4. [Maximum mark: 7]

Consider the functions $f(x) = -(x - h)^2 + 2k$ and $g(x) = e^{x-2} + k$ where $h, k \in \mathbb{R}$.

(a) Find $f'(x)$. [1]

The graphs of f and g have a common tangent at $x = 3$.

(b) Show that $h = \dfrac{e + 6}{2}$. [3]

(c) Hence, show that $k = e + \dfrac{e^2}{4}$. [3]

Turn over

8. [Maximum mark: 5]

Use l'Hôpital's rule to find $\lim\limits_{x \to 0}\left(\dfrac{\arctan 2x}{\tan 3x}\right)$.
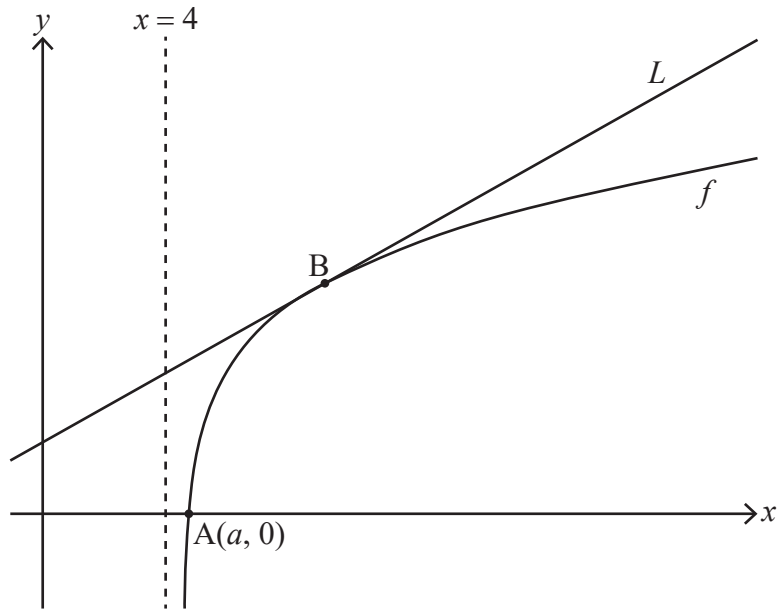
Turn over

4. [Maximum mark: 9]

Consider the function f defined by $f(x) = \ln(x^2 - 16)$ for $x > 4$.

The following diagram shows part of the graph of f which crosses the x-axis at point A, with coordinates $(a, 0)$. The line L is the tangent to the graph of f at the point B.

(a)   Find the exact value of a.   [3]

(b)   Given that the gradient of L is $\dfrac{1}{3}$, find the x-coordinate of B.   [6]

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9. [Maximum mark: 7]

By using the substitution $u = \sin x$, find $\displaystyle\int \frac{\sin x \cos x}{\sin^2 x - \sin x - 2}\, dx$.

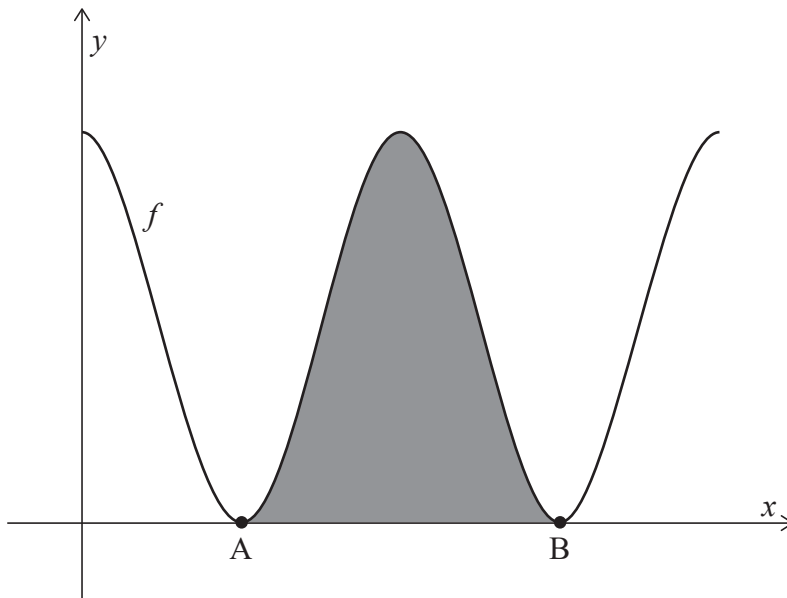**Turn over**

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10.   [Maximum mark: 15]

Consider the function f defined by $f(x) = 6 + 6\cos x$, for $0 \leq x \leq 4\pi$.

The following diagram shows the graph of $y = f(x)$.

The graph of f touches the x-axis at points A and B, as shown. The shaded region is enclosed by the graph of $y = f(x)$ and the x-axis, between the points A and B.

(a)   Find the x-coordinates of A and B.   [3]

(b)   Show that the area of the shaded region is 12π.   [5]

(This question continues on the following page)

Do **not** write solutions on this page.

(Question 10 continued)

The right cone in the following diagram has a total surface area of 12π, equal to the shaded area in the previous diagram.

The cone has a base radius of 2, height h, and slant height l.

diagram not to scale
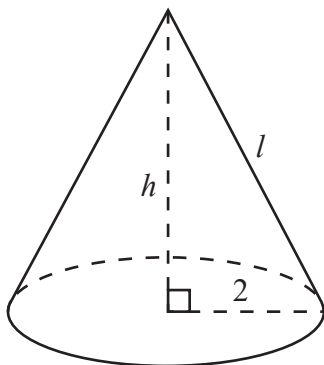
(c)   Find the value of l.   [3]

(d)   Hence, find the volume of the cone.   [4]

9.   [Maximum mark: 7]

