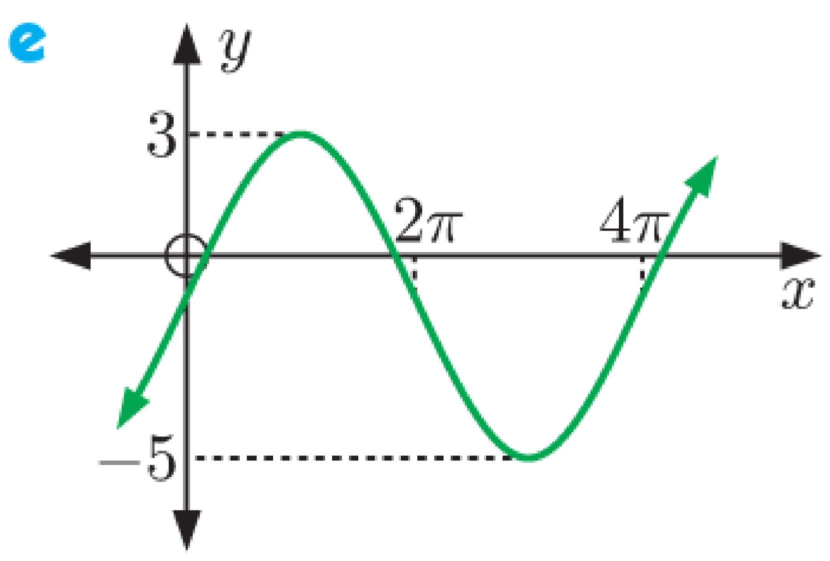
c



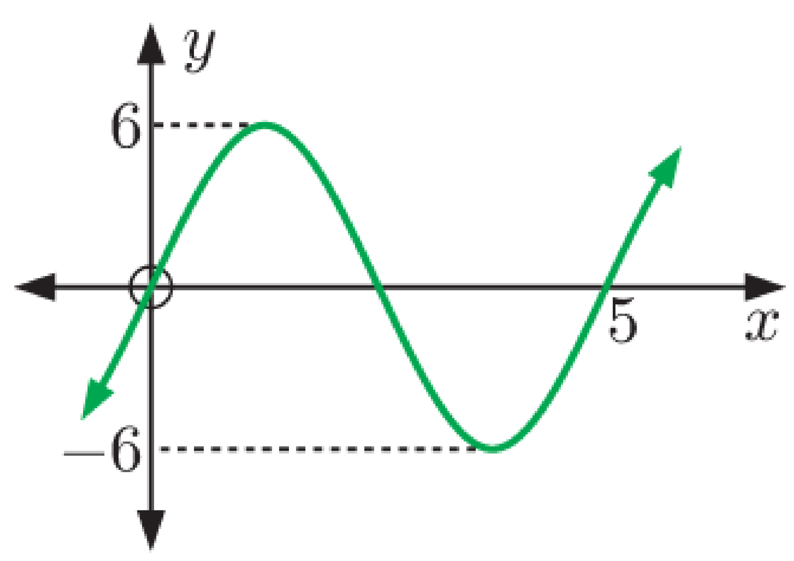
d



e

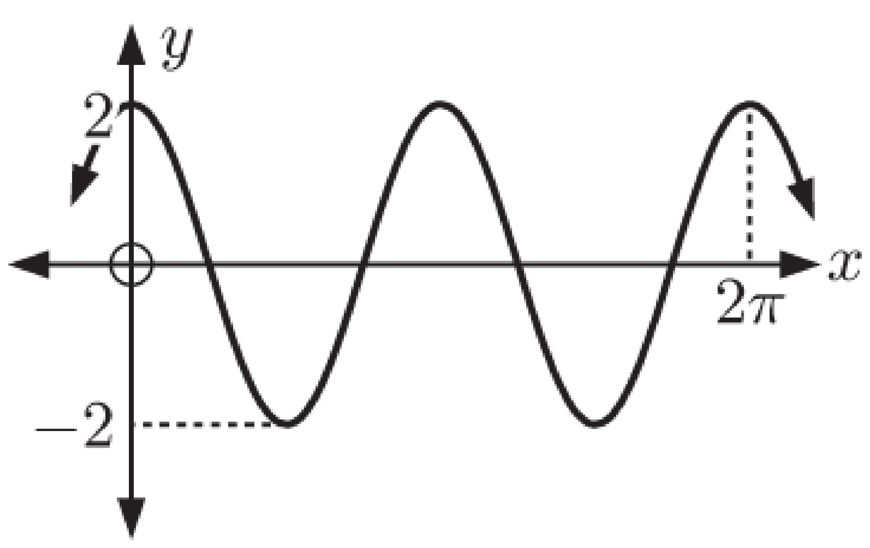


f

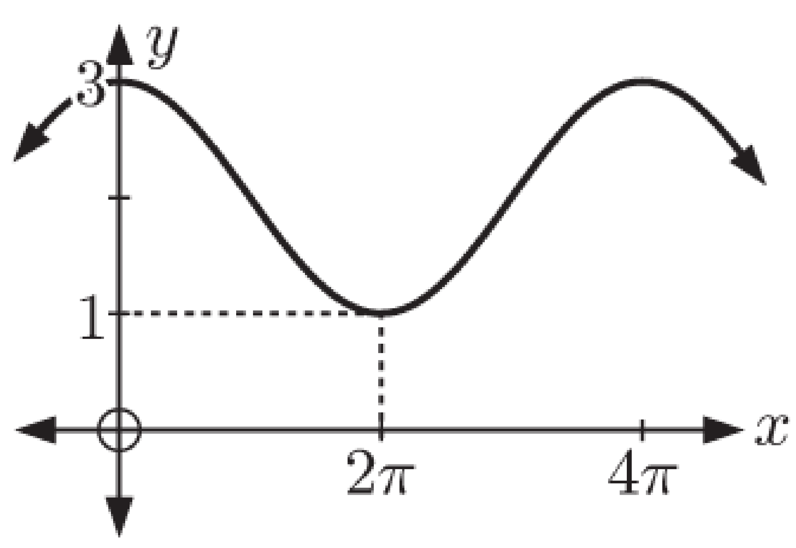


17 Find the cosine function shown in the graph:

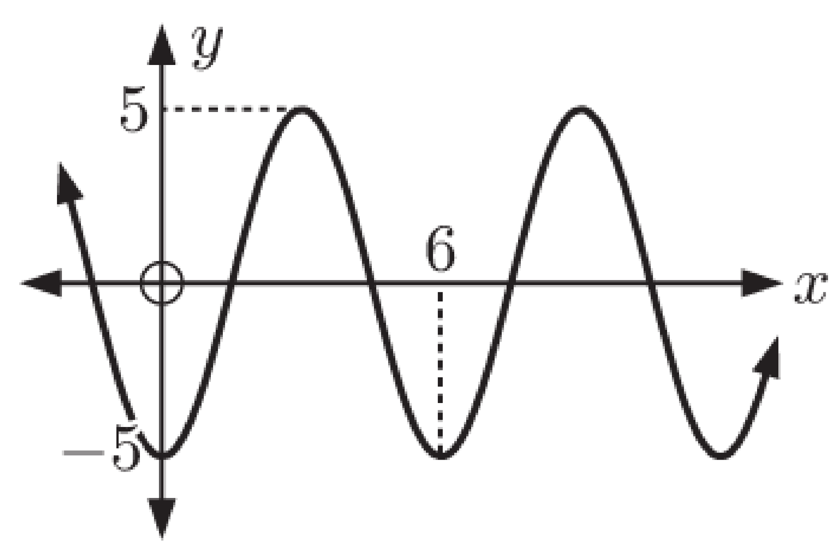
a



b



c



D MODELLING PERIODIC BEHAVIOUR

The sine and cosine functions are both referred to as **sinusoidal functions**. They can be used to model many periodic phenomena in the real world. In some cases, such as the movement of the hands on a clock, the models we find will be almost exact. In other cases, such as the maximum daily temperature of a city over a year, the model will be less accurate.

Example 5 Self Tutor

The average daytime temperature for a city is given by the function $D(t) = 5\cos\left(\frac{\pi}{6}t\right) + 20$ °C, where t is the time in months after January.

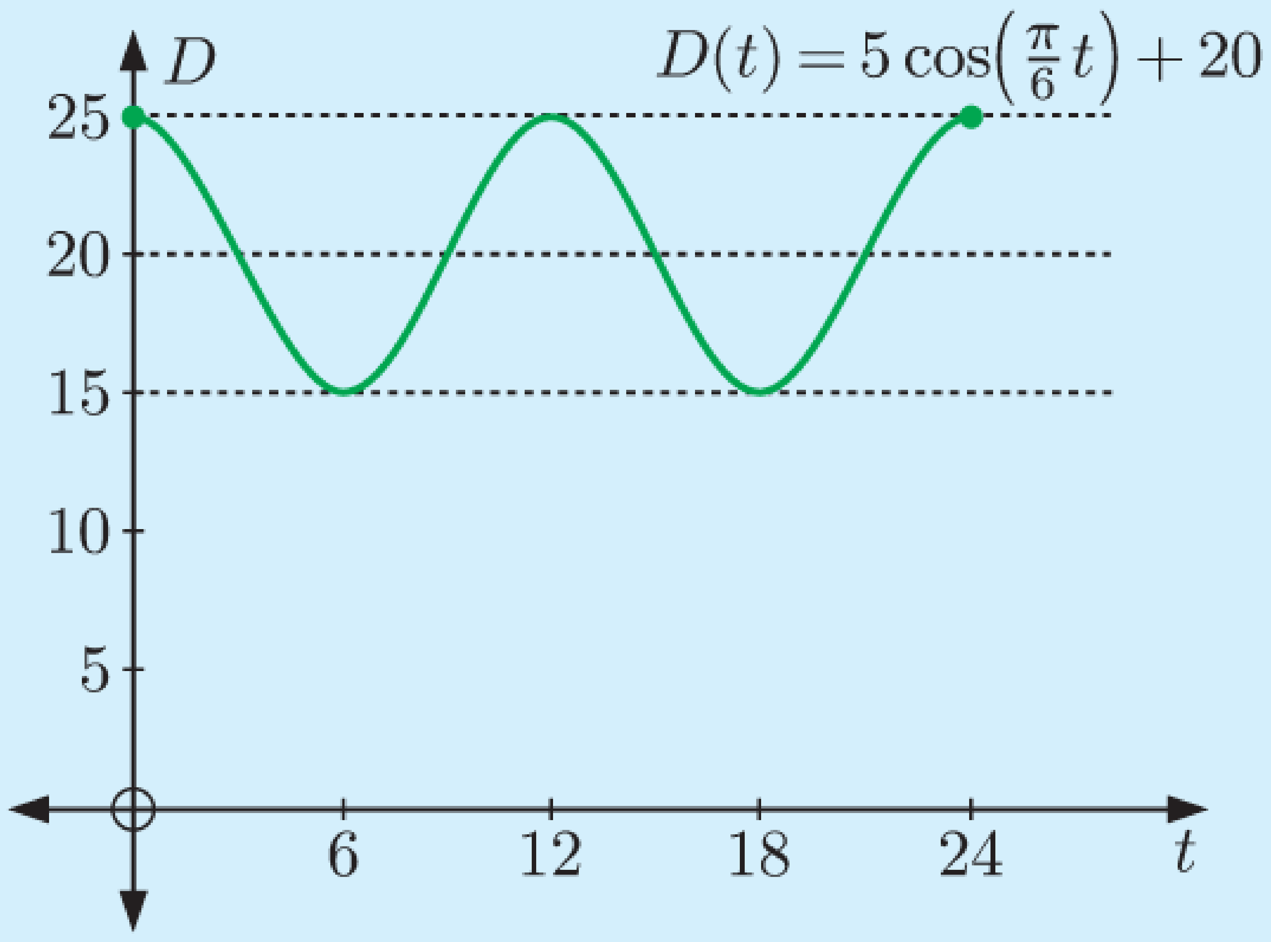
a Sketch the graph of D against t for $0 \leqslant t \leqslant 24$.

b Find the average daytime temperature during May.

c Find the minimum average daytime temperature, and the month in which it occurs.

a For $D(t) = 5\cos\left(\frac{\pi}{6}t\right) + 20$:

- the amplitude is 5
- the period is $\dfrac{2\pi}{\left(\frac{\pi}{6}\right)} = 12$ months
- the principal axis is $D = 20$.



b May is 4 months after January.

When $t = 4$, $D = 5 \times \cos\frac{4\pi}{6} + 20$

$= 5 \times \left(-\frac{1}{2}\right) + 20$

$= 17.5$

So, the average daytime temperature during May is 17.5°C.

c The minimum average daytime temperature is $20 - 5 = 15$°C, which occurs when $t = 6$ or 18.

So, the minimum average daytime temperature occurs during July.

EXERCISE 17D

1 The temperature inside Vanessa's house t hours after midday is given by the function $T(t) = 6\sin\left(\frac{\pi}{12}t\right) + 26$ °C.

a Sketch the graph of T against t for $0 \leqslant t \leqslant 24$.

b Find the temperature inside Vanessa's house at:

i midnight **ii** 2 pm.

c Find the maximum temperature inside Vanessa's house, and the time at which it occurs.

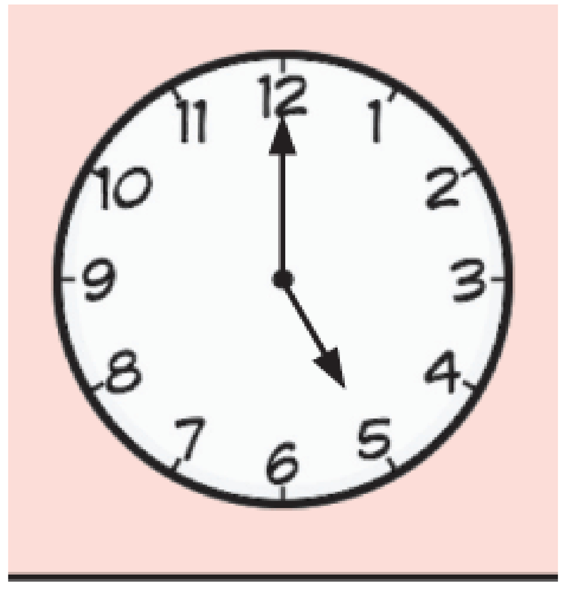
2 The depth of water in a harbour t hours after midnight is $D(t) = 4\cos\left(\frac{\pi}{6}t\right) + 6$ metres.

a Sketch the graph of D against t for $0 \leqslant t \leqslant 24$.

b Find the highest and lowest depths of the water, and the times at which they occur.

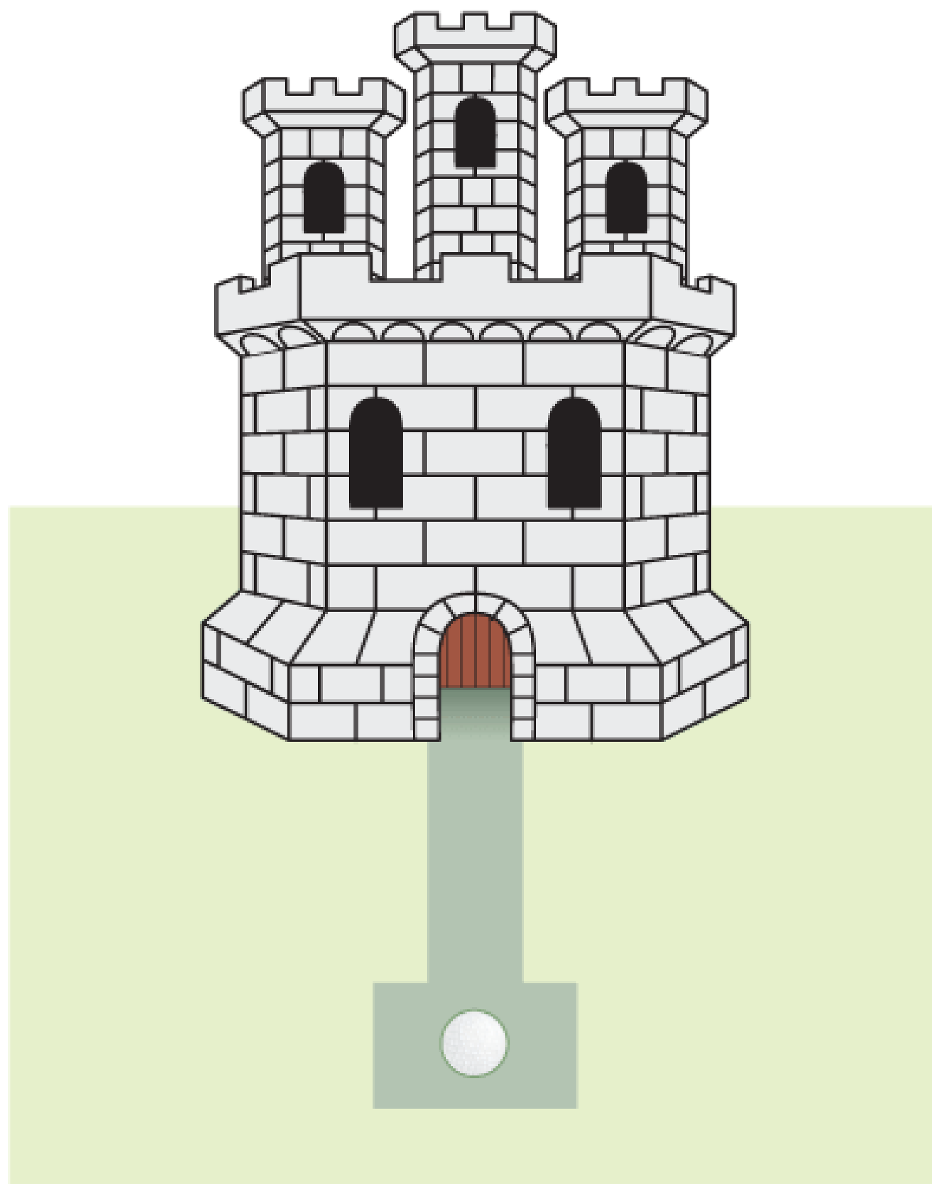
c A boat requires a water depth of 5 metres to sail in. Will the boat be able to enter the harbour at 8 pm?

3 The tip of a clock's minute hand is $H(t) = 15\cos\left(\frac{\pi}{30}t\right) + 150$ cm above ground level, where t is the time in minutes after 5 pm.



- **a** Sketch the graph of H against t for $0 \leqslant t \leqslant 180$.
- **b** Find the length of the minute hand.
- **c** Find, rounded to 1 decimal place, the height of the minute hand's tip at:
  - **i** 5:08 pm **ii** 5:37 pm **iii** 5:51 pm **iv** 6:23 pm

4 On a mini-golf hole, golfers must putt the ball through a castle's entrance. The entrance is protected by a gate which moves up and down.
The height of the gate above the ground t seconds after it touches the ground is $H(t) = 4\sin\left(\frac{\pi}{4}(t-2)\right) + 4$ cm.



- **a** Sketch the graph of H against t for $0 \leqslant t \leqslant 16$.
- **b** Find the height of the gate above the ground 2 seconds after the gate touches the ground.
- **c** Eric is using a golf ball with radius 2.14 cm. He putts the ball 1 second after the gate touches the ground, and the ball takes 5.3 seconds to reach the castle's entrance. Will the ball pass through the entrance?

Example 6 **Self Tutor**

On a hot summer day in Madrid, Antonio pays careful attention to the temperature. The maximum of 41.8°C occurs at 3:30 pm. The minimum was 27.5°C. Suggest a sine function to model the temperature for that day.

The mean temperature $= \dfrac{41.8 + 27.5}{2} = 34.65°\text{C}$, so $d = 34.65$.

The amplitude $= \dfrac{41.8 - 27.5}{2} = 7.15°\text{C}$

$\therefore\ a = 7.15$

The period is 24 hours, so $b = \frac{2\pi}{24} = \frac{\pi}{12}$.

The maximum occurs at 3:30 pm, so we assume the temperature passed its mean value 6 hours earlier, at 9:30 am.

The day begins at midnight, so the function is shifted $9\frac{1}{2}$ hours to the right, thus $c = 9.5$.

If t is the number of hours after midnight, the temperature T is modelled by

$$T(t) = 7.15\sin\left(\tfrac{\pi}{12}(t - 9.5)\right) + 34.65\ °\text{C}.$$

5 On a September day in Moscow, the maximum temperature 15.8°C occurred at 2 pm. The minimum was 5.4°C. Suggest a sine function to model the temperature for that day. Let T be the temperature and t be the time in hours after midnight.

6 The ferry operator at Picton, New Zealand, is studying the tides. High tides occur every 12.4 hours. The first high tide tomorrow will be at 1:30 am. The high tide will be 1.36 m and the low tide will be 0.16 m. Find a cosine function to model the tide height for the day. Let H be the tide height and t be the time in hours after midnight.

7 Answer the **Opening Problem** on page **448**.

8 Some of the largest tides in the world are observed in Canada's Bay of Fundy. The difference between high and low tides is 14 metres, and the average time difference between high tides is about 12.4 hours. On a particular day, the first high tide is 16.2 m, occurring at 9 am.

- **a** Find a sine model for the height of the tide H in terms of the time t.
- **b** Sketch the graph of the function for that day.

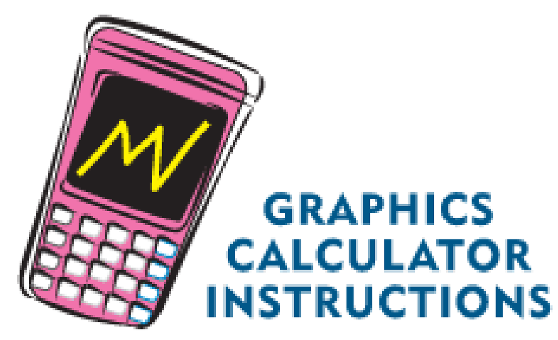
9 On an analogue clock, the hour hand is 6 cm long and the minute hand is 12 cm long. Let t be the time in hours after midnight.

- **a** Write a cosine function for the height of the tip of the hour hand relative to the centre of the clock.
- **b** Write a sine function for the horizontal displacement of the tip of the minute hand relative to the centre of the clock.

E FITTING TRIGONOMETRIC MODELS TO DATA

Suppose we have **data** in which we observe periodic behavior. In such cases, we usually cannot fit an *exact* model. However, we can still apply the same principles to estimate the period, amplitude, and principal axis from the data.

You can check your models using your graphics calculator. Click on this icon for instructions.



Example 7 Self Tutor

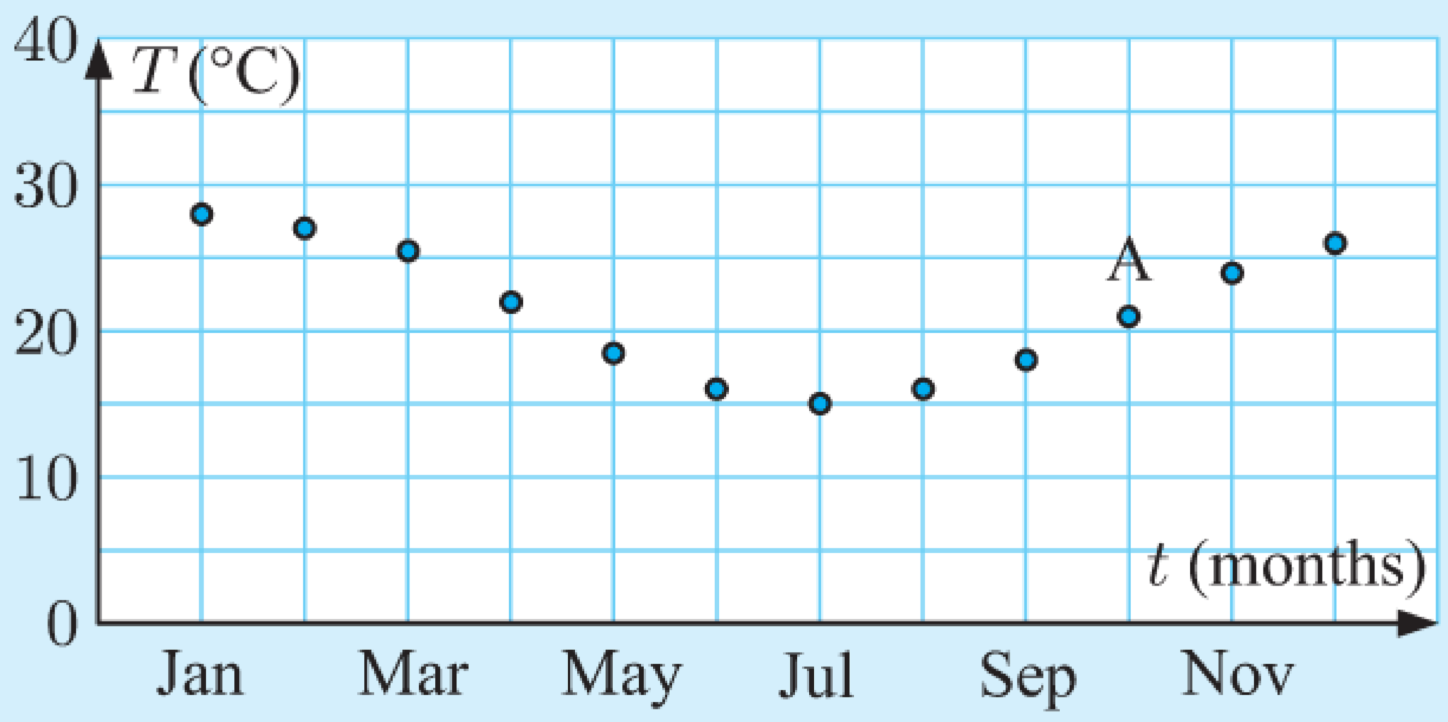
The mean monthly maximum temperatures for Cape Town, South Africa are shown below:

Month (t)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature (T °C)	28	27	25.5	22	18.5	16	15	16	18	21.5	24	26

We want to model the data with a trigonometric function of the form $T = a\sin(b(t-c)) + d$ where Jan $\equiv 1$, Feb $\equiv 2$, and so on.

- **a** Draw a scatter diagram of the data.
- **b** Without using technology, estimate:
  - **i** b
  - **ii** a
  - **iii** d
  - **iv** c
- **c** Check your answers using technology.

a



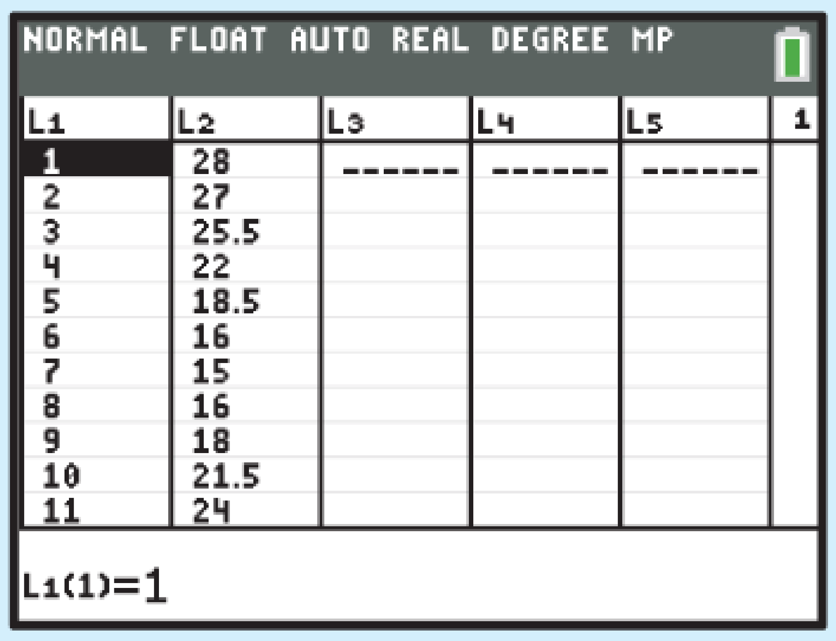
b **i** The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore$ $b = \frac{\pi}{6}$.

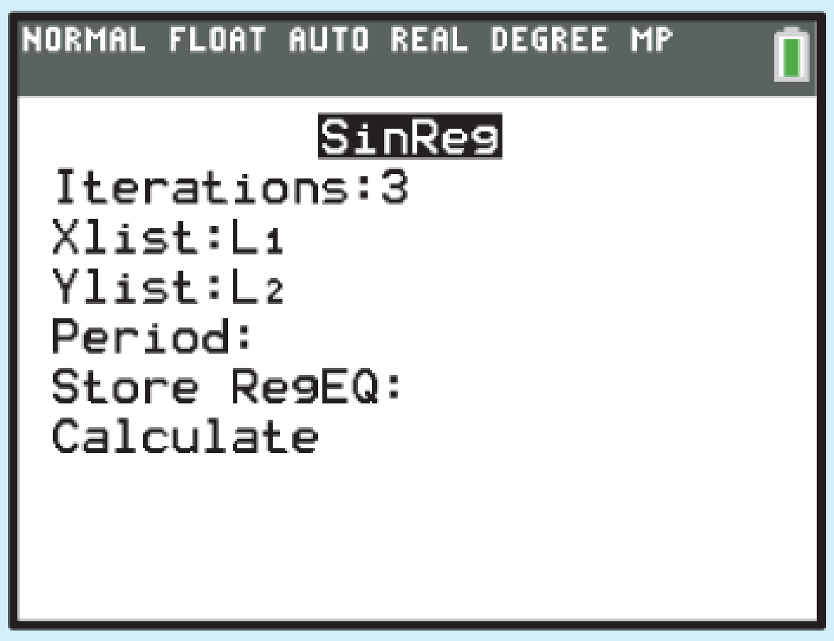
ii The amplitude $= \frac{\text{max} - \text{min}}{2} \approx \frac{28 - 15}{2} \approx 6.5$, so $a \approx 6.5$.

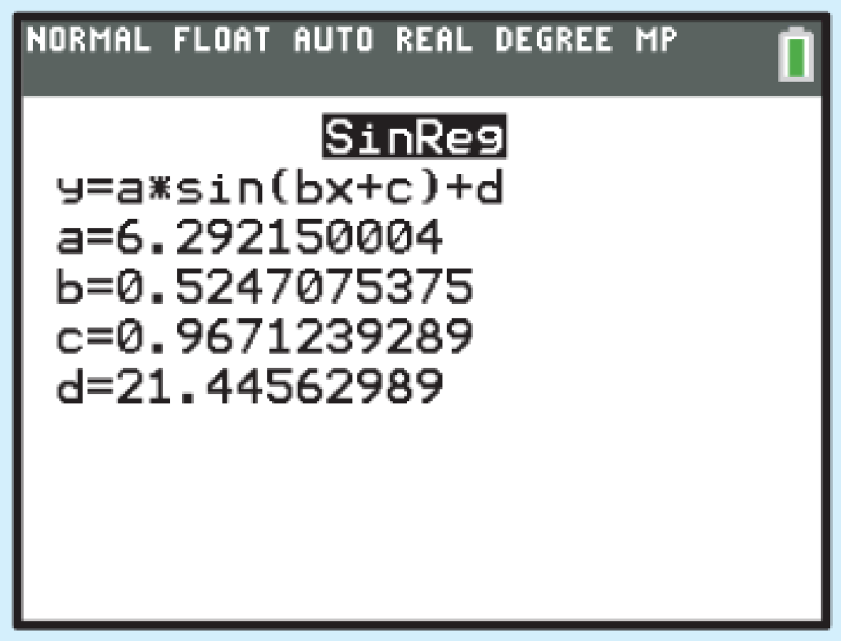
iii The principal axis is midway between the maximum and minimum, so
$d \approx \frac{28 + 15}{2} \approx 21.5$.

iv The model is $T \approx 6.5 \sin\left(\frac{\pi}{6}(t - c)\right) + 21.5$ for some constant c.
On the original graph, point A is the first point shown at which the sine function starts a new period. Since A is at (10, 21.5), $c = 10$.

c From **b**, our model is $T \approx 6.5 \sin\left(\frac{\pi}{6}(t - 10)\right) + 21.5$
$\approx 6.5 \sin(0.524t - 5.24) + 21.5$

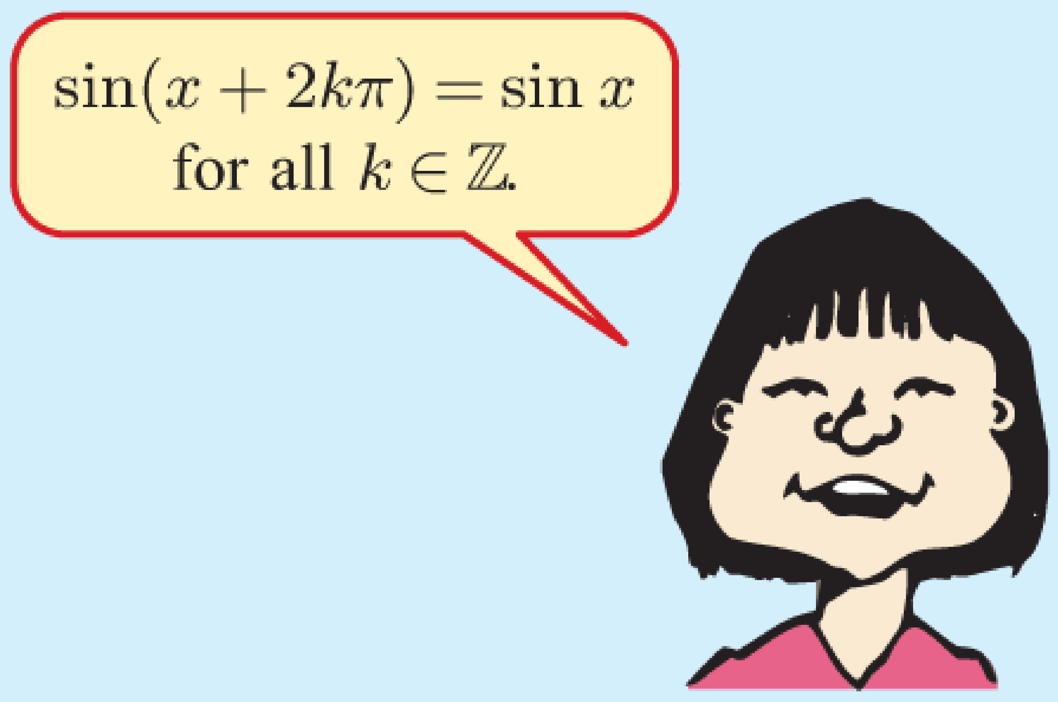






Using technology,

$T \approx 6.29 \sin(0.525t + 0.967) + 21.4$
$\approx 6.29 \sin(0.525t + 0.967 - 2\pi) + 21.4$
$\approx 6.29 \sin(0.525t - 5.32) + 21.4$



EXERCISE 17E

1 Below is a table which shows the mean monthly maximum temperatures for a city in Greece.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	14	15	18	21	25	27	26	24	20	18	16

a Draw a scatter diagram of the data.

b What features of the data suggest a trigonometric model is appropriate?

c Your task is to model the data with a sine function of the form $T \approx a \sin(b(t - c)) + d$, where Jan $\equiv 1$, Feb $\equiv 2$, and so on.
Without using technology, estimate:

i b **ii** a **iii** d **iv** c

d Use technology to check your model. How well does your model fit?

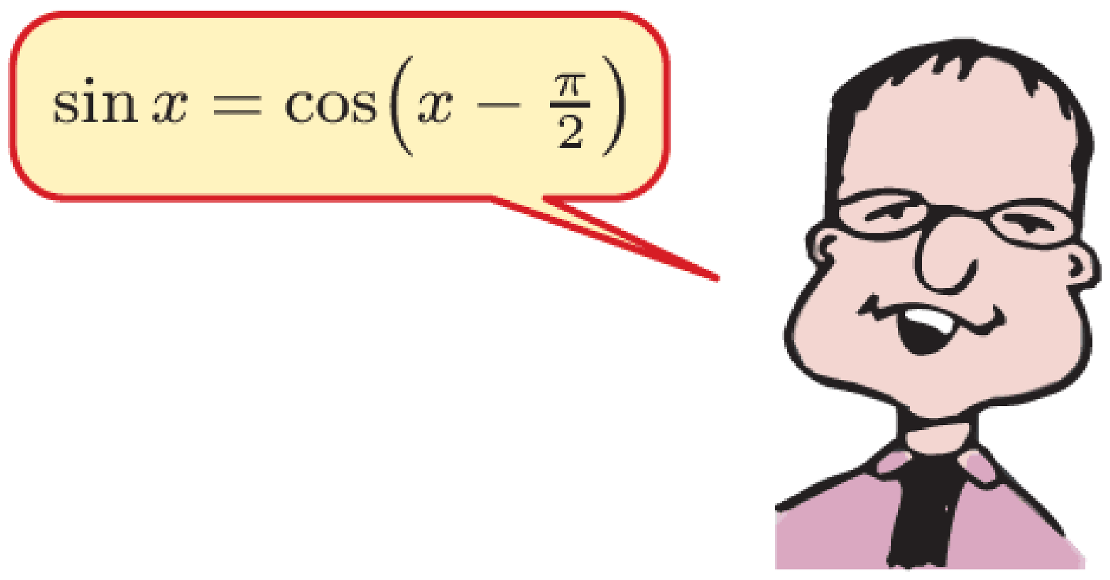
2 The data in the table shows the mean monthly temperatures for Christchurch, New Zealand.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14

a Find a cosine model for this data in the form $T \approx a\cos(b(t-c))+d$ without using technology. Let Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Draw a scatter diagram of the data and sketch the graph of your model on the same set of axes.

c Use technology to check your answer to **a**.



3 At the Mawson base in Antarctica, the mean monthly temperatures for the last 30 years are:

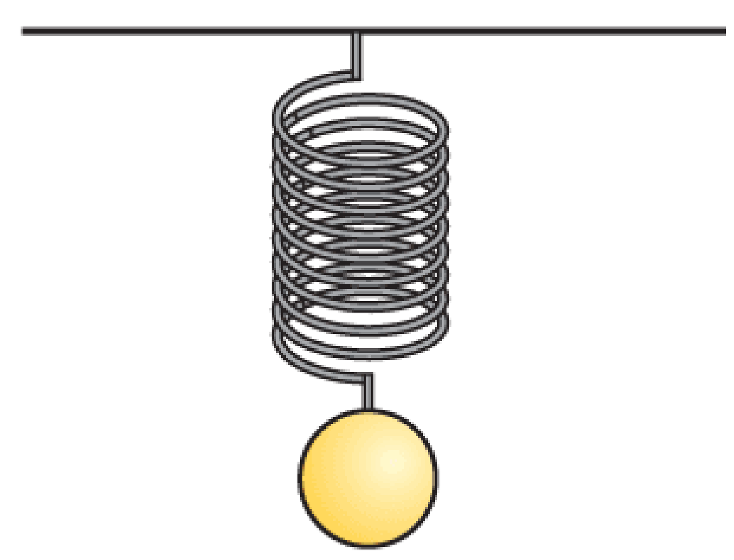
Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	0	0	-4	-9	-14	-17	-18	-19	-17	-13	-6	-2

a Find a sine model for this data without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Draw a scatter diagram of the data and sketch the graph of your model on the same set of axes.

c How appropriate is the model?

4 An object is suspended from a spring. If the object is pulled below its resting position and then released, it will oscillate up and down. The data below shows the height of the object relative to its rest position, at different times.



Time (t seconds)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Height (H cm)	-15	-13	-7.5	0	7.5	13	15	13	7.5	0

Time (t seconds)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Height (H cm)	-7.5	-13	-15	-13	-7.5	0	7.5	13	15	13	7.5

a Draw a scatter diagram of the data.

b Find a trigonometric function which models the height of the object over time.

c Use your model to predict the height of the object after 4.25 seconds.

d What do you think is unrealistic about this model? What would happen differently in reality?

RESEARCH

1 How accurately will a trigonometric function model the phases of the moon?

2 Are there any periodic phenomena which can be modelled by the *sum* of trigonometric functions?

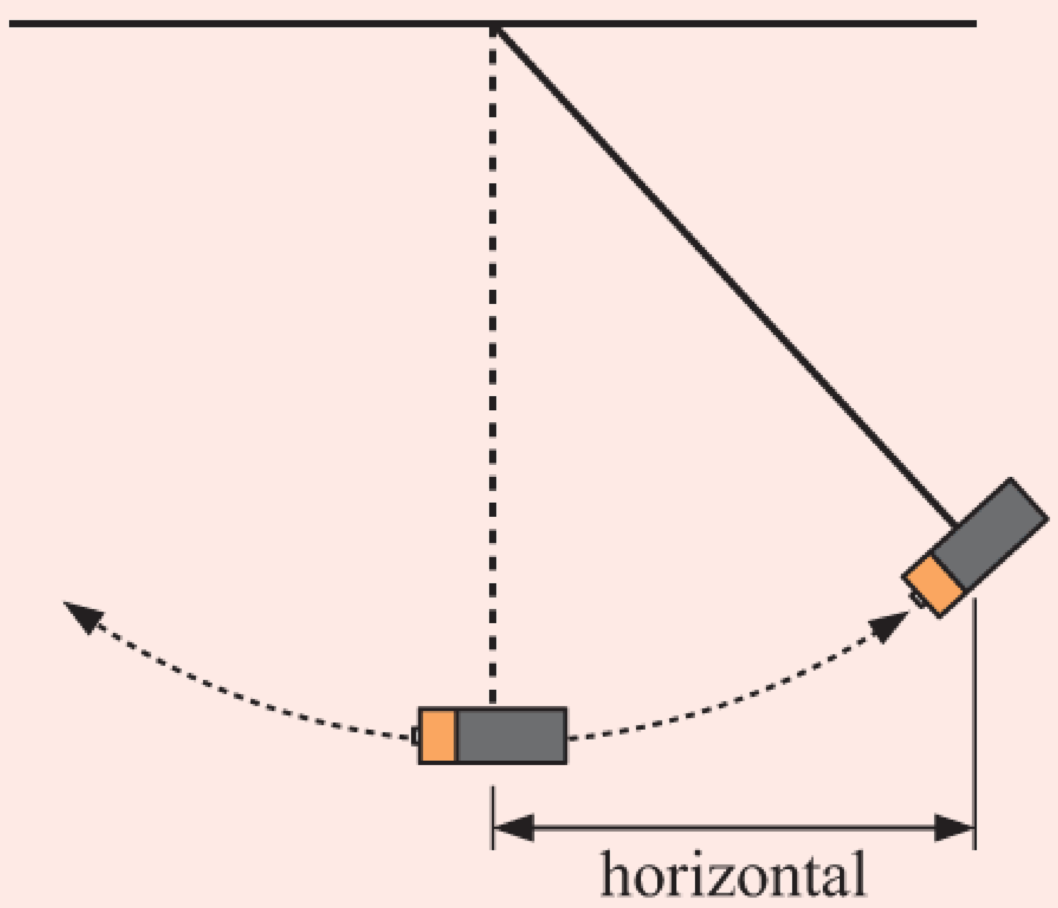
ACTIVITY 2 — THE PENDULUM

In this Activity you will work in small groups to model the behaviour of a pendulum.

You will need: string, sticky tape, a ruler, a stopwatch, and a AA battery.

What to do:

1. Cut a piece of string of length 75 cm. Attach one end of the string to the battery, and the other end to your desk.
2. Hold the battery to one side, then release it, causing the battery to swing back and forth like a pendulum.
3. Using your stopwatch and ruler, measure the maximum and minimum horizontal displacement reached by the battery, and the times at which they occurred. You may need to repeat the experiment several times, but make sure the battery is released from the same position each time.



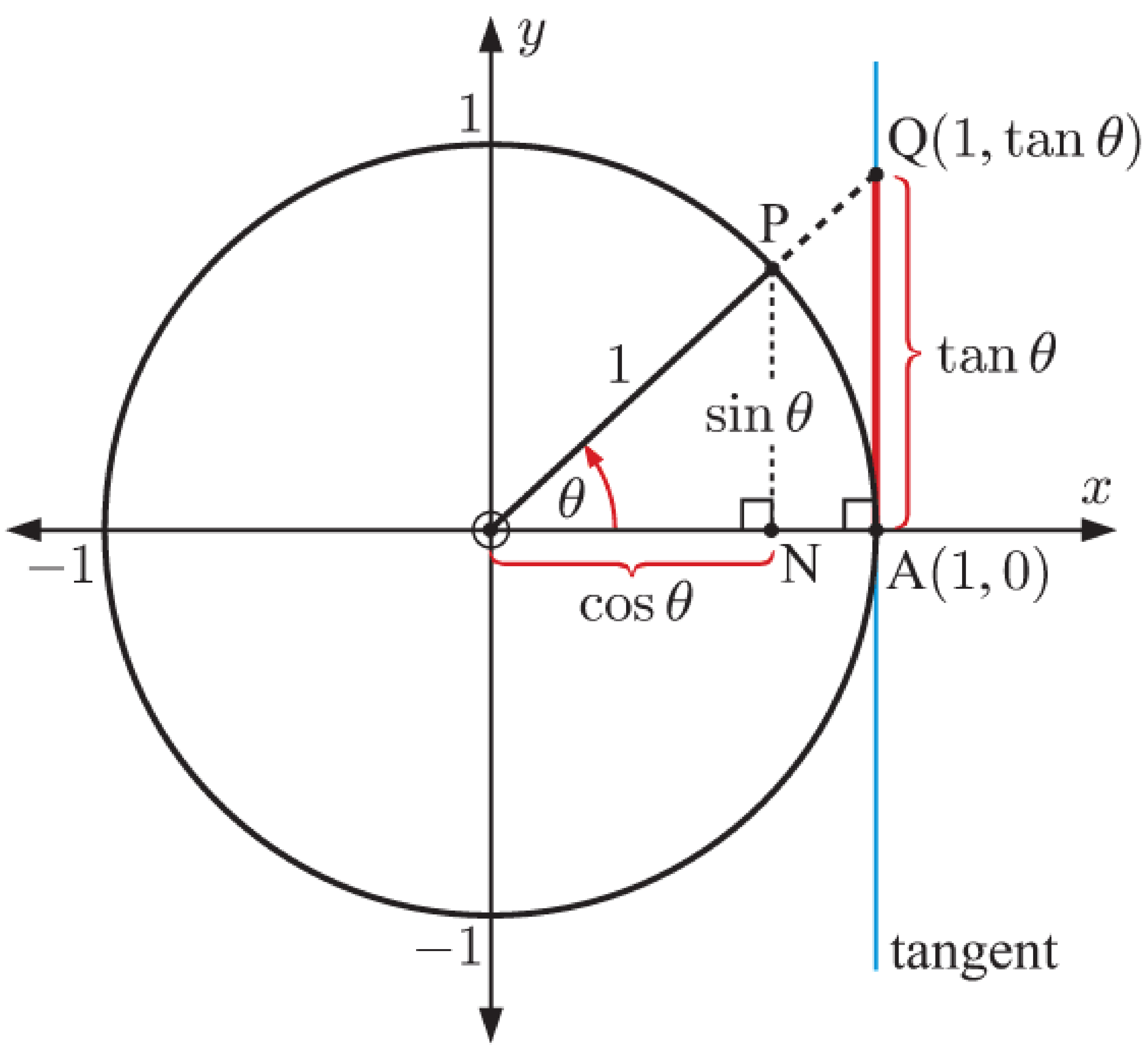
4. Use your data to find a trigonometric function which models the horizontal displacement of the battery over time.
5. What part of the function affects the *period* of the pendulum?
6. Repeat the experiment with strings of different length. Explore the relationship between the length of the string and the period of the pendulum.

F THE TANGENT FUNCTION

We have seen that if $P(\cos\theta, \sin\theta)$ is a point which is free to move around the unit circle, and if [OP] is extended to meet the tangent at $A(1, 0)$, the intersection between these lines occurs at $Q(1, \tan\theta)$.

This enables us to define the **tangent function**

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$



We have also seen that $\tan\theta$ is:

- positive in quadrants 1 and 3
- negative in quadrants 2 and 4
- periodic with period π.

DISCUSSION

What happens to $\tan\theta$ when P is at:

a $(1, 0)$ and $(-1, 0)$

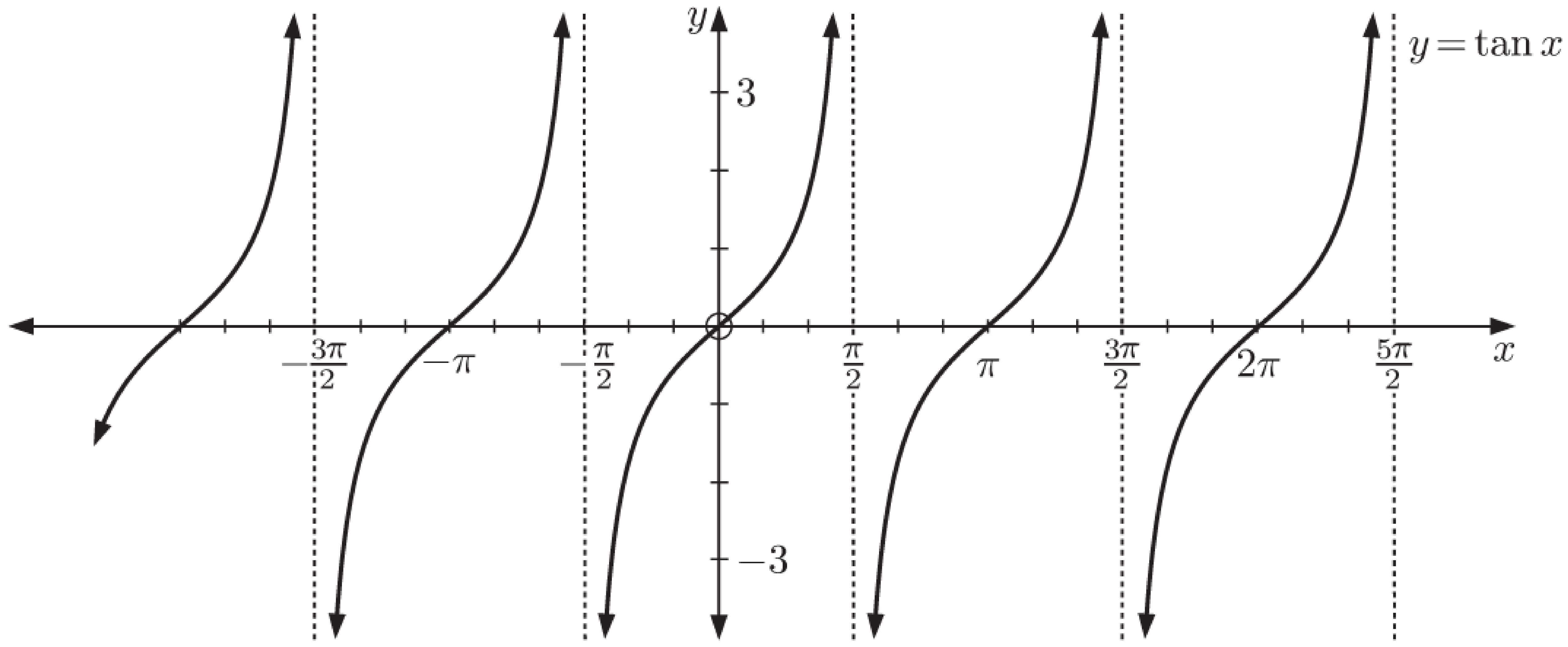
b $(0, 1)$ and $(0, -1)$?

THE GRAPH OF $y = \tan x$

$\tan x$ is zero whenever $\sin x = 0$, so the **zeros** of $y = \tan x$ are $k\pi$, $k \in \mathbb{Z}$.

$\tan x$ is undefined whenever $\cos x = 0$, so the **vertical asymptotes** of $y = \tan x$ are $x = \frac{\pi}{2} + k\pi$ for all $k \in \mathbb{Z}$.

$\tan x$ has period $= \pi$ and range $y \in \mathbb{R}$.



Click on the icon to explore how the tangent function is produced from the unit circle.

TANGENT FUNCTION

THE GENERAL TANGENT FUNCTION

The **general tangent function** is $y = a\tan(b(x-c)) + d$, $a \neq 0$, $b > 0$.

- The **principal axis** is $y = d$.
- The **period** of this function is $\dfrac{\pi}{b}$.
- The **amplitude** of this function is undefined.
- There are infinitely many vertical asymptotes.

DYNAMIC TANGENT FUNCTION

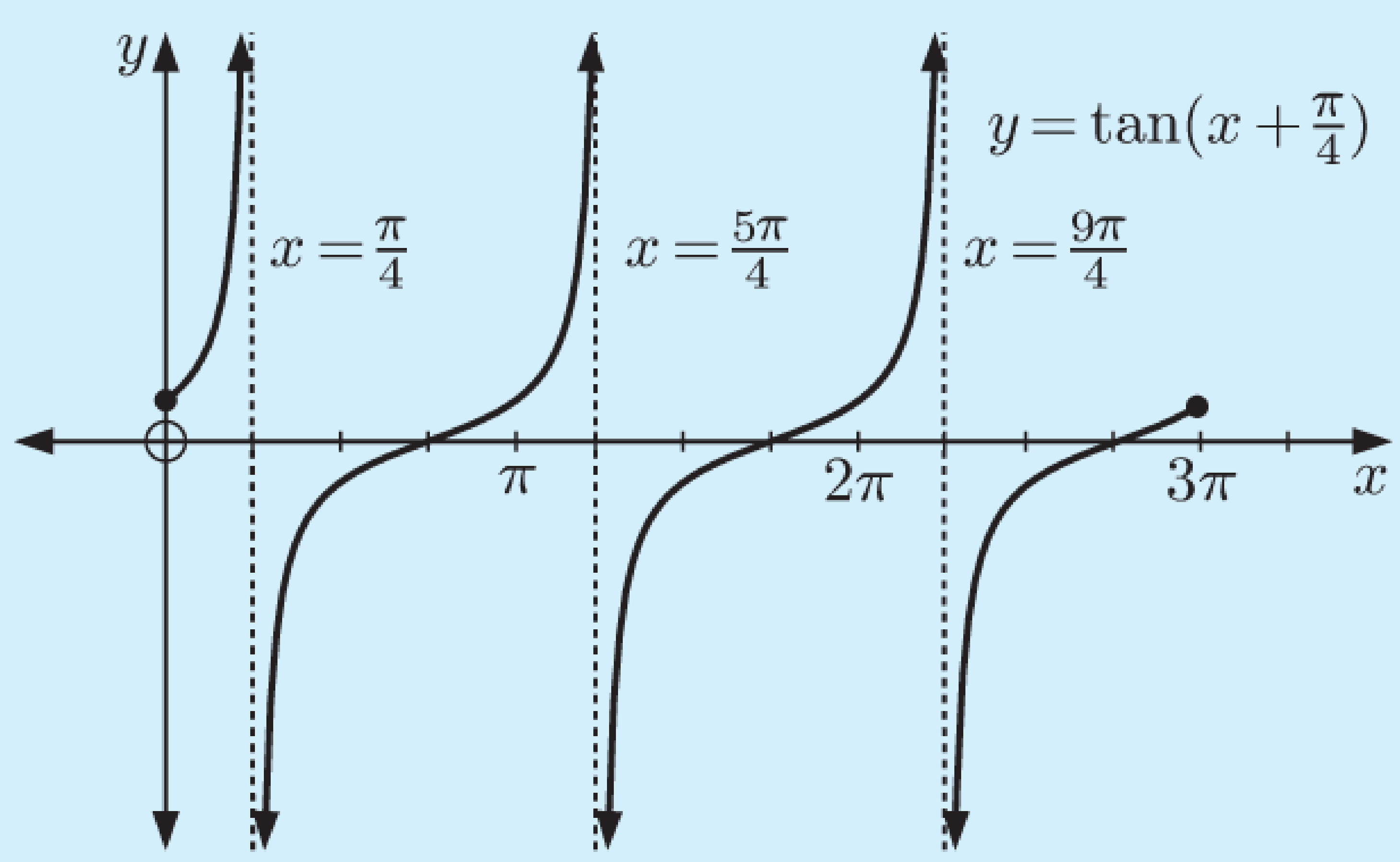
Example 8 Self Tutor

Without using technology, sketch the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$ for $0 \leqslant x \leqslant 3\pi$.

$y = \tan\left(x + \frac{\pi}{4}\right)$ is a horizontal translation of $y = \tan x$ to the left by $\frac{\pi}{4}$ units.

$y = \tan x$ has vertical asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{2}$, and its x-intercepts are 0, π, 2π, and 3π.

$\therefore$ $y = \tan\left(x + \frac{\pi}{4}\right)$ has vertical asymptotes $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$, $x = \frac{9\pi}{4}$, and x-intercepts $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, and $\frac{11\pi}{4}$.



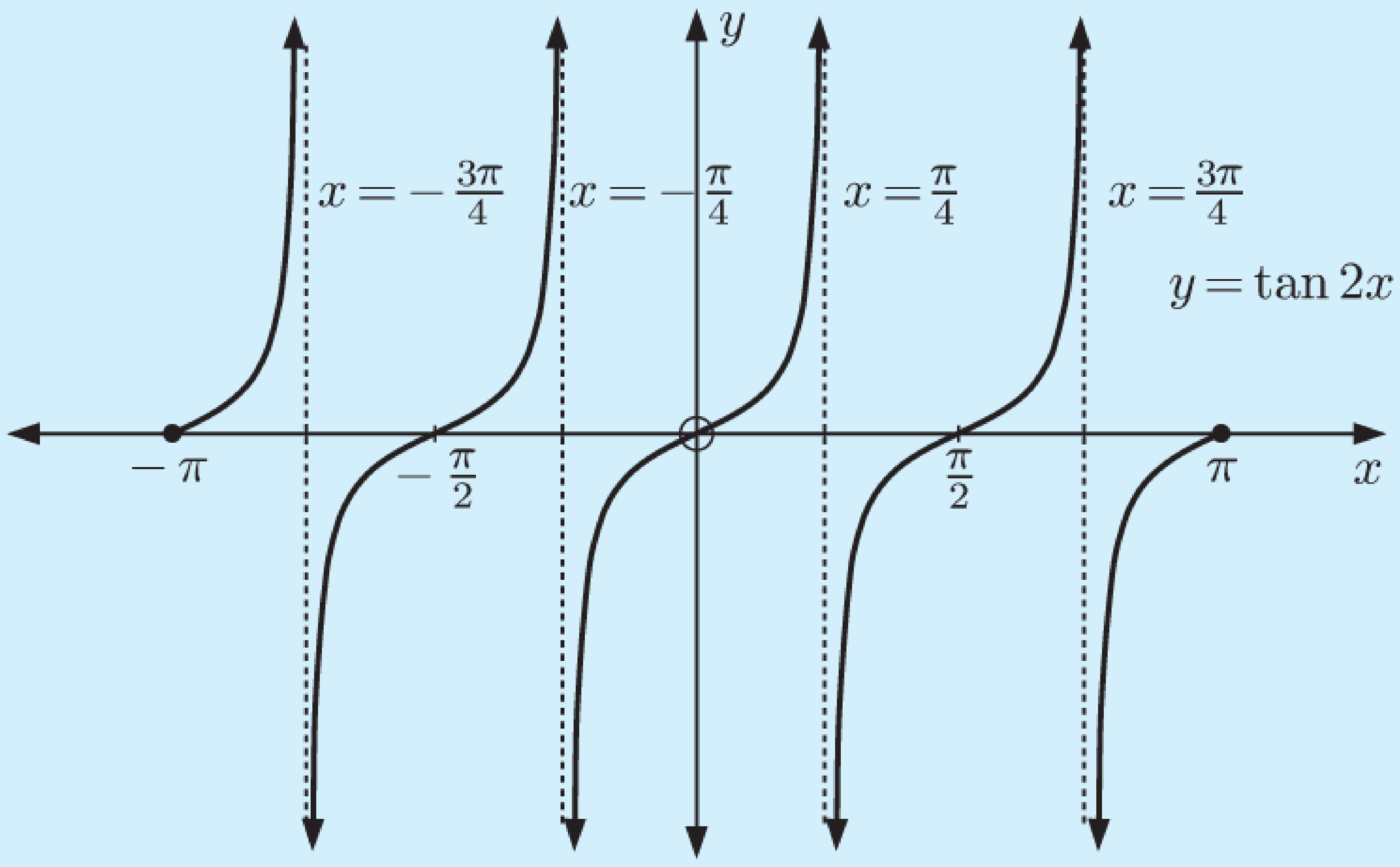
Example 9 **Self Tutor**

Without using technology, sketch the graph of $y = \tan 2x$ for $-\pi \leqslant x \leqslant \pi$.

$y = \tan 2x$ is a horizontal stretch of $y = \tan x$ with scale factor $\frac{1}{2}$.

Since $b = 2$, the period is $\frac{\pi}{2}$.

$y = \tan 2x$ has vertical asymptotes $x = \pm\frac{\pi}{4}$, $x = \pm\frac{3\pi}{4}$, and x-intercepts 0, $\pm\frac{\pi}{2}$, $\pm\pi$.



EXERCISE 17F

1 State the transformations which map $y = \tan x$ onto:

a $y = \tan\left(x - \frac{\pi}{2}\right)$ **b** $y = 4\tan x$ **c** $y = \tan\left(\frac{\pi}{2}x\right)$

d $y = \tan 2x - 1$ **e** $y = -\frac{1}{2}\tan x$ **f** $y = \tan(x + \pi) + 2$

2 State the period of:

a $y = \tan 3x$ **b** $y = \tan \frac{x}{4}$ **c** $y = \tan \pi x$

d $y = -\tan\left(\frac{\pi}{2}x\right)$ **e** $y = \tan\left(\frac{2x}{3} - \frac{\pi}{3}\right)$ **f** $y = \tan nx$, $n \neq 0$

3 For each function, write down the:

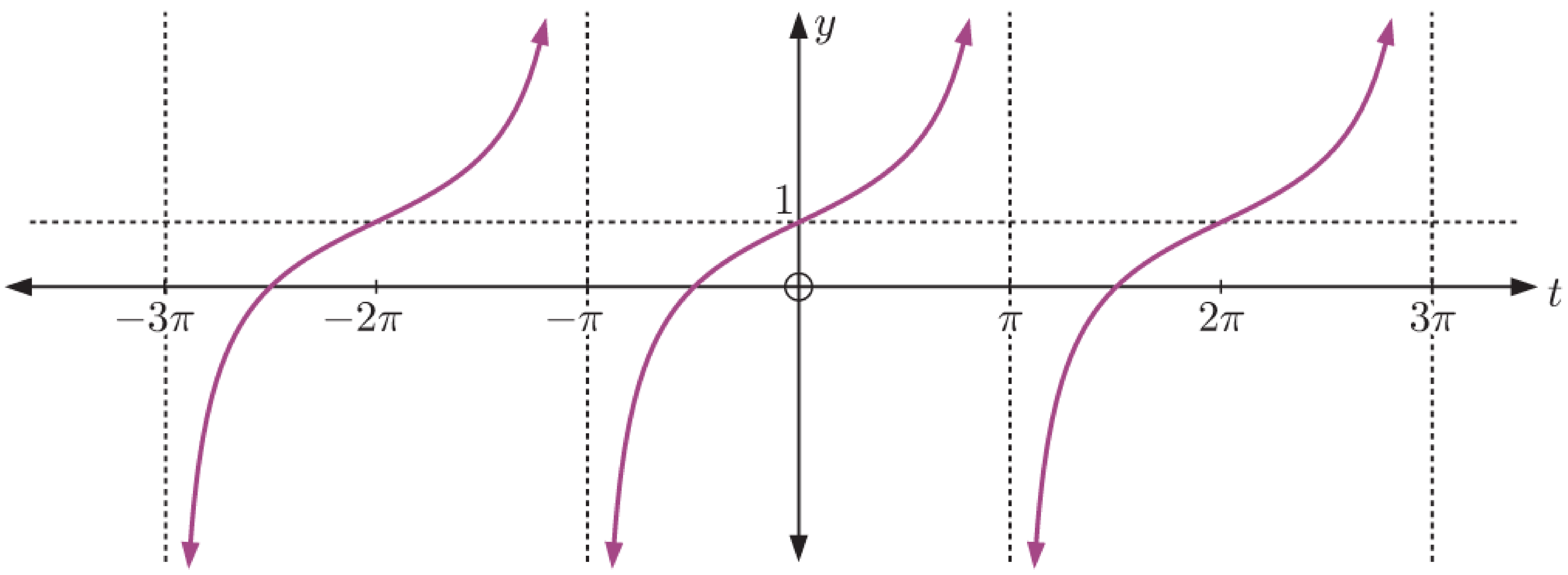
i zeros **ii** vertical asymptotes.

a $y = \tan 2x$ **b** $y = \tan\left(x + \frac{\pi}{3}\right)$ **c** $y = \frac{1}{2}\tan\left(\frac{1}{2}\left(x - \frac{\pi}{6}\right)\right)$

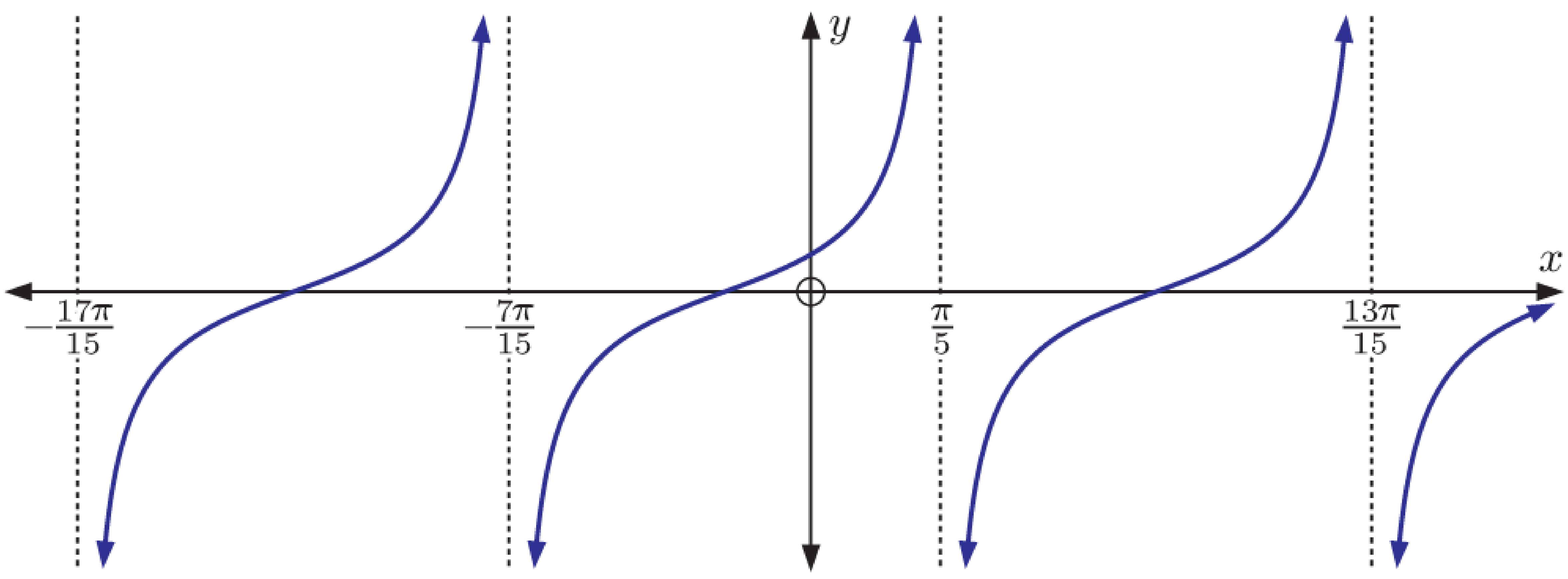
4 Sketch the graph of the following for $-2\pi \leqslant x \leqslant 2\pi$:

a $y = \tan\left(x - \frac{\pi}{4}\right)$ **b** $y = \frac{1}{2}\tan\frac{x}{4}$ **c** $y = 3\tan\left(x - \frac{\pi}{9}\right)$

5 Find p and q given the following graph is of the function $y = \tan pt + q$.



6 Find the possible values of a and b given the following graph is of the function $y = \tan a(x - b)$.



7 **a** Describe the sequence of transformations used to transform $y = \tan x$ into $y = 2\tan\left(x + \frac{\pi}{4}\right) - 1$.

b Sketch $y = 2\tan\left(x + \frac{\pi}{4}\right) - 1$ for $-2\pi \leqslant x \leqslant 2\pi$.

8 Show that the general tangent function $y = a\tan(b(x - c)) + d$, $a \neq 0$, $b > 0$, has asymptotes $x = \dfrac{\pi}{2b}(2k + 1) + c$ for all $k \in \mathbb{Z}$.

9 Consider the functions $f(x) = \tan x$ and $g(x) = 2x - \frac{\pi}{2}$.

a Find:
  i $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

b Find the value of:
  i $(f \circ g)\left(\frac{\pi}{3}\right)$ **ii** $(g \circ f)(\pi)$

c Write down the period and vertical asymptotes of:
  i $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

d Sketch the graphs of $(f \circ g)(x)$ and $(g \circ f)(x)$ for $-2\pi \leqslant x \leqslant 2\pi$.

ACTIVITY 3

Click on the icon to run a card game for trigonometric functions.

CARD GAME

G TRIGONOMETRIC EQUATIONS

Trigonometric equations will often have infinitely many solutions unless a restricted domain such as $0 \leqslant x \leqslant 3\pi$ is given.

In the **Opening Problem**, the height of the green light after t seconds is given by $H(t) = 10\sin\left(\frac{\pi}{50}t\right) + 12$ metres. So, the green light will be 16 metres above the ground when $10\sin\left(\frac{\pi}{50}t\right) + 12 = 16$.

This trigonometric equation has infinitely many solutions provided the wheel keeps rotating. For this reason we would normally specify a time interval for the solution. For example, if we are interested in the first three minutes of its rotation, we specify the domain $0 \leqslant t \leqslant 180$.

We will examine solving trigonometric equations using:

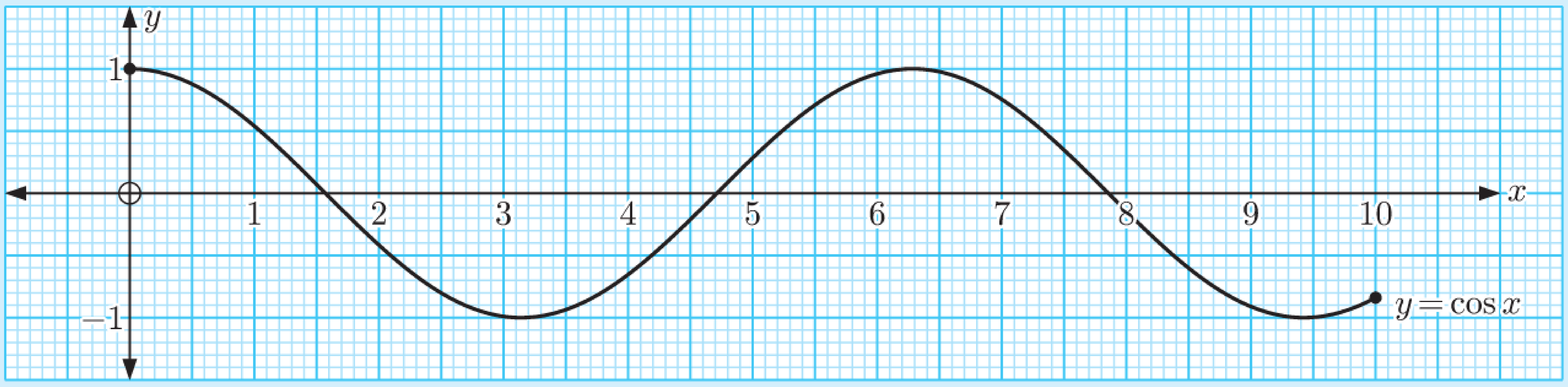
- pre-prepared graphs
- technology
- algebra.

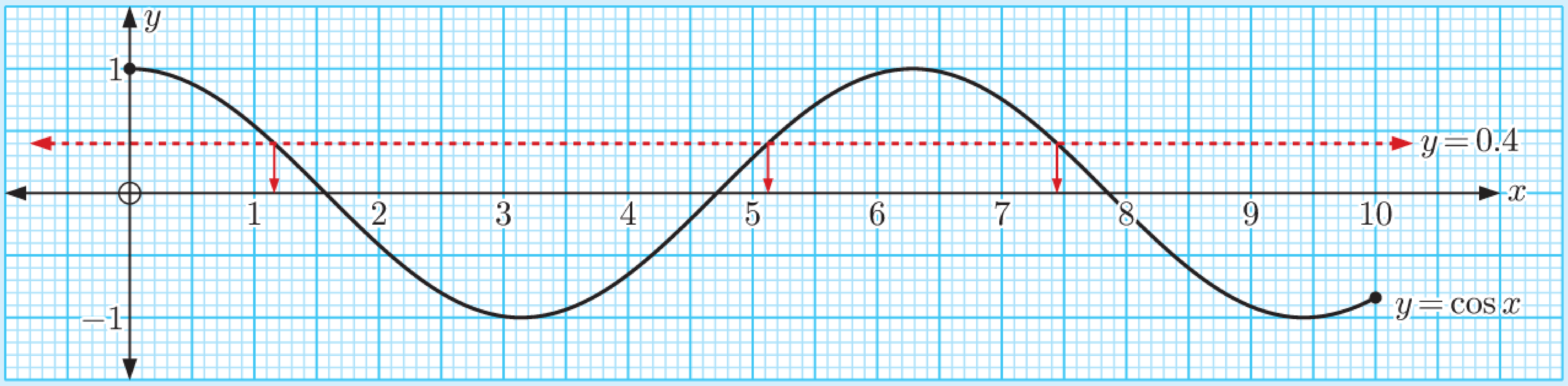
GRAPHICAL SOLUTION OF TRIGONOMETRIC EQUATIONS

If we are given a graph with sufficient accuracy, we can use it to estimate solutions.

Example 10 Self Tutor

Solve $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians using the graph of $y = \cos x$.



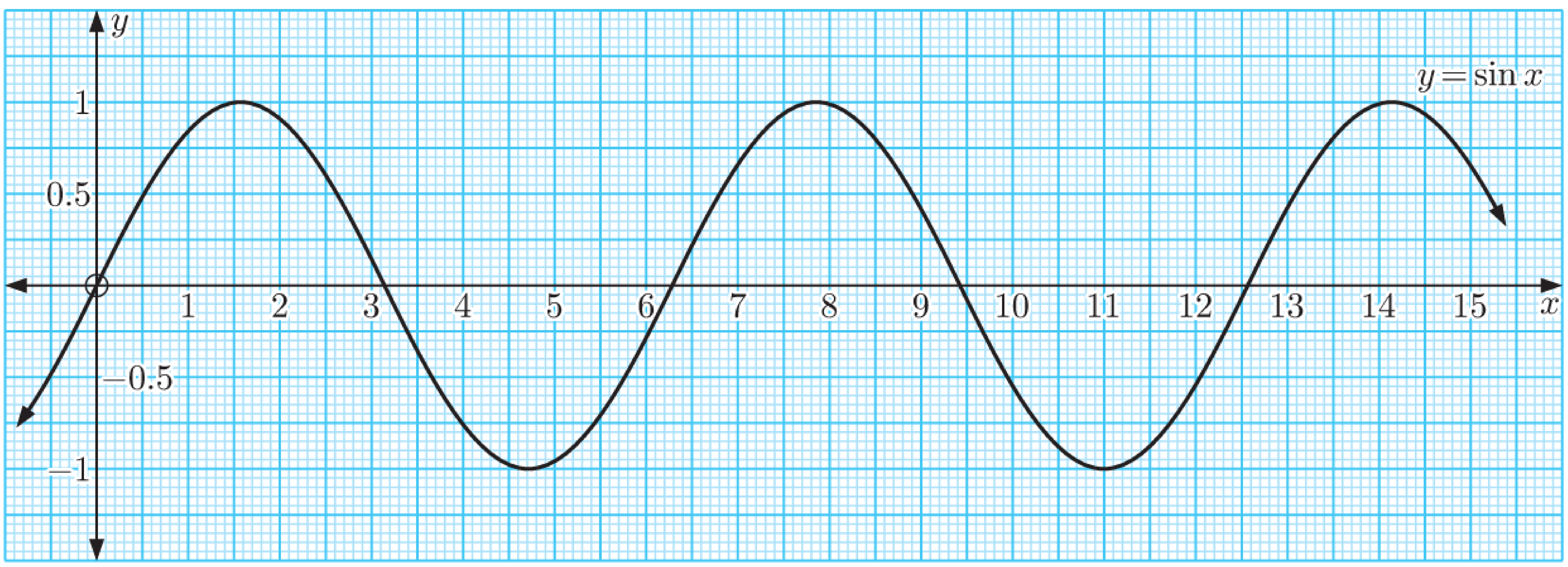


$y = 0.4$ meets $y = \cos x$ when $x \approx 1.2$, 5.1, or 7.4.

The solutions of $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians are 1.2, 5.1, and 7.4.

EXERCISE 17G.1

1



Use the graph of $y = \sin x$ to solve, correct to 1 decimal place:

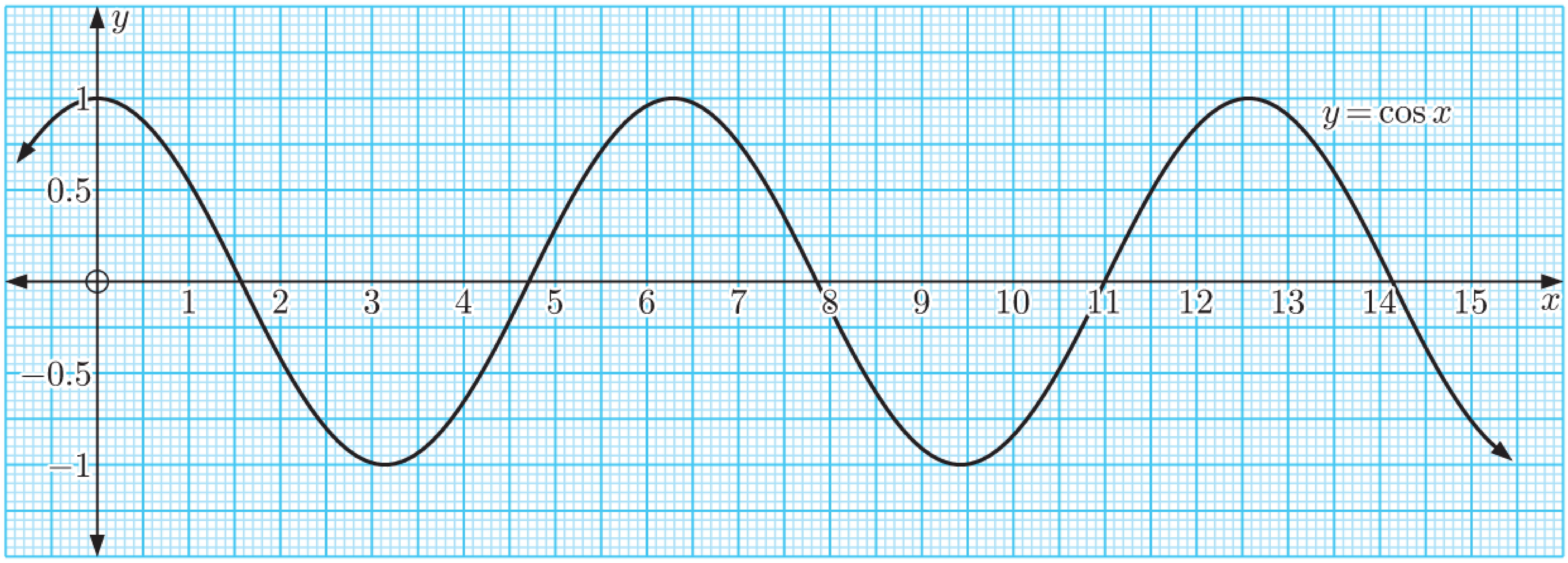
a $\sin x = 0.3$ for $0 \leqslant x \leqslant 15$

b $\sin x = -0.4$ for $5 \leqslant x \leqslant 15$

c $\sin x = 0.3$ or $0 \leqslant x \leqslant 2\pi$

d $\sin x = -0.6$ for $\pi \leqslant x \leqslant 2\pi$

2



Use the graph of $y = \cos x$ to solve, correct to 1 decimal place:

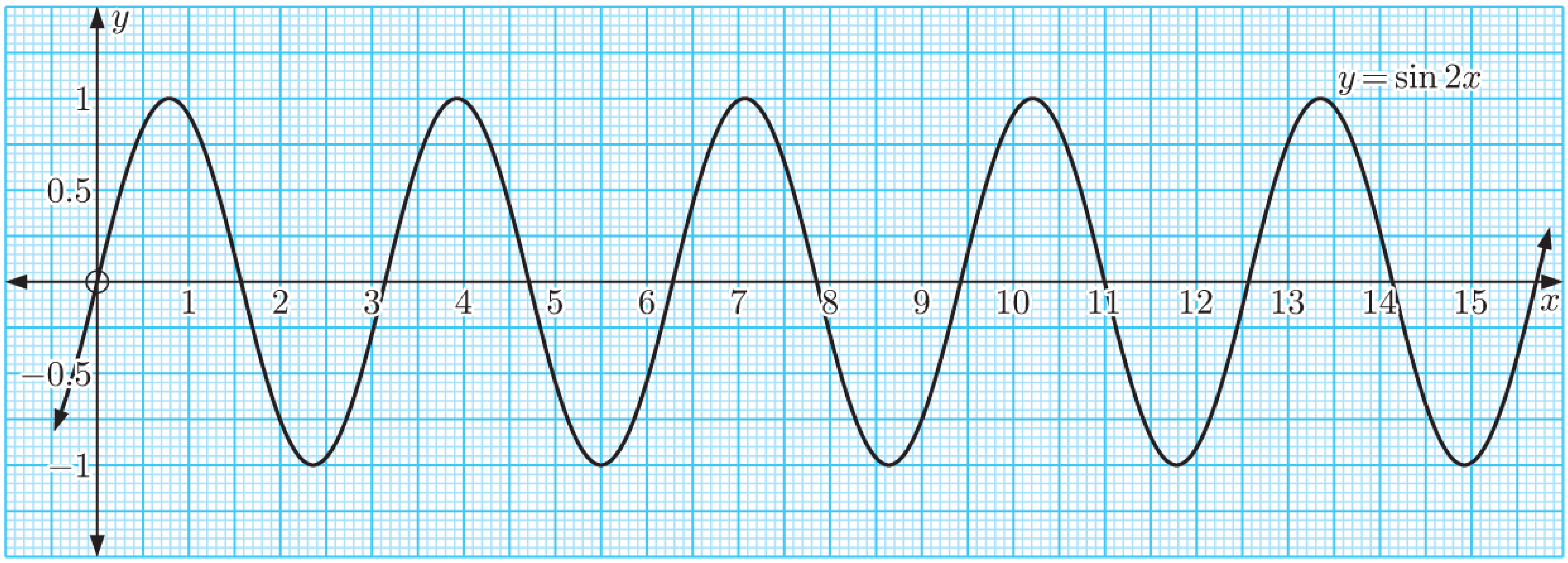
a $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$

b $\cos x = -0.3$ for $4 \leqslant x \leqslant 12$

c $\cos x = 0.5$ for $\pi \leqslant x \leqslant 2\pi$

d $\cos x = -0.8$ for $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$

3



Use the graph of $y = \sin 2x$ to solve, correct to 1 decimal place:

a $\sin 2x = 0.7$ for $0 \leqslant x \leqslant 16$

b $\sin 2x = -0.3$ for $0 \leqslant x \leqslant 16$

c $\sin 2x = 0.2$ for $\pi \leqslant x \leqslant 2\pi$

d $\sin 2x = -0.1$ for $0 \leqslant x \leqslant 2\pi$

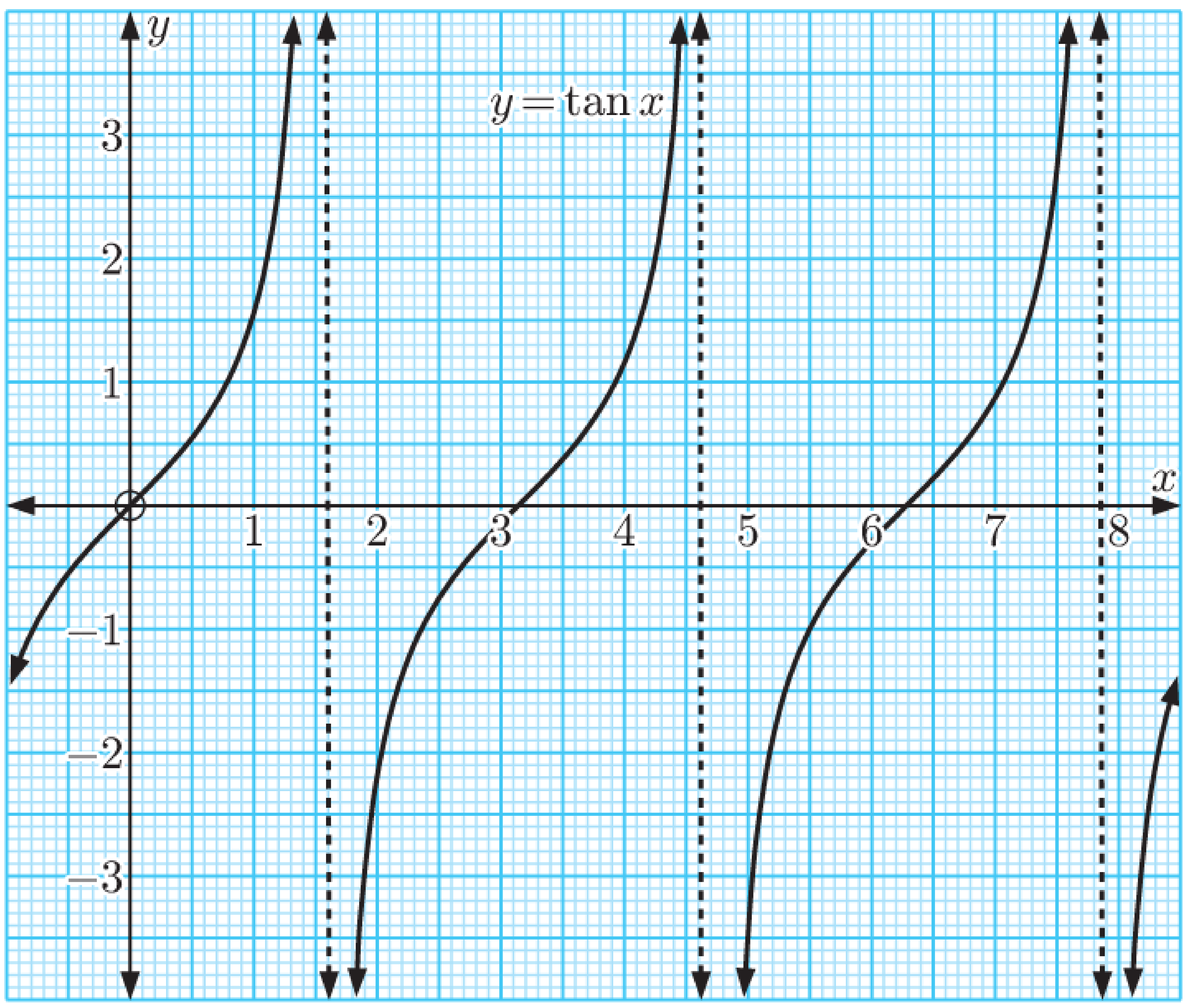
4 Use the graph of $y = \tan x$ to solve, correct to 1 decimal place:

a $\tan x = 2$ for $0 \leqslant x \leqslant 8$

b $\tan x = -1.4$ for $2 \leqslant x \leqslant 7$

c $\tan x = 3.5$ for $0 \leqslant x \leqslant 2\pi$

d $\tan x = -2.4$ for $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$



SOLVING TRIGONOMETRIC EQUATIONS USING TECHNOLOGY

Trigonometric equations may be solved *numerically* using either a **graphing package** or a **graphics calculator**. In most cases the answers will not be exact, but rather a decimal approximation.

GRAPHING PACKAGE

When using a graphics calculator, make sure that the **mode** is set to **radians**.

Example 11 — Self Tutor

Solve $2\sin x - \cos x = 4 - x$ for $0 \leqslant x \leqslant 2\pi$.

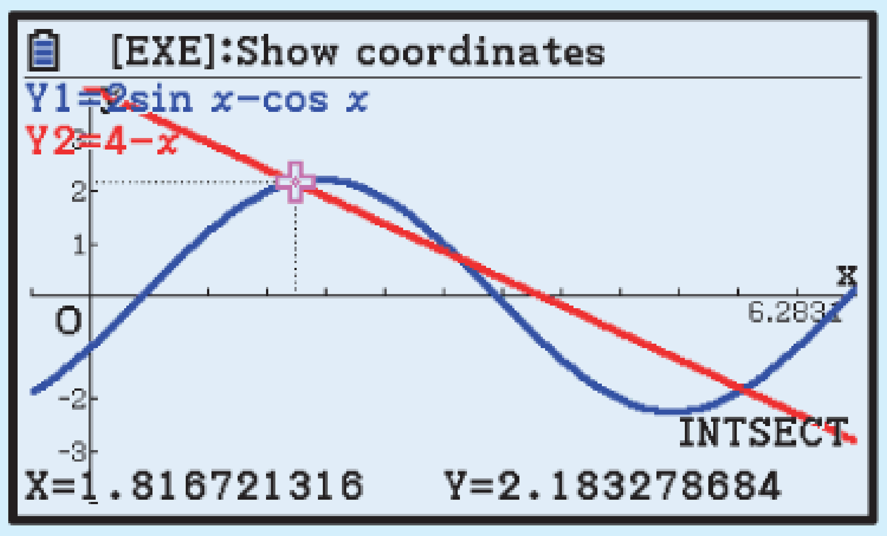
We graph the functions $Y_1 = 2\sin X - \cos X$ and $Y_2 = 4 - X$ on the same set of axes.

GRAPHICS CALCULATOR INSTRUCTIONS

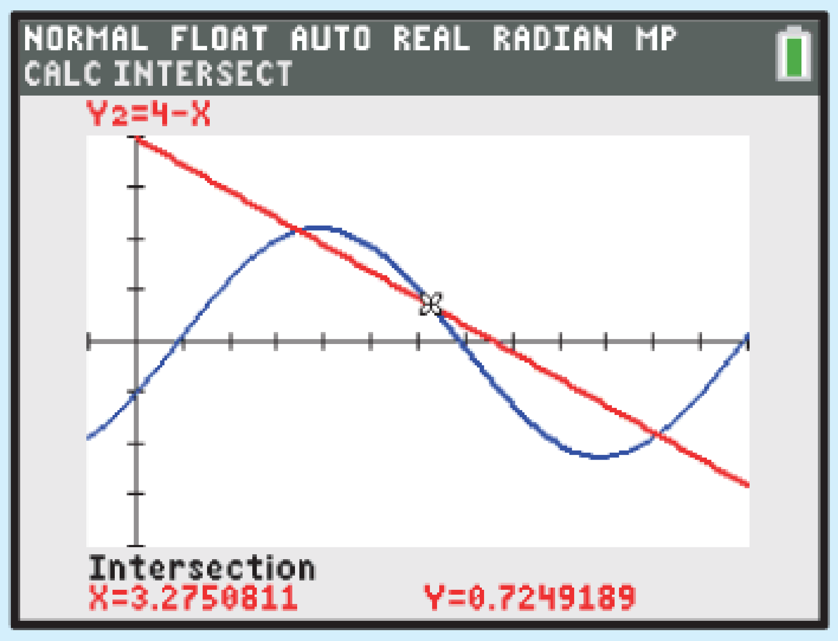
We use **window** settings just larger than the domain:

$X\text{min} = -\frac{\pi}{6}$ $X\text{max} = \frac{13\pi}{6}$ $X\text{scale} = \frac{\pi}{6}$

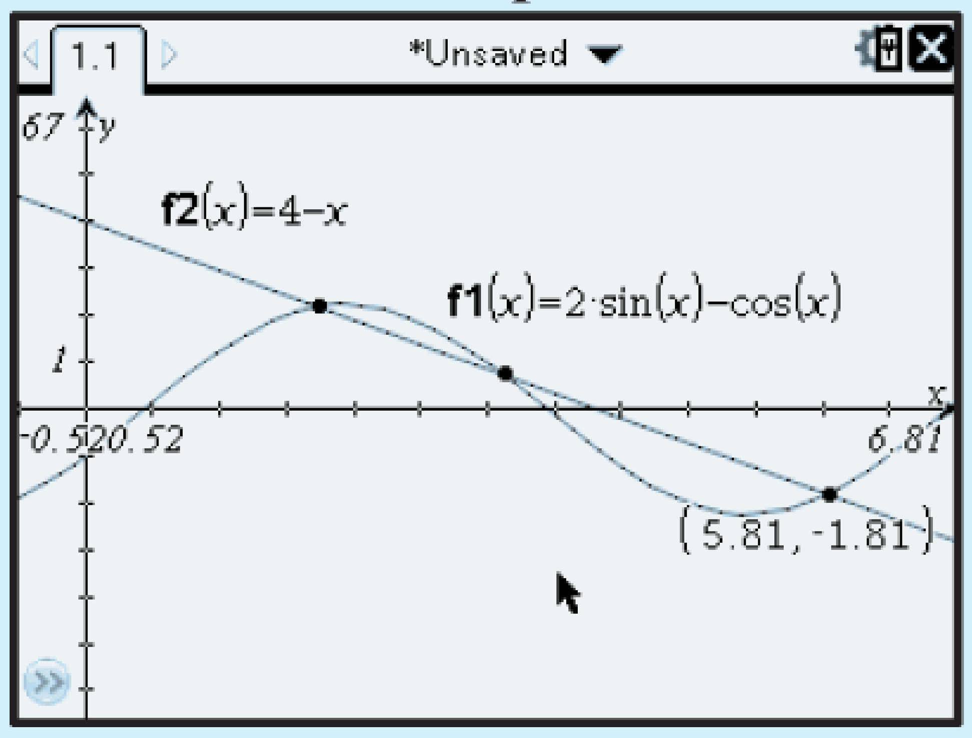
Casio fx-CG50



TI-84 Plus CE



TI-nspire



The solutions are $x \approx 1.82$, 3.28, and 5.81.

EXERCISE 17G.2

1 Solve for x on the domain $0 < x < 12$:

- **a** $\sin x = 0.431$
- **b** $\cos x = -0.814$
- **c** $3\tan x - 2 = 0$

2 Solve for x on the domain $-5 \leqslant x \leqslant 5$:

- **a** $5\cos x - 4 = 0$
- **b** $2\tan x + 13 = 0$
- **c** $8\sin x + 3 = 0$

3 Solve for $0 \leqslant x \leqslant 2\pi$:

- **a** $\sin(x+2) = 0.0652$
- **b** $\sin^2 x + \sin x - 1 = 0$

4 Solve for x: $\cos(x-1) + \sin(x+1) = 6x + 5x^2 - x^3$ for $-2 \leqslant x \leqslant 6$.

5 A goat is tethered by a rope to the edge of a circular grass field. The ratio of the rope length to the radius of the field is x, where $0 < x < 2$.

- **a** Write a function $P(x)$ for the proportion of the field which the goat can graze.
- **b** Sketch $P(x)$.
- **c** Find, to 3 decimal places, the value of x such that the goat can graze exactly half of the field.

This problem was solved for a "hypergoat" in n-dimensional "hyperspace" by Jean Jacquelin in 2003.

SOLVING TRIGONOMETRIC EQUATIONS USING ALGEBRA

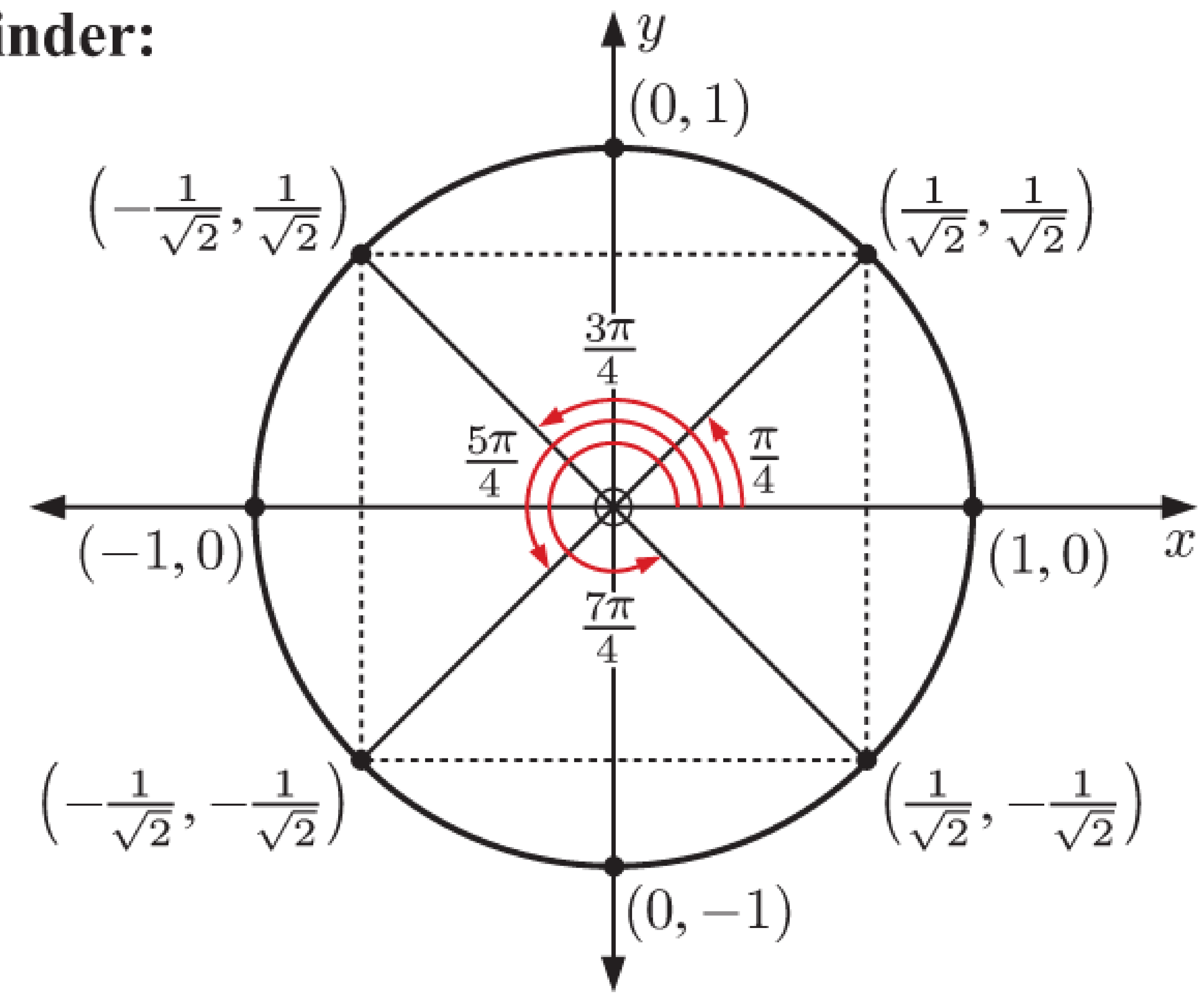
Exact solutions obtained using algebra are called **analytic** solutions. We can find analytic solutions to *some* trigonometric equations, but only if they correspond to angles for which the trigonometric ratios can be expressed exactly.

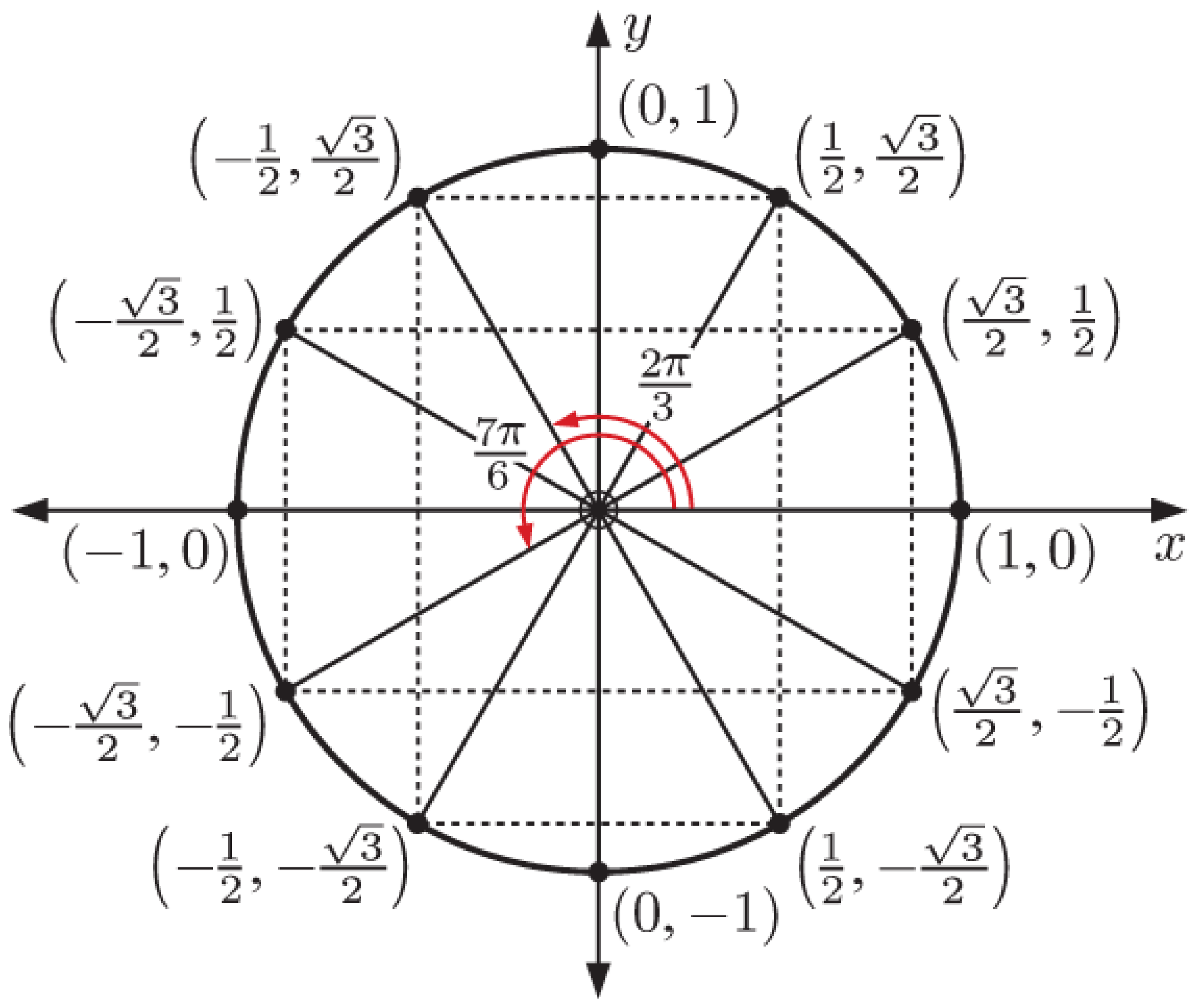
We use the *periodicity* of the trigonometric functions to give us all solutions in the required domain. Remember that $\sin x$ and $\cos x$ both have period 2π, and $\tan x$ has period π.

For an equation such as $\sin 2x = \frac{1}{2}$ on the domain $0 \leqslant x \leqslant 2\pi$, we need to understand that if $0 \leqslant x \leqslant 2\pi$ then $0 \leqslant 2x \leqslant 4\pi$. So, when we consider points on the unit circle with sine $\frac{1}{2}$, we need to consider angles from 0 to 4π.

When solving trigonometric equations, you must find all of the solutions in the required domain.

Reminder:



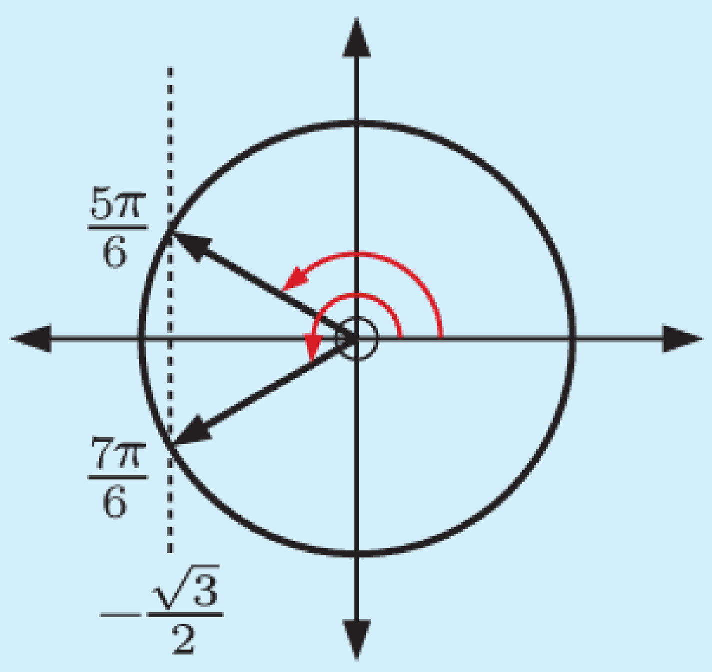


Example 12 | **Self Tutor**

Solve for x on the domain $0 \leqslant x \leqslant 2\pi$:

a $\cos x = -\frac{\sqrt{3}}{2}$ **b** $2\sin x - 1 = 0$ **c** $\tan x + \sqrt{3} = 0$

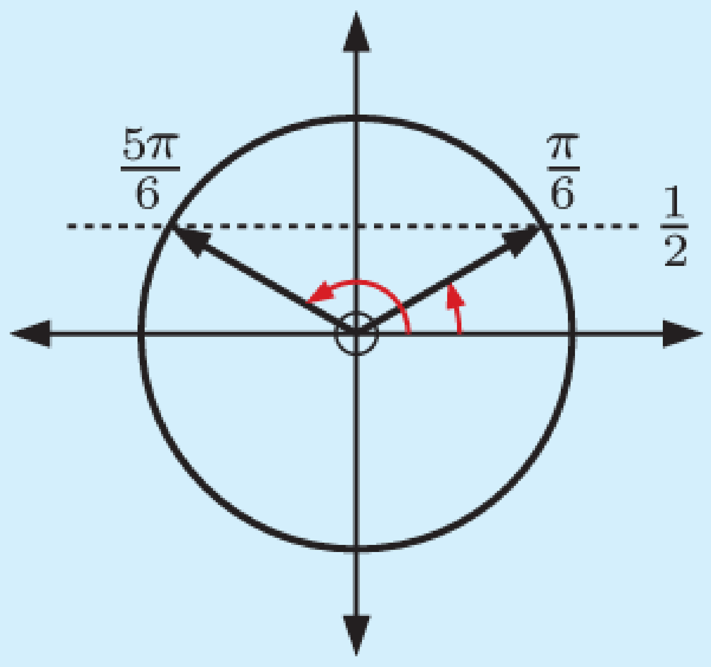
a $\cos x = -\frac{\sqrt{3}}{2}$



$\therefore \quad x = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$

b $2\sin x - 1 = 0$

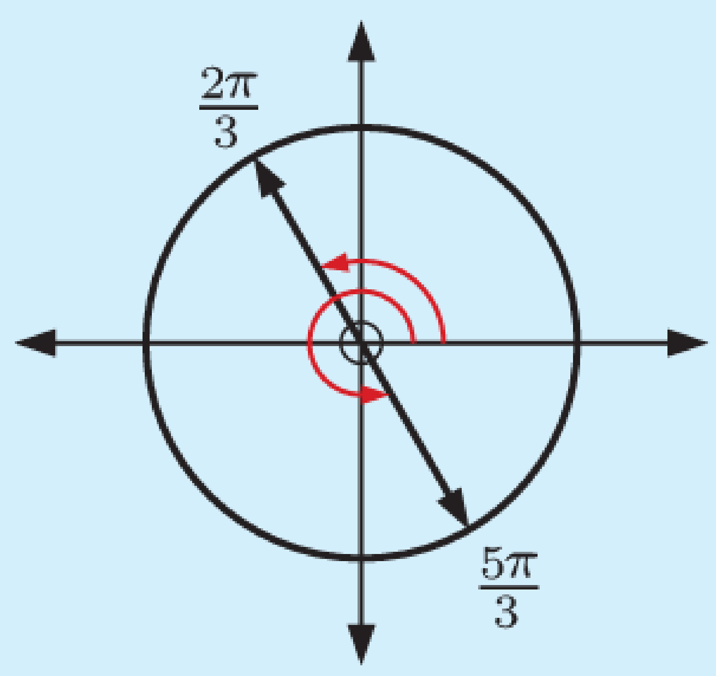
$\therefore \quad \sin x = \frac{1}{2}$



$\therefore \quad x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

c $\tan x + \sqrt{3} = 0$

$\therefore \quad \tan x = -\sqrt{3}$



$\therefore \quad x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$

EXERCISE 17G.3

1 Solve for x on the domain $0 \leqslant x \leqslant 2\pi$:

a $\cos x = \frac{1}{2}$ **b** $\sin x = -\frac{1}{\sqrt{2}}$ **c** $\tan x = \frac{1}{\sqrt{3}}$

d $\sin x = -1$ **e** $\cos x = 0$ **f** $\tan x = 0$

2 Solve for x on the domain $0 \leqslant x \leqslant 2\pi$:

a $2\sin x = \sqrt{3}$ **b** $3\cos x + 3 = 0$ **c** $2\tan x - 2 = 0$

3 Solve for x on the domain $0 \leqslant x \leqslant 4\pi$:

a $2\cos x + 1 = 0$ **b** $\sqrt{2}\sin x = 1$ **c** $\tan x = 1$

4 Solve for x on the domain $-2\pi \leqslant x \leqslant 2\pi$:

a $2\sin x + \sqrt{3} = 0$ **b** $\sqrt{2}\cos x + 1 = 0$ **c** $\tan x = -1$

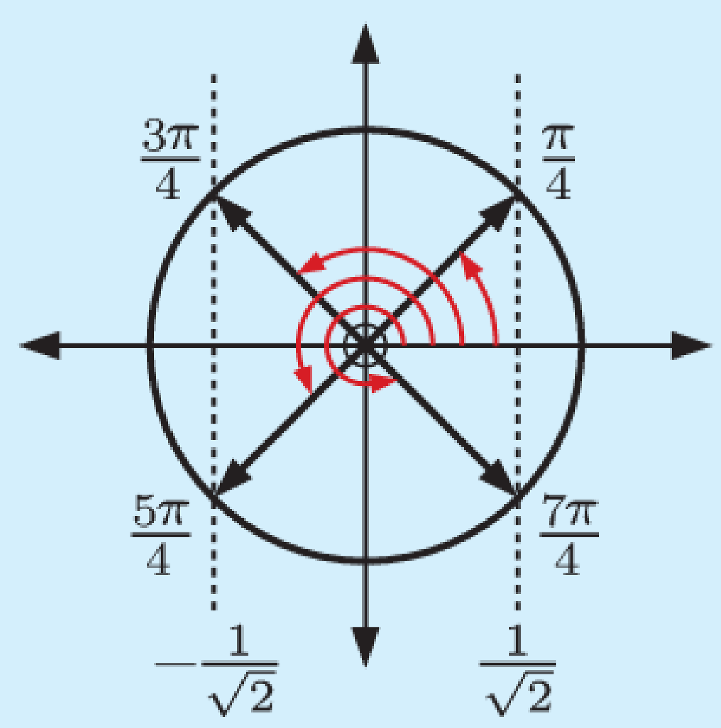
Example 13 **Self Tutor**

Solve $\cos^2 x = \frac{1}{2}$ on $0 \leqslant x \leqslant 2\pi$.

$\cos^2 x = \frac{1}{2}$

$\therefore \cos x = \pm\frac{1}{\sqrt{2}}$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}$



5 Solve for x on $0 \leqslant x \leqslant 2\pi$:

a $\cos^2 x = \frac{3}{4}$ **b** $\sin^2 x = 1$ **c** $\tan^2 x = 3$

Example 14 **Self Tutor**

Solve exactly for $0 \leqslant x \leqslant 3\pi$:

a $\sin x = -\frac{1}{2}$ **b** $\sin 2x = -\frac{1}{2}$ **c** $\sin\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$

The three equations all have the form $\sin\theta = -\frac{1}{2}$.

a $0 \leqslant x \leqslant 3\pi$

$\therefore x = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$

b In this case θ is $2x$.

If $0 \leqslant x \leqslant 3\pi$ then $0 \leqslant 2x \leqslant 6\pi$.

$\therefore 2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}, \text{ or } \frac{35\pi}{6}$

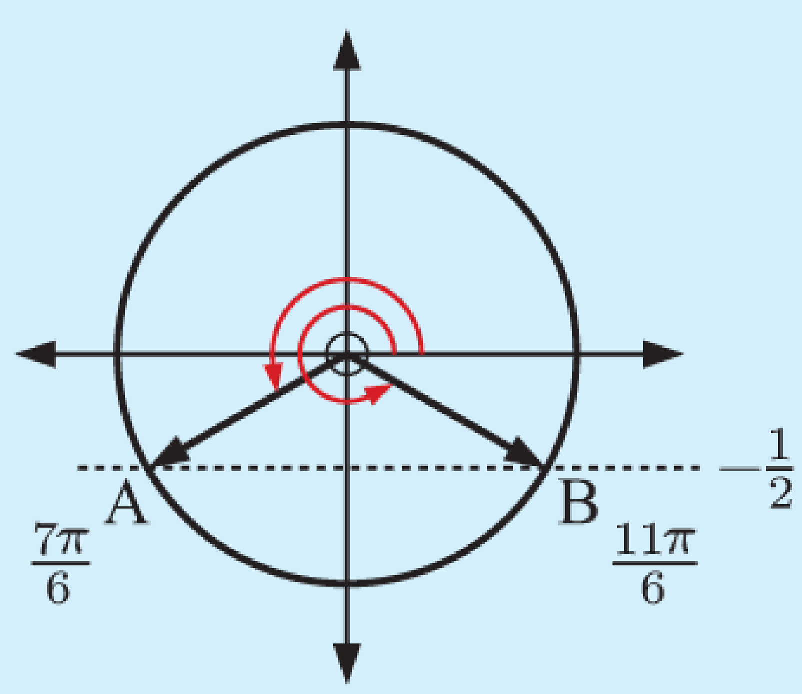
$\therefore x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{31\pi}{12}, \text{ or } \frac{35\pi}{12}$

c In this case θ is $x - \frac{\pi}{6}$.

If $0 \leqslant x \leqslant 3\pi$ then $-\frac{\pi}{6} \leqslant x - \frac{\pi}{6} \leqslant \frac{17\pi}{6}$.

$\therefore x - \frac{\pi}{6} = -\frac{\pi}{6}, \frac{7\pi}{6}, \text{ or } \frac{11\pi}{6}$

$\therefore x = 0, \frac{4\pi}{3}, \text{ or } 2\pi$



Start at $-\frac{\pi}{6}$ and work around to $\frac{17\pi}{6}$, recording the angle every time you reach points A and B.

6 If $0 \leqslant x \leqslant 2\pi$, state the domain of:

a $2x$ **b** $\frac{x}{4}$ **c** $x + \frac{\pi}{2}$ **d** $x - \frac{\pi}{6}$ **e** $2\left(x - \frac{\pi}{4}\right)$ **f** $-x$

7 If $-\pi \leqslant x \leqslant \pi$, state the domain of:

a $3x$ **b** $\frac{x}{4}$ **c** $x - \frac{\pi}{2}$ **d** $2x + \frac{\pi}{2}$ **e** $-2x$ **f** $\pi - x$

8 Solve exactly for $0 \leqslant x \leqslant 3\pi$:

a $\cos x = \frac{1}{2}$ **b** $\cos 2x = \frac{1}{2}$ **c** $\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

9 Solve for x on $0 \leqslant x \leqslant 2\pi$:

a $\sin 2x = -\frac{1}{2}$ **b** $\cos 3x = \frac{\sqrt{3}}{2}$ **c** $\tan 2x - \sqrt{3} = 0$

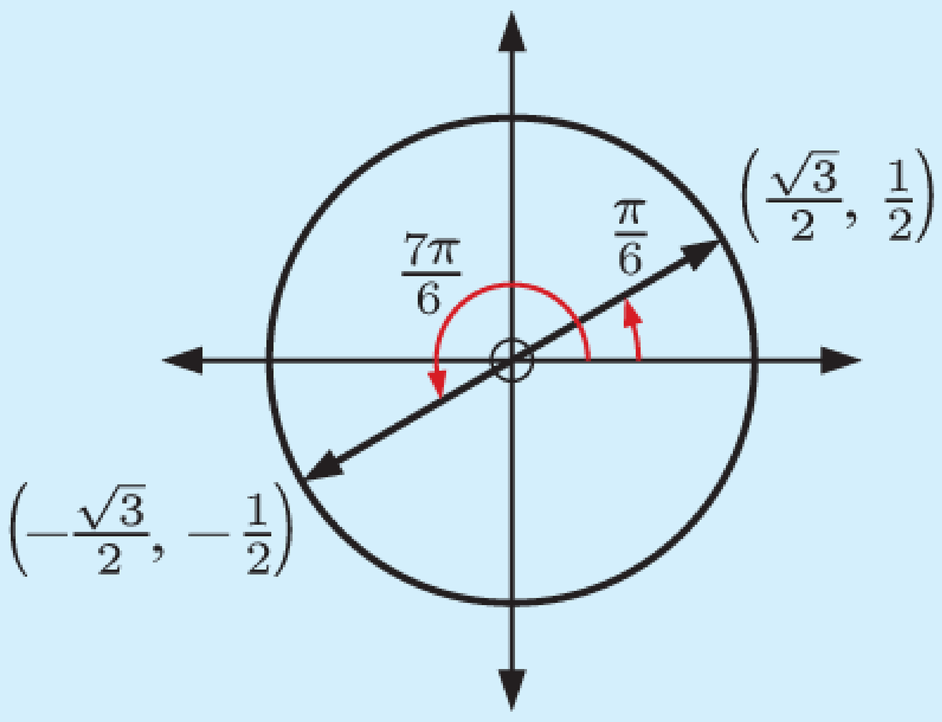
d $\sin \frac{x}{2} = \frac{1}{\sqrt{2}}$ **e** $2\cos \frac{x}{2} + 1 = 0$ **f** $3\tan \frac{x}{3} - 3 = 0$

10 Solve for x on $0 \leqslant x \leqslant 2\pi$:

a $\cos^2 3x = \frac{1}{4}$ **b** $\sin^2 2x = 1$ **c** $\tan^2\left(\frac{x}{2}\right) = \frac{1}{3}$

Example 15 Self Tutor

Find the exact solutions of $\sqrt{3}\sin x = \cos x$ for $0 \leqslant x \leqslant 2\pi$.



$\sqrt{3}\sin x = \cos x$

$\therefore \frac{\sin x}{\cos x} = \frac{1}{\sqrt{3}}$ {dividing both sides by $\sqrt{3}\cos x$}

$\therefore \tan x = \frac{1}{\sqrt{3}}$

$\therefore x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$

11 Find the exact solutions for $0 \leqslant x \leqslant 2\pi$:

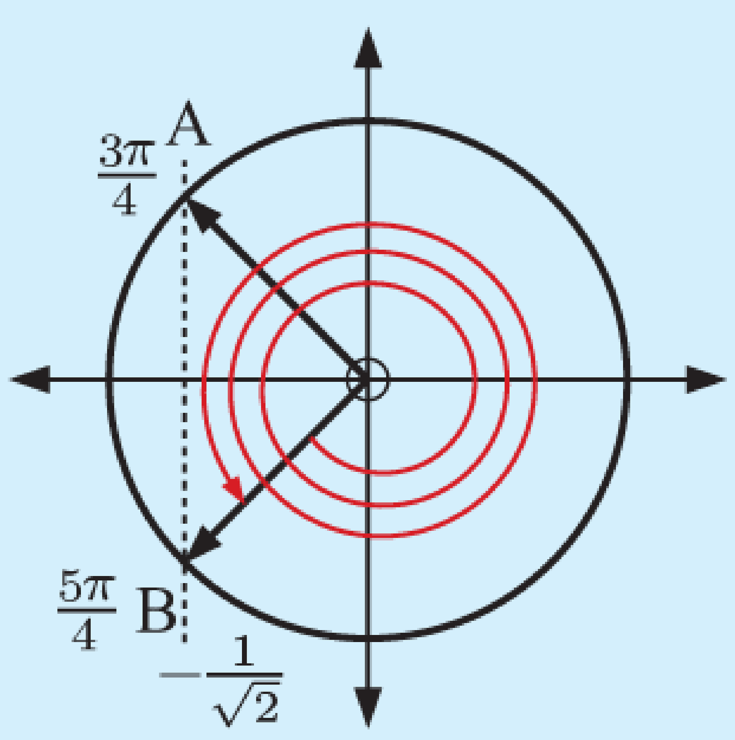
a $\sin x = -\cos x$ **b** $\sin 3x = \cos 3x$ **c** $\sin 2x = \sqrt{3}\cos 2x$

Example 16 Self Tutor

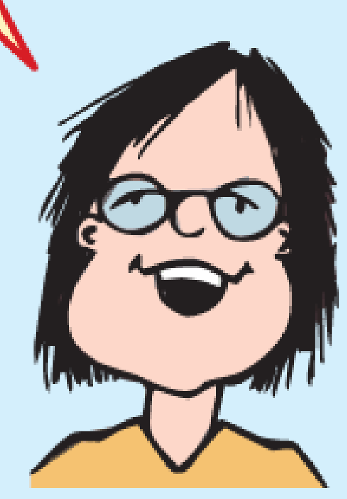
Solve $\sqrt{2}\cos\left(x - \frac{3\pi}{4}\right) + 1 = 0$ for $0 \leqslant x \leqslant 6\pi$.

$\sqrt{2}\cos\left(x - \frac{3\pi}{4}\right) + 1 = 0$

$\therefore \cos\left(x - \frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.



Start at $-\frac{3\pi}{4}$ and work around to $\frac{21\pi}{4}$, recording the angle every time you reach points A and B.



Since $0 \leqslant x \leqslant 6\pi$,

$-\frac{3\pi}{4} \leqslant x - \frac{3\pi}{4} \leqslant \frac{21\pi}{4}$

So, $x - \frac{3\pi}{4} = -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{19\pi}{4}$, or $\frac{21\pi}{4}$

$\therefore x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, 4\pi, \frac{11\pi}{2}$, or 6π

12 Solve exactly:

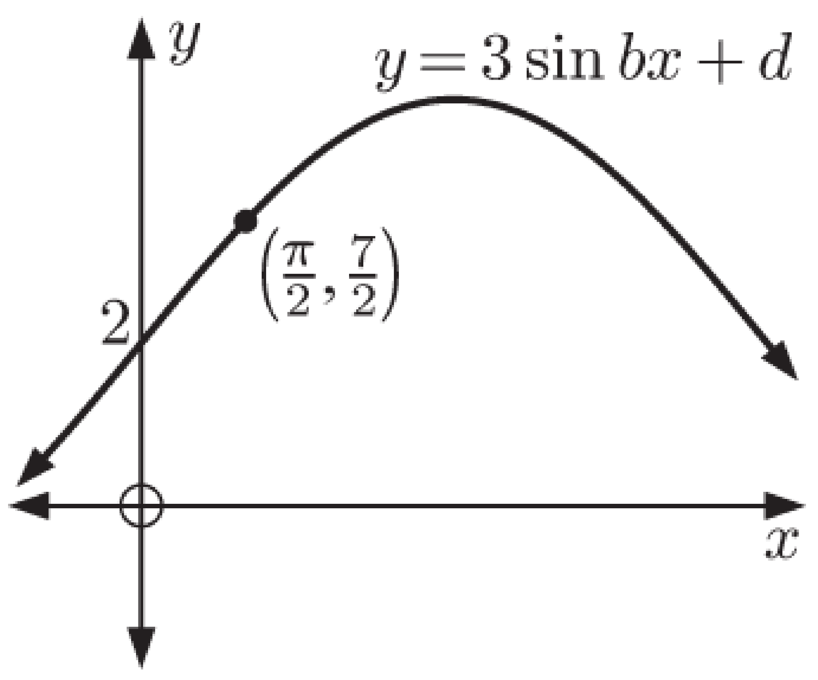
a $\cos\left(x - \frac{2\pi}{3}\right) = \frac{1}{2}, \quad -2\pi \leqslant x \leqslant 2\pi$

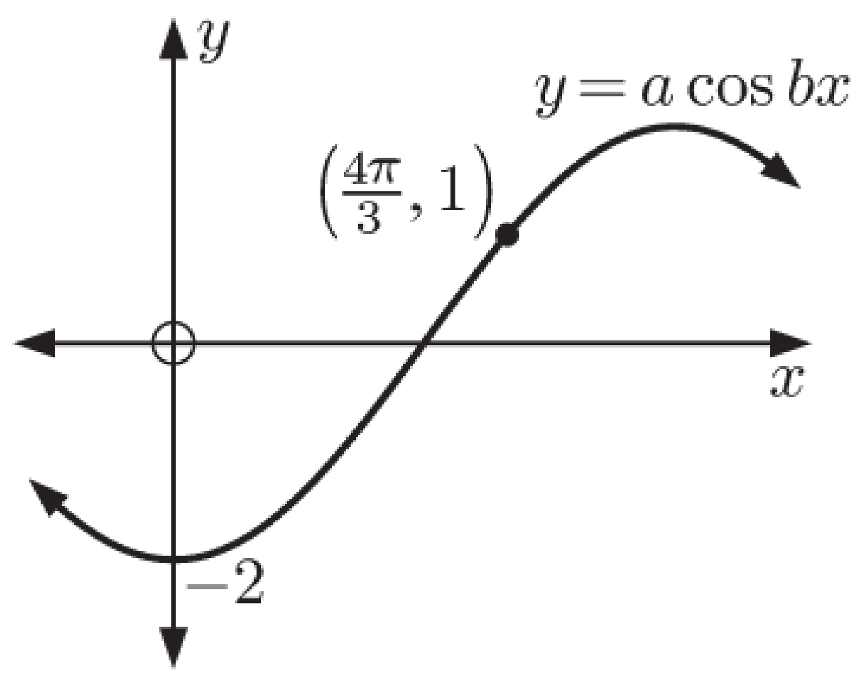
b $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1 = 0, \quad 0 \leqslant x \leqslant 3\pi$

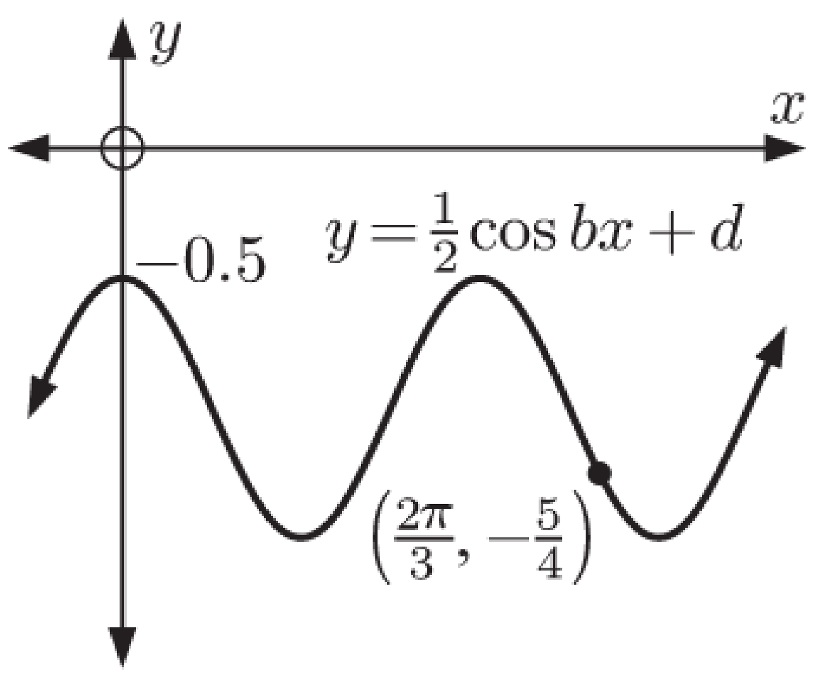
c $\sin\left(4\left(x - \frac{\pi}{4}\right)\right) = 0, \quad 0 \leqslant x \leqslant \pi$

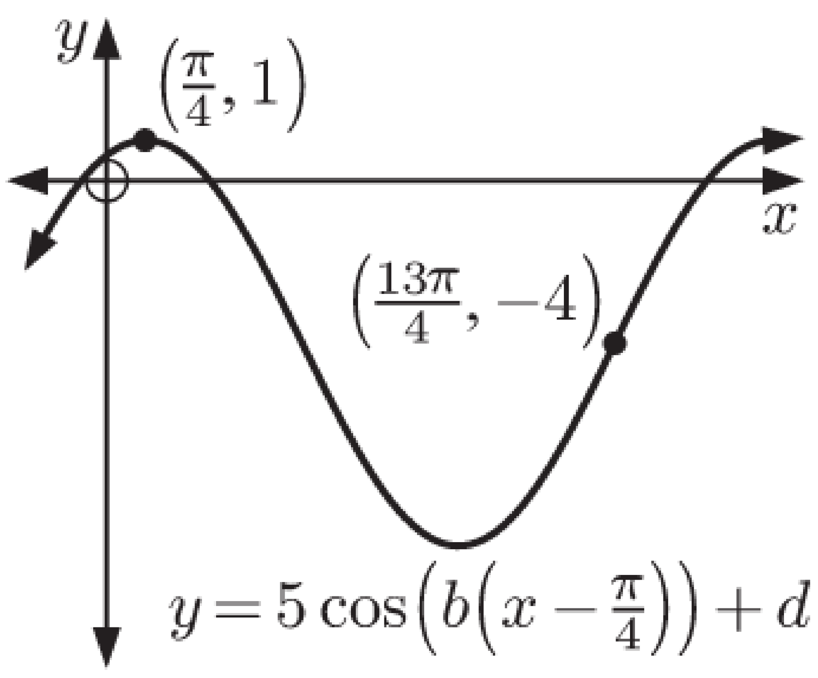
d $2\sin\left(2\left(x - \frac{\pi}{3}\right)\right) = -\sqrt{3}, \quad 0 \leqslant x \leqslant 2\pi$

13 Find the unknowns in each function:

a


b


c


d


14 Find the exact solutions of $\tan x = \sqrt{3}$ for $0 \leqslant x \leqslant 2\pi$. Hence solve the following equations for $0 \leqslant x \leqslant 2\pi$:

a $\tan\left(x - \frac{\pi}{6}\right) = \sqrt{3}$

b $\tan 4x = \sqrt{3}$

c $\tan^2 x = 3$

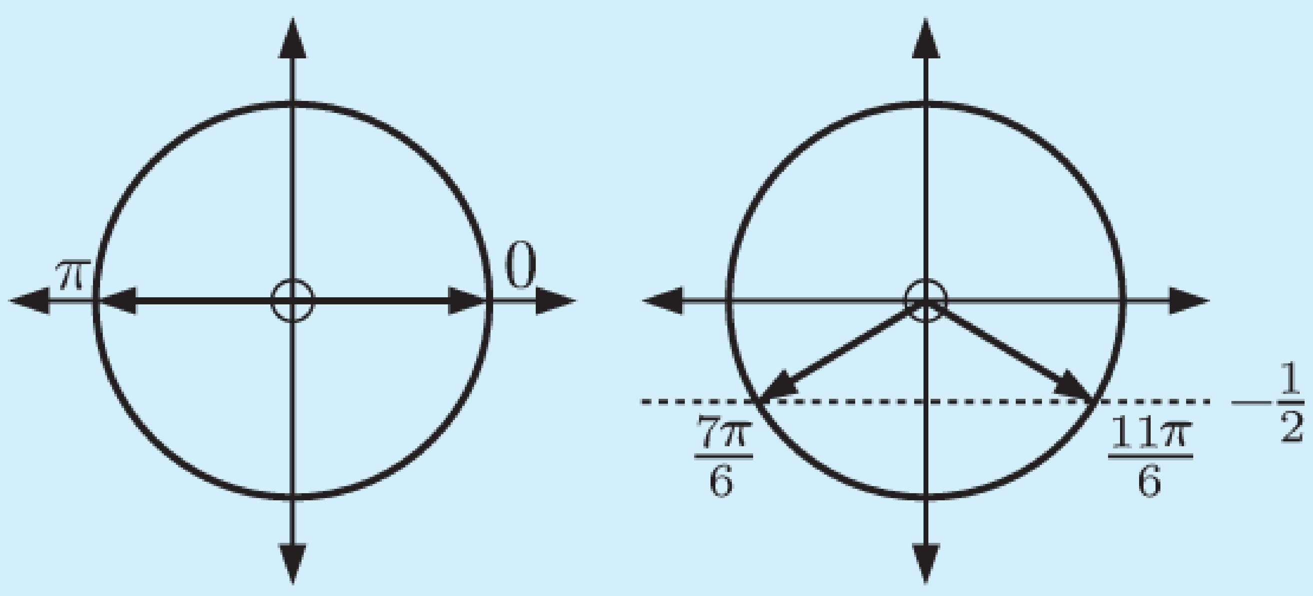
Example 17 — Self Tutor

Solve for x on $0 \leqslant x \leqslant 2\pi$, giving your answers as exact values:

a $2\sin^2 x + \sin x = 0$

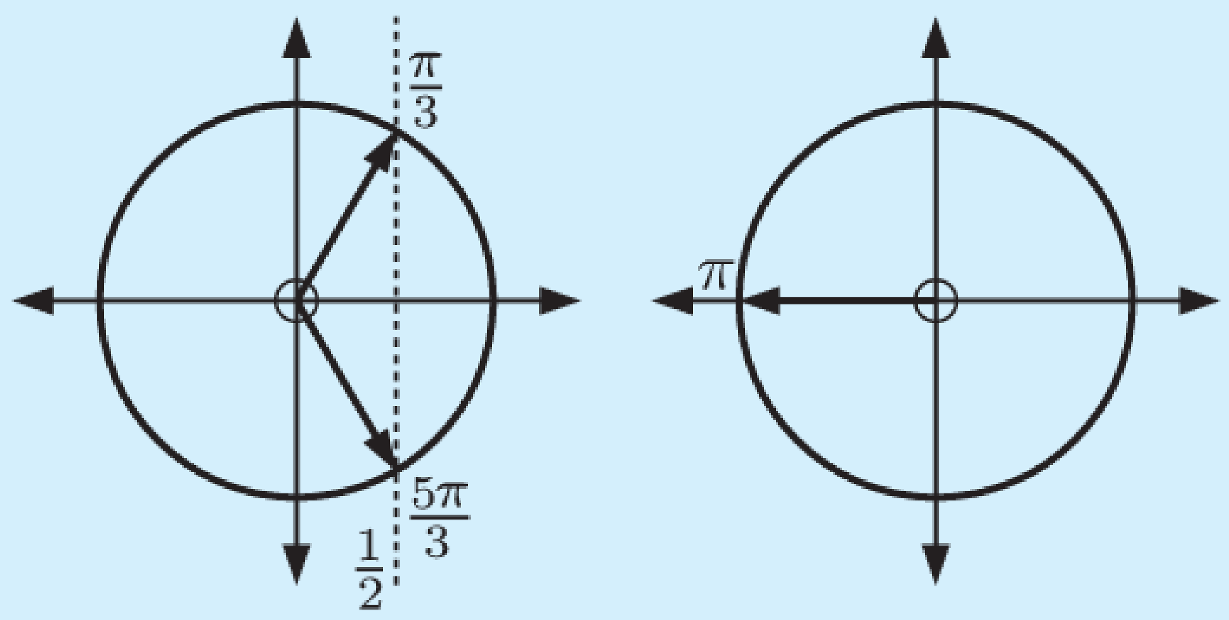
b $2\cos^2 x + \cos x - 1 = 0$

a
$$2\sin^2 x + \sin x = 0$$
$$\therefore \sin x(2\sin x + 1) = 0$$
$$\therefore \sin x = 0 \text{ or } -\tfrac{1}{2}$$



$\therefore \quad x = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, \text{ or } 2\pi.$

b
$$2\cos^2 x + \cos x - 1 = 0$$
$$\therefore (2\cos x - 1)(\cos x + 1) = 0$$
$$\therefore \cos x = \tfrac{1}{2} \text{ or } -1$$



$\therefore \quad x = \frac{\pi}{3}, \pi, \text{ or } \frac{5\pi}{3}.$

15 Solve for $0 \leqslant x \leqslant 2\pi$ giving your answers as exact values:

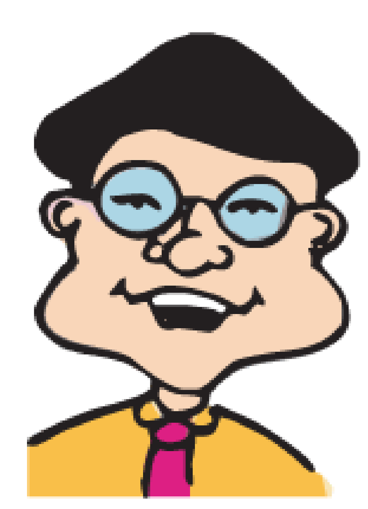
a $2\sin^2 x - \sin x = 0$

b $2\cos^2 x = \cos x$

c $2\cos^2 x - \cos x - 1 = 0$

d $2\sin^2 x + 3\sin x + 1 = 0$

In **a** we cannot simply divide through by $\sin x$, or we will lose the solutions corresponding to $\sin x = 0$.

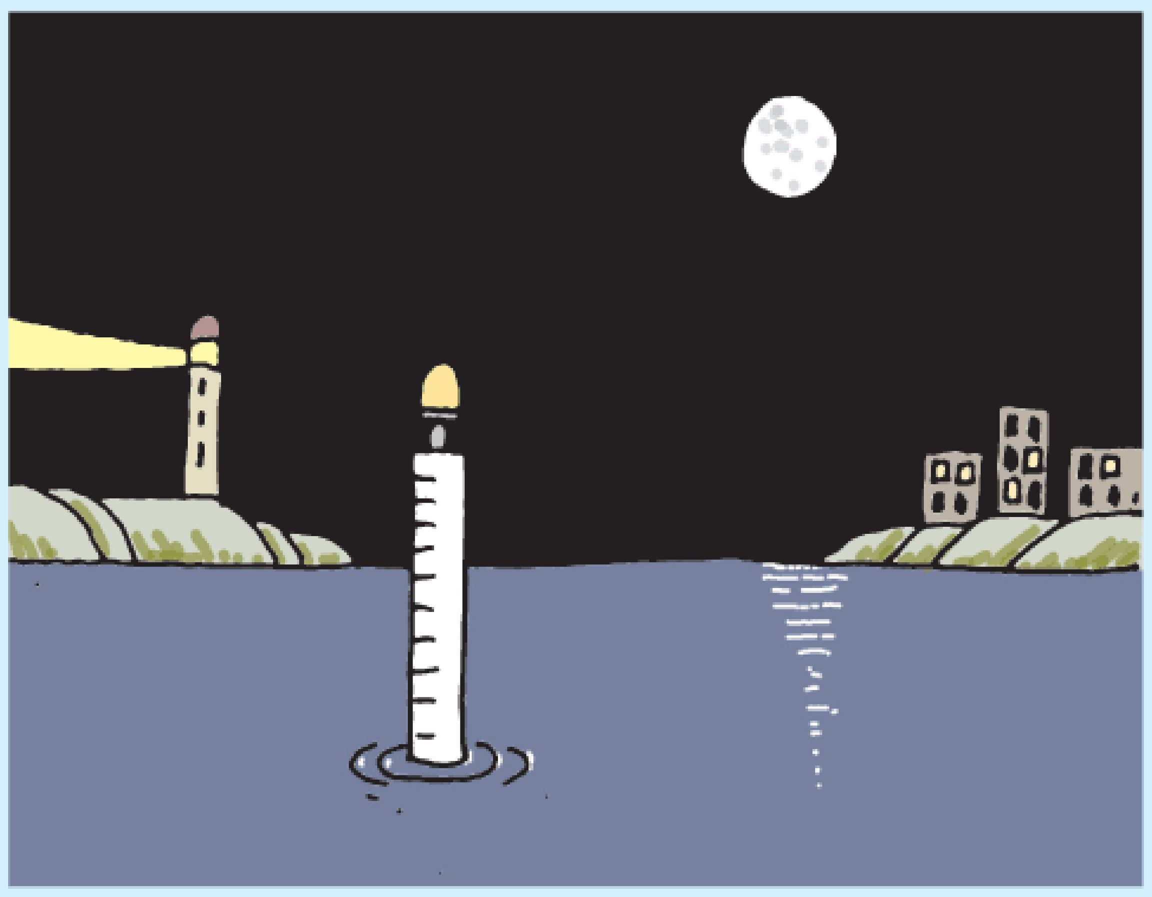


H USING TRIGONOMETRIC MODELS

Having studied trigonometric equations, we can now apply them to the trigonometric models studied in **Section D**.

Example 18 Self Tutor

The height of the tide above mean sea level on January 24th at Cape Town is modelled by $h(t) = 3\sin\frac{\pi t}{6}$ metres, where t is the number of hours after midnight.



a Graph $y = h(t)$ for $0 \leqslant t \leqslant 24$.

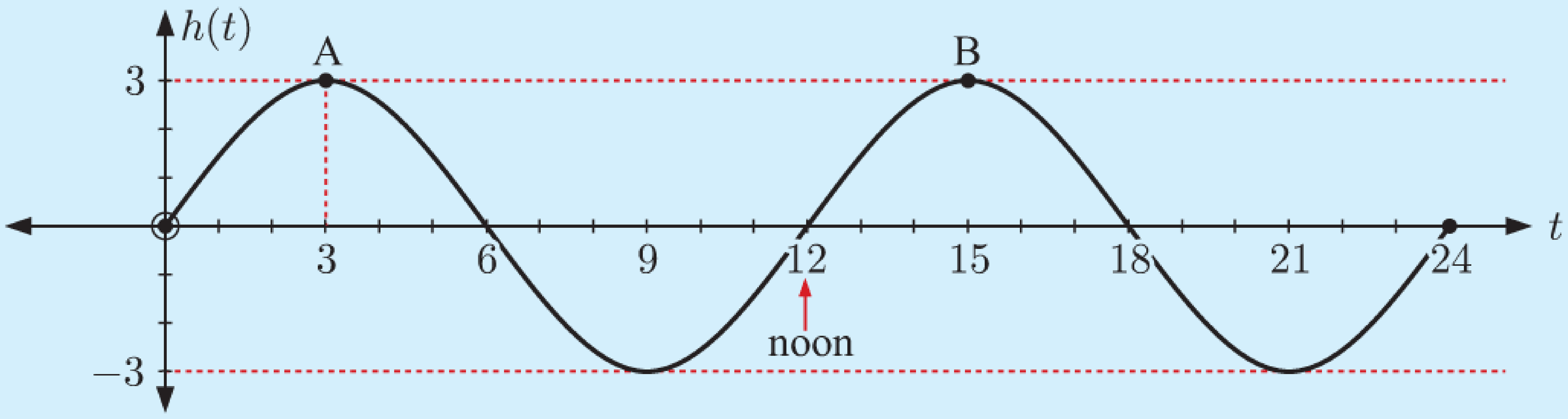
b When is high tide and what is the maximum height?

c What is the height of the tide at 2 pm?

d A ship can cross the harbour provided the tide is at least 2 m above mean sea level. When is crossing possible on January 24th?

a $h(0) = 0$

$h(t) = 3\sin\frac{\pi t}{6}$ has period $= \frac{2\pi}{\frac{\pi}{6}} = 2\pi \times \frac{6}{\pi} = 12$ hours

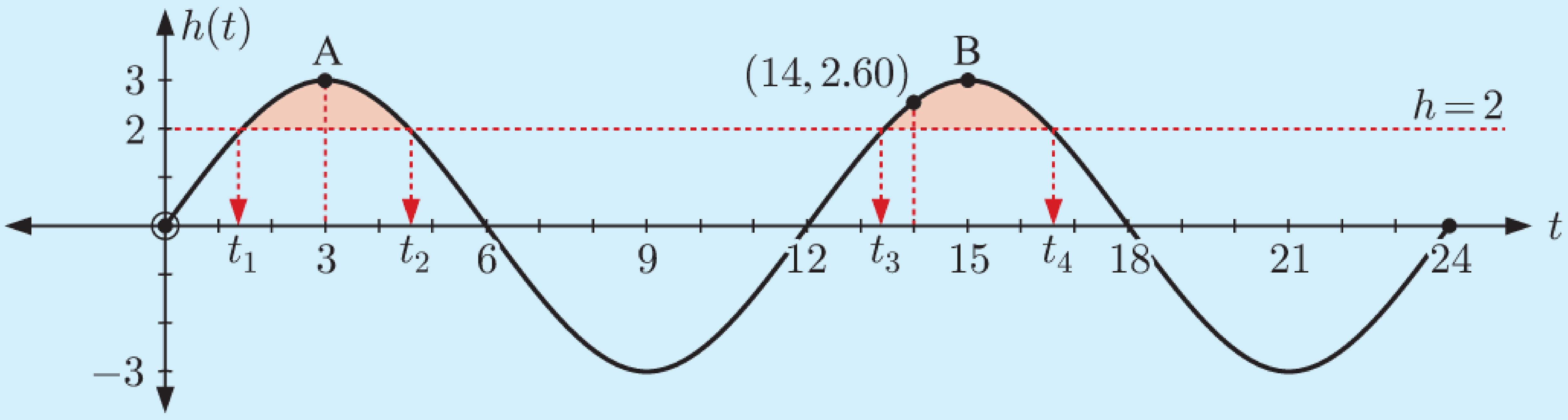


b High tide is at 3 am and 3 pm. The maximum height is 3 m above the mean as seen at points A and B.

c At 2 pm, $t = 14$ and $h(14) = 3\sin\frac{14\pi}{6} \approx 2.60$ m.

So, the tide is 2.6 m above the mean.

d



We need to solve $h(t) = 2$, so $3\sin\frac{\pi t}{6} = 2$.

Using a graphics calculator with $Y_1 = 3\sin\frac{\pi X}{6}$ and $Y_2 = 2$

we obtain $t_1 \approx 1.39$, $t_2 \approx 4.61$, $t_3 \approx 13.39$, $t_4 \approx 16.61$

Now 1.39 hours = 1 hour 23 minutes, and so on.

So, the ship can cross between 1:23 am and 4:37 am or 1:23 pm and 4:37 pm.

EXERCISE 17H

1 The population of grasshoppers after t weeks is $P(t) = 7500 + 3000\sin\frac{\pi t}{8}$ for $0 \leqslant t \leqslant 12$.

a Find:

i the initial population **ii** the population after 5 weeks.

b What is the greatest population size over this interval and when does it occur?

c When is the population: **i** 9000 **ii** 6000?

d During what time interval(s) does the population size exceed 10 000?

2 The model for the height of a passenger on a Ferris wheel is $H(t) = 20 - 19\cos\frac{2\pi t}{3}$, where H is the height in metres above the ground, and t is in minutes.

a Where is the passenger at time $t = 0$?

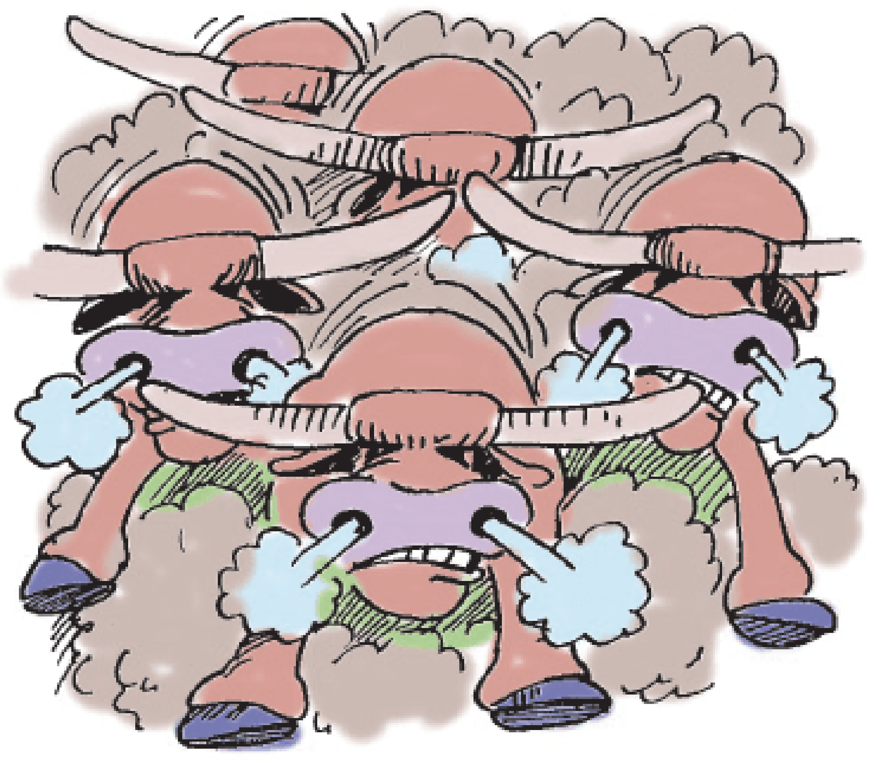
b At what time is the passenger at the maximum height in the first revolution of the wheel?

c How long does the wheel take to complete one revolution?

d Sketch the graph of the function $H(t)$ over one revolution.

e The passenger can see his friend when he is at least 13 m above the ground. During what times in the first revolution can the passenger see his friend?

3 The population of water buffalo is given by $P(t) = 400 + 250\sin\frac{\pi t}{2}$ where t is the number of years since the first estimate was made.



a What was the initial estimate?

b What was the population size after:

i 6 months **ii** two years?

c Find $P(1)$. What is the significance of this value?

d Find the smallest population size and when it first occurred.

e Find the first time when the herd exceeded 500.

4 Over a 28 day period, the cost per litre of petrol was modelled by $C(t) = 9.2\sin\left(\frac{\pi}{7}(t-4)\right) + 107.8$ cents L^{-1}.

a True or false?

i "The cost per litre oscillates about 107.8 cents with maximum price \$1.17 per litre."

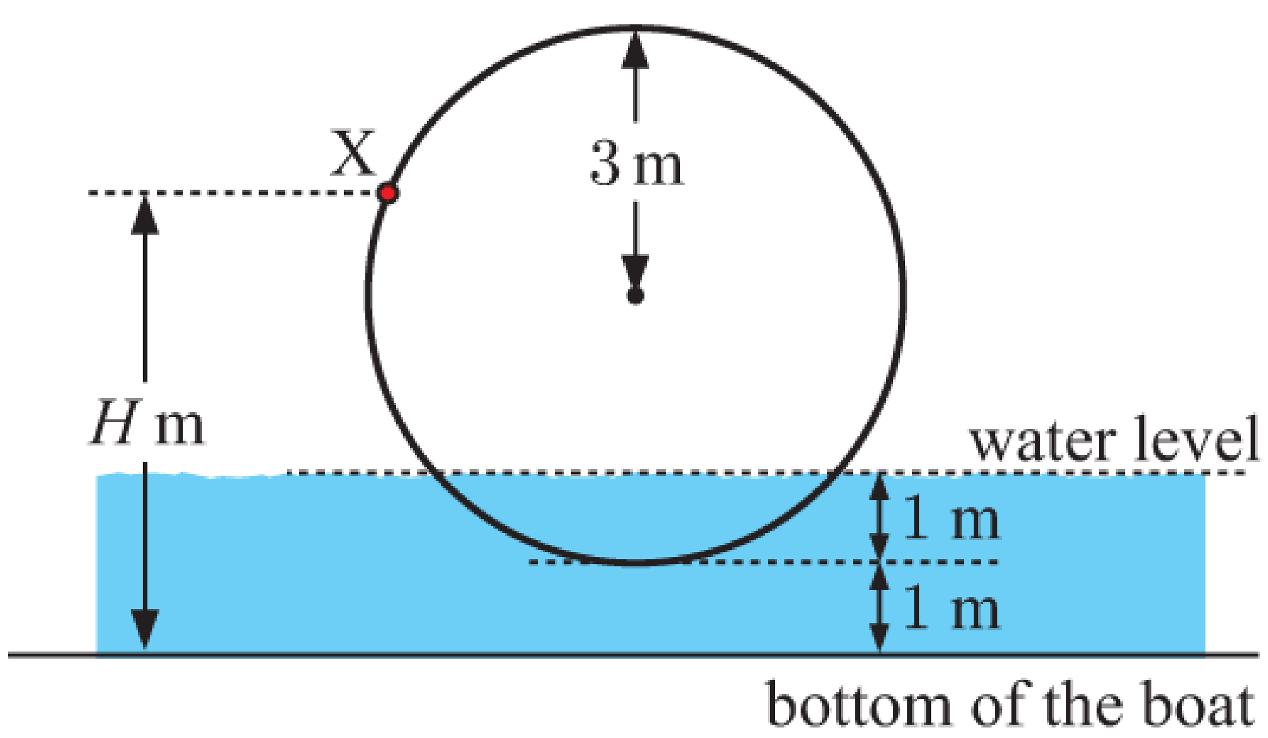
ii "Every 14 days, the cycle repeats itself."

b What was the cost of petrol on day 7, to the nearest tenth of a cent per litre?

c On which days was the petrol priced at \$1.10 per litre?

d What was the minimum cost per litre and when did it occur?

5 A paint spot X lies on the outer rim of the wheel of a paddle-steamer. The wheel has radius 3 m and rotates anticlockwise at a constant rate. X is seen entering the water every 4 seconds.
H is the distance of X above the bottom of the boat. At time $t = 0$, X is at its highest point.



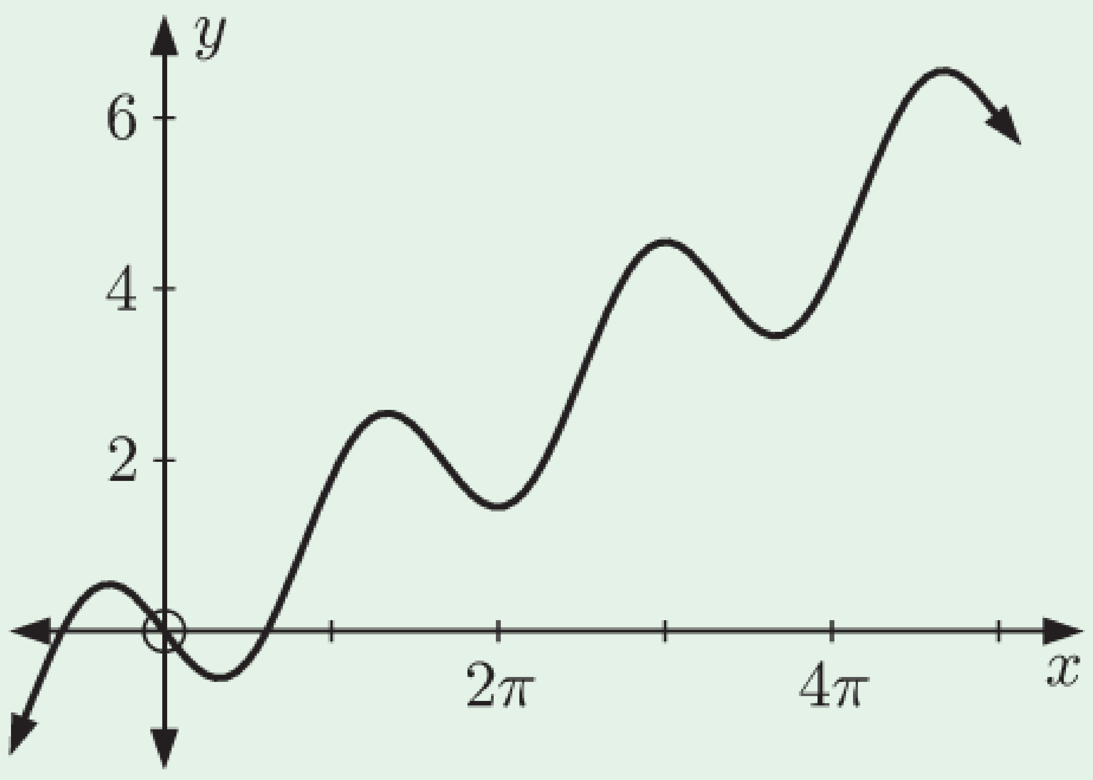
a Find a cosine model for H in the form $H(t) = a\cos(b(t-c)) + d$.

b At what time t does X first enter the water?

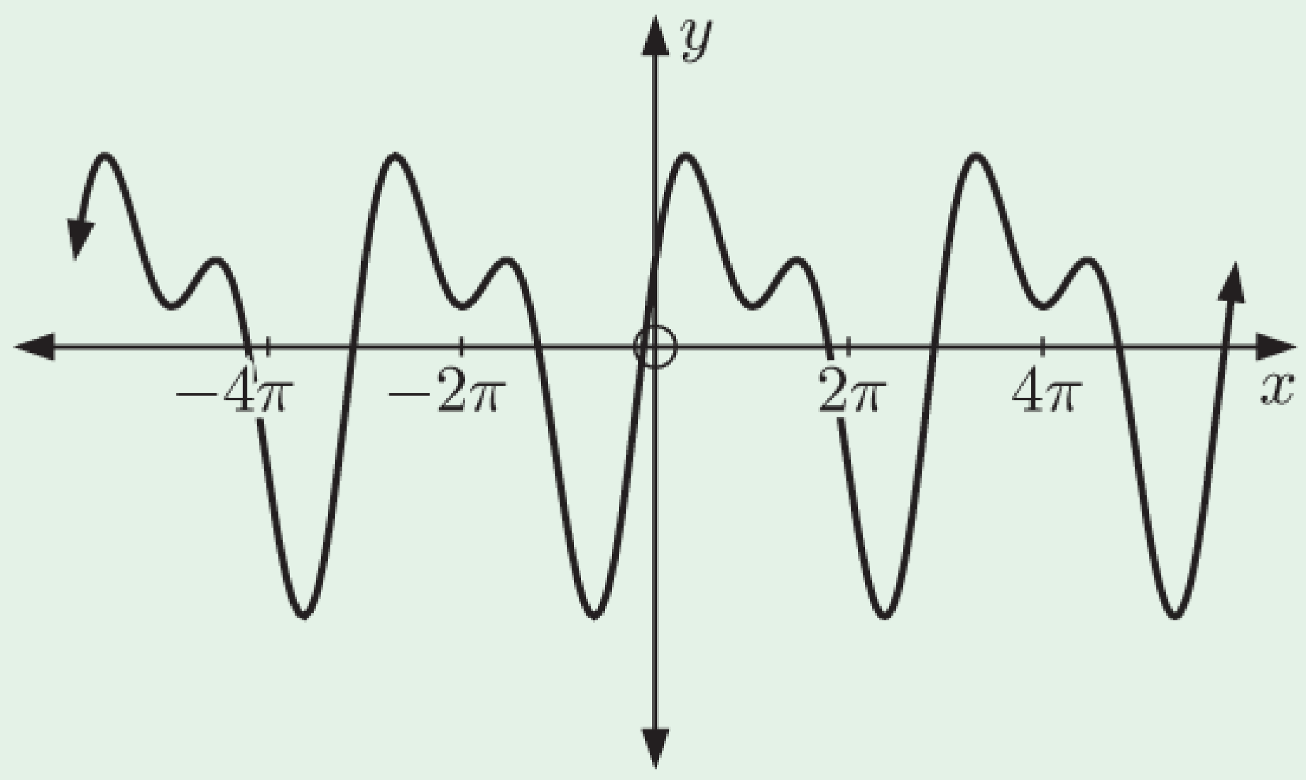
REVIEW SET 17A

1 Which of the following graphs display periodic behaviour?

a



b



2 State the minimum and maximum values of:

a $1 + \sin x$ **b** $-2\cos 3x$

3 State the period of:

a $y = 4\sin\frac{x}{5}$ **b** $y = -2\cos 4x$

c $y = 4\cos\frac{x}{2} + 4$ **d** $y = \frac{1}{2}\tan 3x$

4 Copy and complete:

Function	*Period*	*Amplitude*	*Range*
$y = -3\sin\frac{x}{4} + 1$			
$y = 3\cos\pi x$			

5 **a** Draw the graph of $y = \cos 3x$ for $0 \leqslant x \leqslant 2\pi$.

b Find the value of y when $x = \frac{3\pi}{4}$. Mark this point on your graph.

6 Sketch the graphs of the following for $-2\pi \leqslant x \leqslant 2\pi$:

a $y = 4\sin x$ **b** $y = \sin\left(x - \frac{\pi}{3}\right) + 2$ **c** $y = \sin\frac{3x}{2}$

d $y = \cos\left(x + \frac{\pi}{4}\right)$ **e** $y = \frac{3}{4}\cos x$ **f** $y = \cos 4x$

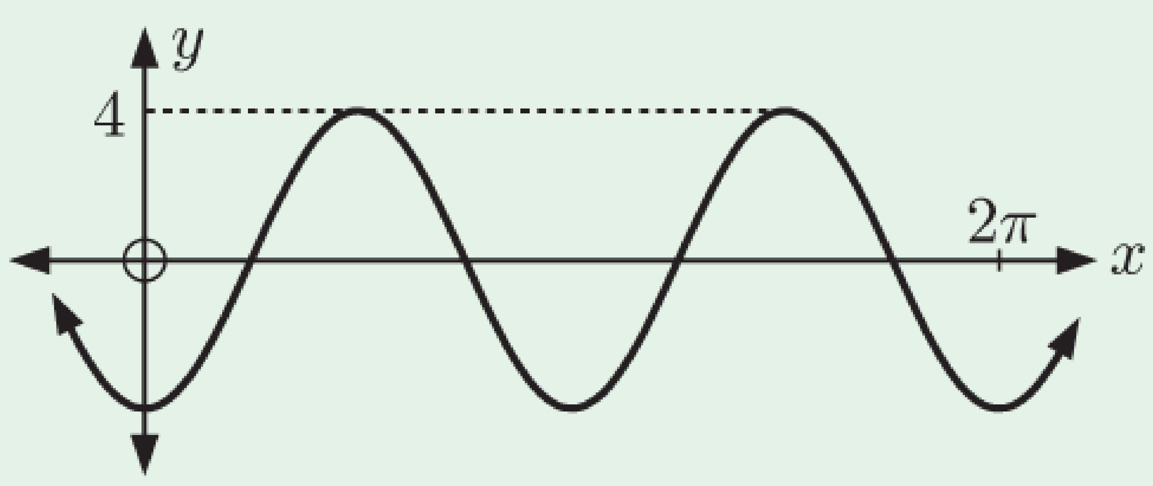
7 State the transformations which map:

a $y = \sin x$ onto $y = 3\sin 2x$ **b** $y = \cos x$ onto $y = \cos\left(x - \frac{\pi}{3}\right) - 1$

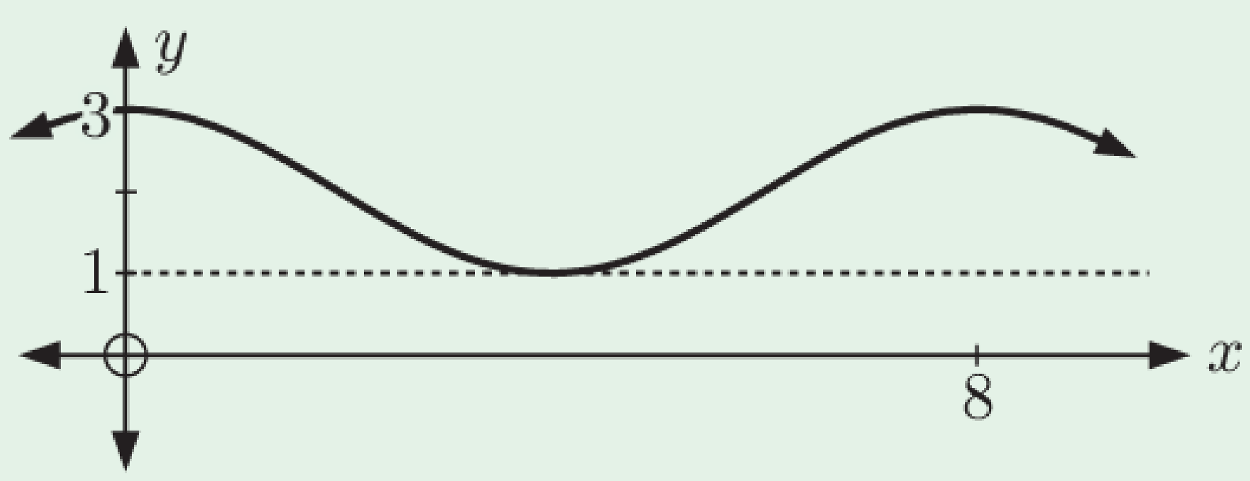
c $y = \tan x$ onto $y = -\tan 2x$ **d** $y = \sin x$ onto $y = 2\sin\left(\frac{x}{2} - \frac{\pi}{4}\right) + \frac{1}{2}$.

8 Find the cosine function represented in each of the following graphs:

a



b



9 Sketch for $0 \leqslant x \leqslant 4\pi$:

a $y = \tan\frac{x}{4}$ **b** $y = \frac{1}{4}\tan\frac{x}{2}$

10 **a** Describe the sequence of transformations which maps $y = \tan x$ onto $y = \tan 3x + 2$.

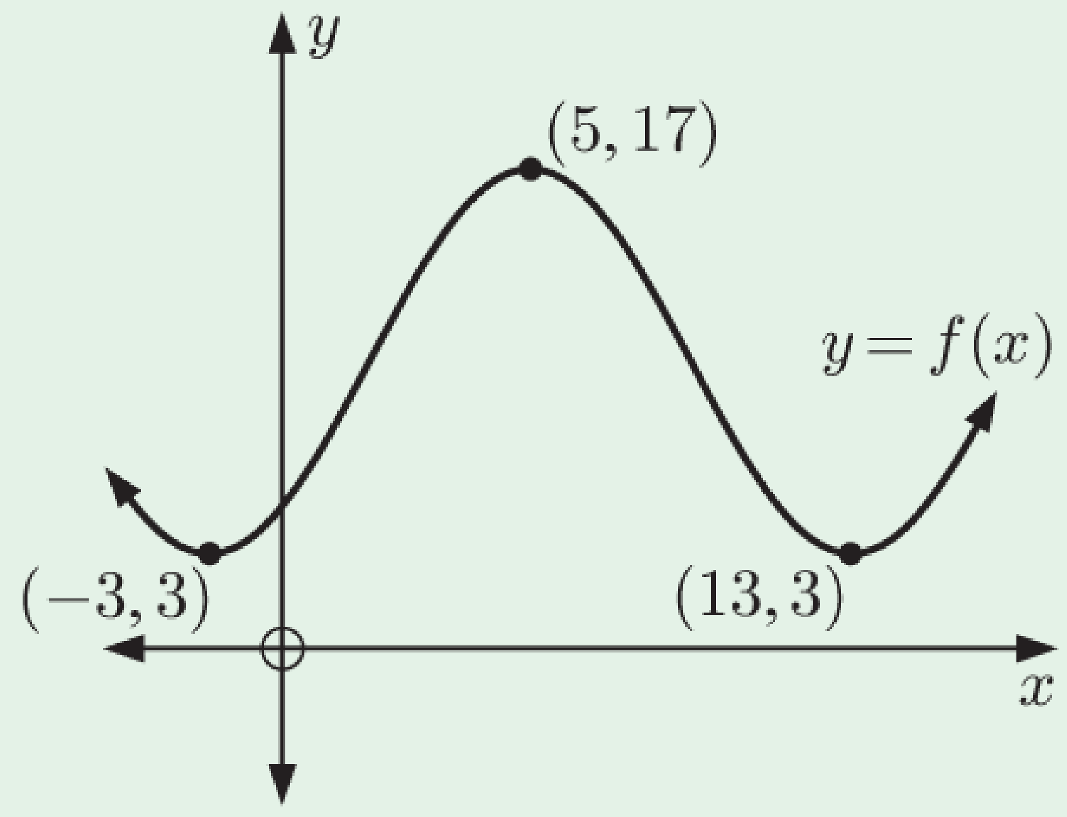
b State the period of $y = \tan 3x + 2$.

c Sketch $y = \tan 3x + 2$ for $-\pi \leqslant x \leqslant \pi$.

11 The graph of $f(x) = a\sin(b(x - c)) + d$ is shown alongside.

a Find the values of a, b, c, and d.

b The function $g(x)$ is obtained by translating $f(x)$ 2 units right and 3 units down, followed by a vertical stretch with scale factor 2.
Find $g(x)$ in the form $g(x) = p\sin(q(x-r))+s$.



12 The proportion of the Moon which is illuminated each night is given by the function $M(t) = \frac{1}{2}\cos\left(\frac{\pi}{15}t\right) + \frac{1}{2}$, where t is the time in days after January 1st.

a Sketch the graph of M against t for $0 \leqslant t \leqslant 60$.

b Find the proportion of the Moon which is illuminated on the night of:

i January 6th **ii** January 21st **iii** January 27th **iv** February 19th.

c How often does a full moon occur?

d On what dates during January and February is the Moon not illuminated at all?

13 On an April day in Kyoto, the maximum temperature 14.1°C occurred at 2:30 pm. The minimum was 6.7°C.

a Suggest a sine function to model the temperature for that day. Let T be the temperature and t be the time in hours after midnight.

b Graph $T(t)$ for $0 \leqslant t \leqslant 24$.

14 A robot on Mars records the temperature every Mars day. A summary series, showing every one hundredth Mars day, is shown in the table below.

Number of Mars days (n)	0	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300
Temp. (°C)	−43	−15	−5	−21	−59	−79	−68	−50	−27	−8	−15	−70	−78	−68

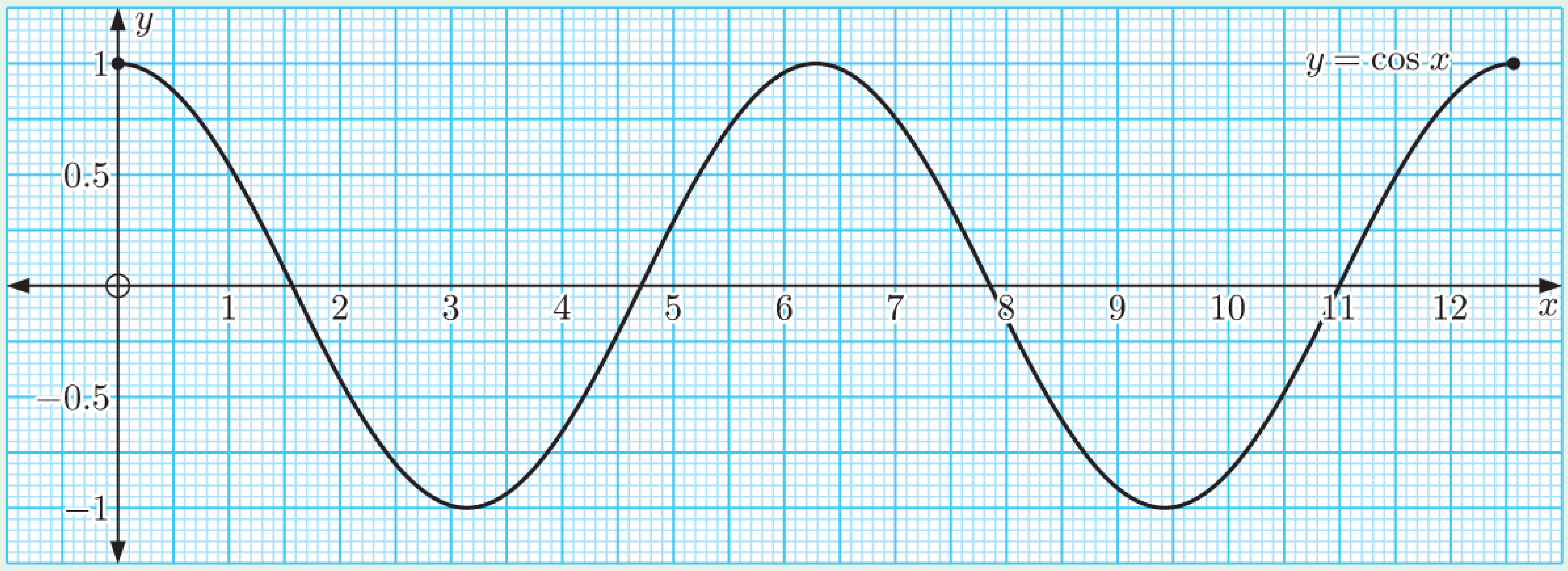
a Find the maximum and minimum temperatures recorded by the robot.

b Use the data to estimate the length of a Mars year.

c Without using technology, find a sine model for the temperature T in terms of the number of Mars days n.

d Draw a scatter diagram of the data and sketch the graph of your model on the same set of axes.

e Check your answer to **c** using technology. How well does your model fit?

15 

Use the graph of $y = \cos x$ on the domain $0 \leqslant x \leqslant 4\pi$ to solve, correct to 1 decimal place:

a $\cos x = -0.4$ **b** $\cos x = 0.9$

16 Solve for $0 \leqslant x \leqslant 8$:

a $\sin x = 0.382$ **b** $\tan \frac{x}{2} = -0.458$

17 Solve:

a $\cos x = 0.4379$ for $0 \leqslant x \leqslant 10$ **b** $\cos(x - 2.4) = -0.6014$ for $0 \leqslant x \leqslant 6$

18 Solve for $0 \leqslant x \leqslant 2\pi$:

a $2\sin x = -1$ **b** $\sqrt{2}\cos x - 1 = 0$ **c** $2\cos 2x + 1 = 0$

19 Solve algebraically for $0 \leqslant x \leqslant 2\pi$, giving answers in terms of π:

a $\tan^2 2x = 1$ **b** $\sin^2 x - \sin x - 2 = 0$ **c** $4\sin^2 x = 1$

20 Find the exact solutions of:

a $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) - 1 = 0, \ 0 \leqslant x \leqslant 4\pi$ **b** $\tan 2x - \sqrt{3} = 0, \ 0 \leqslant x \leqslant 2\pi$

21 An ecologist studying a species of water beetle estimates the population of a colony over an eight week period. If t is the number of weeks after the initial estimate is made, then the population in thousands can be modelled by $P(t) = 5 + 2\sin \frac{\pi t}{3}$ where $0 \leqslant t \leqslant 8$.

a What was the initial population?

b What were the smallest and largest populations?

c During what time interval(s) did the population exceed 6000?

REVIEW SET 17B

1 Consider the graph alongside.

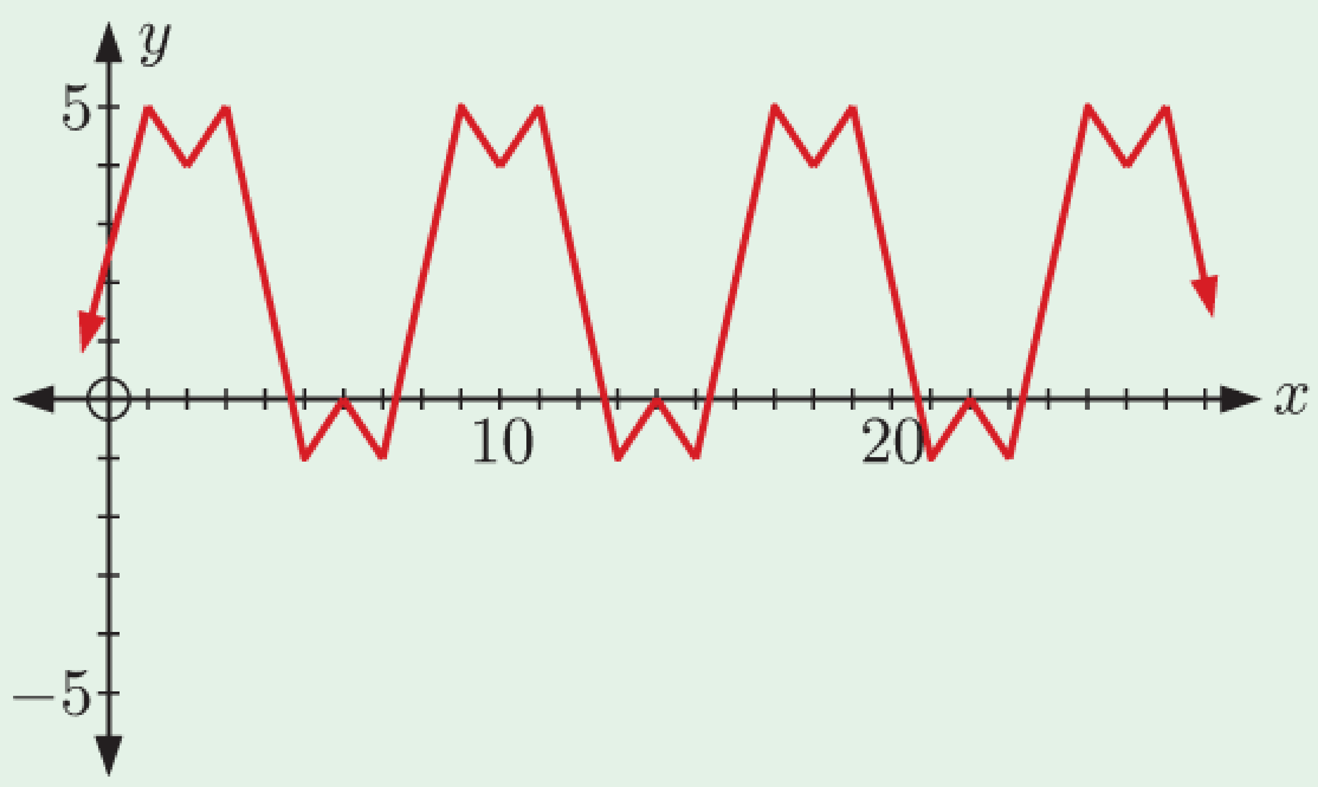
a Explain why this graph shows periodic behaviour.

b State:

i the period

ii the maximum value

iii the minimum value.



2 State the transformation which maps:

a $y = \cos x$ onto $y = \cos\left(x - \frac{\pi}{3}\right) + 1$ **b** $y = \sin x$ onto $y = \sin 3x$.

3 State the period of:

a $y = 4\sin\frac{x}{3}$ **b** $y = \tan 4x$

4 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a 6π **b** $\frac{\pi}{12}$ **c** 9

5 State the minimum and maximum values of:

a $y = 5\sin x - 3$ **b** $y = \frac{1}{3}\cos x + 1$

6 Find the principal axis of:

a $y = -\frac{1}{3}\sin\left(x - \frac{\pi}{4}\right) + 5$ **b** $y = 2\cos\frac{x}{3} - 4$

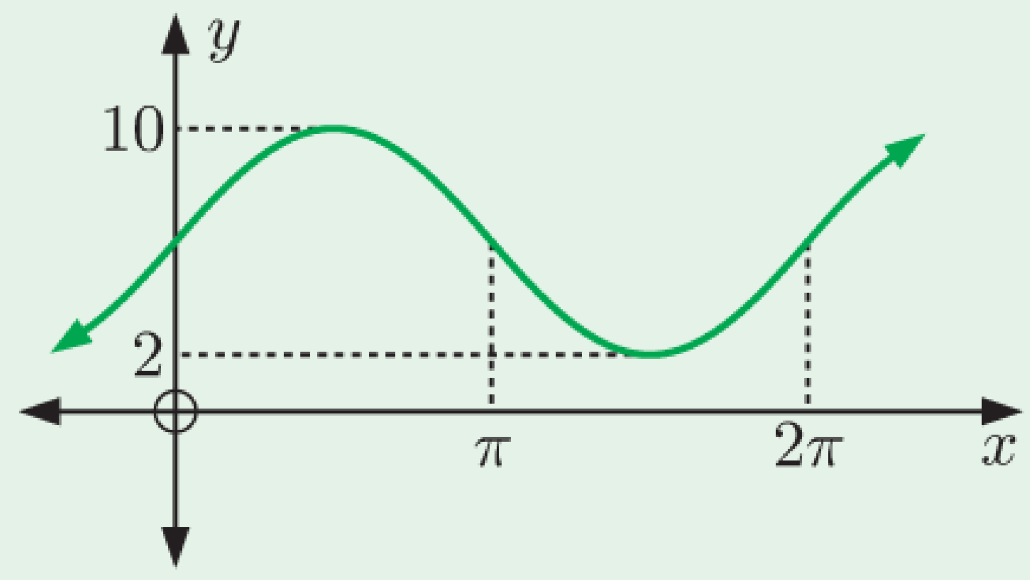
7 Sketch the graphs of the following for $0 \leqslant x \leqslant 2\pi$:

a $y = 2\cos 3x$ **b** $y = 2\sin\left(x - \frac{\pi}{3}\right) + 3$ **c** $y = -\cos\left(x + \frac{\pi}{4}\right)$

d $y = 2\sin x - \frac{1}{2}$ **e** $y = \frac{3}{2}\tan\left(x - \frac{\pi}{6}\right)$ **f** $y = 2\tan\frac{x}{2}$

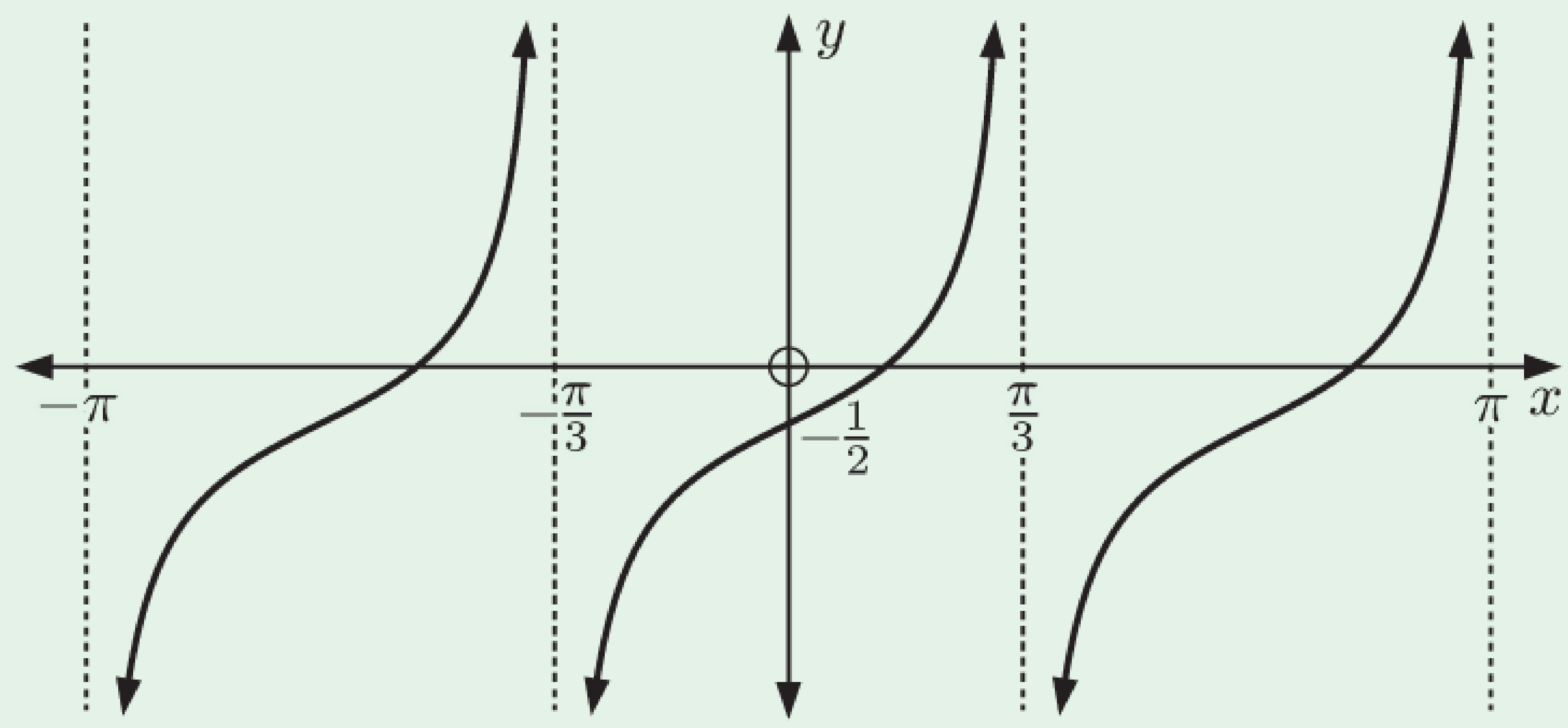
8 **a** Find the sine function shown in this graph.

b Write down the equivalent cosine function for this graph.



9 Draw the graph of $y = 0.6\cos(2.3x)$ for $0 \leqslant x \leqslant 5$.

10 Find a and b given the graph of $y = \tan ax + b$ shown.

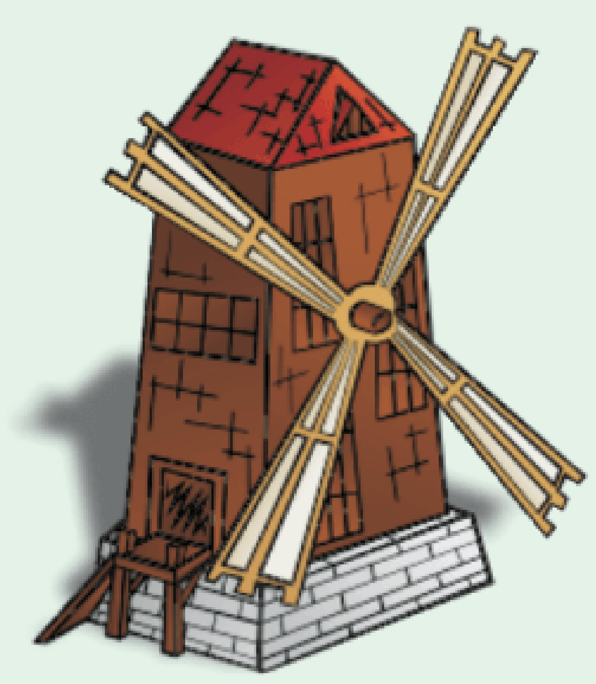


11 State the transformations which map:

a $y = \tan x$ onto $y = -\tan 2x$

b $y = \sin x$ onto $y = 2\sin\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) + \frac{1}{2}$.

12 As the tip of a windmill's blade rotates, its height above ground is given by $H(t) = 10\cos\left(\frac{\pi}{6}t\right) + 20$ metres, where t is the time in seconds.



a Sketch the graph of H against t for $0 \leqslant t \leqslant 36$.

b Find the height of the blade's tip after 9 seconds.

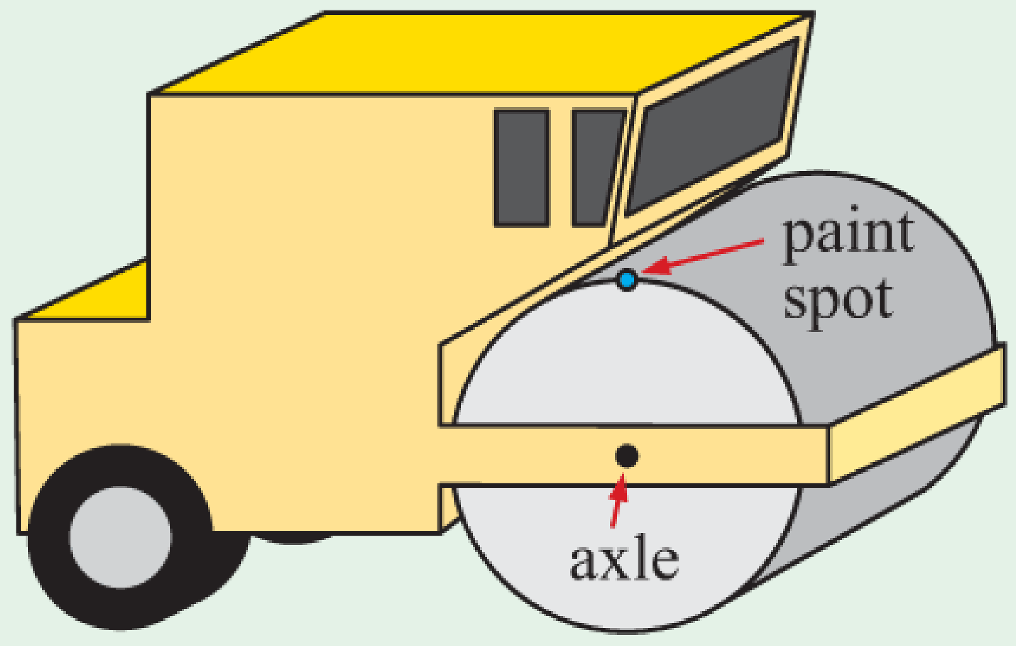
c Find the minimum height of the blade's tip.

d How long does the blade take to complete a full revolution?

13 A steamroller has a spot of paint on its roller. As the steamroller moves, the spot rotates around the axle. The roller has radius 1 metre and completes one full revolution every 2 seconds.

a What does the graph of the spot's height over time look like?

b What function gives the height of the paint spot over time?



14 The table below gives the mean monthly maximum temperature for Perth Airport in Australia.

Month (t)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature (°C)	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8

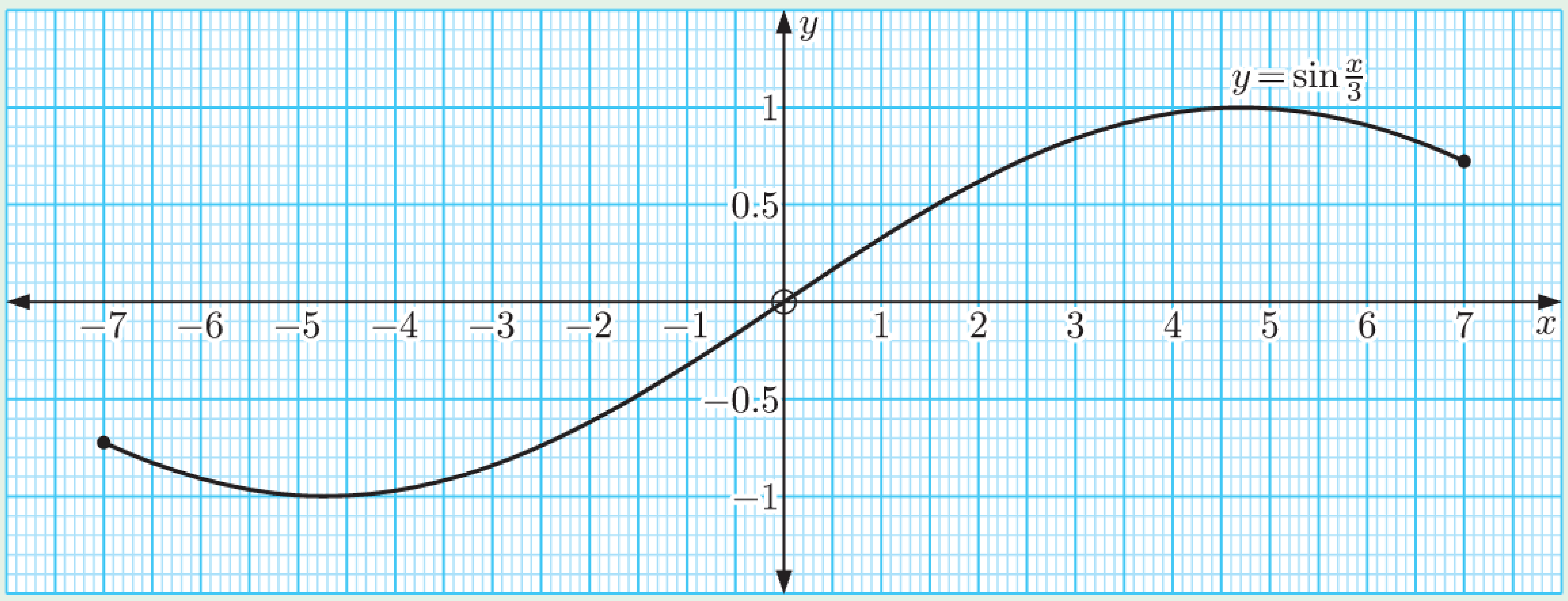
a A sine function of the form $T \approx a\sin(b(t-c)) + d$ is used to model the data. Find good estimates of the constants a, b, c, and d without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Draw a scatter diagram of the data and the graph of your model on the same set of axes.

c Check your answer to **a** using technology. How well does your model fit?

15 Consider $y = \sin\frac{x}{3}$ on the domain $-7 \leqslant x \leqslant 7$. Use the graph to solve, correct to 1 decimal place:

a $\sin\frac{x}{3} = -0.9$ **b** $\sin\frac{x}{3} = \frac{1}{4}$



16 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\cos x = 0.3$ **b** $43 + 8\sin x = 50.1$

17 Solve for $0 \leqslant x \leqslant 10$:

a $\tan x = 4$ **b** $\tan\frac{x}{4} = 4$ **c** $\tan(x-1.5) = 4$

18 Solve for $0 \leqslant x \leqslant 2\pi$:

a $2\sin 3x = -\sqrt{3}$ **b** $\sqrt{3}\tan\frac{x}{2} = -1$ **c** $\cos 2x = \sqrt{3}\sin 2x$

19 Suppose $f(x) = \cos x$ and $g(x) = 2x$. Solve for $0 \leqslant x \leqslant 2\pi$:

a $(f \circ g)(x) = 1$ **b** $(g \circ f)(x) = 1$

20 Find exact solutions for x given $-\pi \leqslant x \leqslant \pi$:

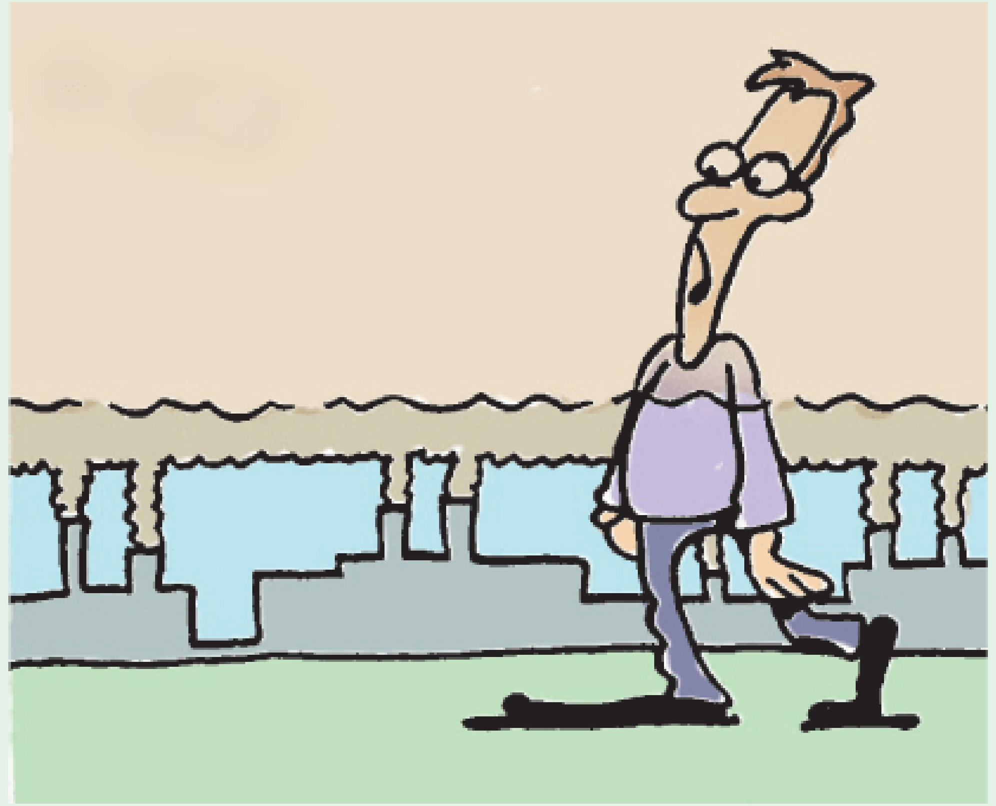
a $\tan\left(x + \frac{\pi}{6}\right) = -\sqrt{3}$ **b** $\tan 2x = -\sqrt{3}$ **c** $\tan^2 x - 3 = 0$

21 Find the x-intercepts of:

a $y = 2\sin 3x + \sqrt{3}$ for $0 \leqslant x \leqslant 2\pi$

b $y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ for $0 \leqslant x \leqslant 3\pi$

22 In an industrial city, the amount of pollution in the air becomes greater during the working week when factories are operating, and lessens over the weekend. The number of milligrams of pollutants in a cubic metre of air is given by $P(t) = 40 + 12\sin\left(\frac{2\pi}{7}\left(t - \frac{37}{12}\right)\right)$ where t is the number of days after midnight on Saturday night.

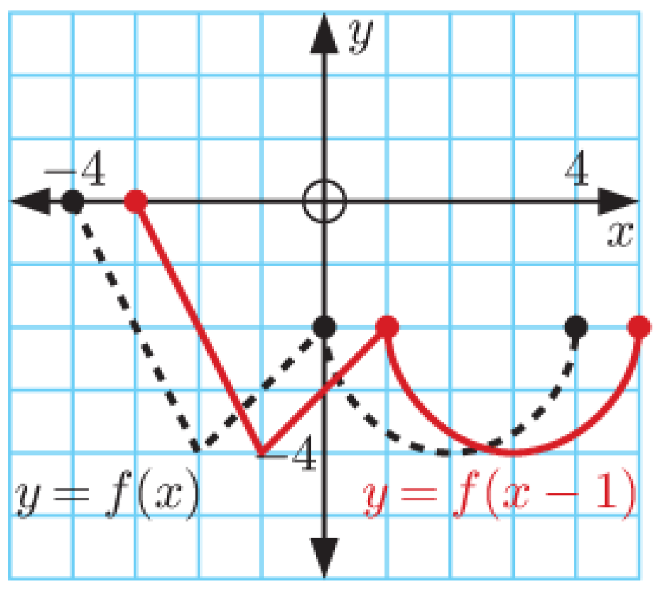


a What is the minimum level of pollution?

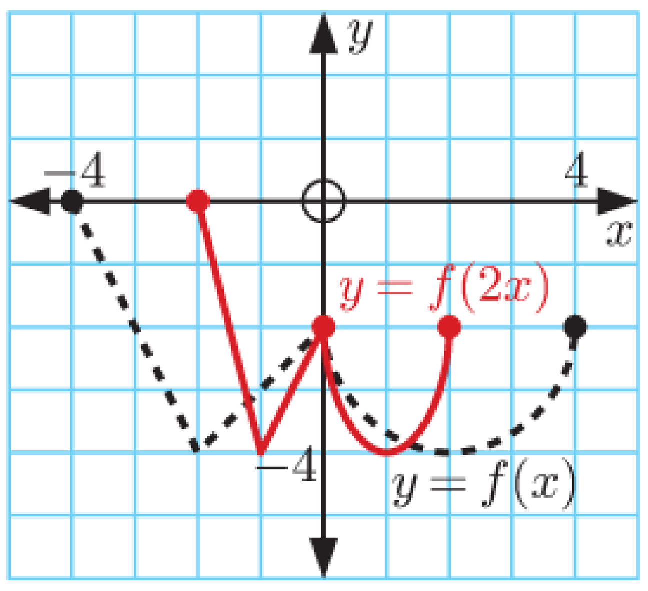
b At what time during the week does this minimum level occur?

REVIEW SET 16B

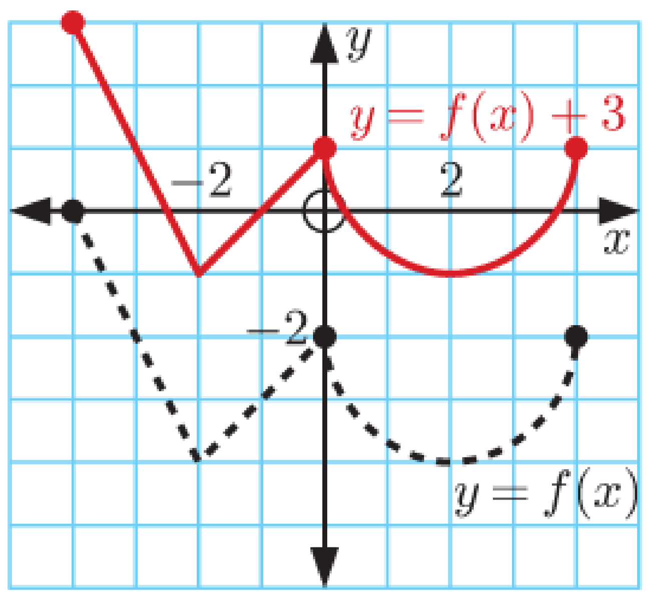
1 **a**



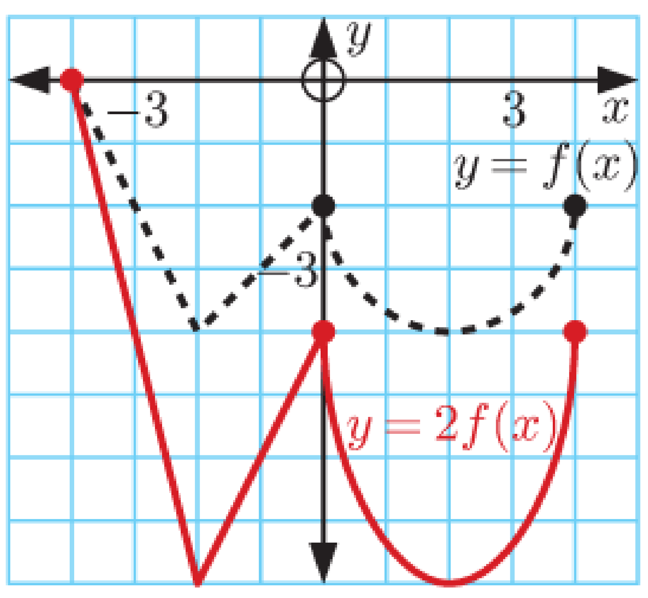
b



c



d



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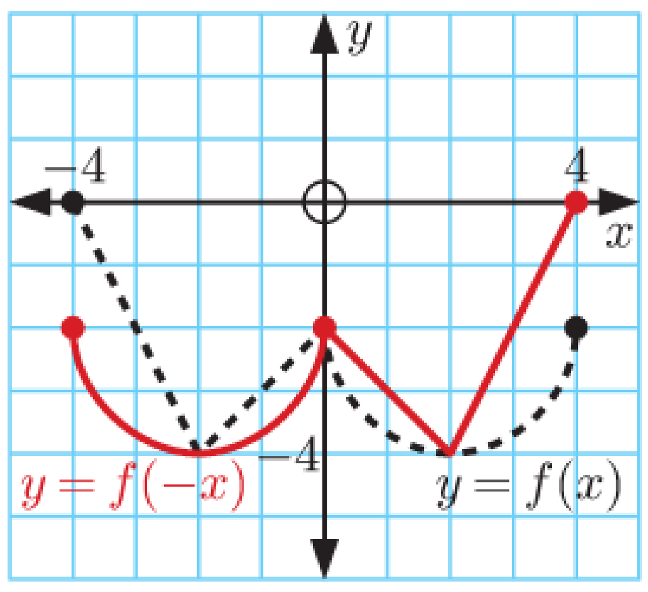

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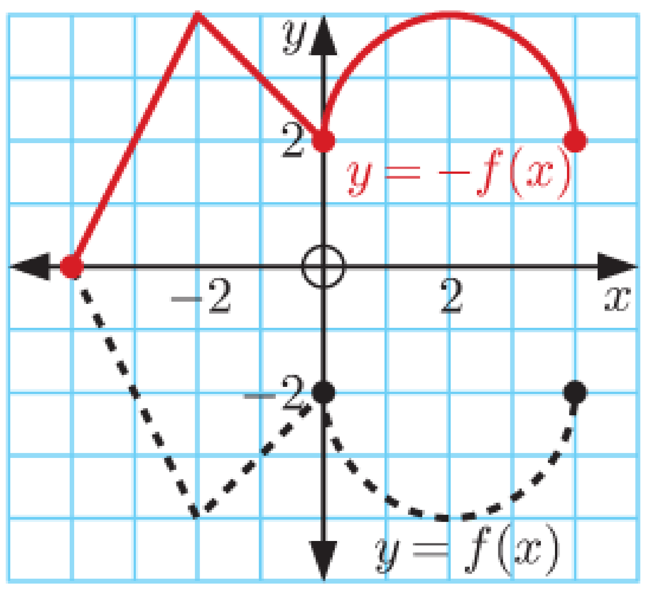
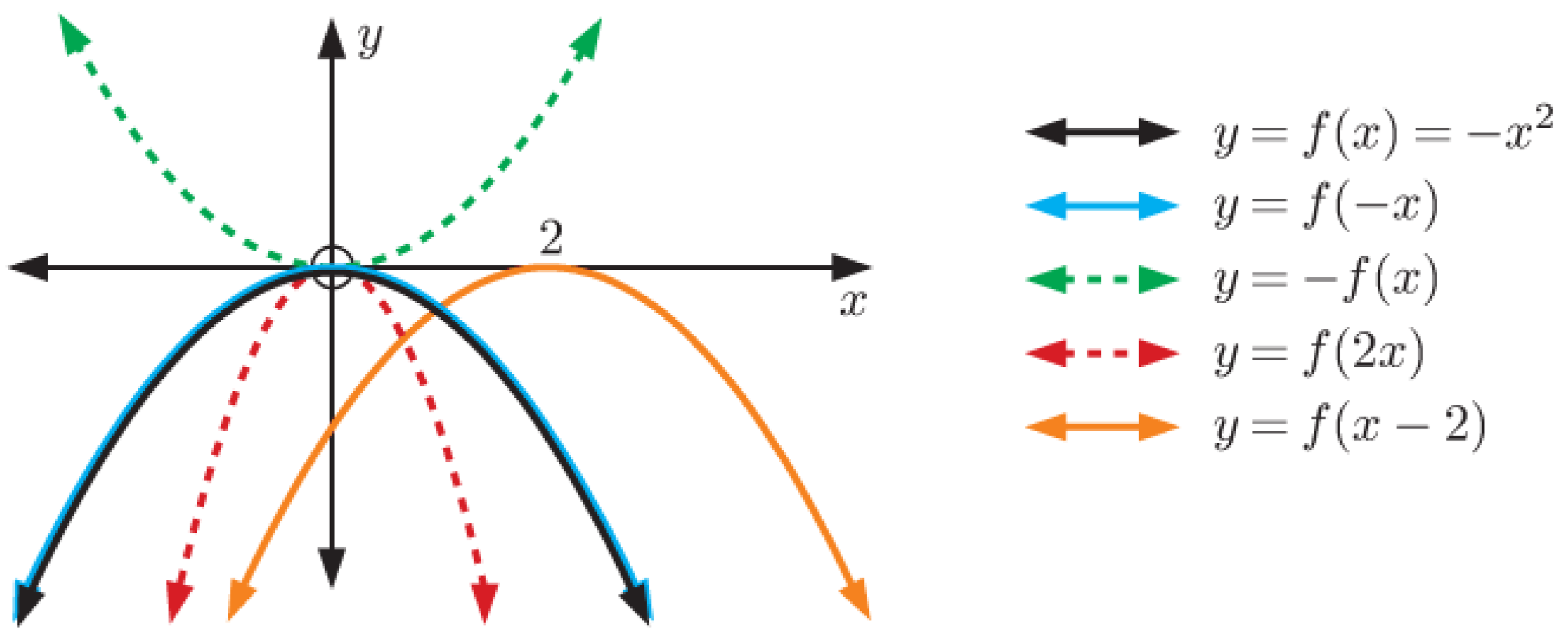


2 **a** $g(x) = 3x - x^2$ **b** $g(x) = 16 - x$

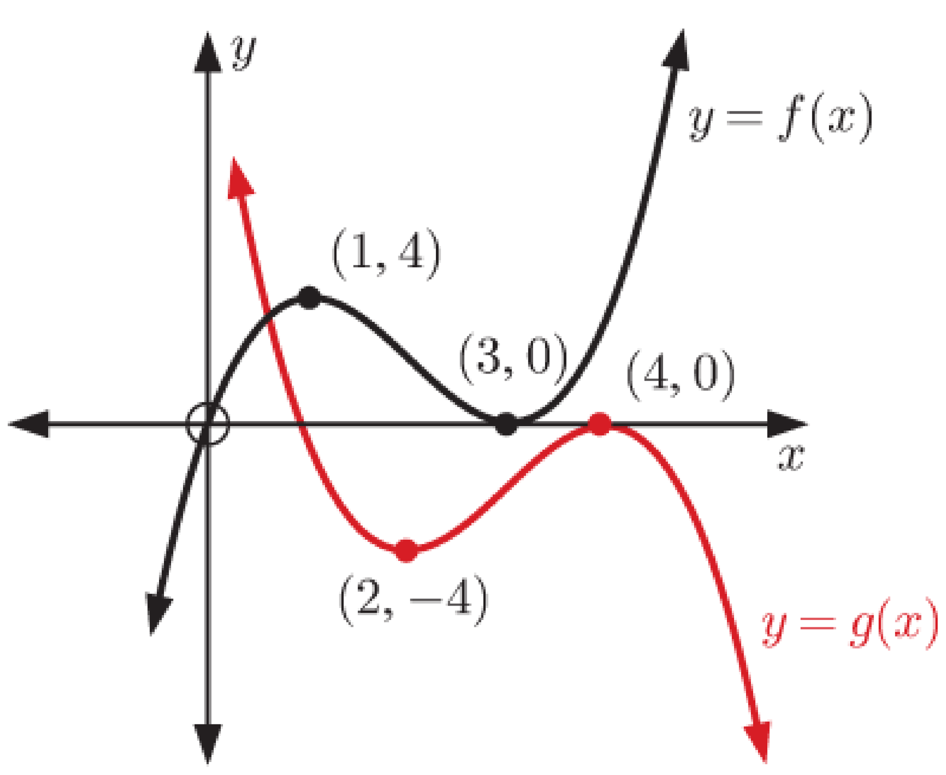
c $g(x) = \frac{1}{12}x + 2$

3 $g(x) = -x^2 - 6x - 7$

4



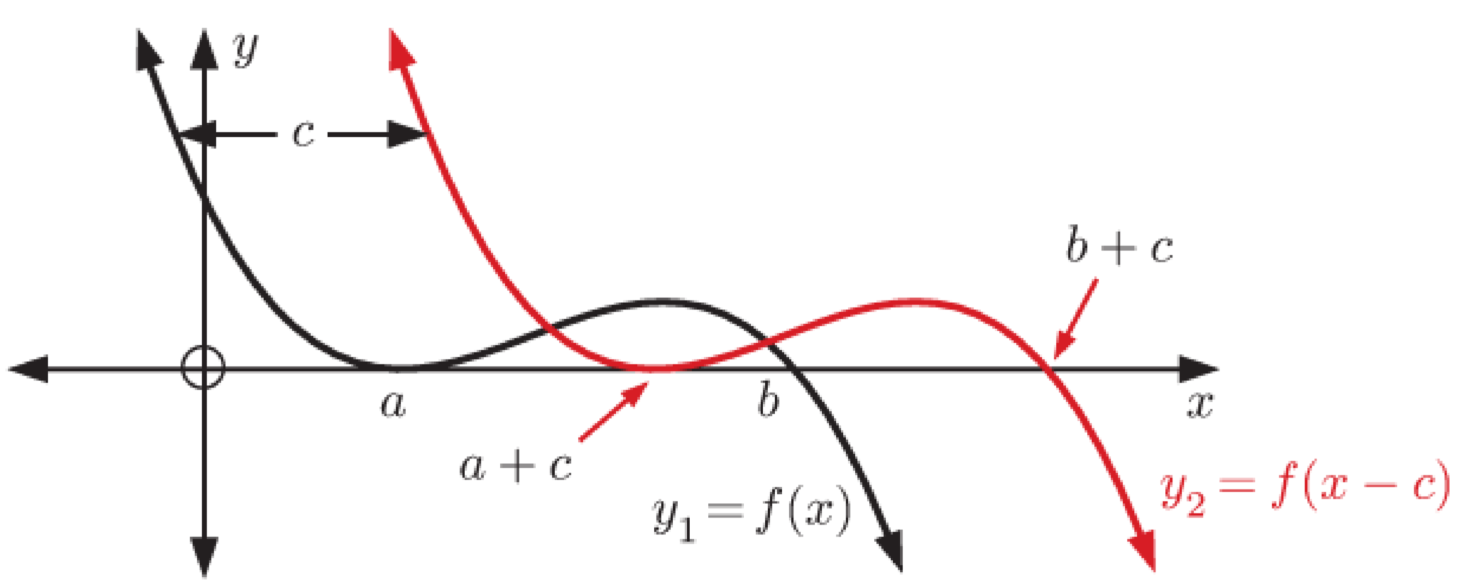
5 **a**



b $(2, -4)$ and $(4, 0)$

6 $y = -2x^2 + 5x - 3$

7



8 A reflection in the x-axis, then a translation through $\begin{pmatrix} \frac{5}{2} \\ -\frac{7}{2} \end{pmatrix}$.

9 $(1, 6)$

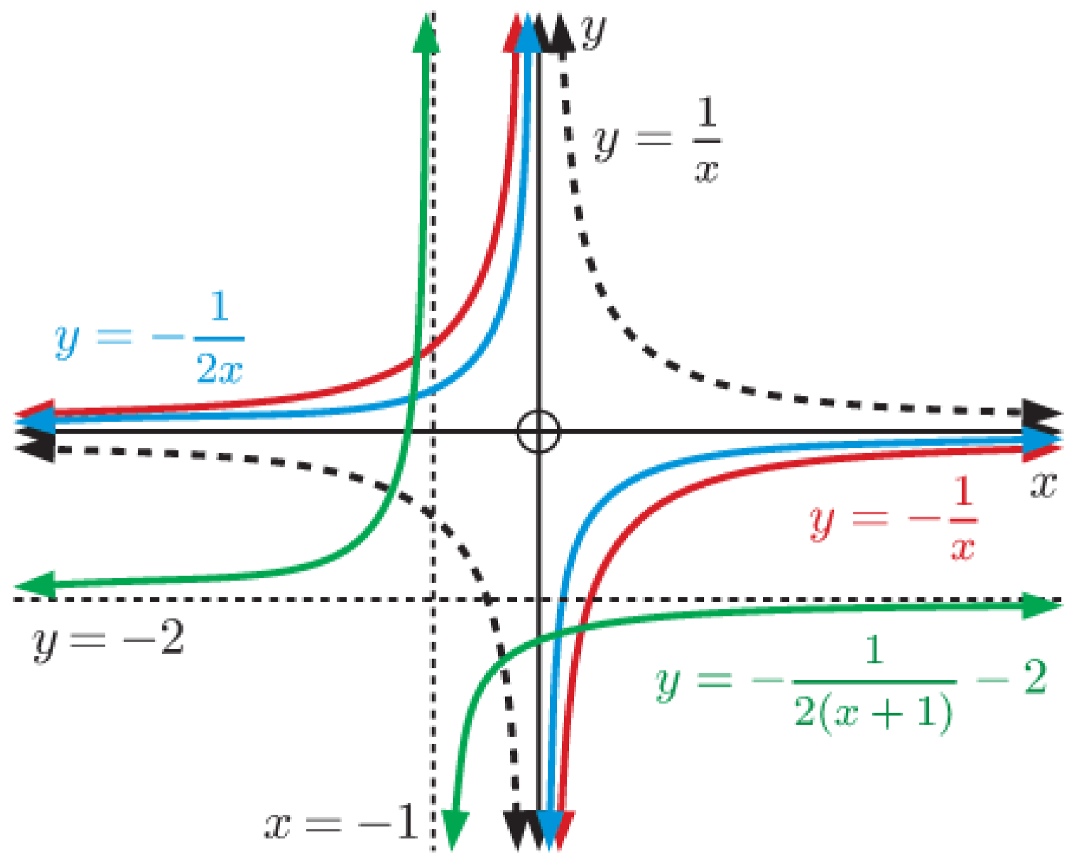
10 **a** A vertical stretch with scale factor 2, then a translation through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

b A reflection in the x-axis, a horizontal stretch with scale factor $\frac{3}{2}$, then a translation through $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$.

c A vertical stretch with scale factor $\frac{1}{3}$, a reflection in the y-axis, then a translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

11 $b = 8$, $c = -20$

12 **a**

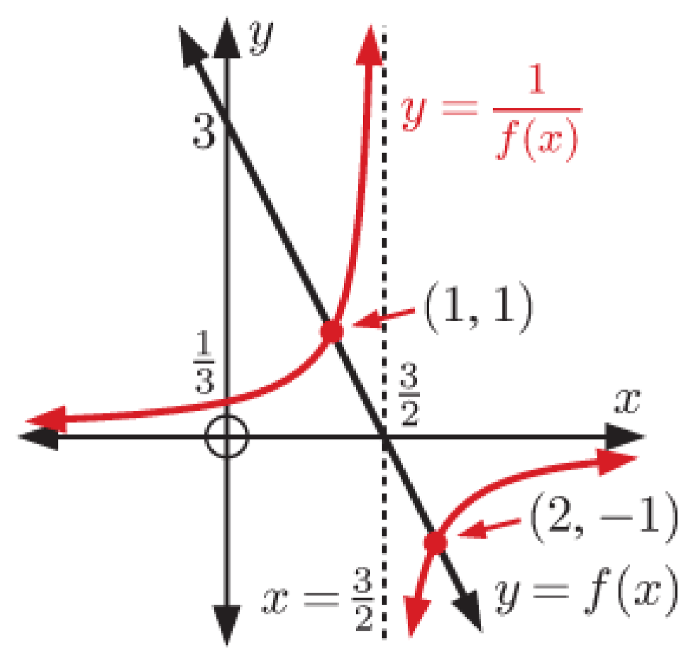


b A reflection in the x-axis, a vertical stretch with scale factor $\frac{1}{2}$, then a translation through $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

c $y = \dfrac{-4x - 5}{2x + 2}$

Domain is $\{x \mid x \neq -1\}$, Range is $\{y \mid y \neq -2\}$

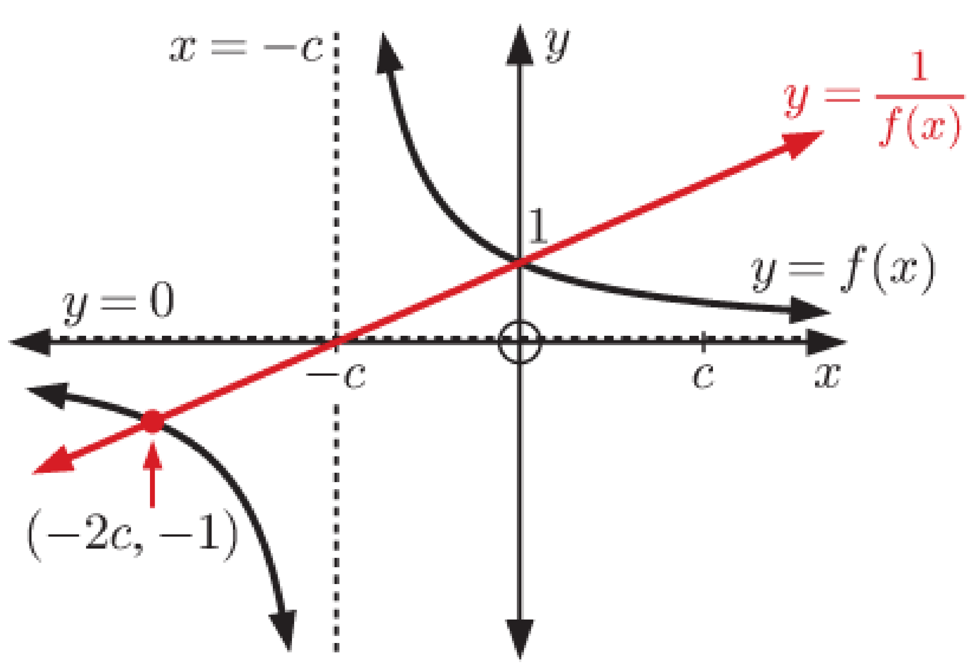
13 **a, d**



b $(1, 1)$ and $(2, -1)$

c y-intercept $\frac{1}{3}$ and vertical asymptote $x = \frac{3}{2}$

14

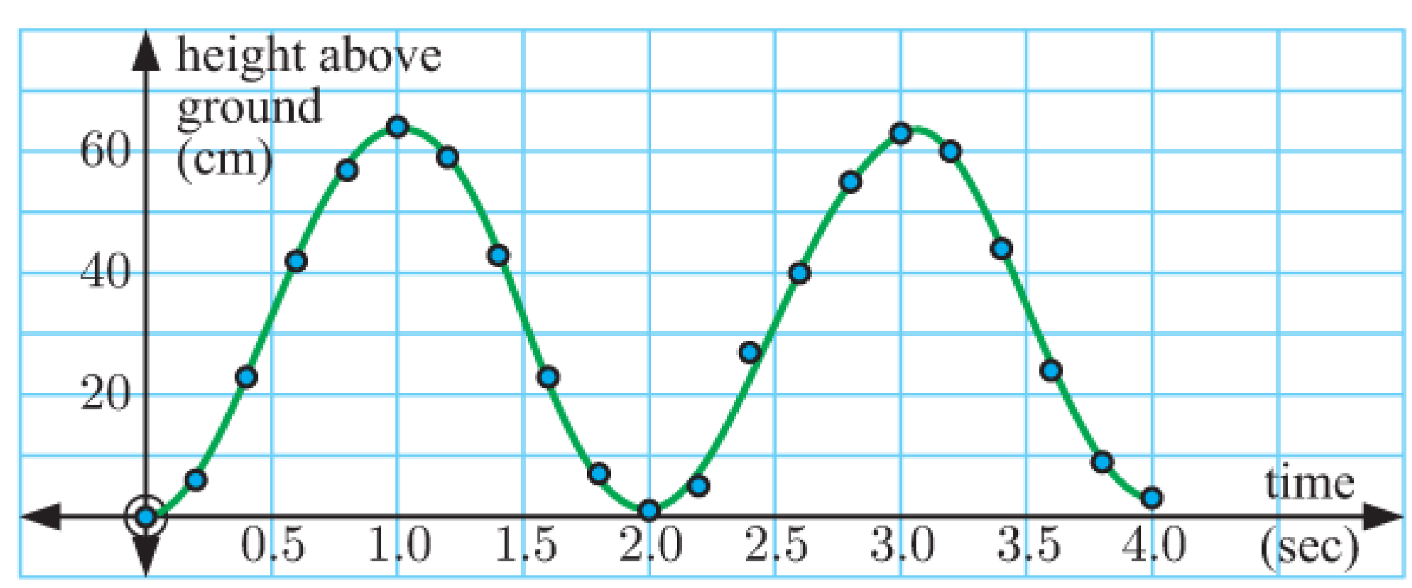


EXERCISE 17A

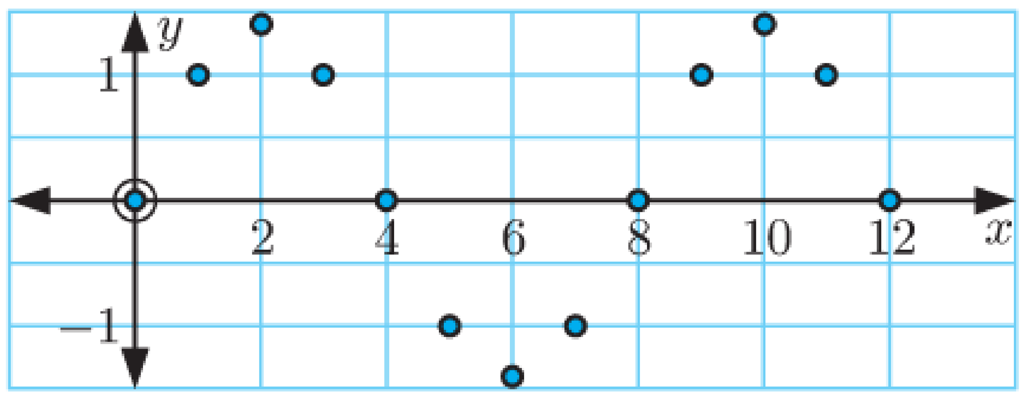
1 **a** periodic **b** periodic **c** periodic **d** not periodic

e periodic **f** periodic **g** not periodic **h** not periodic

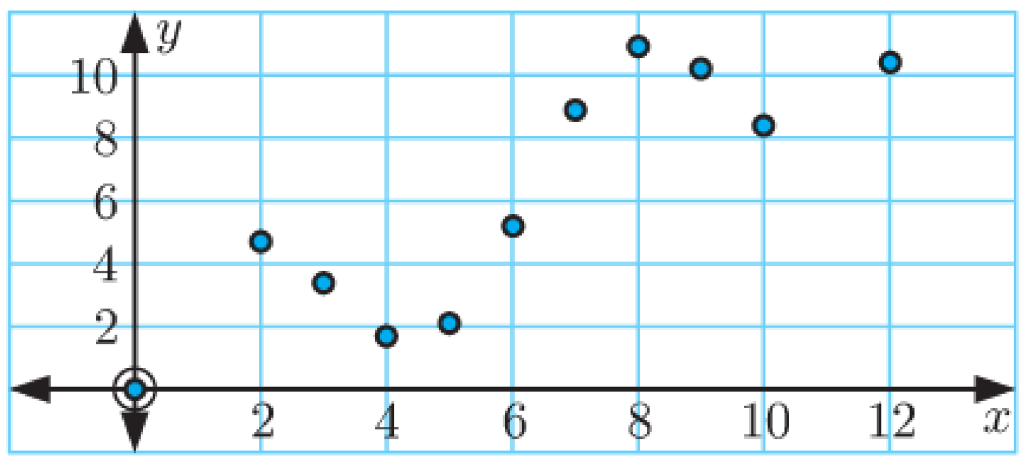
2 **a**



b A curve can be fitted to the data.
c The data is periodic.
i $y = 32$ (approximately) **ii** ≈ 64 cm
iii ≈ 2 seconds **iv** ≈ 32 cm

3 **a**


Data exhibits periodic behaviour.

b


Not enough information to say data is periodic.

EXERCISE 17B

1 **a** 0
b **i** $\theta = 0, \pi, 2\pi, 3\pi, 4\pi$ **ii** $\theta = \frac{3\pi}{2}, \frac{7\pi}{2}$
iii $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ **iv** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$
c **i** $0 < \theta < \pi, \quad 2\pi < \theta < 3\pi$ **d** $\{y \mid -1 \leqslant y \leqslant 1\}$
ii $\pi < \theta < 2\pi, \quad 3\pi < \theta < 4\pi$

2 **a** 1
b **i** $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ **ii** $\theta = 0, 2\pi, 4\pi$
iii $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}$ **iv** $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}$
c **i** $0 \leqslant \theta < \frac{\pi}{2}, \quad \frac{3\pi}{2} < \theta < \frac{5\pi}{2}, \quad \frac{7\pi}{2} < \theta \leqslant 4\pi$
ii $\frac{\pi}{2} < \theta < \frac{3\pi}{2}, \quad \frac{5\pi}{2} < \theta < \frac{7\pi}{2}$
d $\{y \mid -1 \leqslant y \leqslant 1\}$

EXERCISE 17C

1 **a** vertical translation 1 unit downwards
b horizontal translation $\frac{\pi}{4}$ units to the right
c vertical stretch, scale factor 2
d horizontal stretch, scale factor $\frac{1}{4}$
e horizontal stretch, scale factor 4
f translation $\frac{\pi}{3}$ units right and 2 units upwards

2 **a** vertical stretch, scale factor $\frac{1}{2}$ **b** reflection in the x-axis
c translation $\frac{\pi}{6}$ units left and 2 units downwards

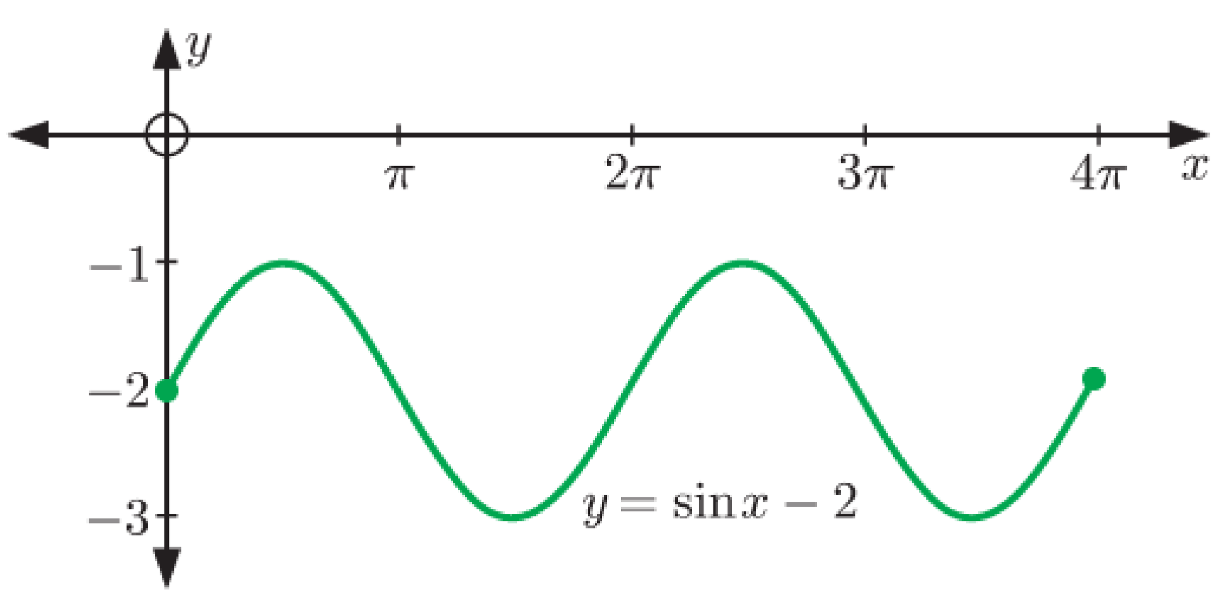
3 **a** $\frac{2\pi}{5}$ **b** $\frac{10\pi}{3}$ **c** 2 **d** $\frac{2\pi}{3}$ **e** 6π **f** 100

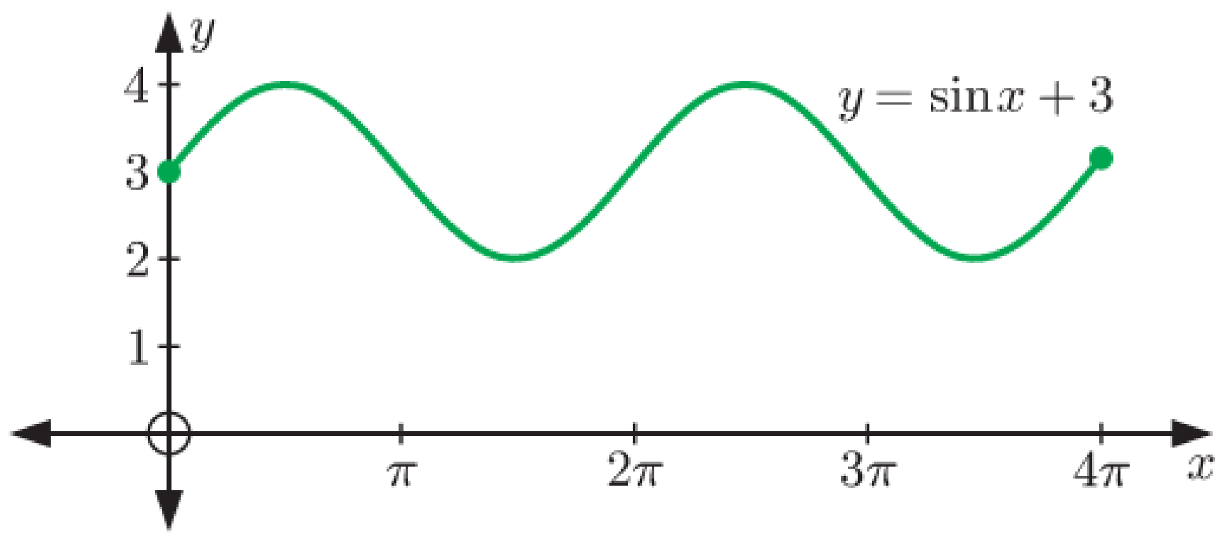
4 **a** $b = \frac{2}{5}$ **b** $b = 3$ **c** $b = \frac{1}{6}$ **d** $b = \frac{\pi}{2}$ **e** $b = \frac{\pi}{50}$

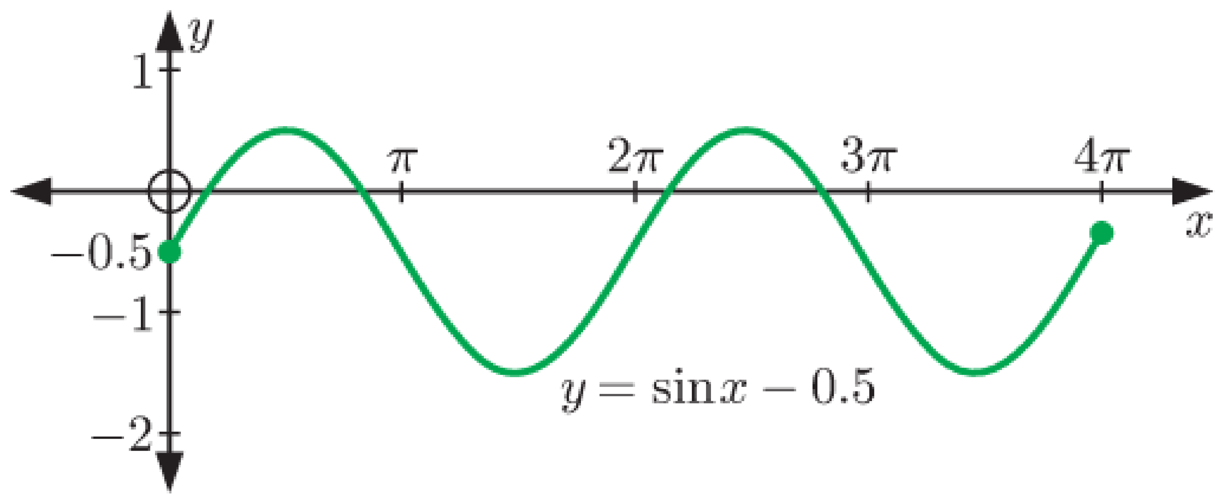
5 **a** maximum 4, minimum -4 **b** maximum 8, minimum 2
c maximum -2, minimum -6

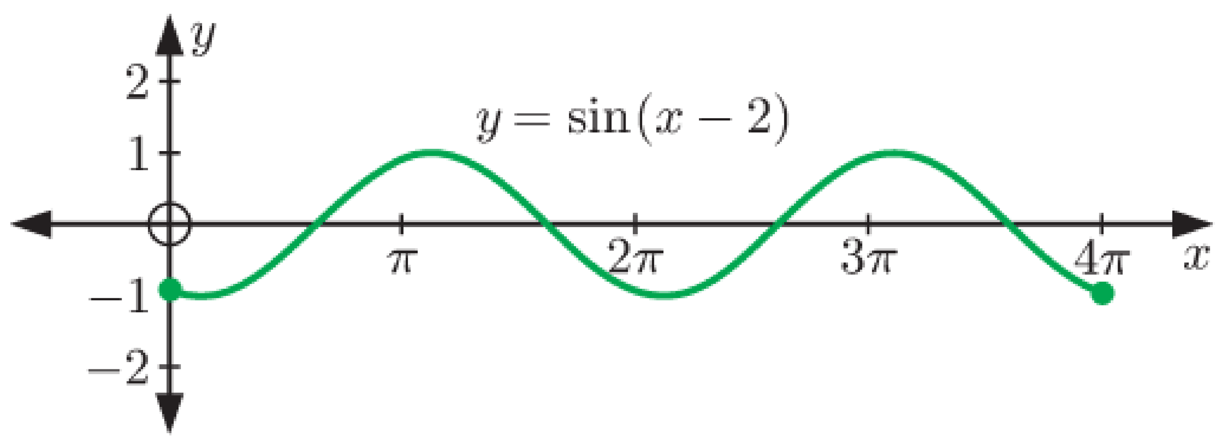
6 **a** 4 **b** $\frac{2\pi}{3}$ **c** $\{y \mid -2 \leqslant y \leqslant 6\}$

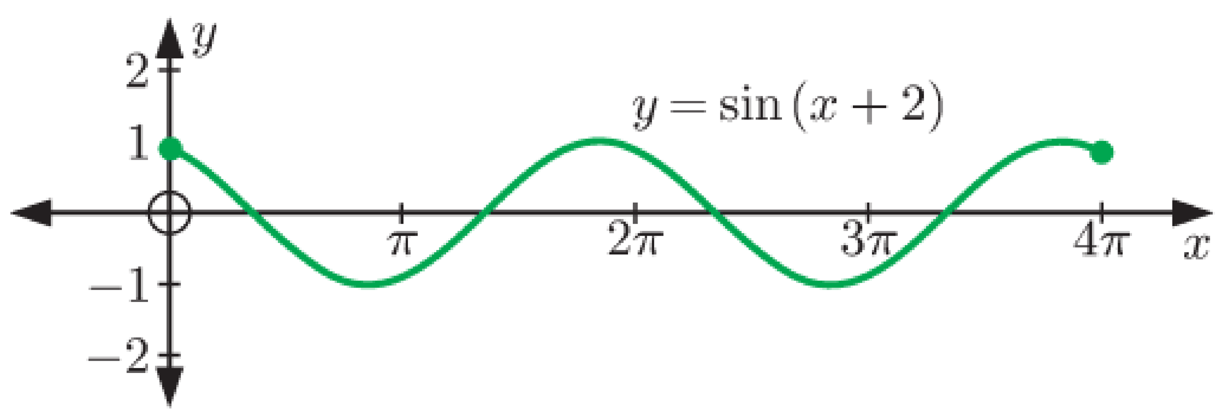
7 $|a| =$ amplitude, $b = \dfrac{2\pi}{\text{period}}$, $c =$ horizontal translation,
$d =$ vertical translation

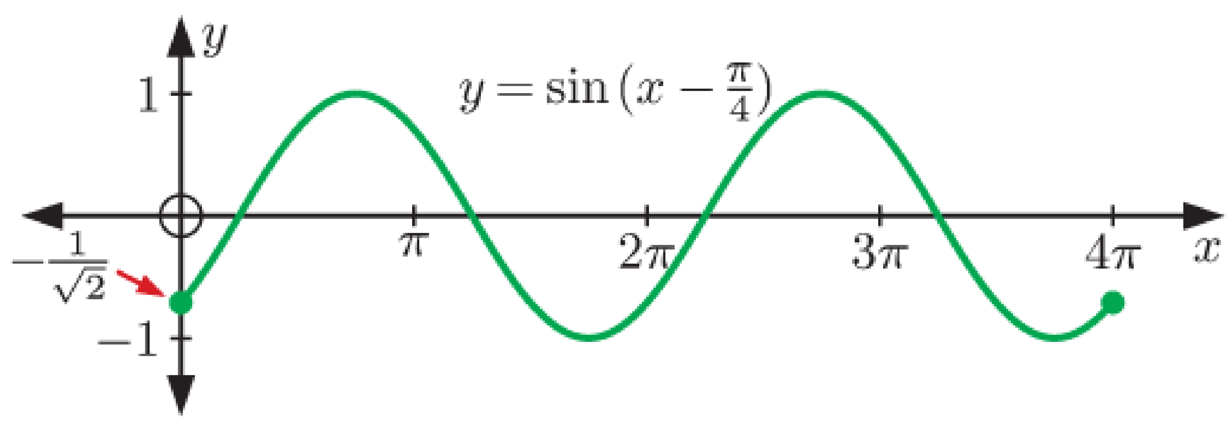
8 **a**


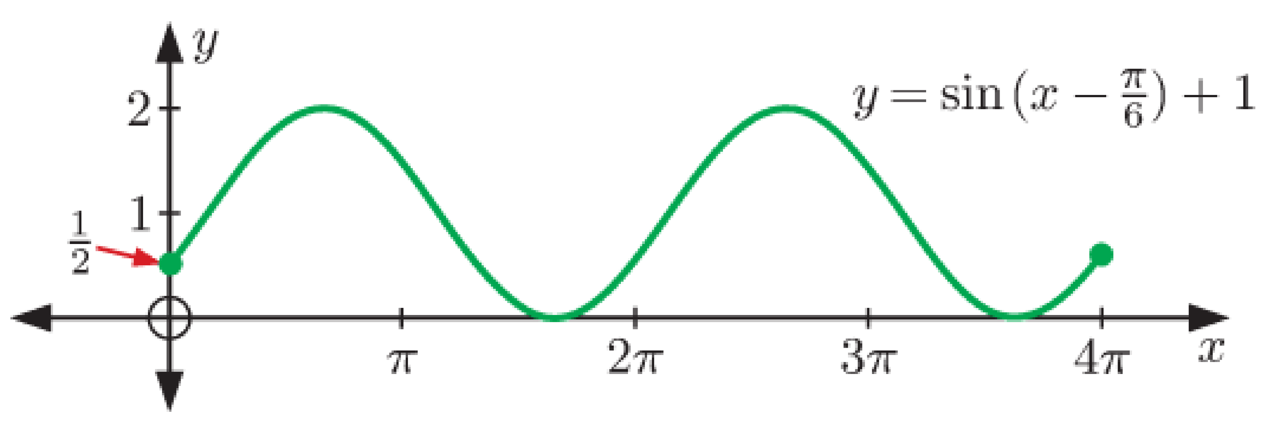
b


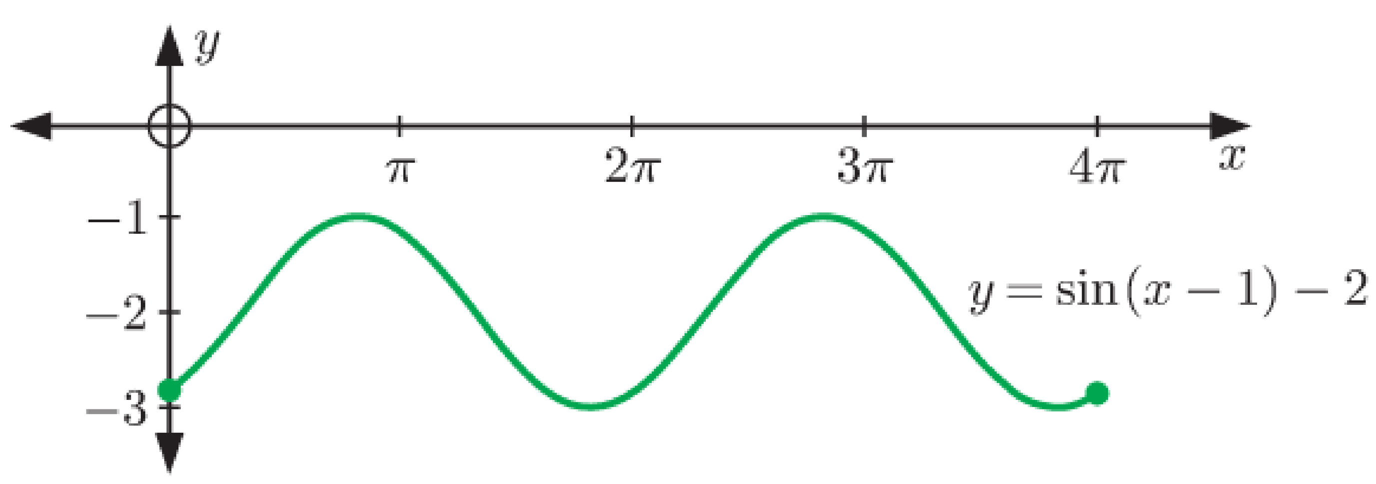
c


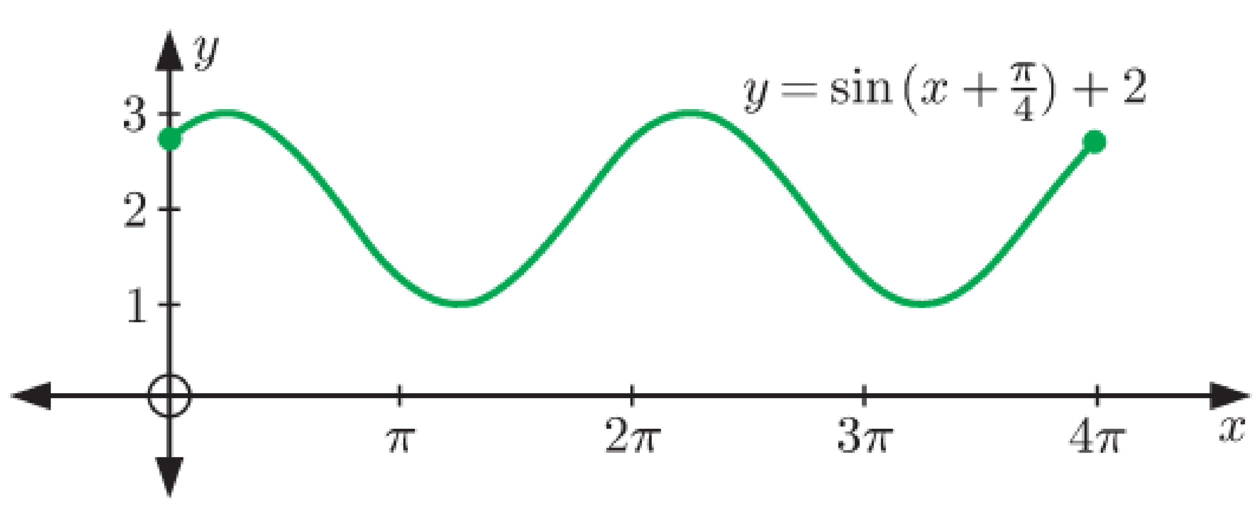
d


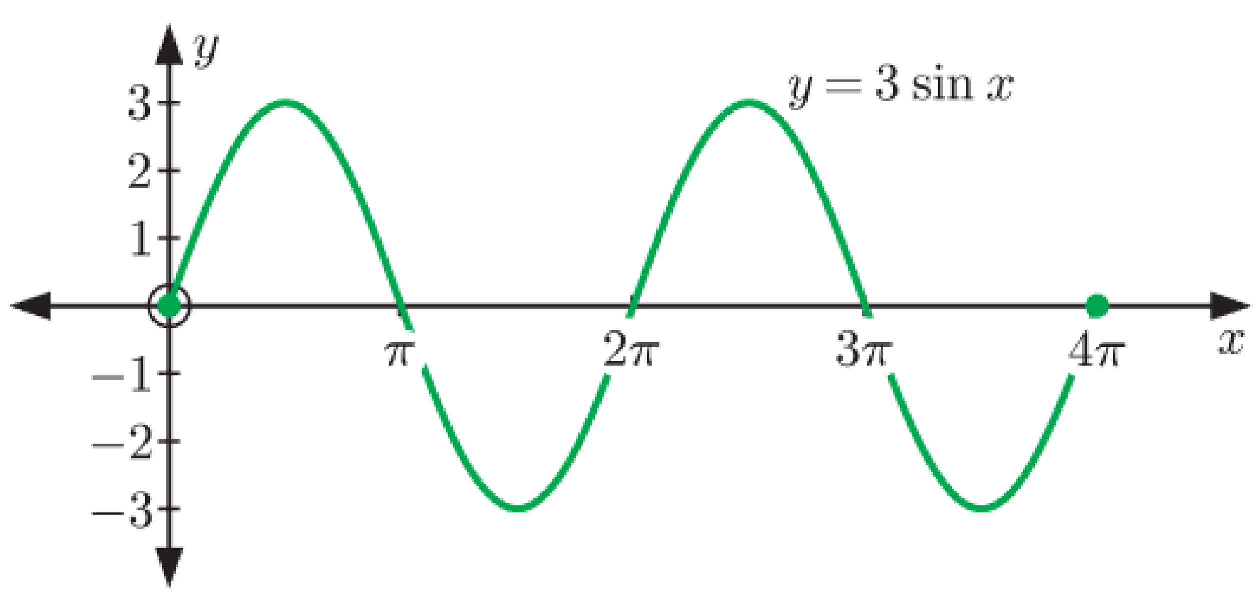
e


f


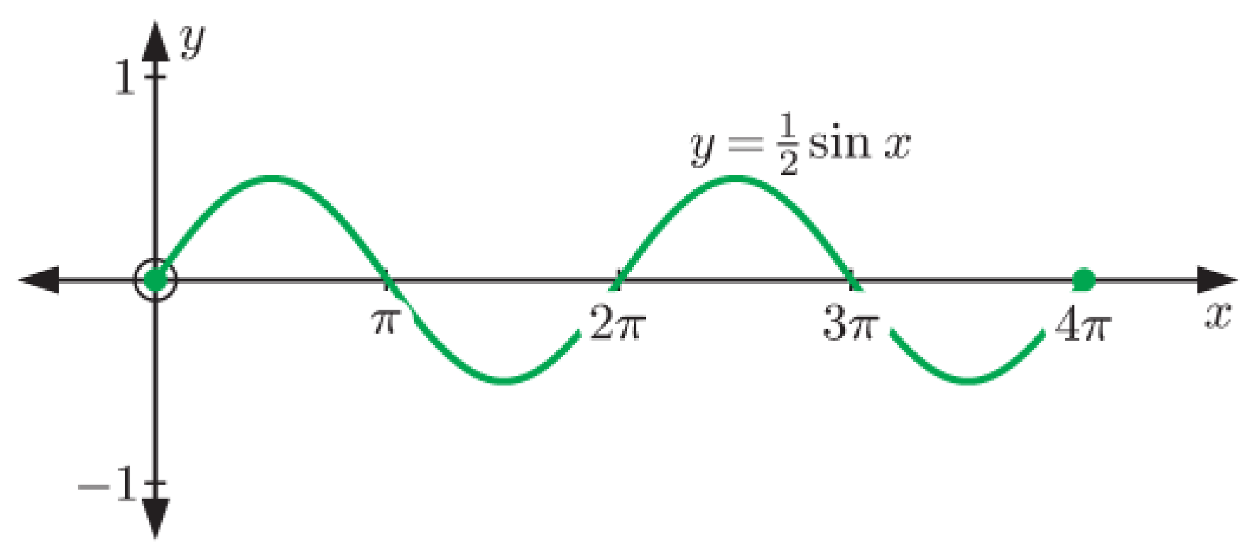
g


h


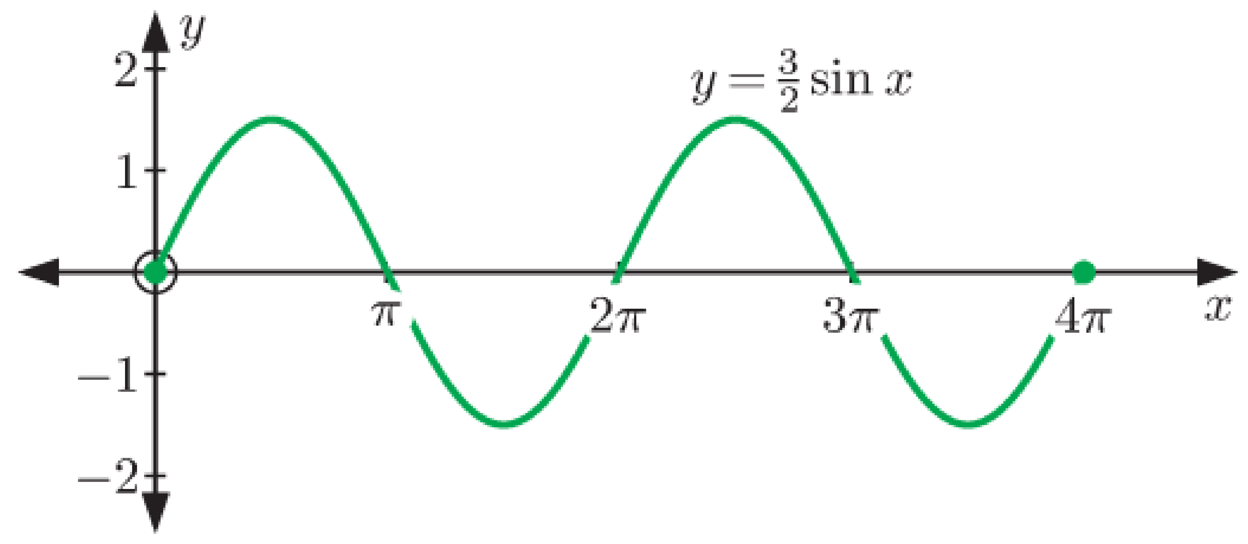
i


j


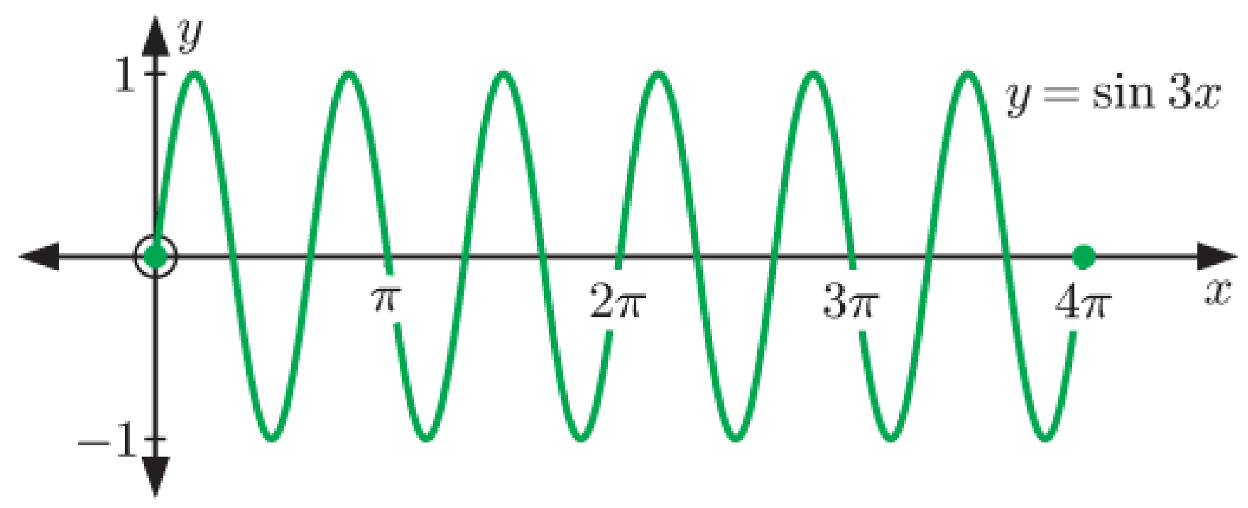
k



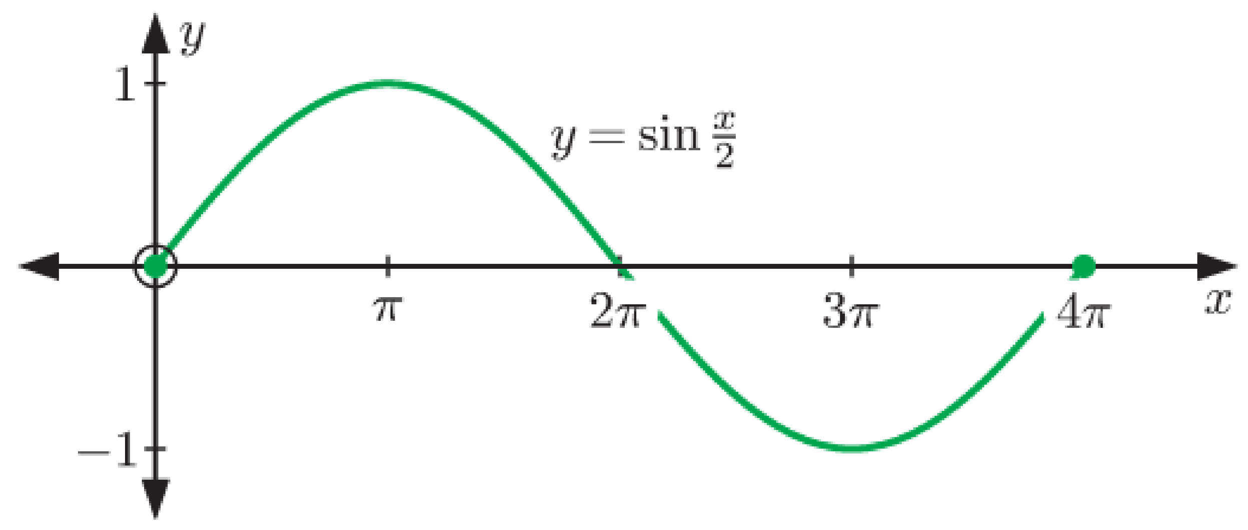
l



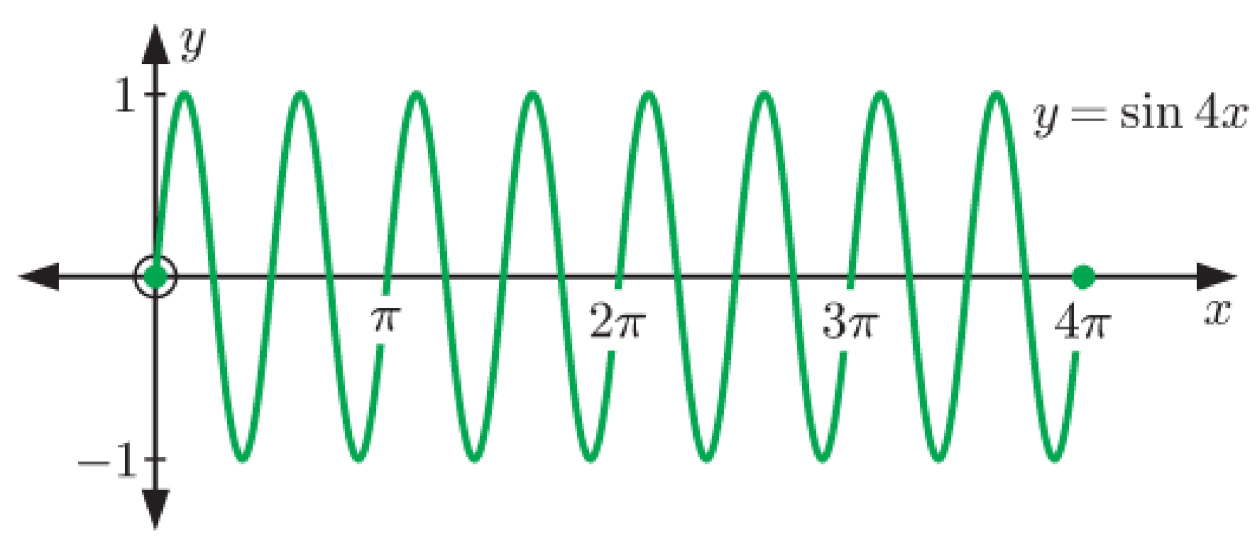
m



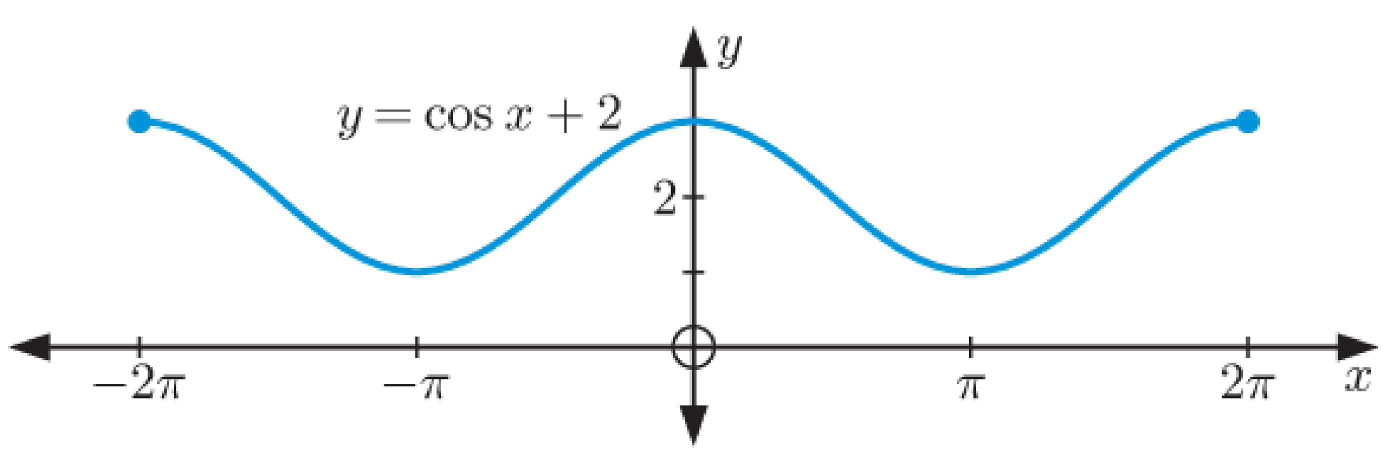
n



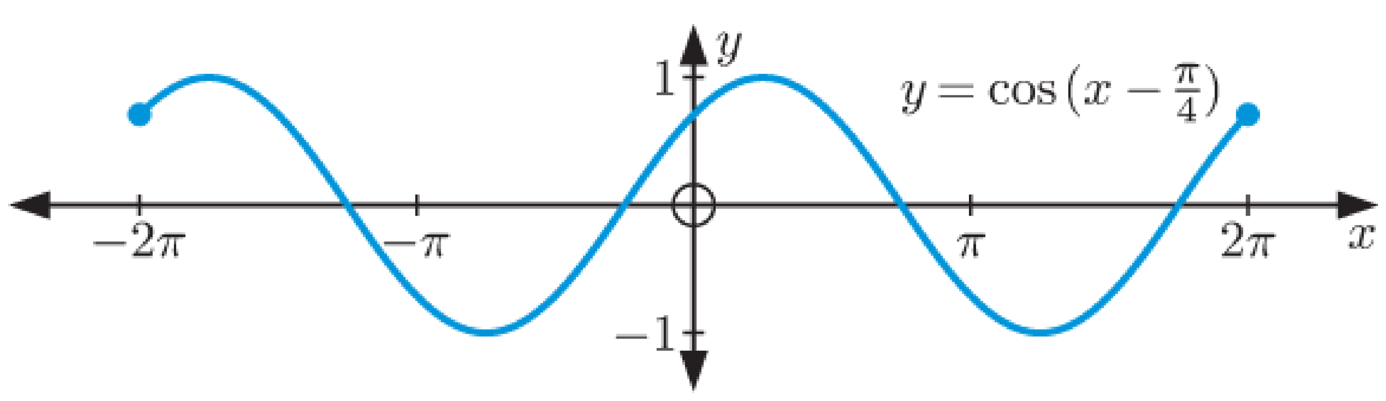
o



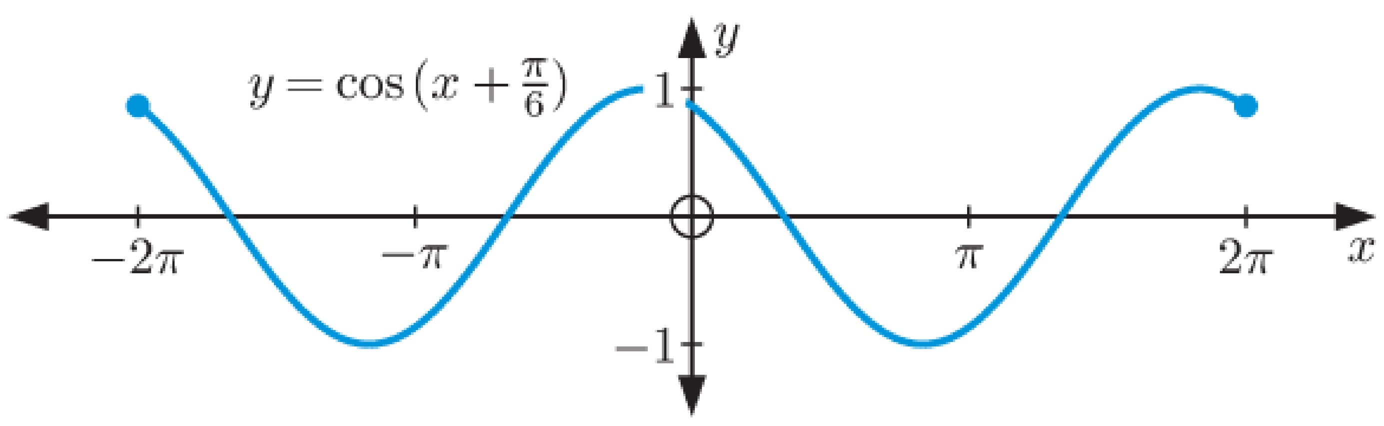
9 **a**



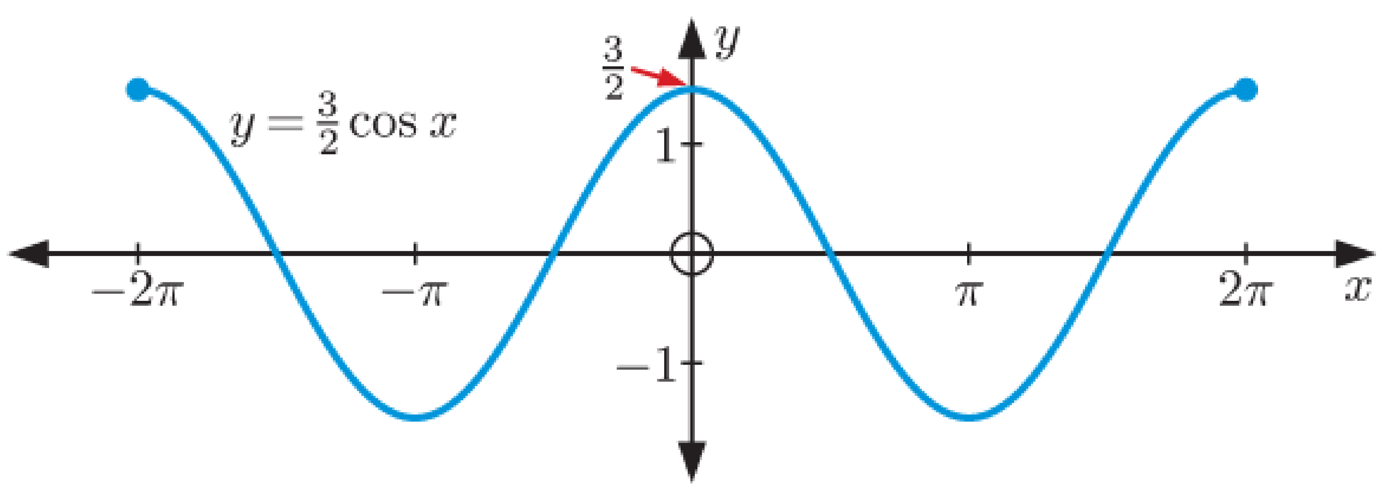
b



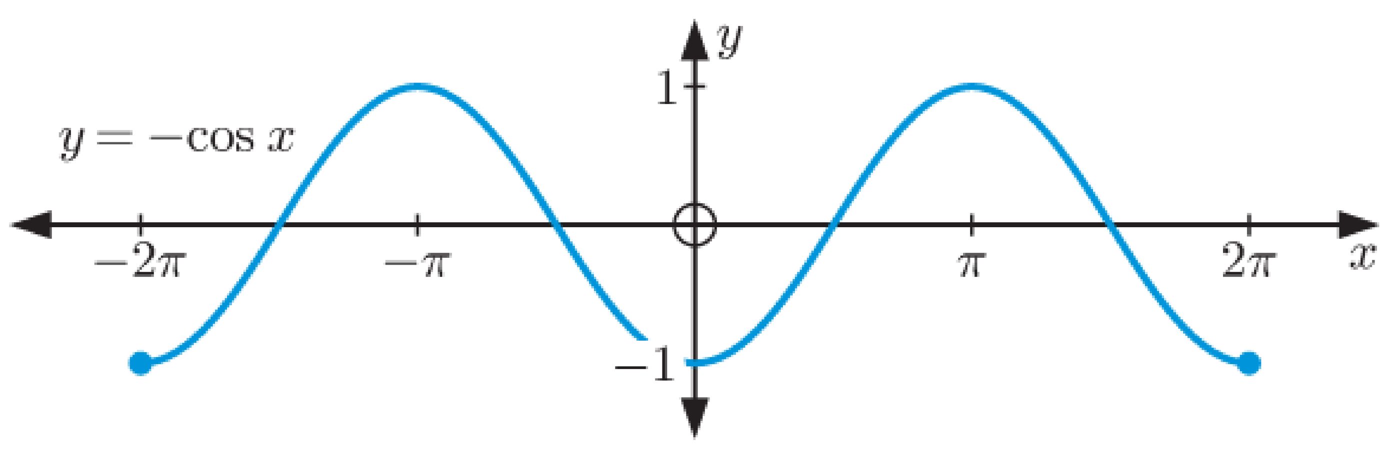
c



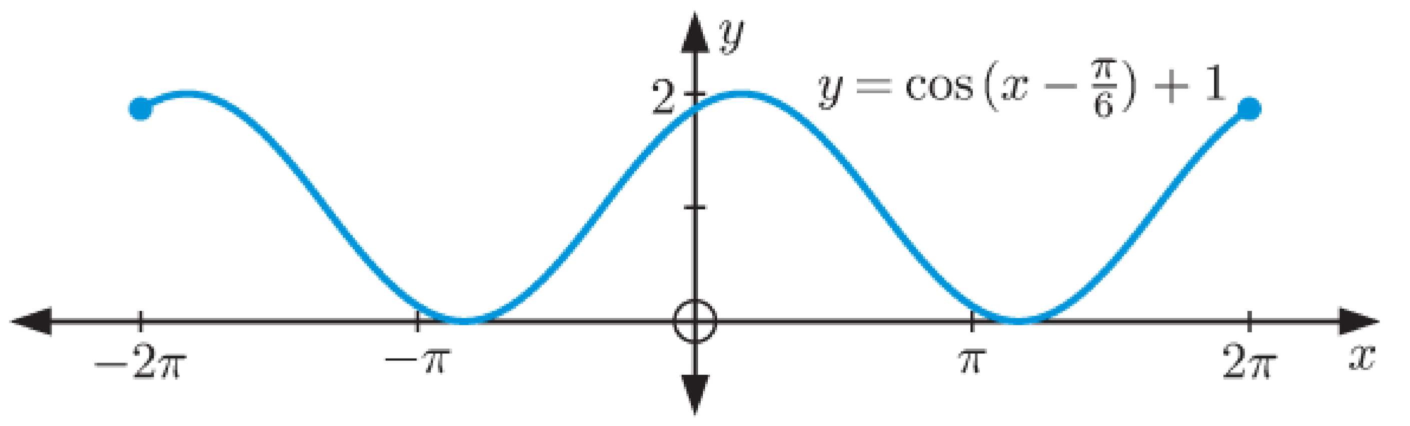
d



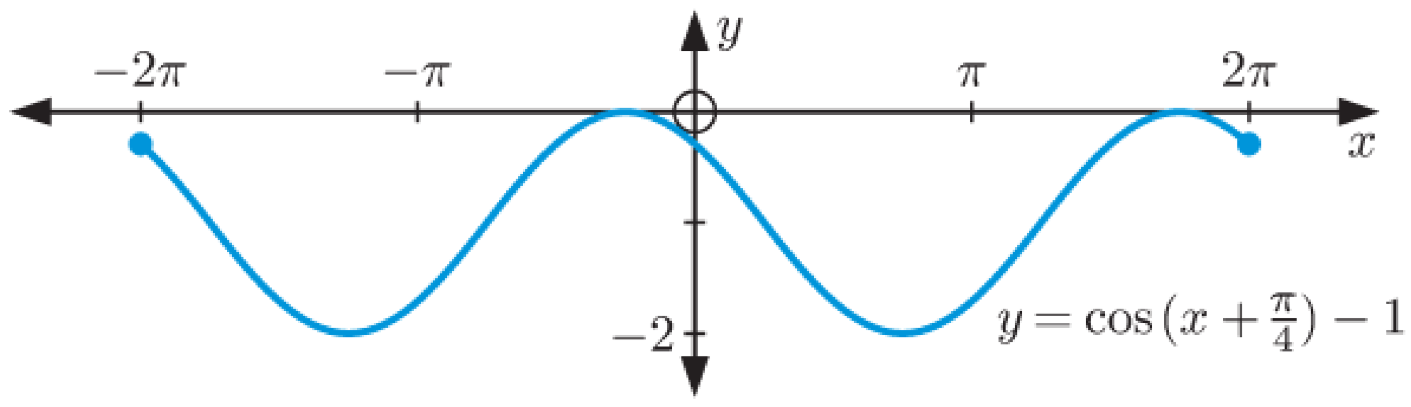
e



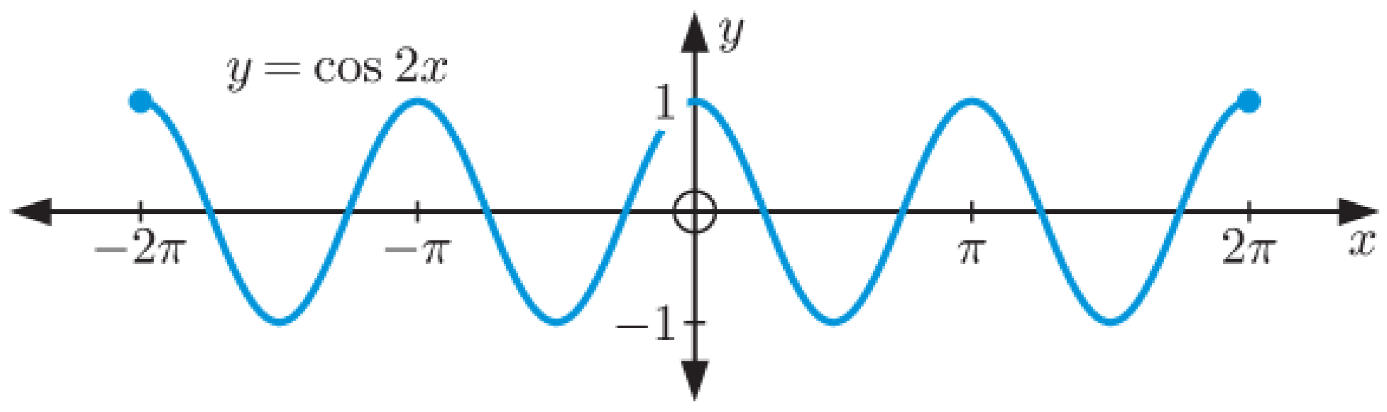
f



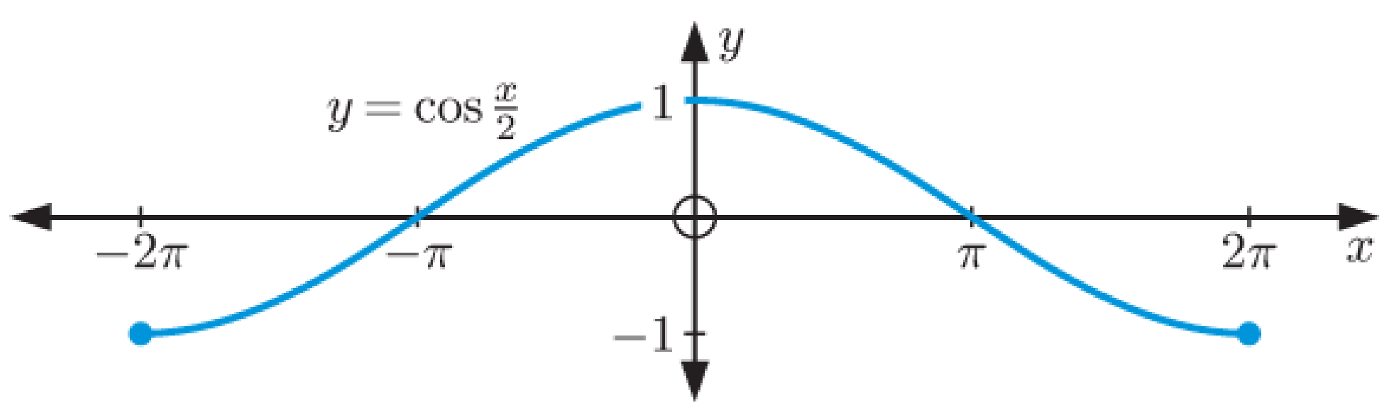
g



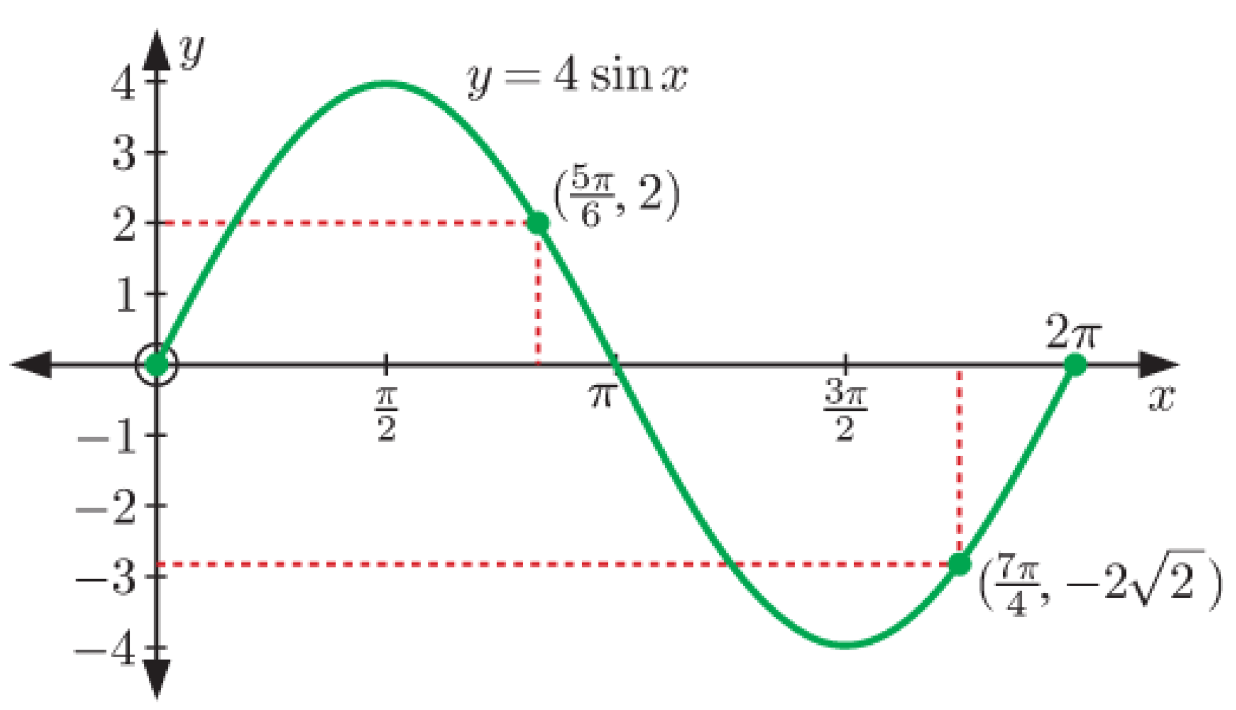
h



i



10 **a**



b **i** $y = 2$ **ii** $y = -2\sqrt{2} \approx -2.83$

11 **a** $d > 3$ **b** $d < -3$ **c** $-3 < d < 3$

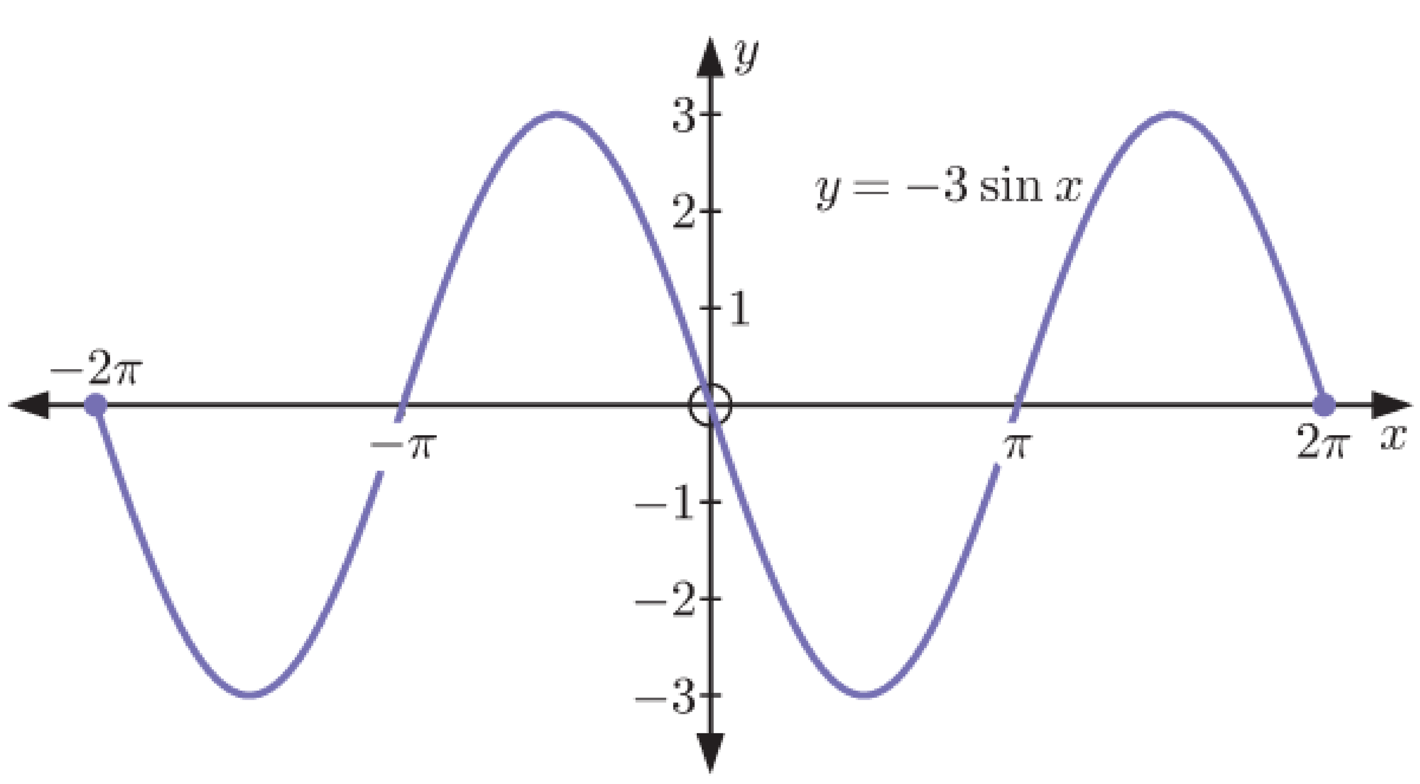
12 **a** A horizontal stretch with scale factor $\frac{1}{3}$, then a vertical stretch with scale factor 2.

b A vertical stretch with scale factor 2, then a reflection in the x-axis.

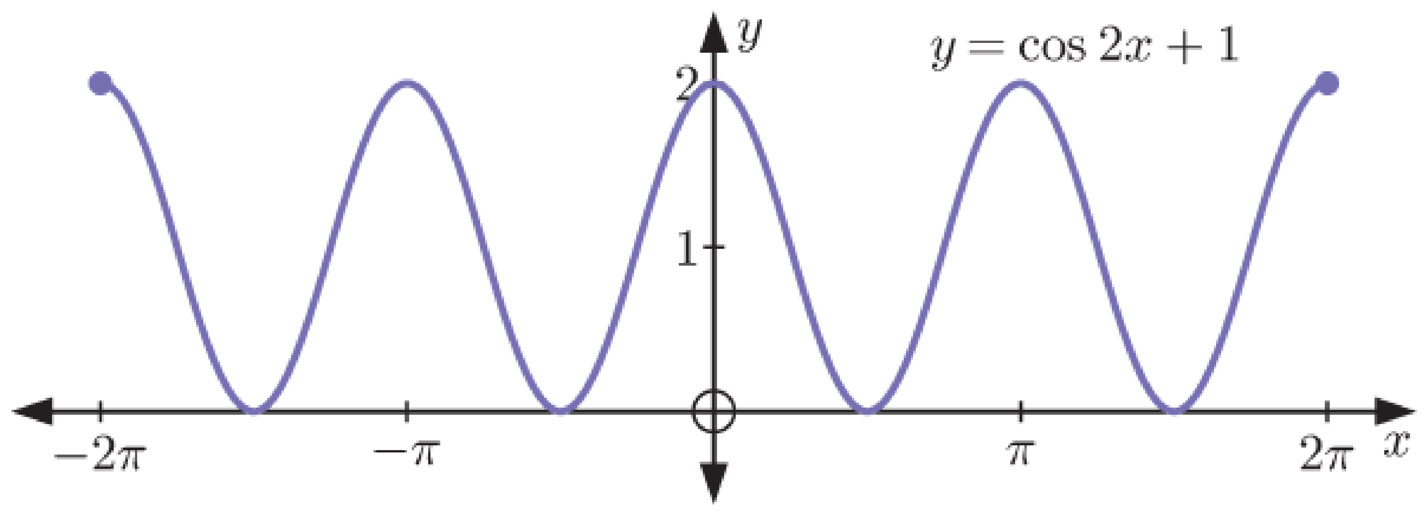
c A vertical stretch with scale factor 3, then a translation 5 units downwards.

d A horizontal stretch with scale factor $\frac{1}{2}$, then a translation $\frac{\pi}{6}$ units left.

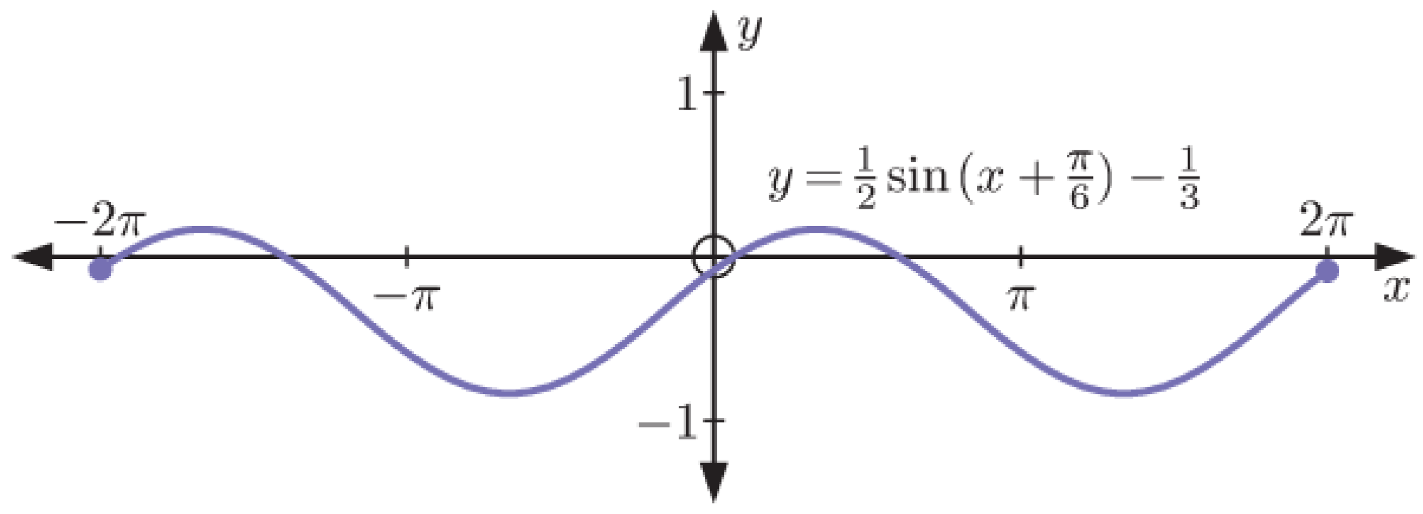
13 **a**



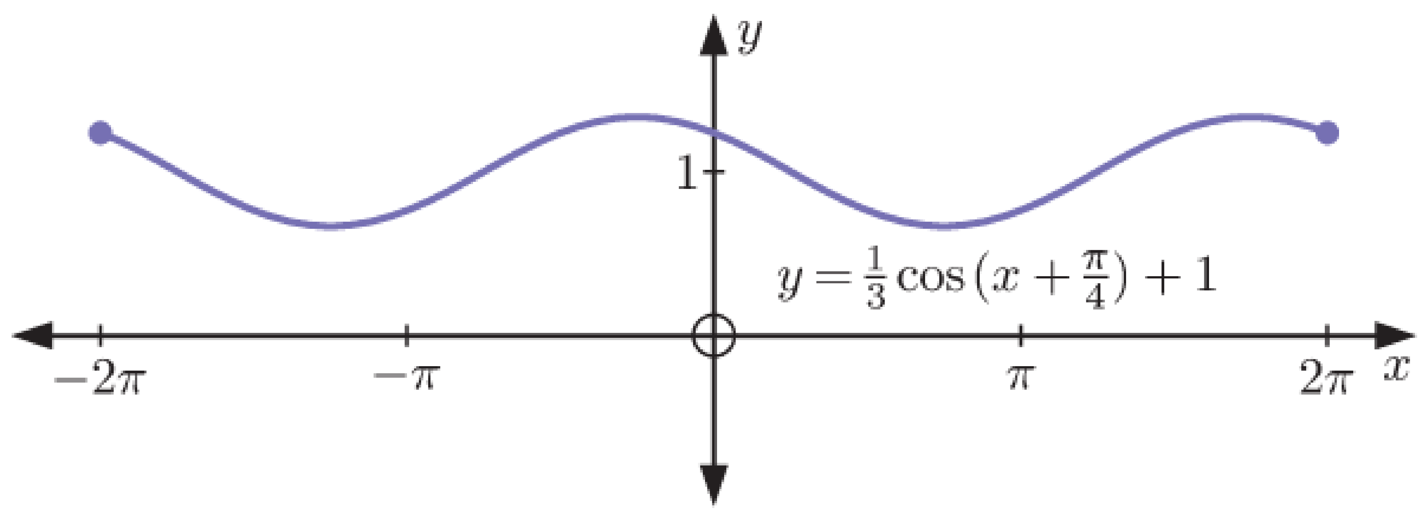
b



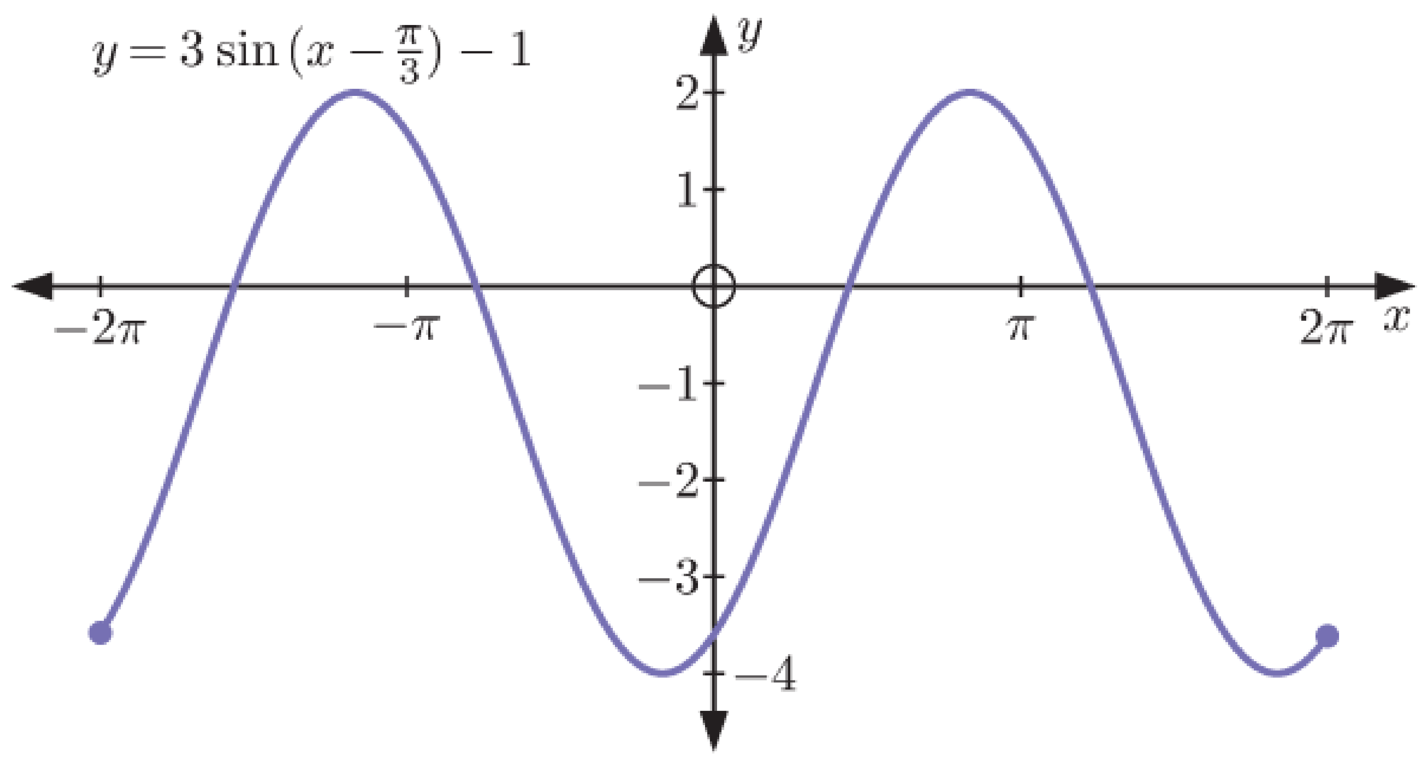
c



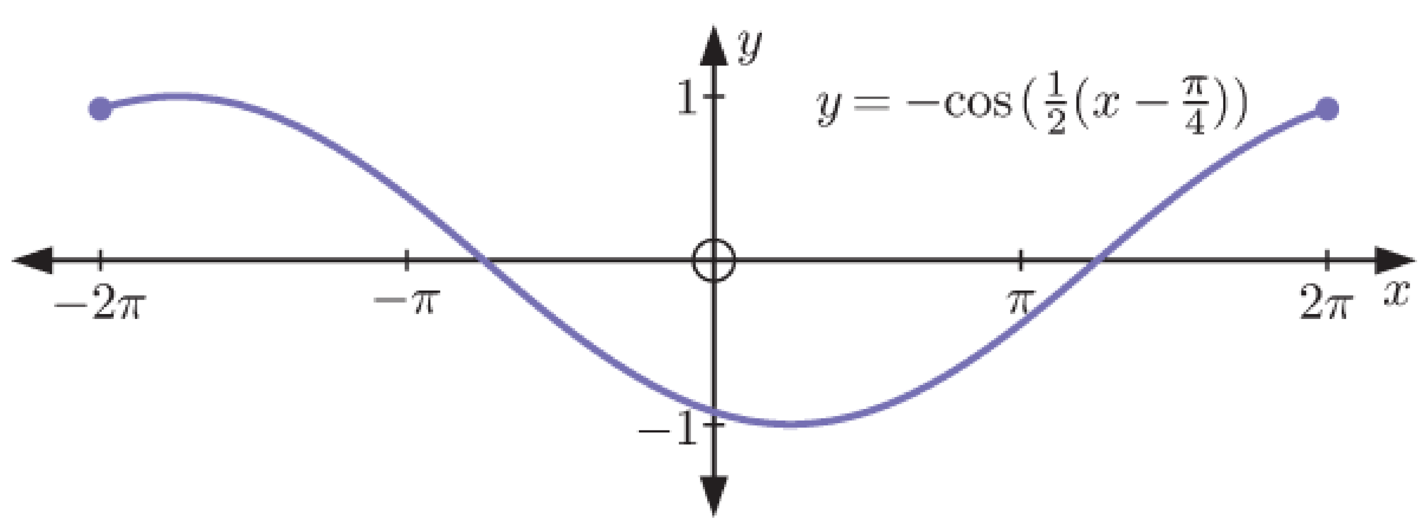
d



e



f



14 **a** b, c, d (provided the function has x-intercepts)
b d **c** d

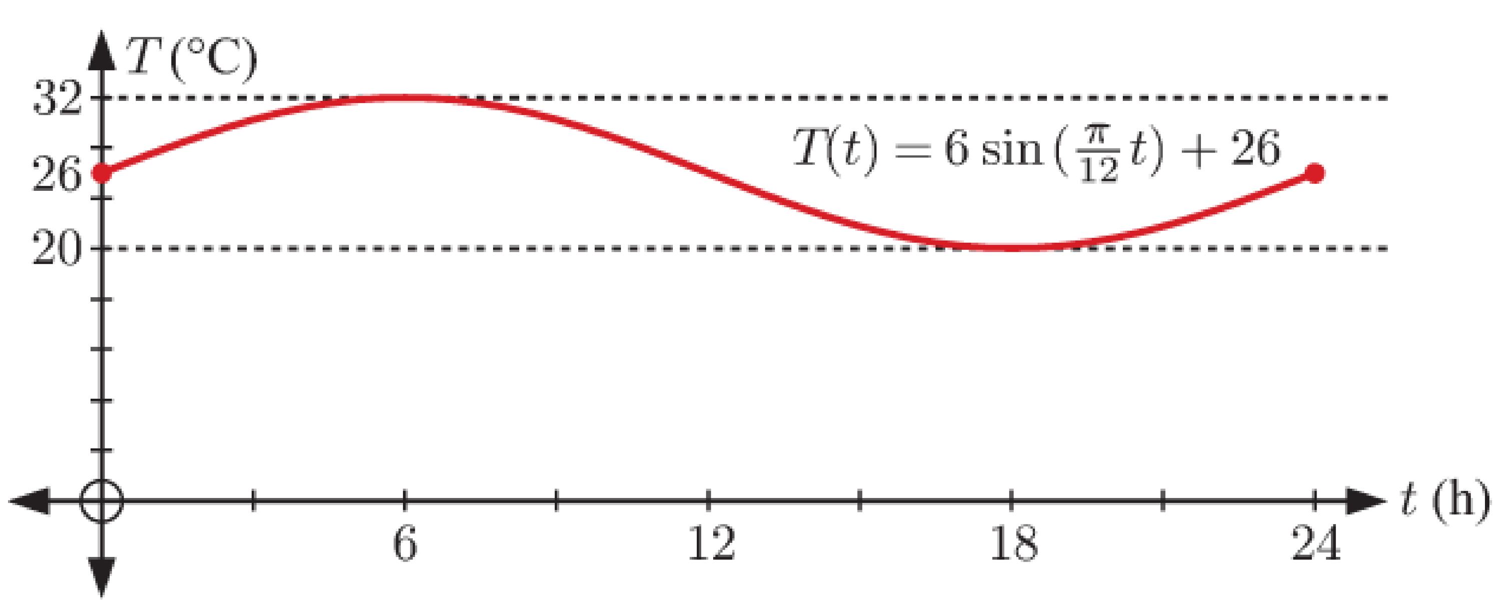
15 **a** $a = 4,\ d = 1$ **b** $a = -2,\ d = 3$ **c** $a = \frac{1}{3},\ d = \frac{4}{3}$

16 **a** $y = \sin x - 2$ **b** $y = \sin 3x$ **c** $y = \sin\left(x + \frac{\pi}{2}\right)$
d $y = 2\sin x + 1$ **e** $y = 4\sin\frac{x}{2} - 1$ **f** $y = 6\sin\frac{2\pi x}{5}$

17 **a** $y = 2\cos 2x$ **b** $y = \cos\frac{x}{2} + 2$ **c** $y = -5\cos\frac{\pi x}{3}$

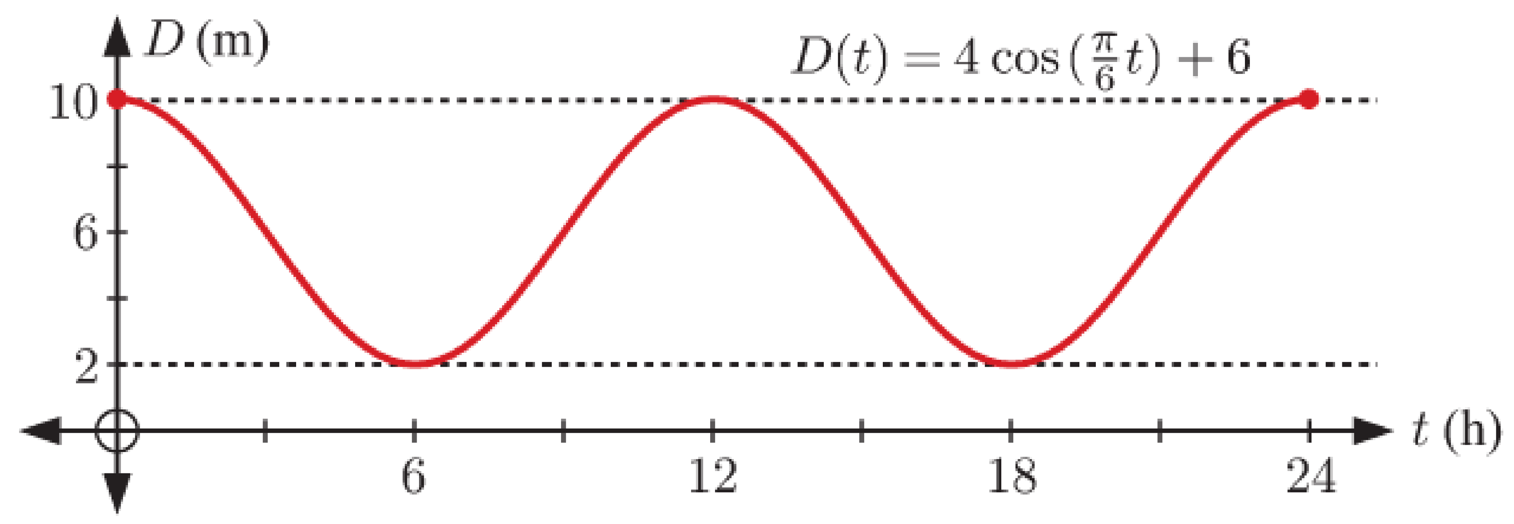
EXERCISE 17D

1 **a**



b **i** 26°C **ii** 29°C **c** 32°C, at 6 pm

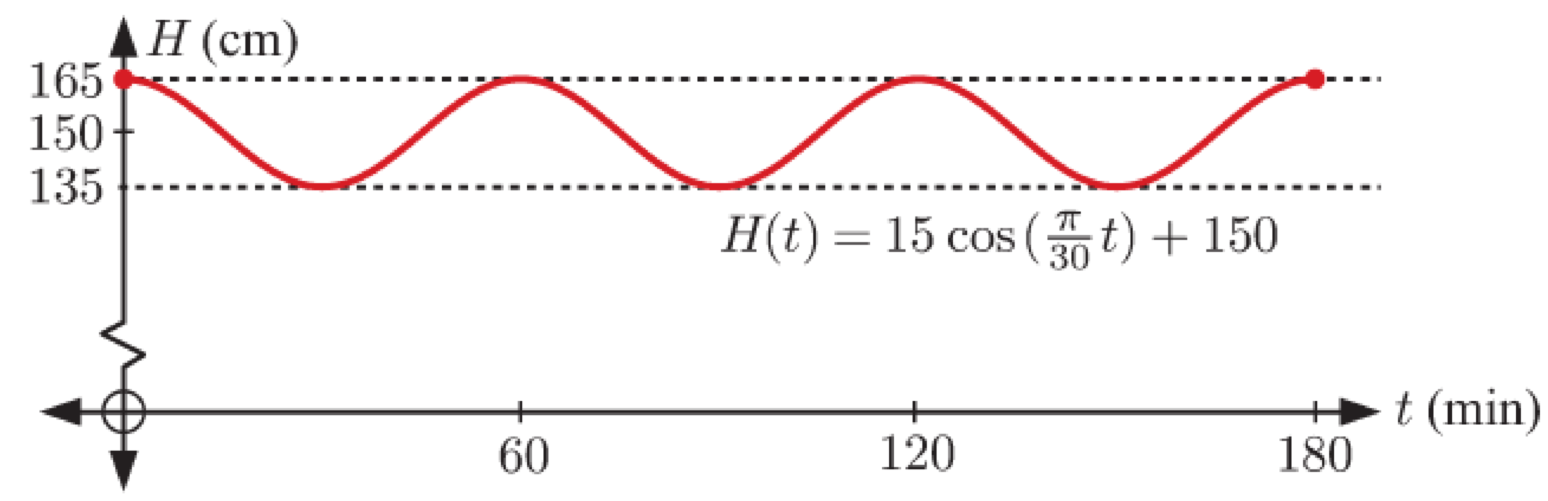
2 **a**



b highest = 10 m, at midnight, midday, and midnight the next day
lowest = 2 m, at 6 am and 6 pm

c no (water height is 4 m)

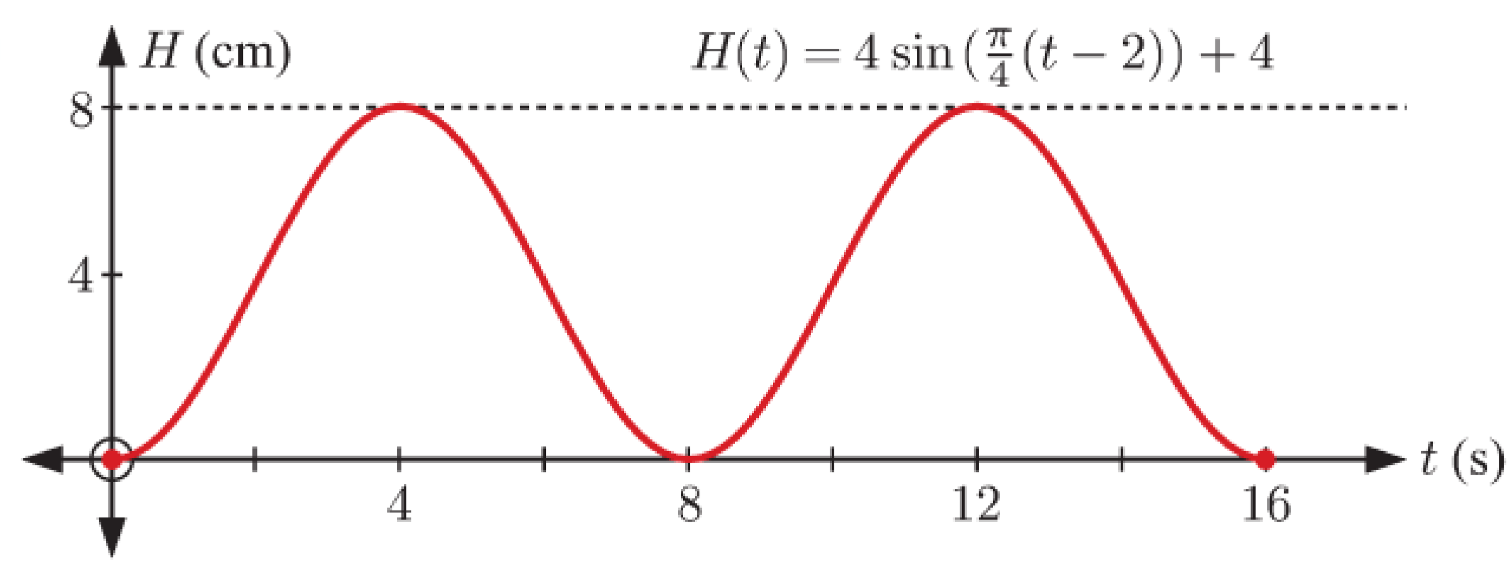
3 **a**



b 15 cm

c **i** ≈ 160.0 cm **ii** ≈ 138.9 cm **iii** ≈ 158.8 cm
iv ≈ 138.9 cm

4 **a**



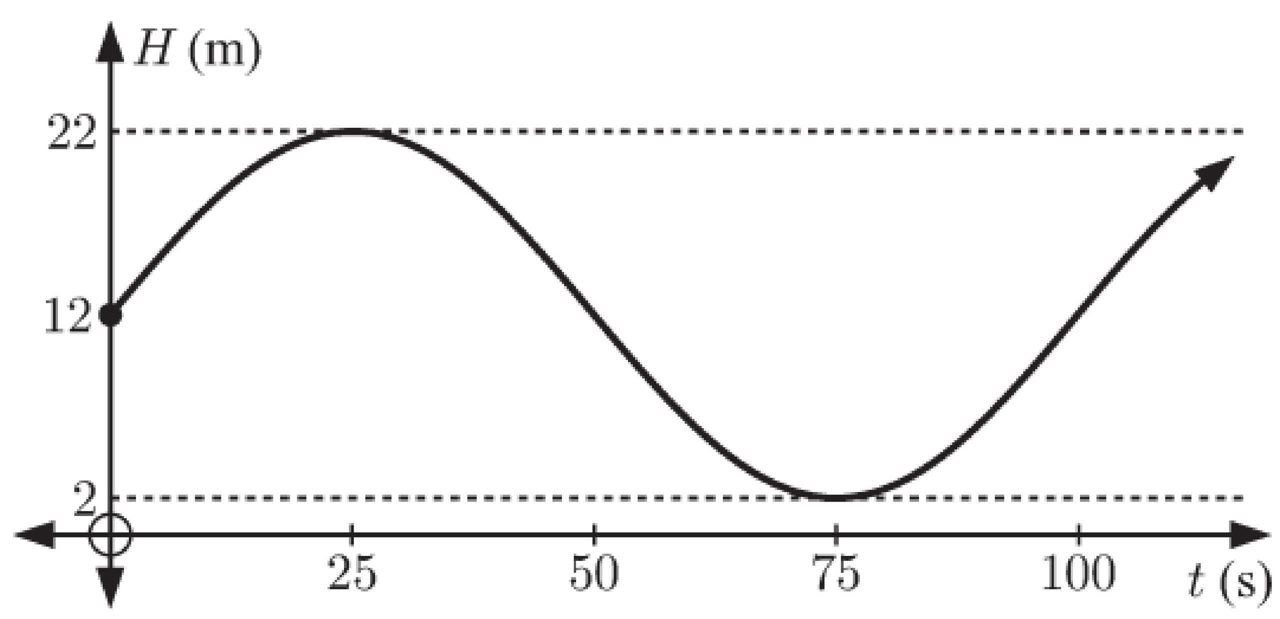
b 4 cm

c no (ball diameter is 4.28 cm, gate height is ≈ 3.07 cm)

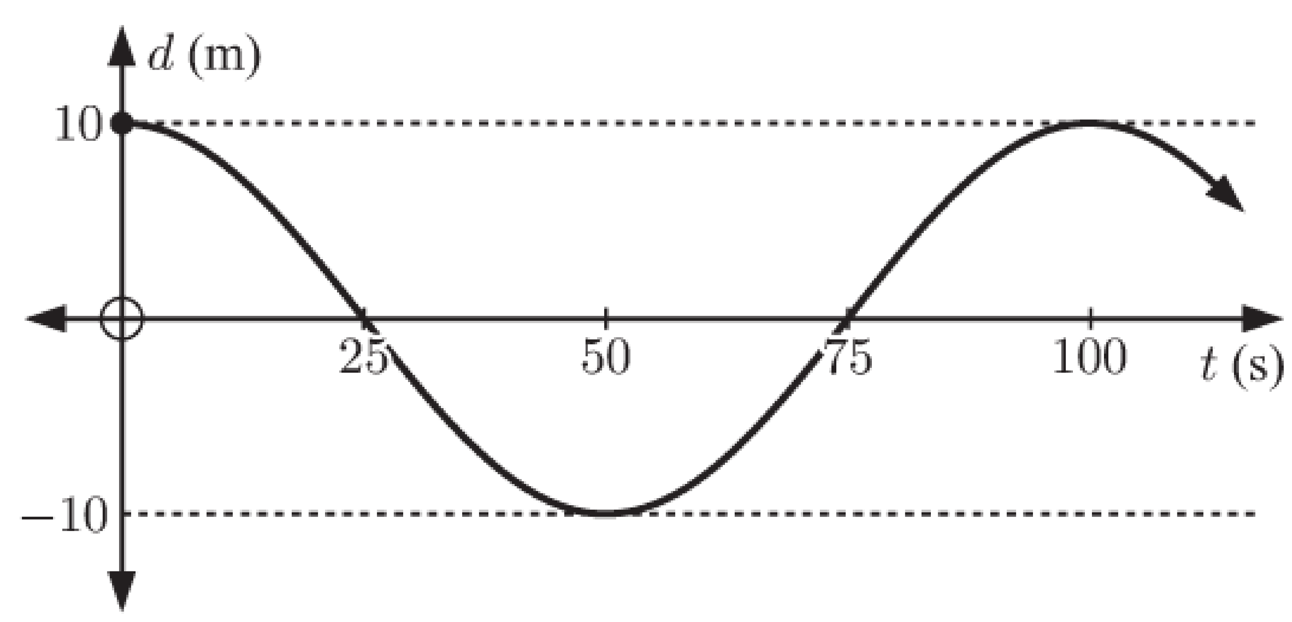
5 $T(t) = 5.2\sin\left(\frac{\pi}{12}(t - 8)\right) + 10.6$ °C

6 $H(t) = 0.6\cos\left(\frac{5\pi}{31}(t - 1.5)\right) + 0.76$ m

7 **a**



b



c Both graphs are periodic with an amplitude of 10 m and a period of 100 s. The graphs differ by a horizontal translation of 25 s and the principal axis is also translated by 12 m.

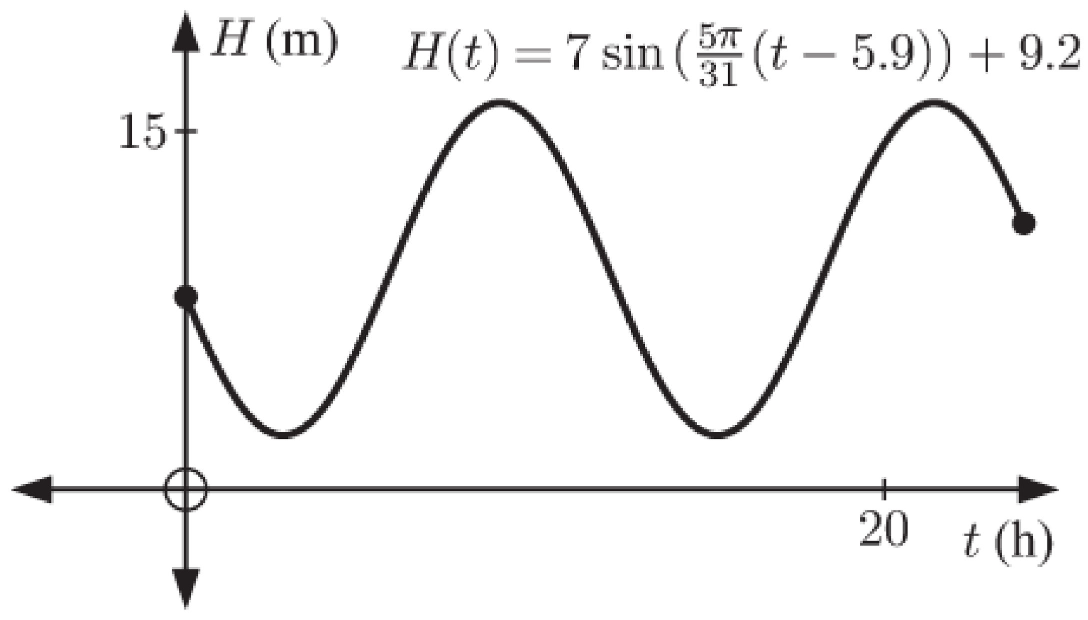
d **i** $H(t) = 10\sin\left(\frac{\pi}{50}t\right) + 12$ m

ii $d(t) = 10\sin\left(\frac{\pi}{50}(t + 25)\right)$ m

Note: The function of horizontal displacement of the light will be different depending on how the coordinate system is defined.

8 **a** $H(t) = 7\sin\left(\frac{5\pi}{31}(t - 5.9)\right) + 9.2$ m

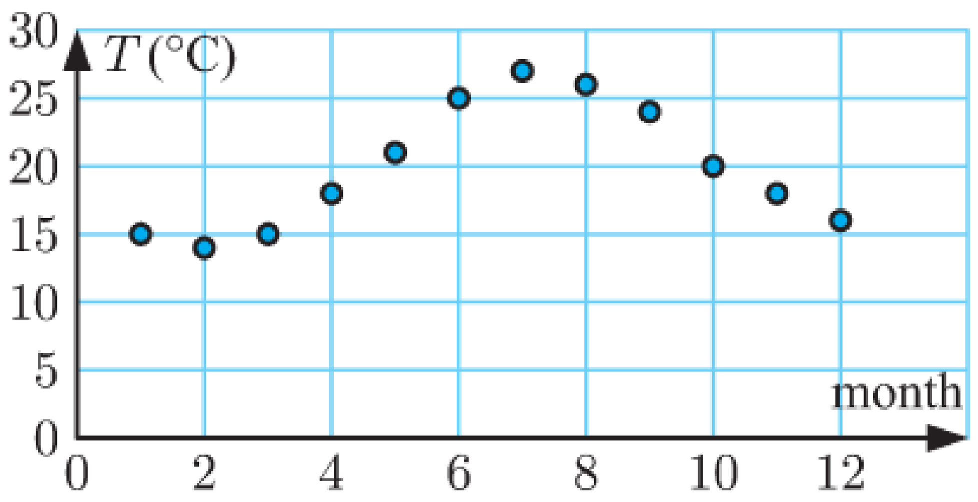
b



9 **a** $H(t) = 6\cos\left(\frac{\pi}{6}t\right)$ **b** $d(t) = 12\sin 2\pi t$

EXERCISE 17E

1 **a**



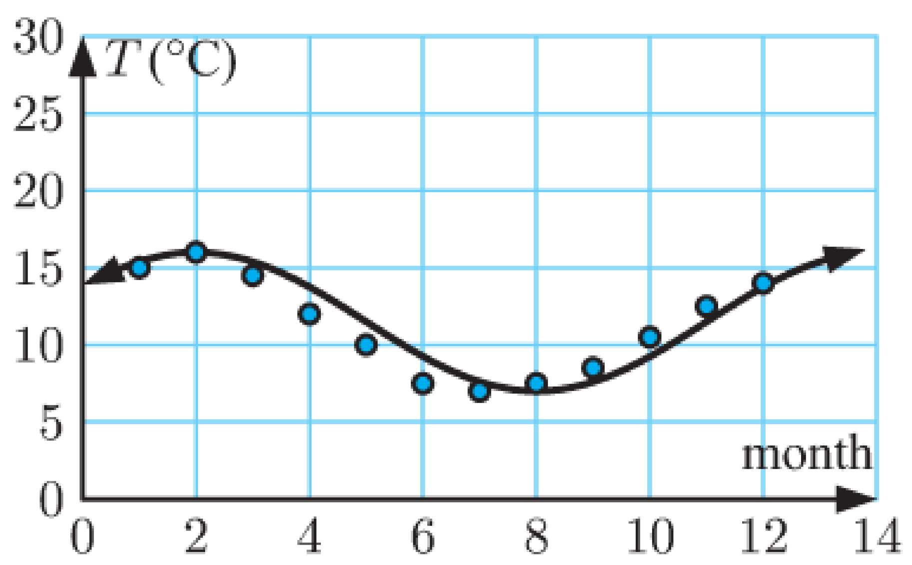
b The data appears to be periodic.

c **i** $b \approx \frac{\pi}{6}$ **ii** $a \approx 6.5$ **iii** $d \approx 20.5$ **iv** $c \approx 4.5$

d Using technology, $T \approx 6.15\sin(0.575t - 2.69) + 20.4$. Our model was a reasonable fit.

2 **a** $T \approx 4.5\cos\left(\frac{\pi}{6}(t-2)\right) + 11.5$

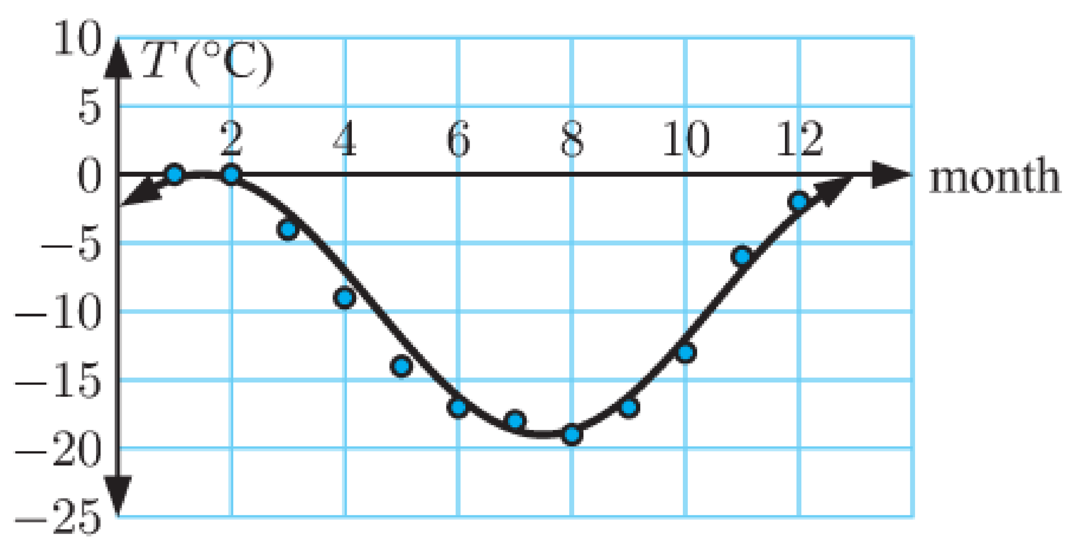
b



c Using technology, $T \approx 4.29\cos(0.533t - 0.805) + 11.2$.

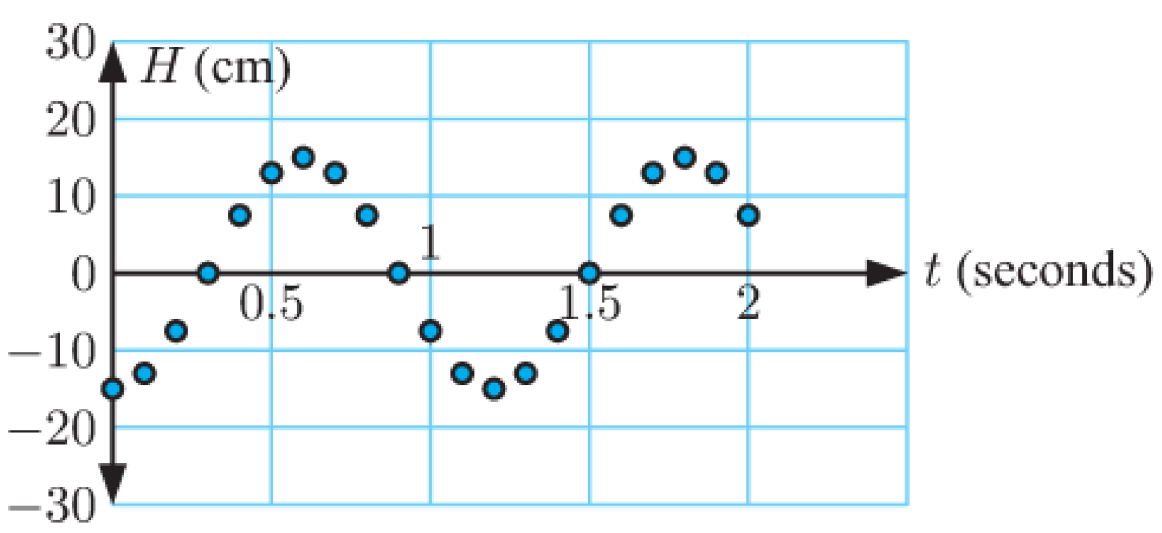
3 **a** $T \approx 9.5\sin\left(\frac{\pi}{6}(t - 10.5)\right) - 9.5$

b



c The model is a reasonable fit, but not perfect.

4 **a**



b $H \approx 15.0\sin(5.24t - 1.57) + 0.000\,170$ **c** ≈ 14.5 cm

d The spring will not oscillate indefinitely at the same rate.

EXERCISE 17F

1 **a** A horizontal translation $\frac{\pi}{2}$ units to the right.

b A vertical stretch with scale factor 4.

c A horizontal stretch with scale factor $\frac{2}{\pi}$.

d A horizontal stretch with scale factor $\frac{1}{2}$, then a translation 1 unit downwards.

e A vertical stretch with scale factor $\frac{1}{2}$, then a reflection in the x-axis.

f A translation 2 units upwards.

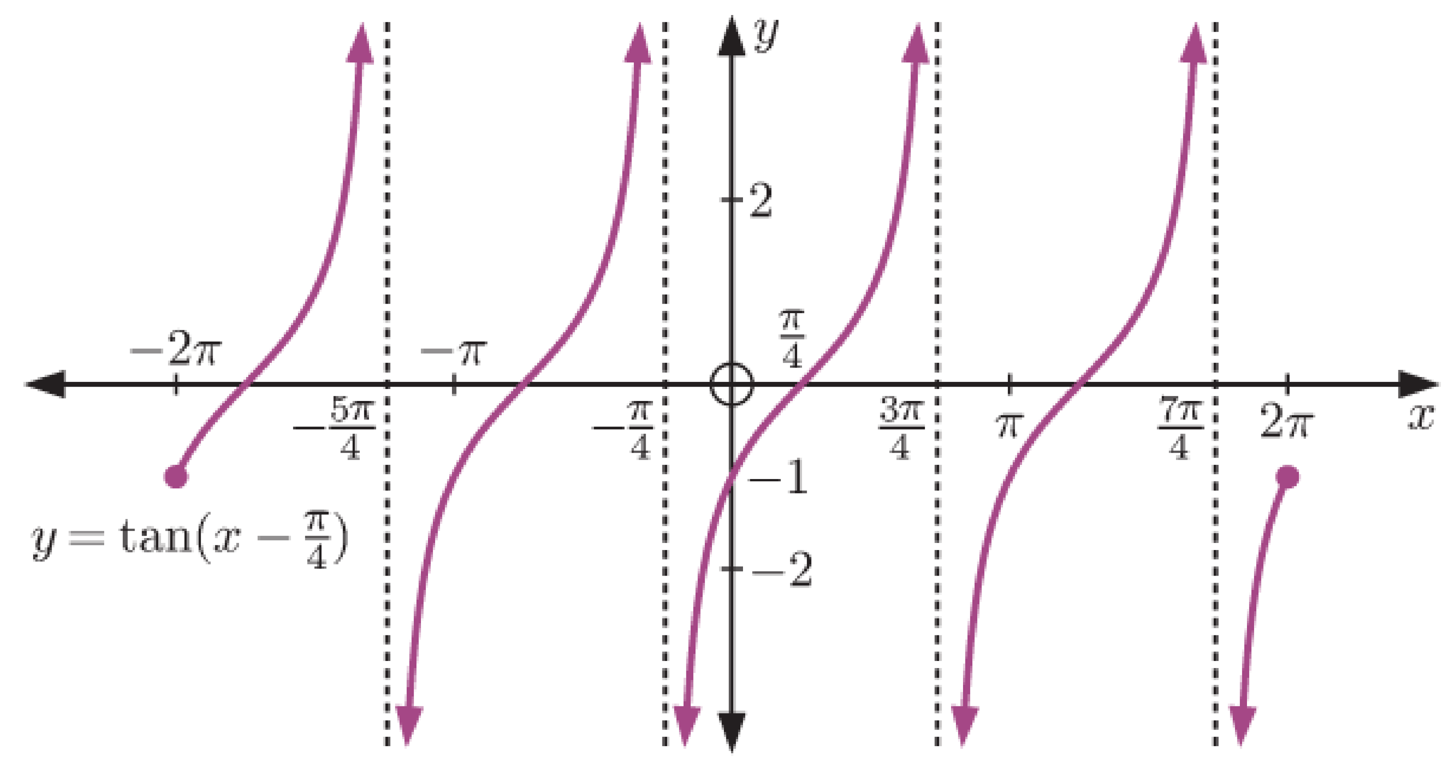
2 **a** $\frac{\pi}{3}$ **b** 4π **c** 1 **d** 2 **e** $\frac{3\pi}{2}$ **f** $\frac{\pi}{n}$

3 **a** **i** $\frac{k\pi}{2}$, $k \in \mathbb{Z}$ **ii** $x = \frac{\pi}{4} + \frac{k\pi}{2}$, $k \in \mathbb{Z}$

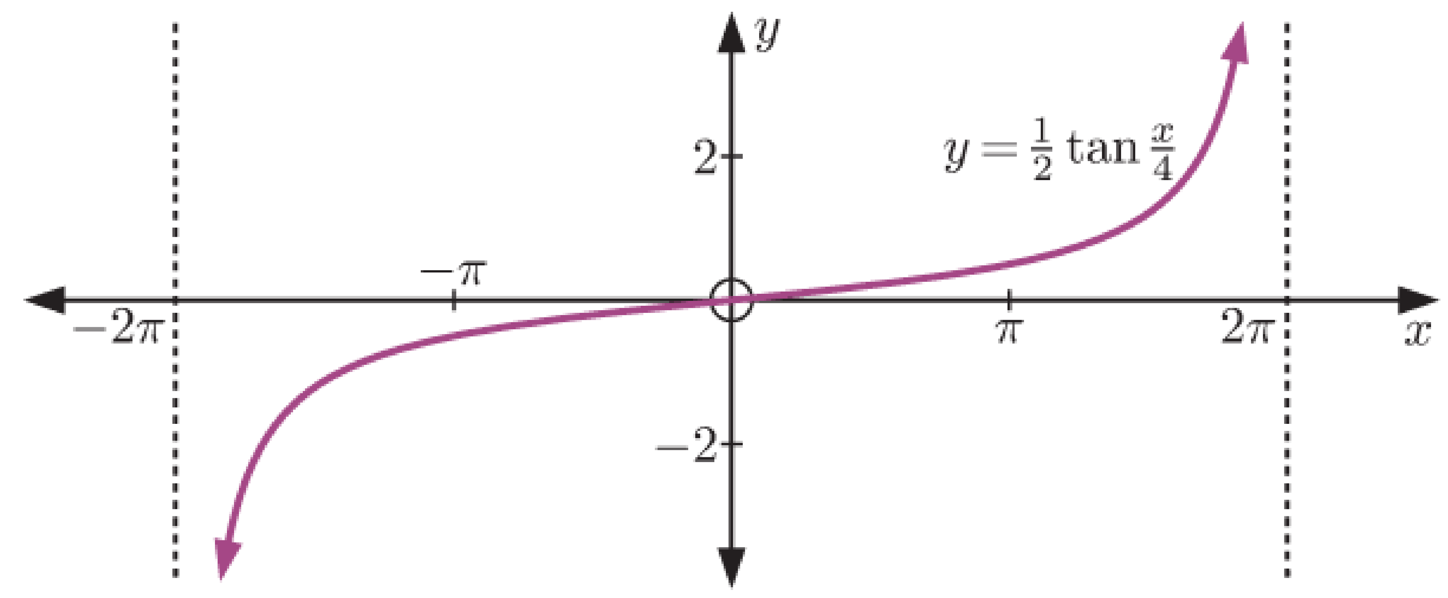
b **i** $\frac{2\pi}{3} + k\pi$, $k \in \mathbb{Z}$ **ii** $x = \frac{\pi}{6} + k\pi$, $k \in \mathbb{Z}$

c **i** $\frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$ **ii** $x = \frac{7\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$

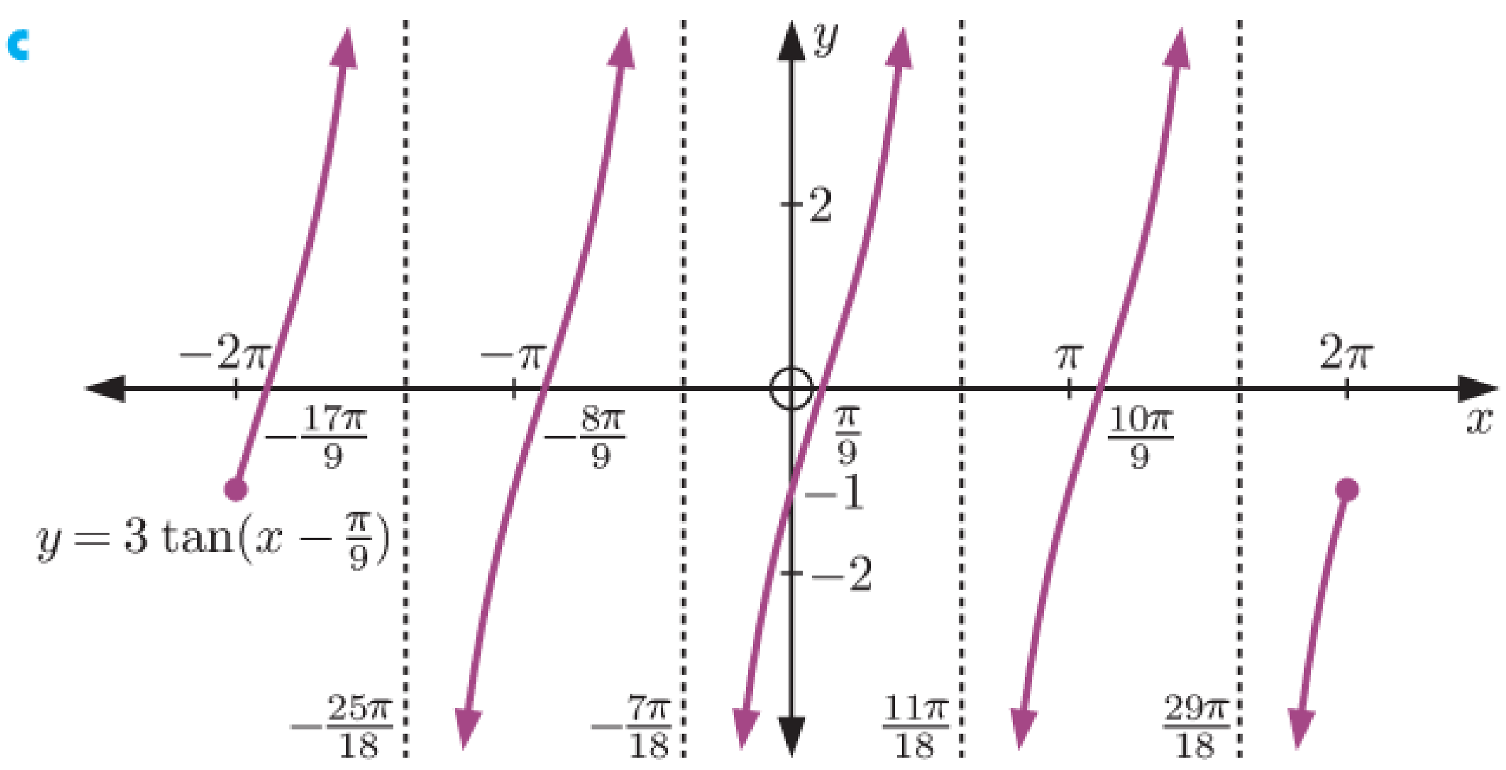
4 **a**



b



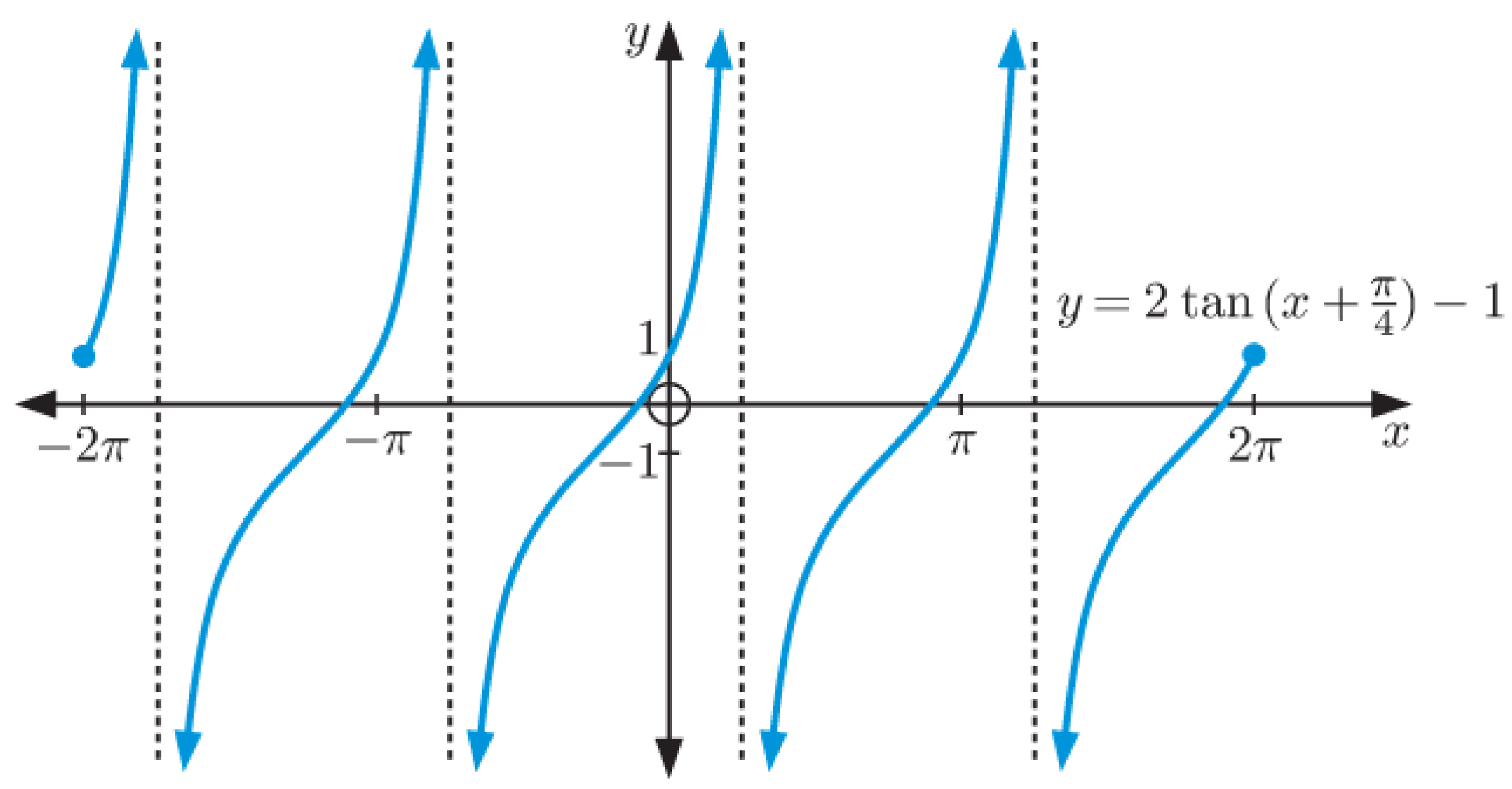
c



5 $p = \frac{1}{2}$, $q = 1$ **6** $a = \frac{3}{2}$, $b = -\frac{2\pi}{15} + \frac{2k\pi}{3}$, $k \in \mathbb{Z}$

7 **a** A vertical stretch with scale factor 2, then a translation $\frac{\pi}{4}$ units left and 1 unit downwards.

b



8 **Hint:** The function is undefined when $b(x-c) = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

9 **a** **i** $(f \circ g)(x) = \tan\left(2x - \frac{\pi}{2}\right)$

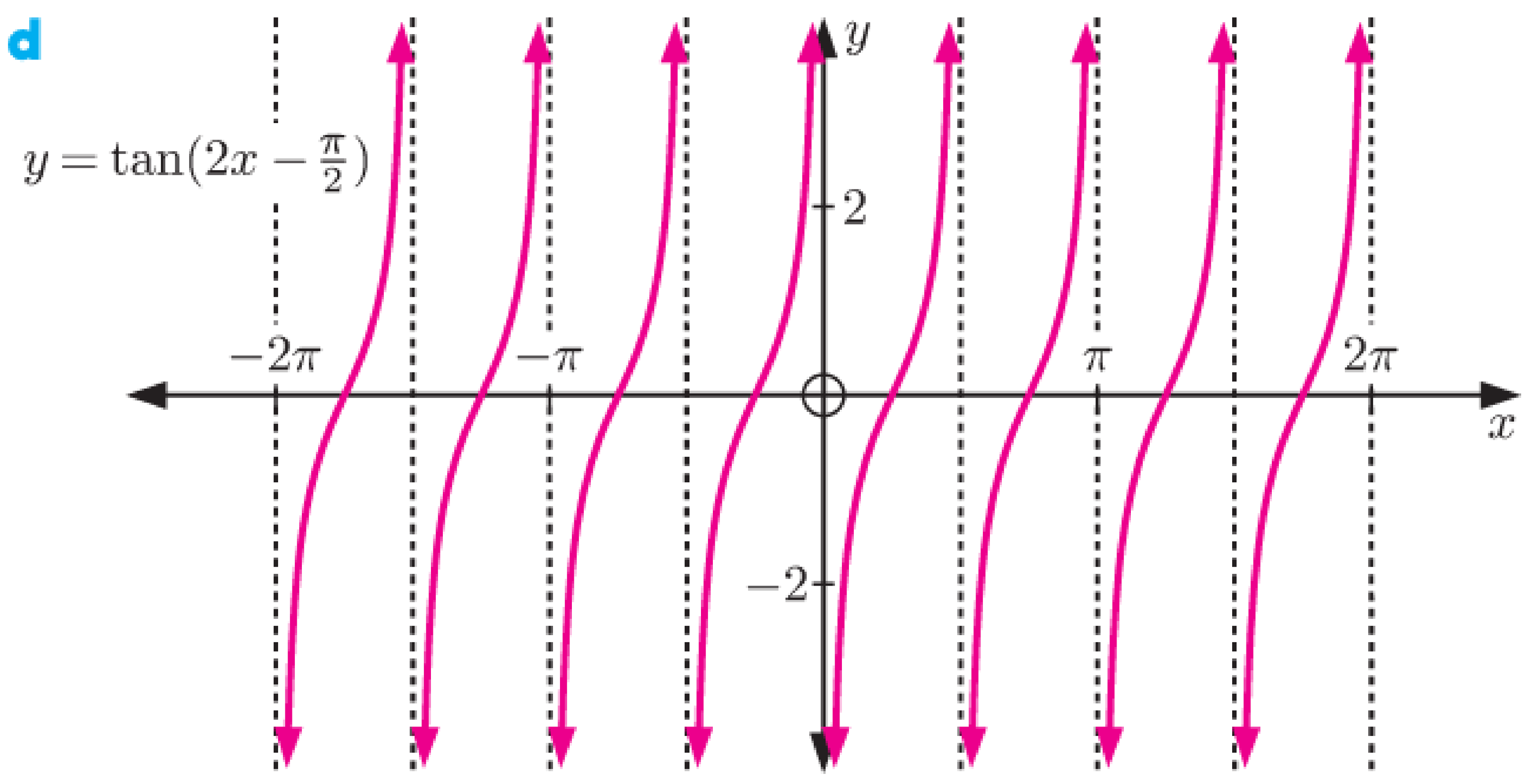
ii $(g \circ f)(x) = 2\tan x - \frac{\pi}{2}$

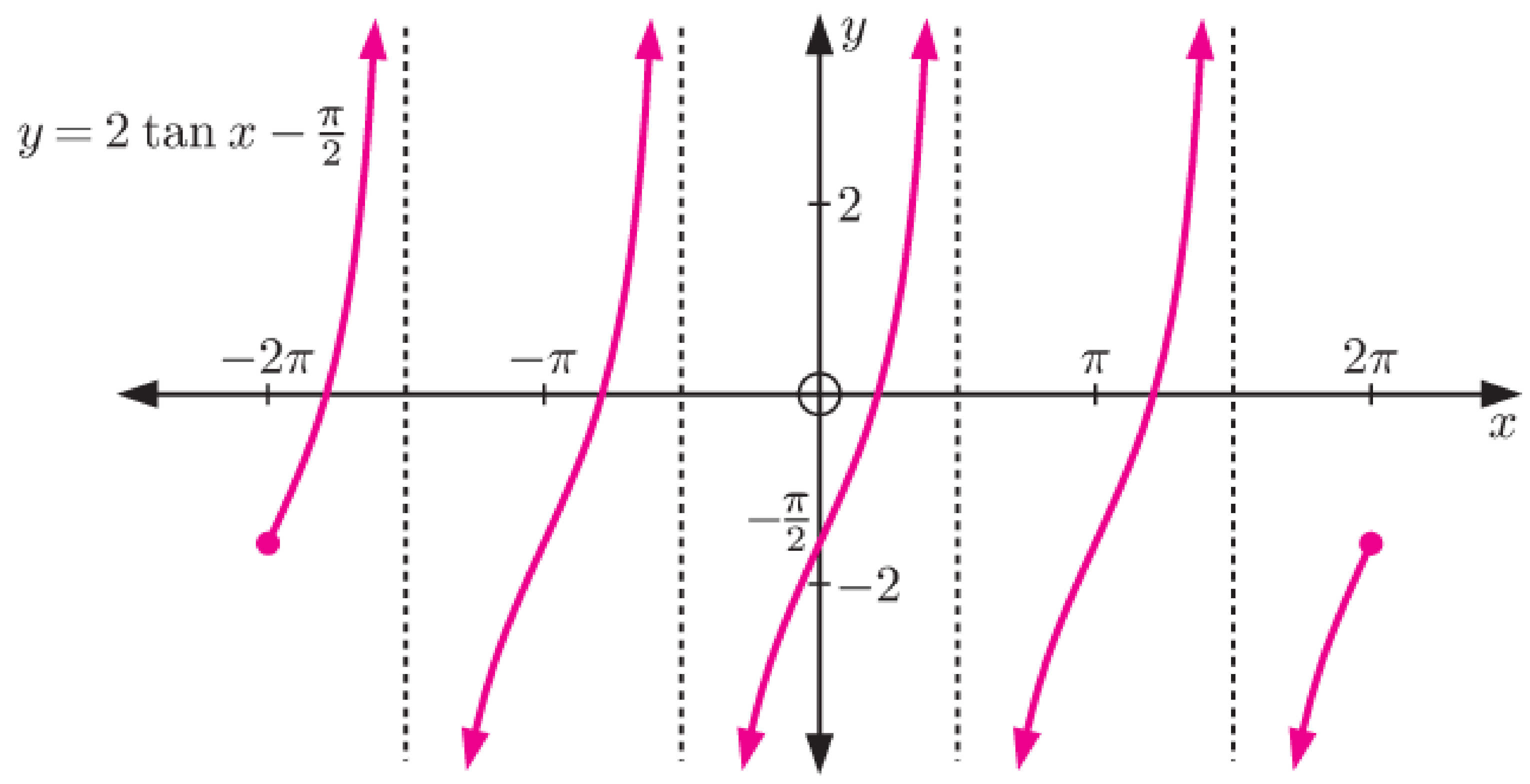
b **i** $\frac{1}{\sqrt{3}}$ **ii** $-\frac{\pi}{2}$

c **i** period $\frac{\pi}{2}$, vertical asymptotes $x = \dfrac{k\pi}{2}$, $k \in \mathbb{Z}$

ii period π, vertical asymptotes $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

d





EXERCISE 17G.1

1 **a** $x \approx 0.3, 2.8, 6.6, 9.1, 12.9$ **b** $x \approx 5.9, 9.8, 12.2$
c $x \approx 0.3, 2.8$ **d** $x \approx 3.8, 5.6$

2 **a** $x \approx 1.2, 5.1, 7.4$ **b** $x \approx 4.4, 8.2, 10.7$
c $x \approx 5.2$ **d** $x \approx 2.5, 3.8$

3 **a** $x \approx 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.7$
b $x \approx 1.7, 3.0, 4.9, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3, 15.6$
c $x \approx 3.2, 4.6$ **d** $x \approx 1.6, 3.1, 4.8, 6.2$

4 **a** $x \approx 1.1, 4.2, 7.4$ **b** $x \approx 2.2, 5.3$
c $x \approx 1.3, 4.4$ **d** $x \approx 2.0$

EXERCISE 17G.2

1 **a** $x \approx 0.446, 2.70, 6.73, 8.98$
b $x \approx 2.52, 3.76, 8.80, 10.0$
c $x \approx 0.588, 3.73, 6.87, 10.0$

2 **a** $x \approx -0.644, 0.644$ **b** $x \approx -4.56, -1.42, 1.72, 4.87$
c $x \approx -2.76, -0.384, 3.53$

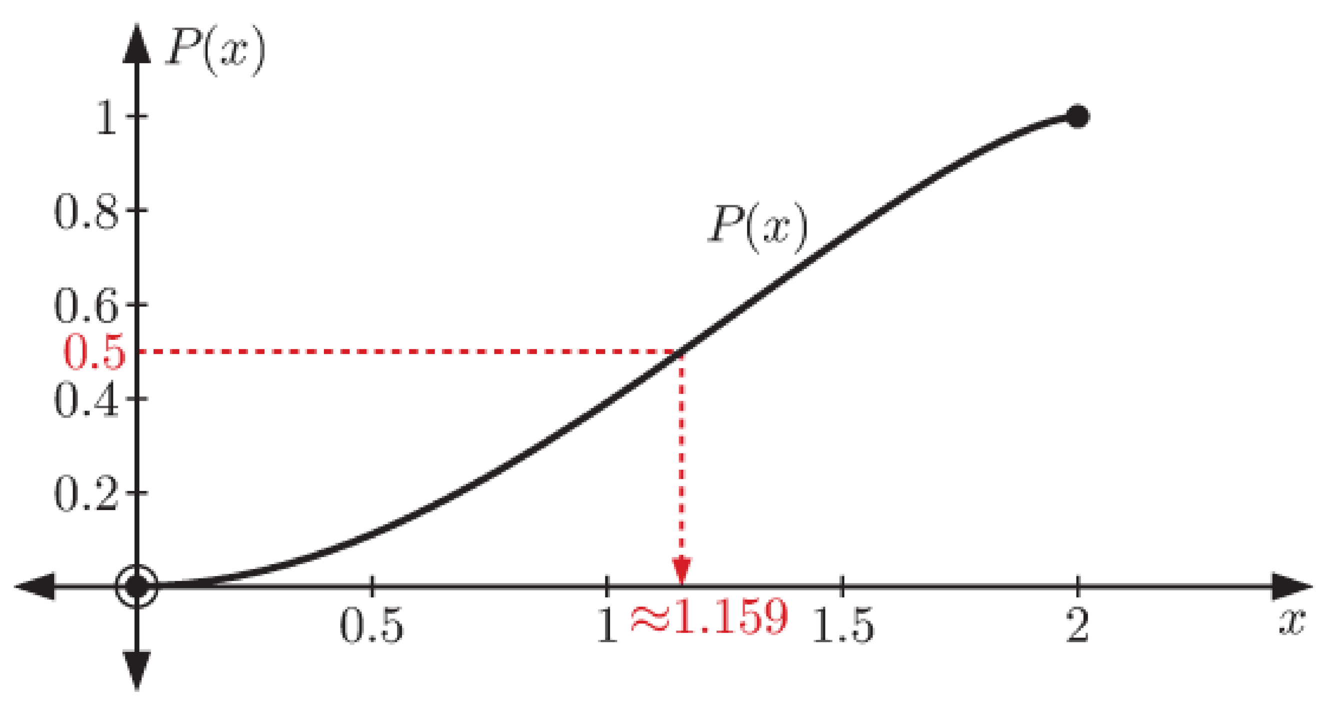
3 **a** $x \approx 1.08, 4.35$ **b** $x \approx 0.666, 2.48$

4 $x \approx -0.951, 0.234, 5.98$

5 **a** **Note:** The function $P(x)$ can be written in many different ways.

$$P(x) = 1 + \frac{x^2 - 2}{\pi}\arccos\frac{x}{2} - \frac{x}{2\pi}\sqrt{4 - x^2}$$

b



c $x \approx 1.159$

EXERCISE 17G.3

1 **a** $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ **b** $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$ **c** $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$
d $x = \frac{3\pi}{2}$ **e** $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ **f** $x = 0$, π, or 2π

2 **a** $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ **b** $x = \pi$ **c** $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

3 **a** $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$, or $\frac{10\pi}{3}$ **b** $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$, or $\frac{11\pi}{4}$
c $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, or $\frac{13\pi}{4}$

4 **a** $x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$
b $x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}$, or $\frac{5\pi}{4}$
c $x = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}$, or $\frac{7\pi}{4}$

5 **a** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$ **b** $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$

6 **a** $0 \leqslant 2x \leqslant 4\pi$ **b** $0 \leqslant \dfrac{x}{4} \leqslant \frac{\pi}{2}$
c $\frac{\pi}{2} \leqslant x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$ **d** $-\frac{\pi}{6} \leqslant x - \frac{\pi}{6} \leqslant \frac{11\pi}{6}$
e $-\frac{\pi}{2} \leqslant 2\left(x - \frac{\pi}{4}\right) \leqslant \frac{7\pi}{2}$ **f** $-2\pi \leqslant -x \leqslant 0$

7 **a** $-3\pi \leqslant 3x \leqslant 3\pi$ **b** $-\frac{\pi}{4} \leqslant \dfrac{x}{4} \leqslant \frac{\pi}{4}$
c $-\frac{3\pi}{2} \leqslant x - \frac{\pi}{2} \leqslant \frac{\pi}{2}$ **d** $-\frac{3\pi}{2} \leqslant 2x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$
e $-2\pi \leqslant -2x \leqslant 2\pi$ **f** $0 \leqslant \pi - x \leqslant 2\pi$

8 **a** $x = \frac{\pi}{3}, \frac{5\pi}{3}$, or $\frac{7\pi}{3}$
b $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$
c $x = 0, \frac{4\pi}{3}$, or 2π

9 **a** $x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$, or $\frac{23\pi}{12}$
b $x = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}$, or $\frac{35\pi}{18}$
c $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$, or $\frac{5\pi}{3}$ **d** $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
e $x = \frac{4\pi}{3}$ **f** $x = \frac{3\pi}{4}$

10 **a** $x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$, or $\frac{17\pi}{9}$
b $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$ **c** $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

11 **a** $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ **b** $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}$, or $\frac{7\pi}{4}$
c $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$, or $\frac{5\pi}{3}$

12 **a** $x = -\frac{5\pi}{3}, -\pi, \frac{\pi}{3}$, or π **b** $x = 0, \frac{3\pi}{2}$, or 2π
c $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, or π **d** $x = 0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}$, or 2π

13 **a** $b = \frac{1}{3}$, $d = 2$ **b** $a = -2$, $b = \frac{1}{2}$
c $b = 4$, $d = -1$ **d** $b = \frac{1}{2}$, $d = -4$

14 $x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$
a $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
b $x = \frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}, \frac{13\pi}{12}, \frac{4\pi}{3}, \frac{19\pi}{12}$, or $\frac{11\pi}{6}$
c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$

15 **a** $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$, or 2π **b** $x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}$, or $\frac{5\pi}{3}$
c $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, or 2π **d** $x = \frac{7\pi}{6}, \frac{3\pi}{2}$, or $\frac{11\pi}{6}$

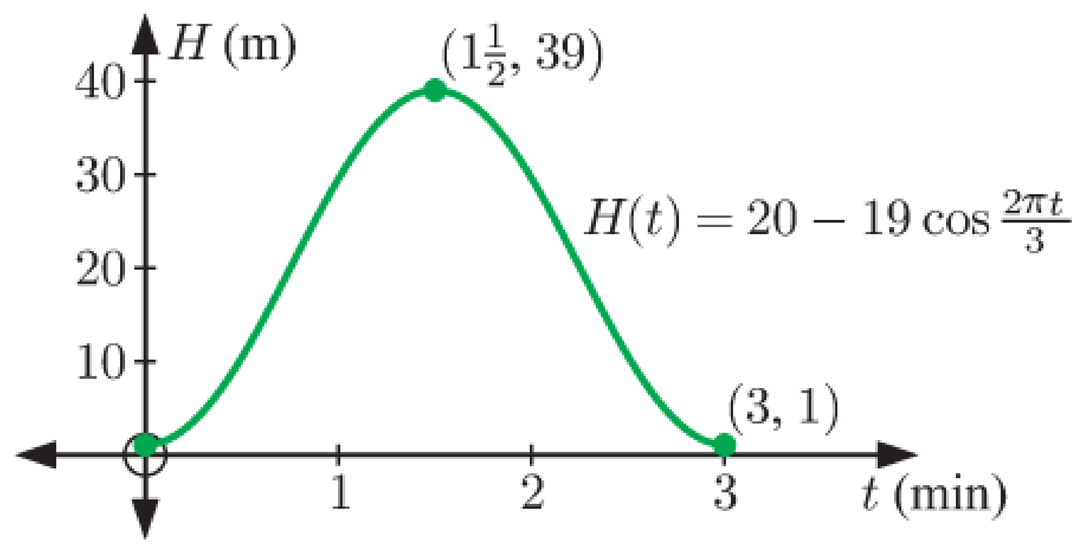
EXERCISE 17H

1 **a** **i** 7500 grasshoppers **ii** $\approx 10\,300$ grasshoppers
b 10 500 grasshoppers, when $t = 4$ weeks
c **i** at $t = 1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks
ii at $t = 9\frac{1}{3}$ weeks

d $2.51 \leqslant t \leqslant 5.49$

2 **a** 1 m above ground **b** at $t = 1\frac{1}{2}$ min **c** 3 min

d



e $0.570 \leqslant t \leqslant 2.43$ min

3 **a** 400 water buffalo

b **i** 577 water buffalo **ii** 400 water buffalo

c 650, which is the maximum population.

d 150, after 3 years **e** $t \approx 0.262$ years

4 **a** **i** true **ii** true **b** 116.8 cents L^{-1}

c on the 5th, 11th, 19th, and 25th days

d 98.6 cents L^{-1} on the 1st and 15th days

5 **a** $H(t) = 3\cos\left(\frac{\pi}{2}t\right) + 4$ **b** $t \approx 1.46$ s

REVIEW SET 17A

1 **a** not periodic **b** periodic

2 **a** minimum = 0, maximum = 2

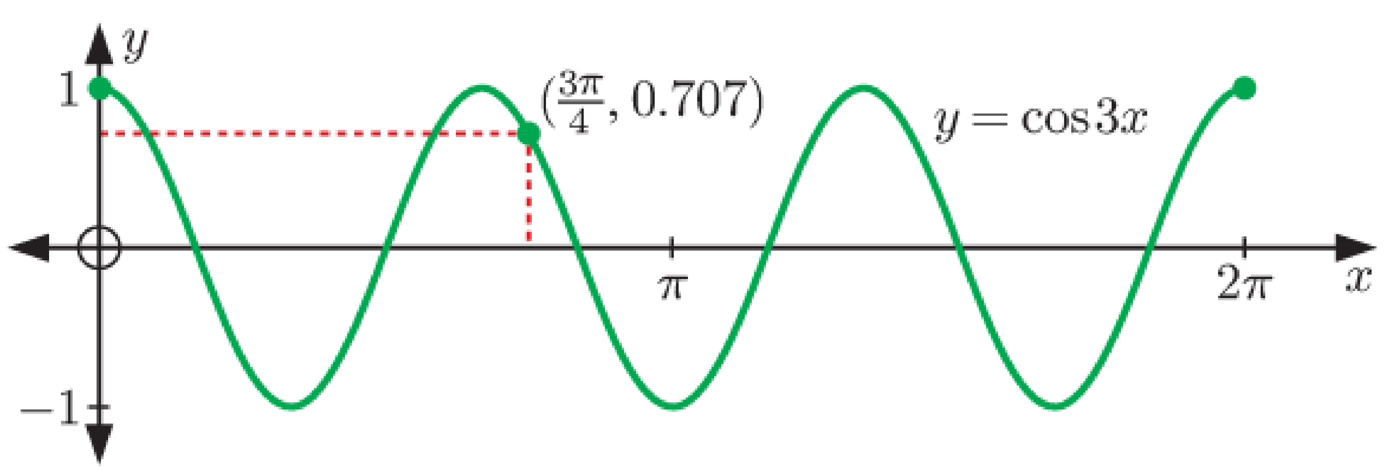
b minimum = -2, maximum = 2

3 **a** 10π **b** $\frac{\pi}{2}$ **c** 4π **d** $\frac{\pi}{3}$

4

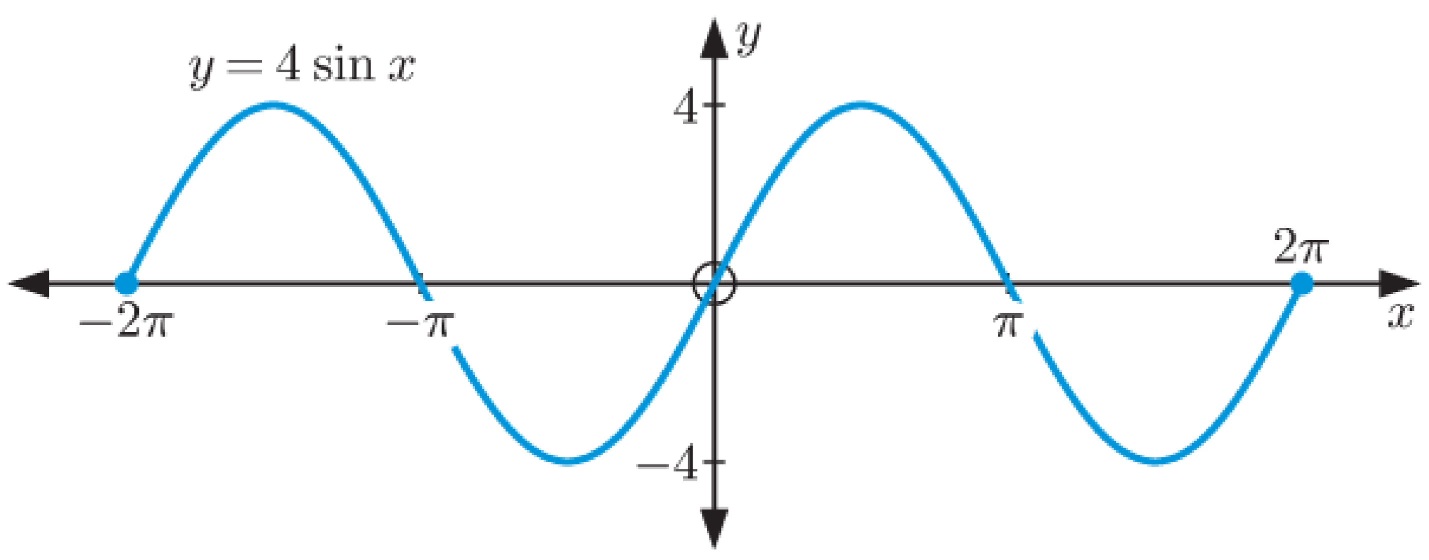
Function	*Period*	*Amplitude*	*Range*
$y = -3\sin\frac{x}{4} + 1$	8π	3	$-2 \leqslant y \leqslant 4$
$y = 3\cos\pi x$	2	3	$-3 \leqslant y \leqslant 3$

5 **a**



b $y = \frac{1}{\sqrt{2}} \approx 0.707$

6 **a**



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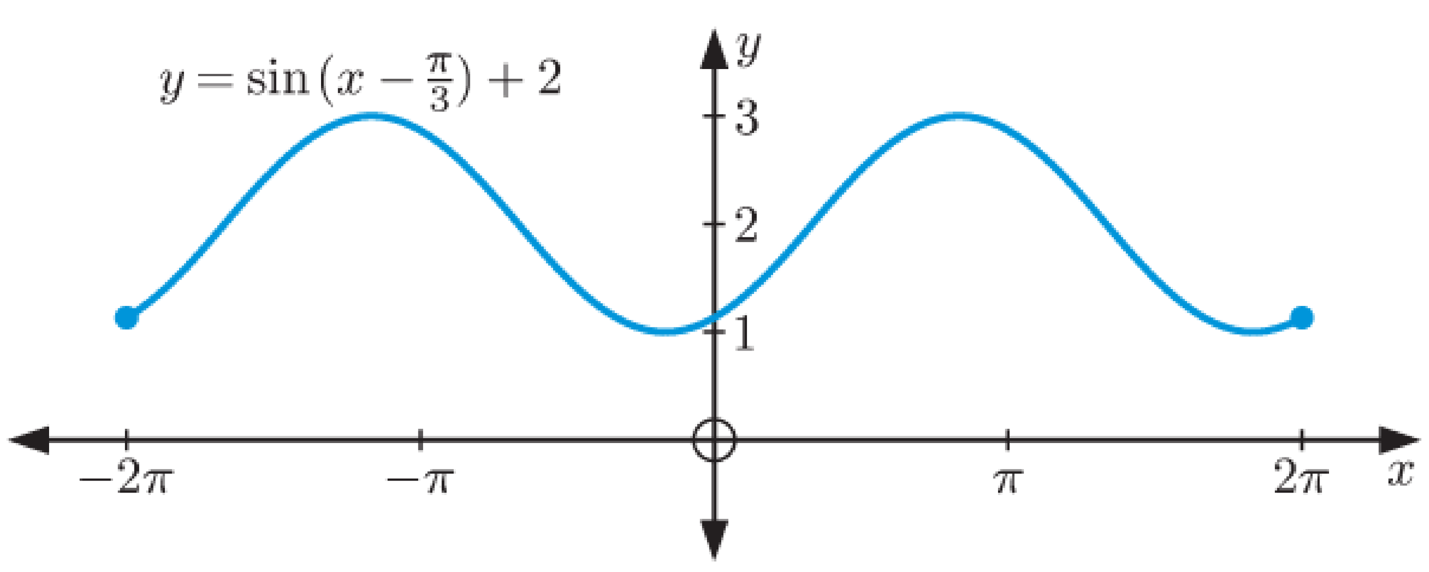
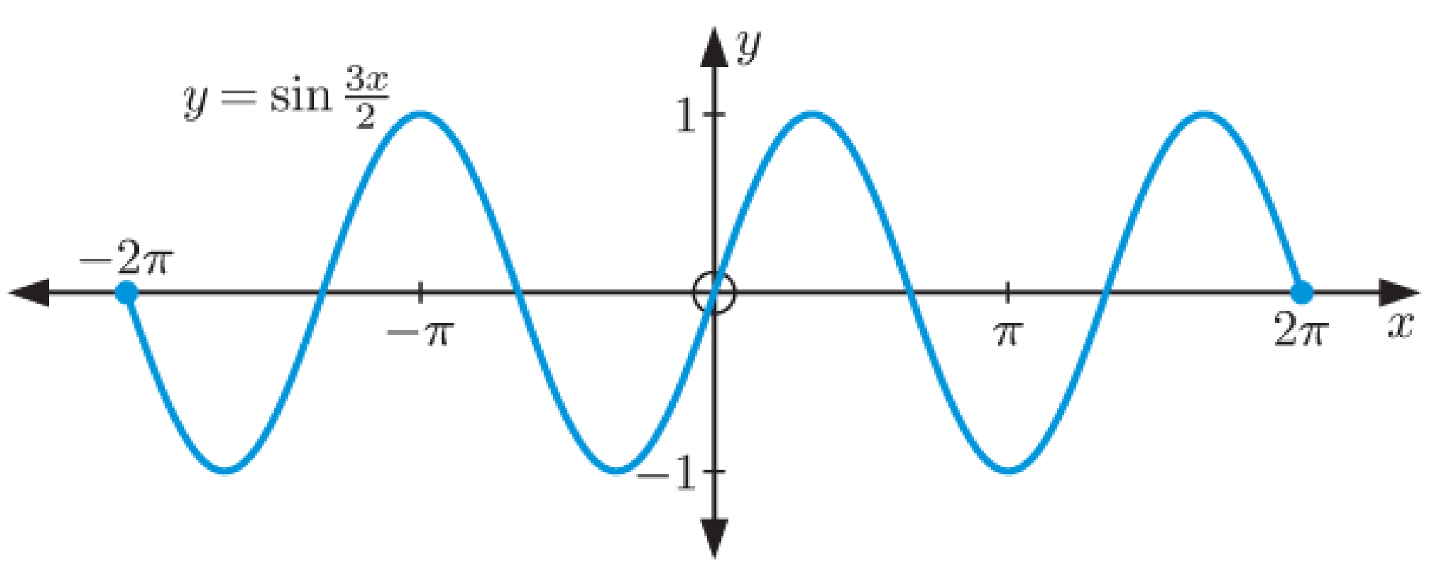
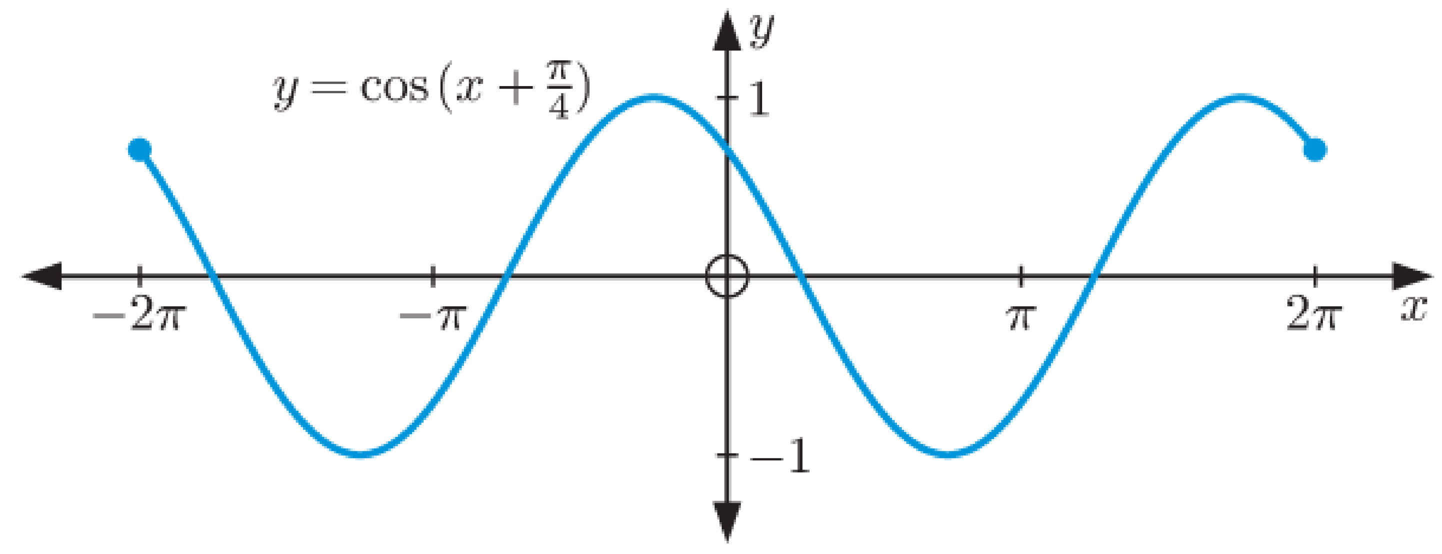


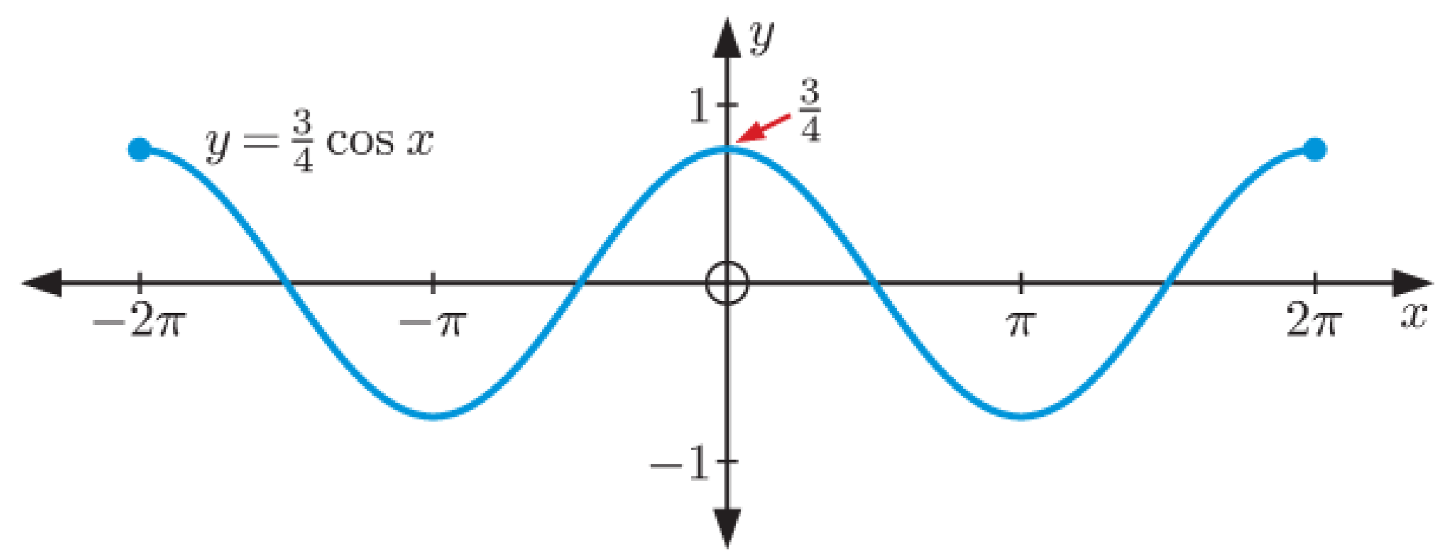
c



d



e



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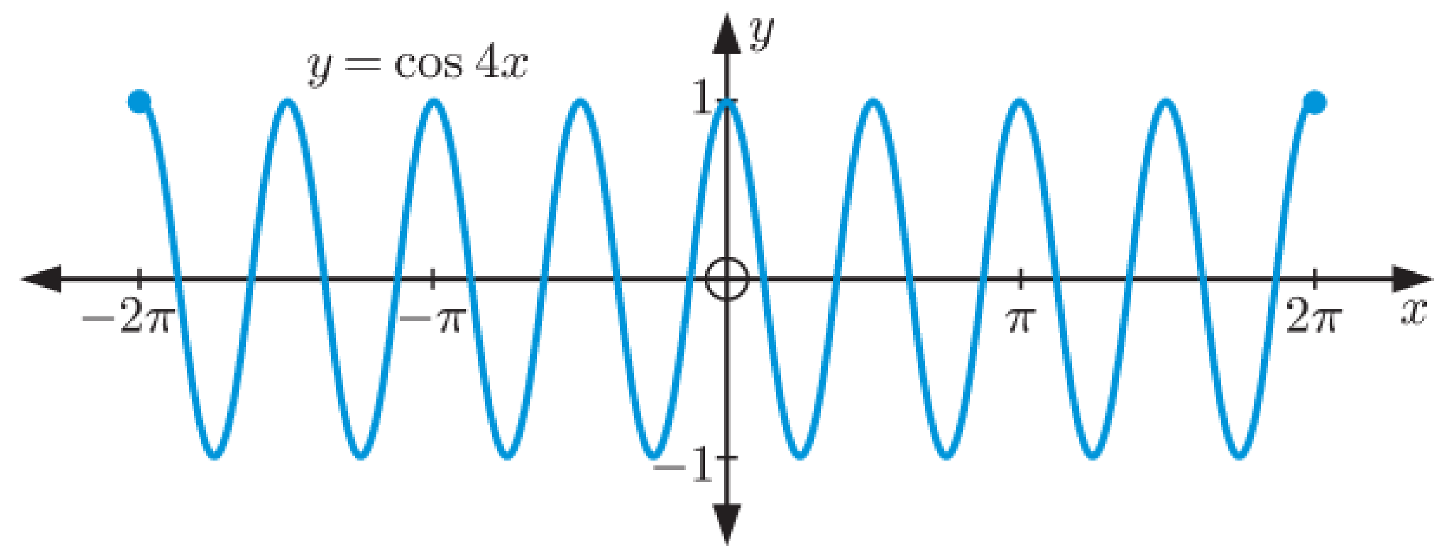


7 **a** A vertical stretch with scale factor 3, then a horizontal stretch with scale factor $\frac{1}{2}$.

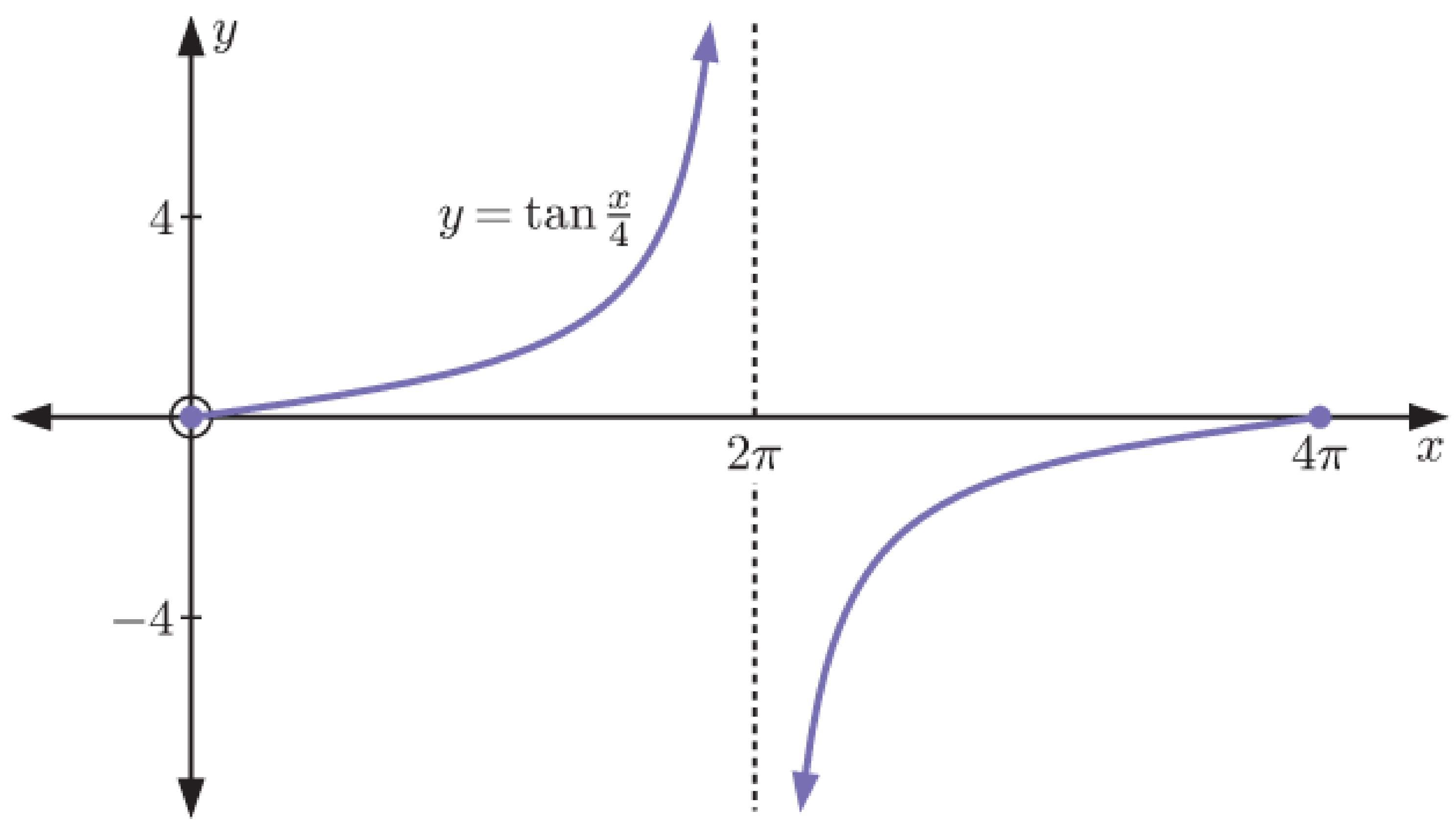
b A translation $\frac{\pi}{3}$ units right and 1 unit downwards.

c A reflection in the x-axis, then a horizontal stretch with scale factor $\frac{1}{2}$.

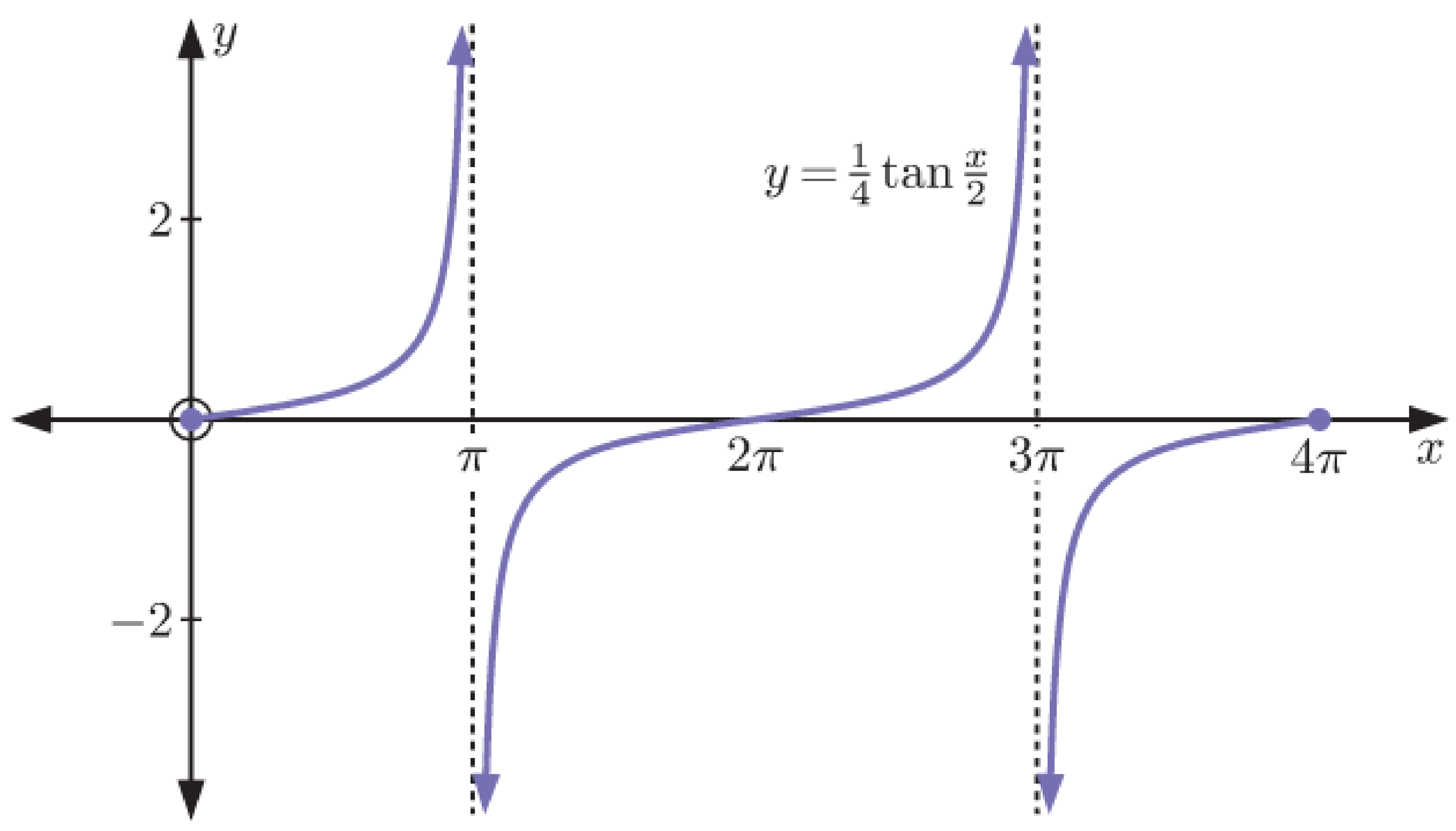
d A vertical stretch with scale factor 2, then a horizontal stretch with scale factor 2, then a translation $\frac{\pi}{2}$ units right and $\frac{1}{2}$ unit upwards.

8 **a** $y = -4\cos 2x$ **b** $y = \cos\frac{\pi x}{4} + 2$

9 **a**



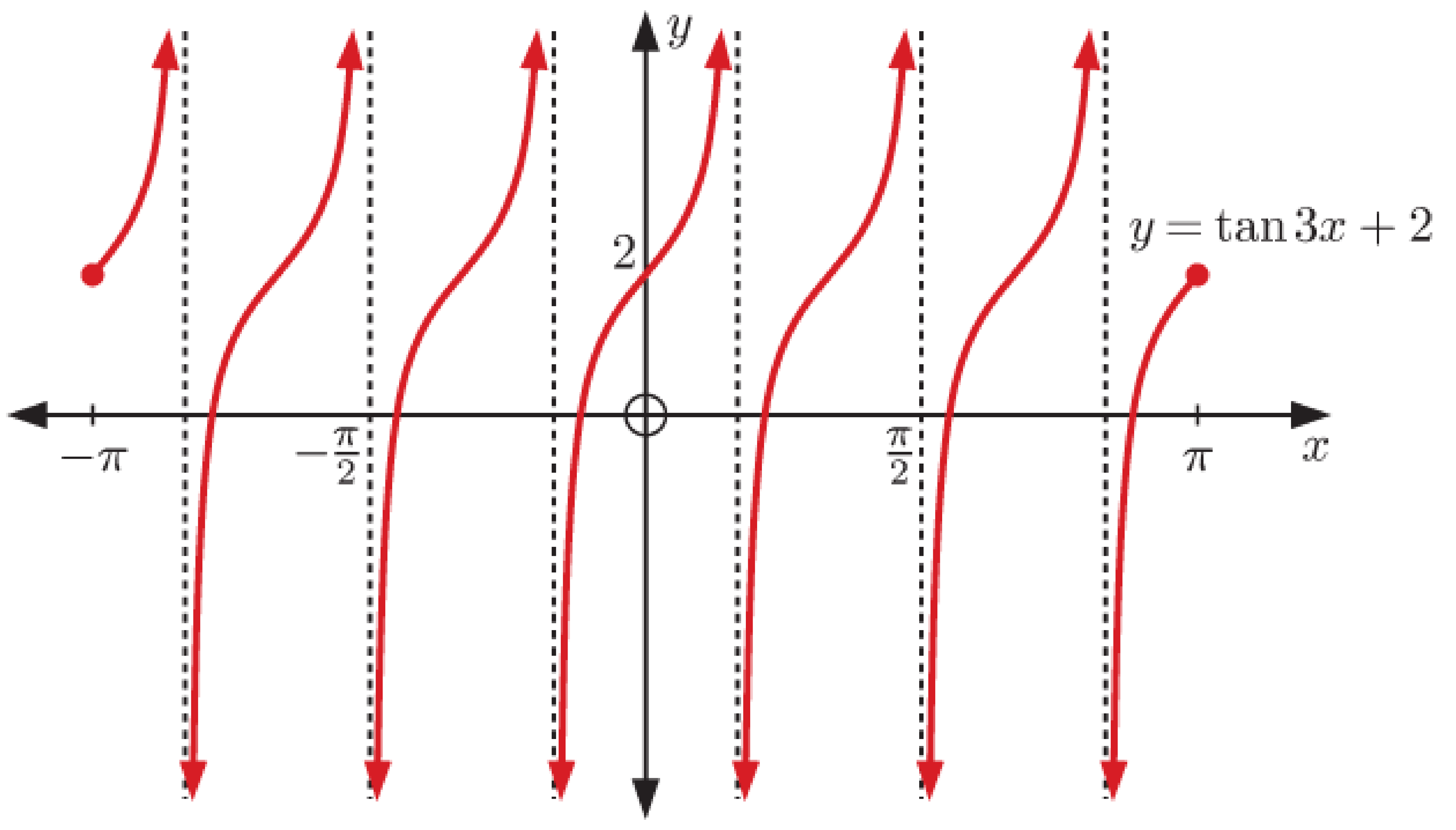
b



10 **a** A horizontal stretch with scale factor $\frac{1}{3}$, then a vertical translation 2 units upwards.

b $\frac{\pi}{3}$

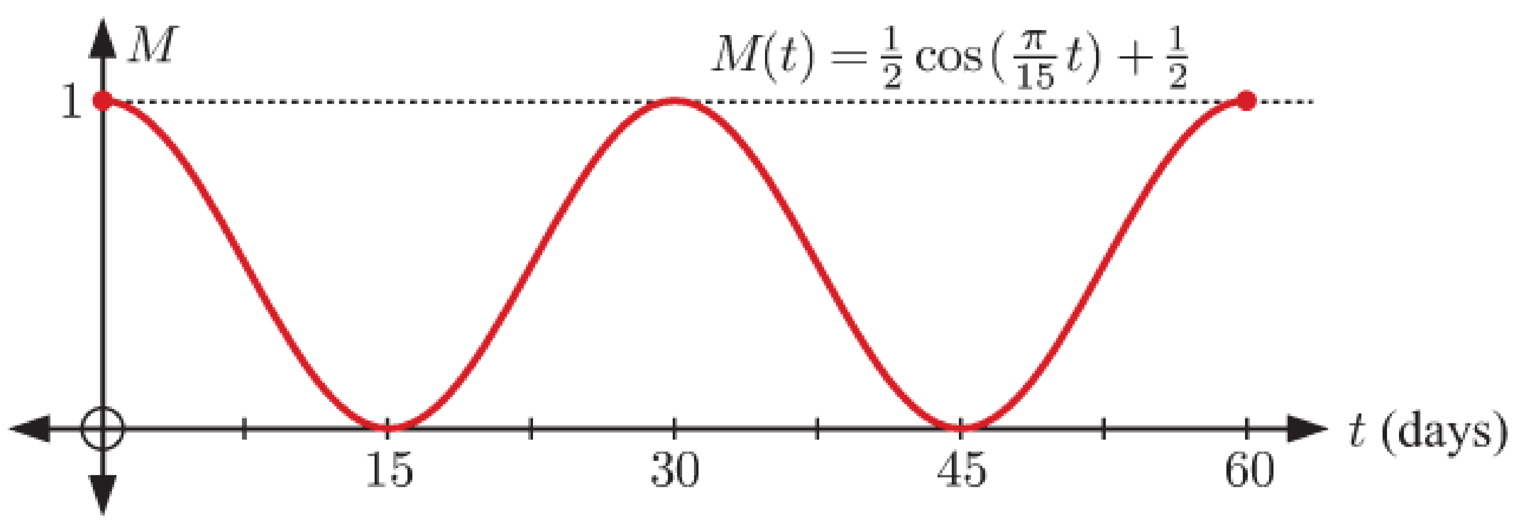
c



11 a $a = 7,\ b = \frac{\pi}{8},\ c = 1,\ d = 10$

b $g(x) = 14\sin\left(\frac{\pi}{8}(x-3)\right) + 14$

12 a

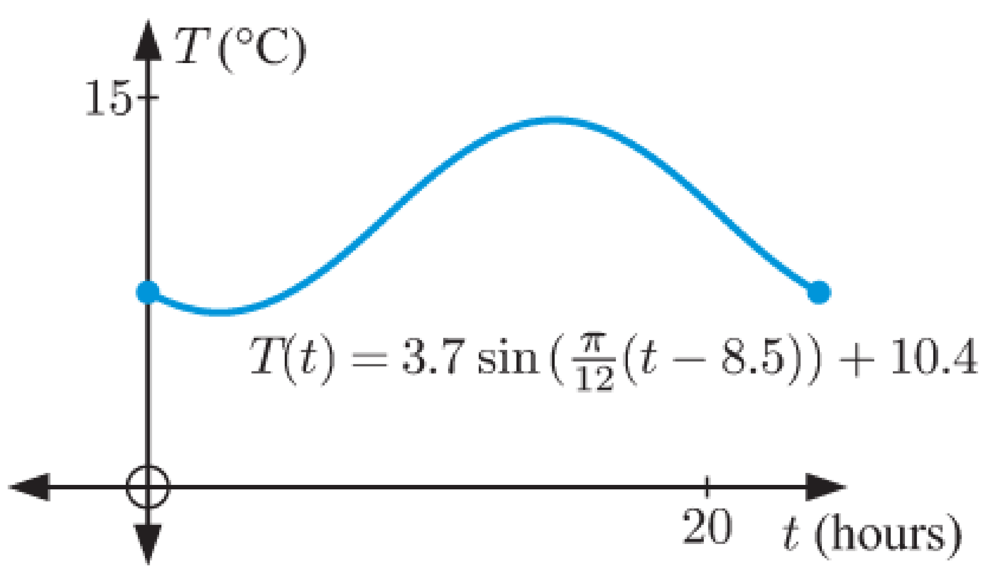


b i 0.75 ii 0.25 iii ≈ 0.835 iv ≈ 0.165

c once every 30 days d January 16, February 15

13 a $T(t) = 3.7\sin\left(\frac{\pi}{12}(t-8.5)\right) + 10.4$ °C

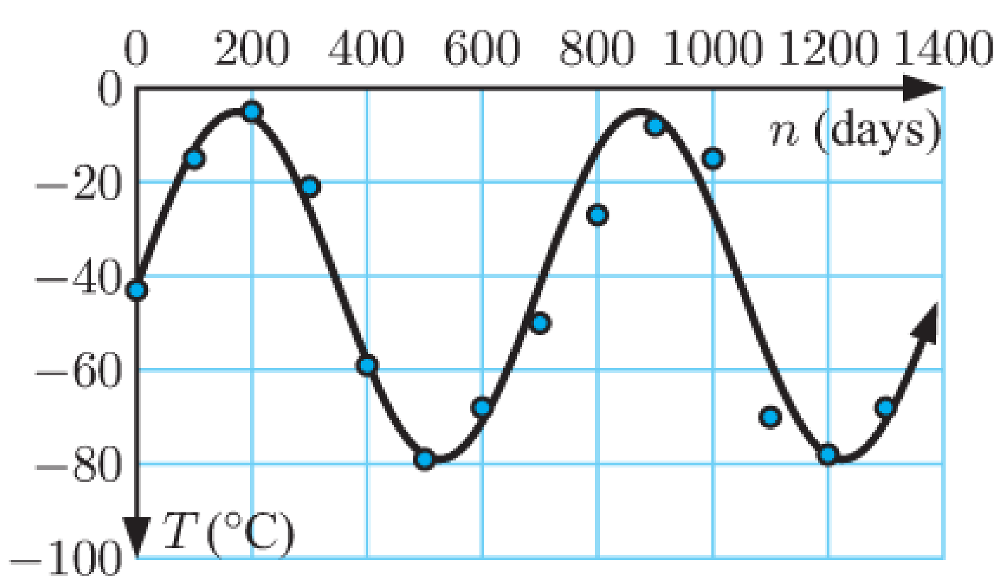
b



14 a maximum: -5°C, minimum: -79°C

b ≈ 700 Mars days c $T \approx 37\sin(0.008\,98n) - 42$

d



e Using technology,
$T \approx 36.5\sin(0.009\,01x - 0.0903) - 43.2$.
Our model fits the data well.

15 a $x \approx 2.0, 4.3, 8.3, 10.6$ b $x \approx 0.5, 5.8, 6.7, 12.1$

16 a $x \approx 0.392, 2.75, 6.68$ b $x \approx 5.42$

17 a $x \approx 1.12, 5.17, 7.40$ b $x \approx 0.184, 4.62$

18 a $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$ b $x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$

19 a $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}$, or $\frac{15\pi}{8}$

b $x = \frac{3\pi}{2}$ c $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$

20 a $x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}$, or 4π b $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$, or $\frac{5\pi}{3}$

21 a 5000 beetles b smallest 3000, largest 7000

c $0.5 < t < 2.5$ and $6.5 < t \leqslant 8$

REVIEW SET 17B

1 a The function repeats itself over and over in a horizontal direction, in intervals of length 8 units.

b i 8 ii 5 iii -1

2 a A translation $\frac{\pi}{3}$ units right and 1 unit upwards.

b A horizontal stretch with scale factor $\frac{1}{3}$.

3 a 6π b $\frac{\pi}{4}$

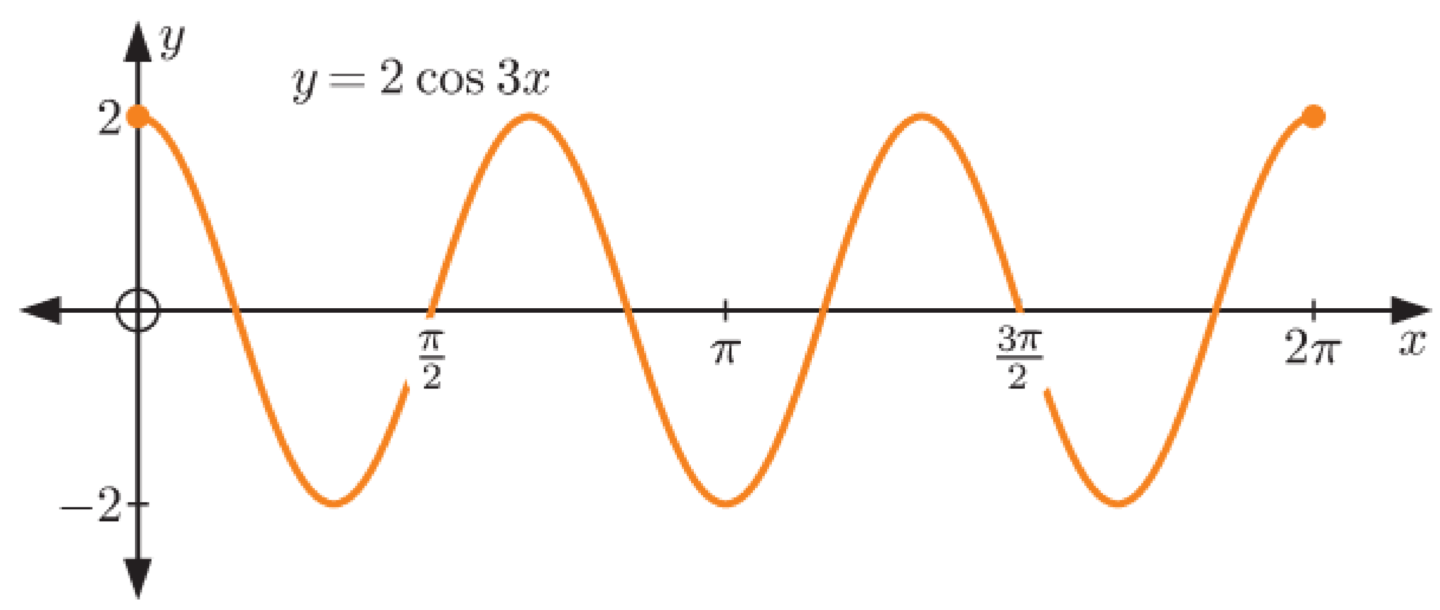
4 a $b = \frac{1}{3}$ b $b = 24$ c $b = \frac{2\pi}{9}$

5 a minimum $= -8$, maximum $= 2$

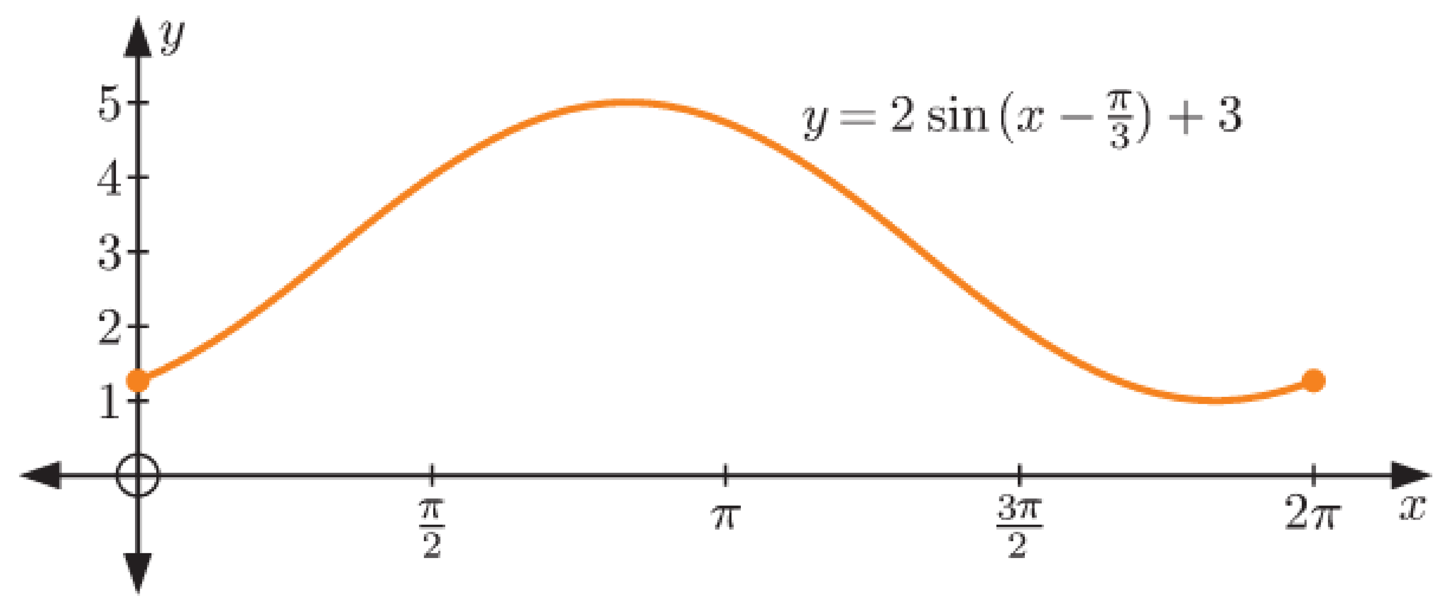
b minimum $= \frac{2}{3}$, maximum $= 1\frac{1}{3}$

6 a $y = 5$ b $y = -4$

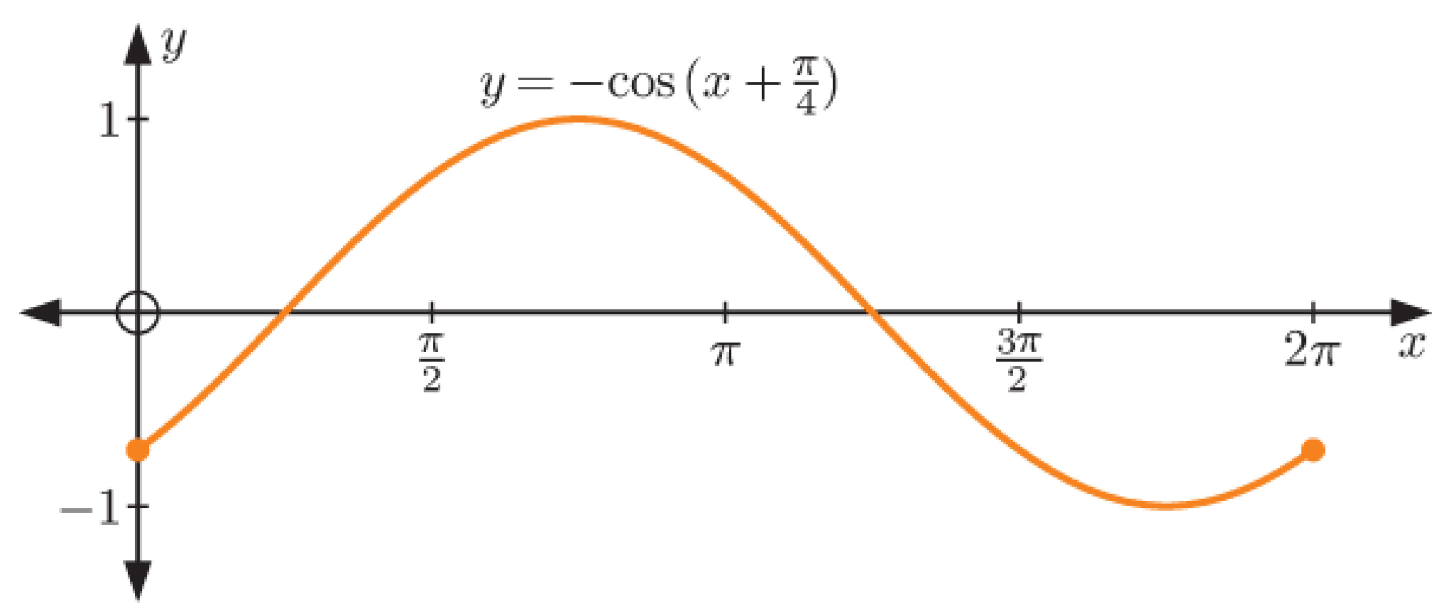
7 a



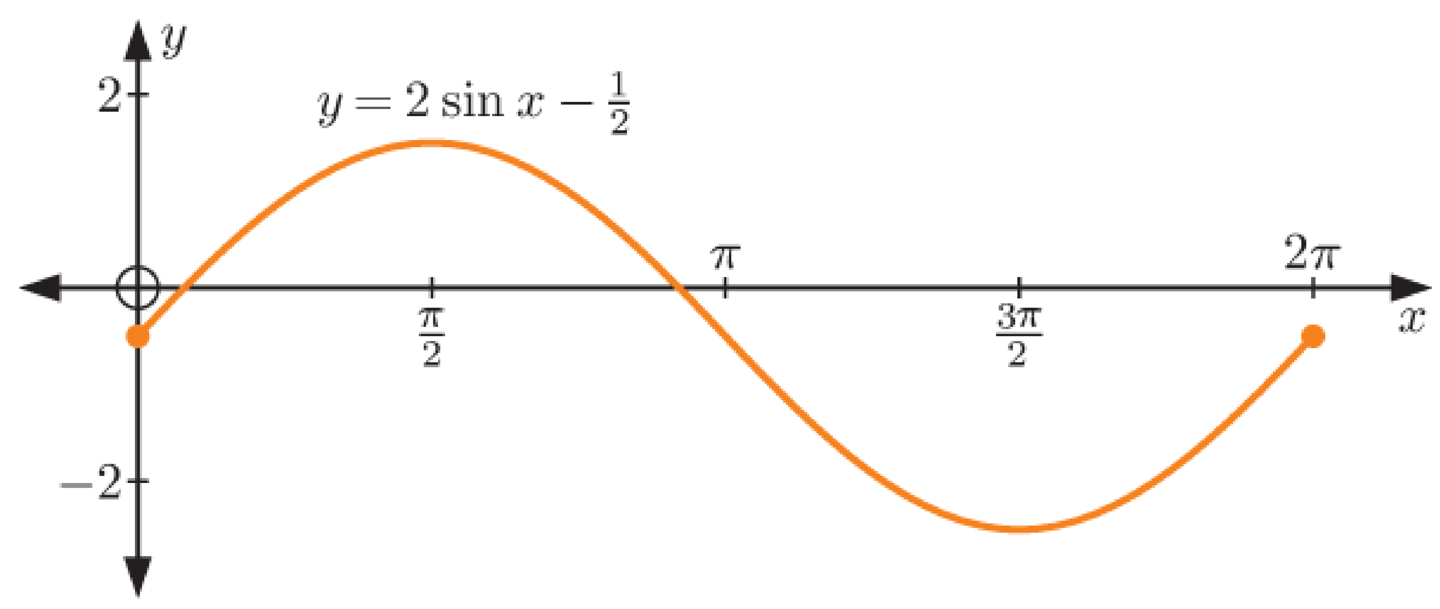
b



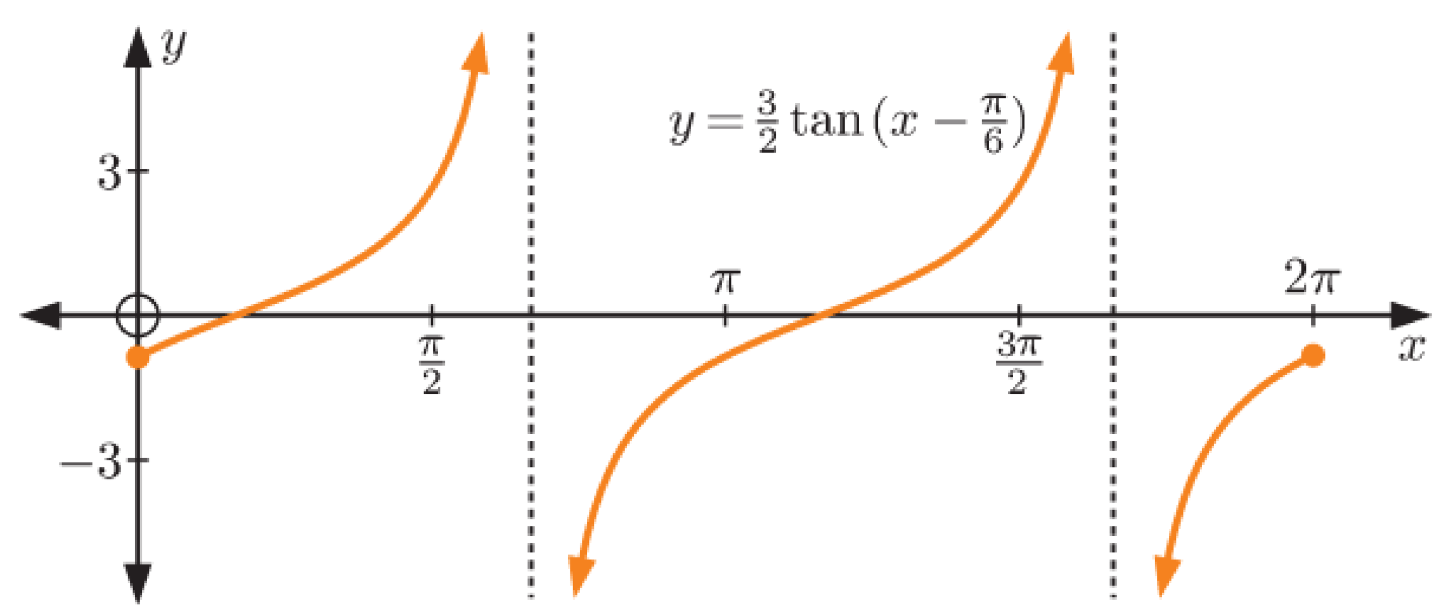
c



d



e



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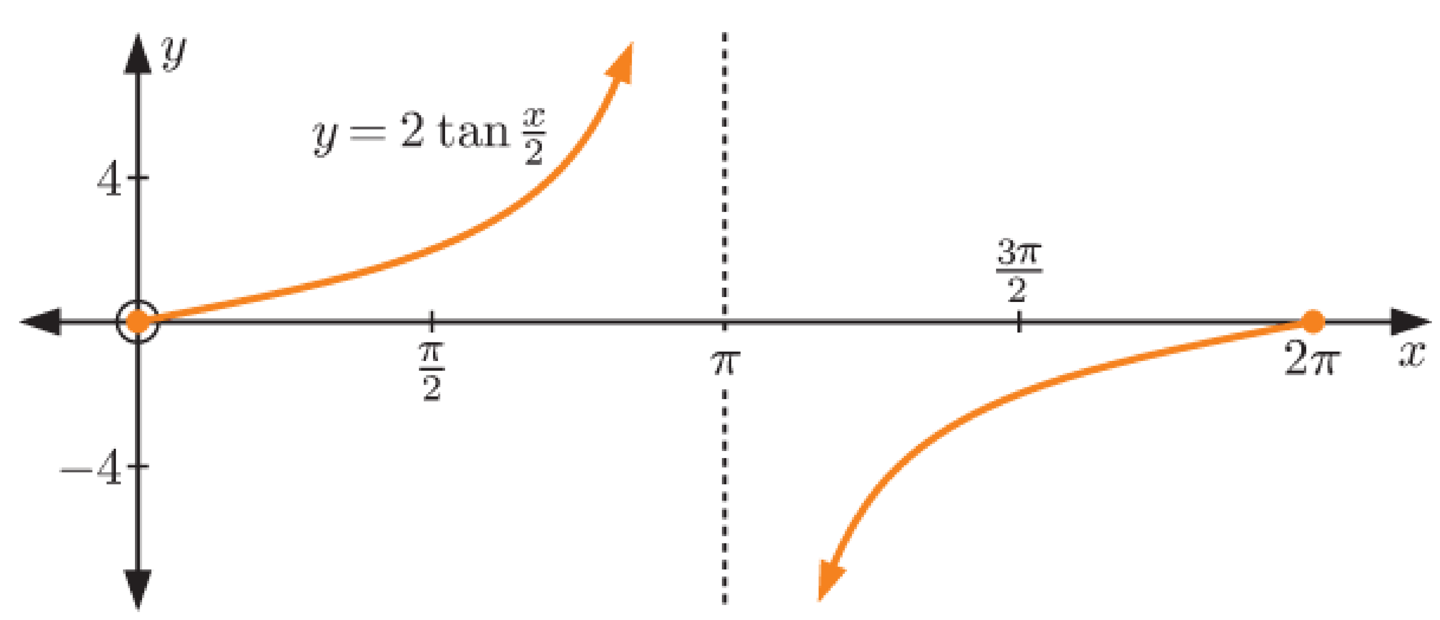
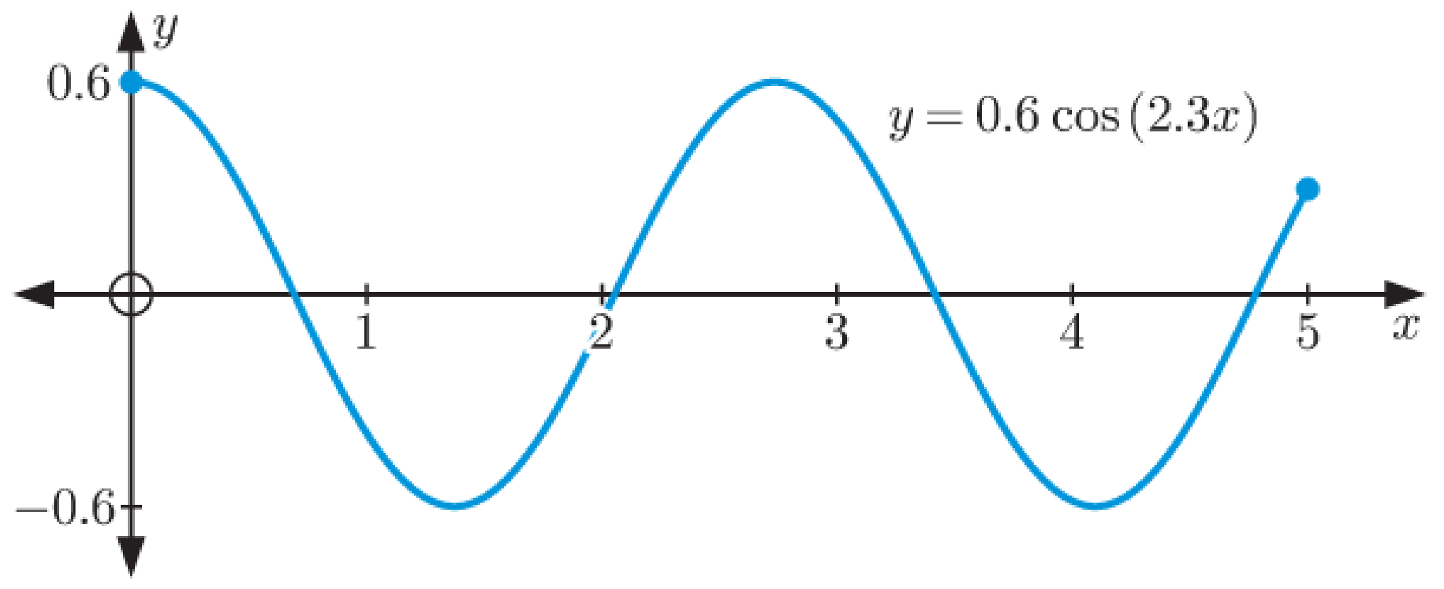


8 **a** $y = 4\sin x + 6$ **b** $y = 4\cos\left(x - \frac{\pi}{2}\right) + 6$

9

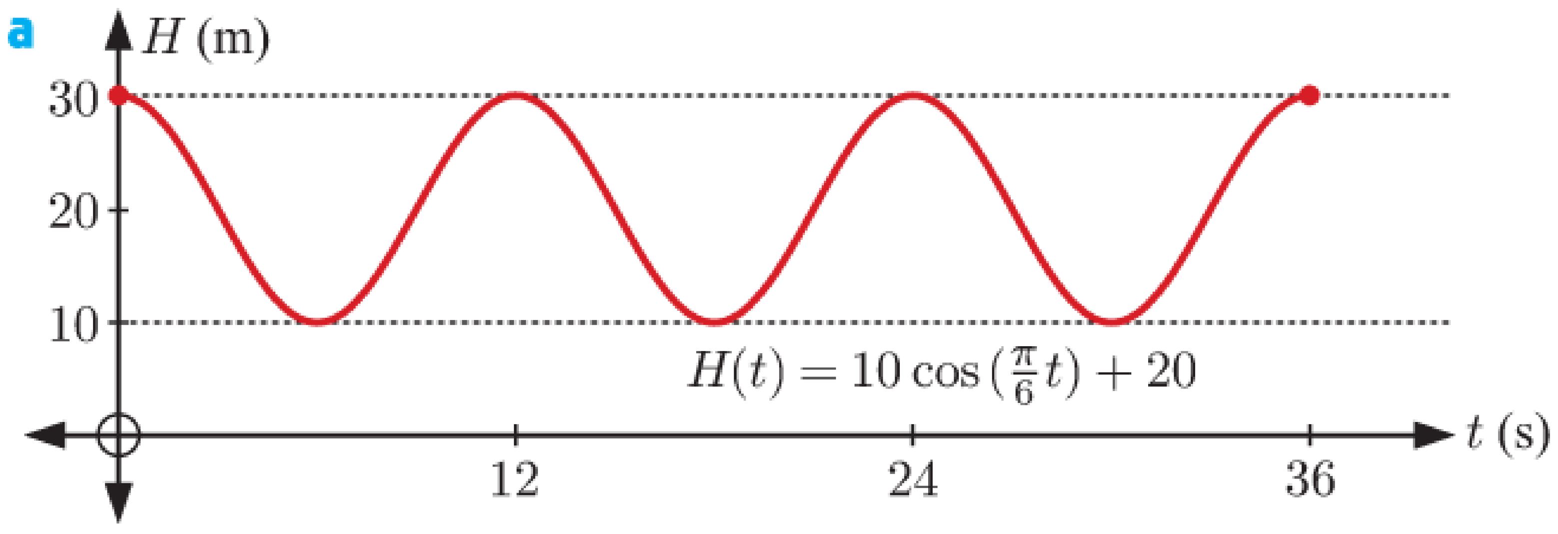


10 $a = \frac{3}{2}$, $b = -\frac{1}{2}$

11 **a** A reflection in the x-axis, then a horizontal stretch with scale factor $\frac{1}{2}$.

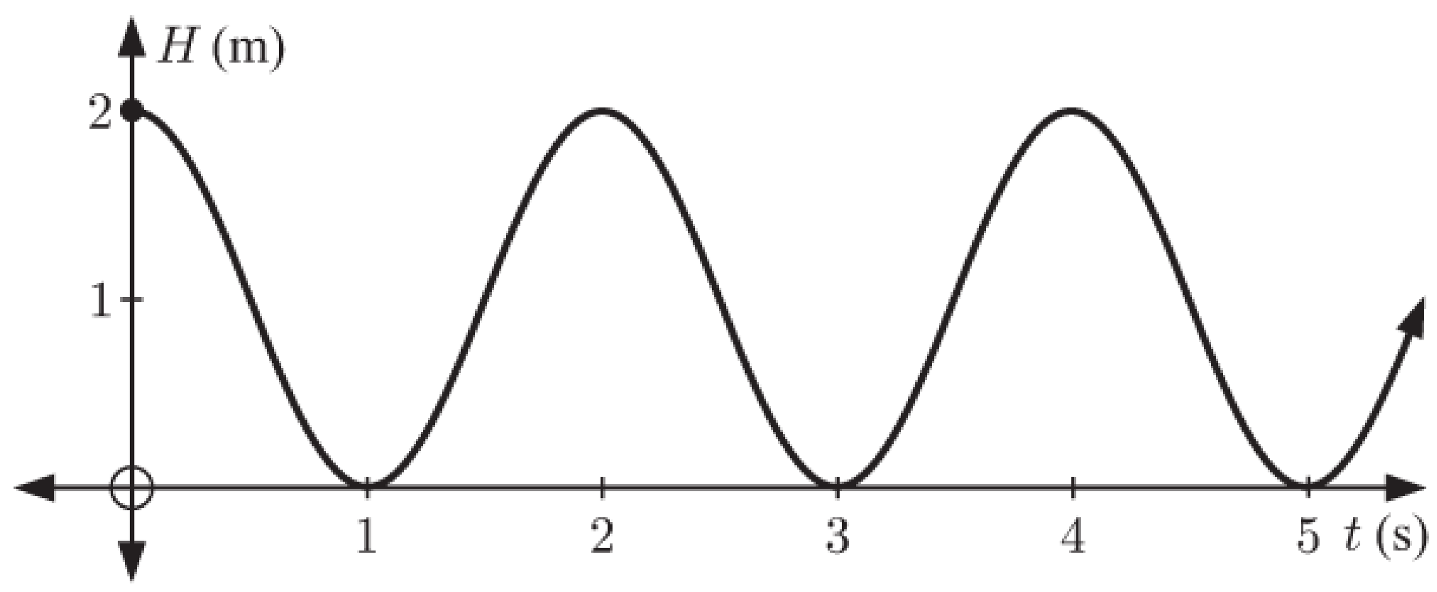
b A vertical stretch with scale factor 2, then a horizontal stretch with scale factor 2, then a translation $\frac{\pi}{2}$ units right and $\frac{1}{2}$ unit upwards.

12 **a**



b 20 m **c** 10 m **d** 12 seconds

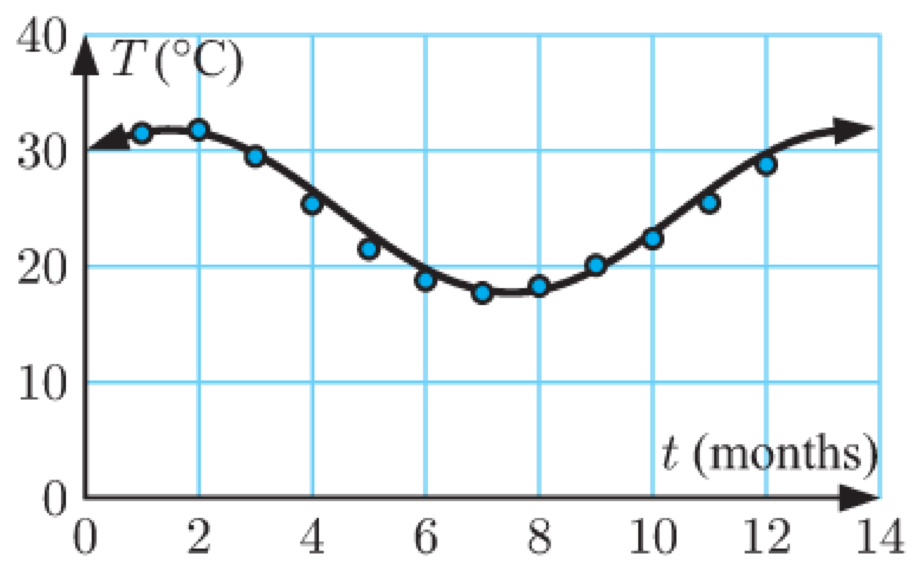
13 **a**



b $H(t) = \sin(\pi(t - 1.5)) + 1$

14 **a** $a \approx 7.05$, $b \approx \frac{\pi}{6}$, $c \approx 10.5$, $d \approx 24.75$

b



c Using technology, $T \approx 7.20\sin(0.488t - 1.08) + 24.7$. The model fits reasonably well but not perfectly.

15 **a** $x \approx -6.1, -3.4$ **b** $x \approx 0.8$

16 **a** $x \approx 1.27, 5.02$ **b** $x \approx 1.09, 2.05$

17 **a** $x \approx 1.33, 4.47, 7.61$ **b** $x \approx 5.30$

c $x \approx 2.83, 5.97, 9.11$

18 **a** $x = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}$, or $\frac{17\pi}{9}$ **b** $x = \frac{5\pi}{3}$

c $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$, or $\frac{19\pi}{12}$

19 **a** $x = 0, \pi$, or 2π **b** $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

20 **a** $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$ **b** $x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}$, or $\frac{5\pi}{6}$

c $x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$, or $\frac{2\pi}{3}$

21 **a** $\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$

22 **a** 28 milligrams per m^3 **b** 8:00 am Monday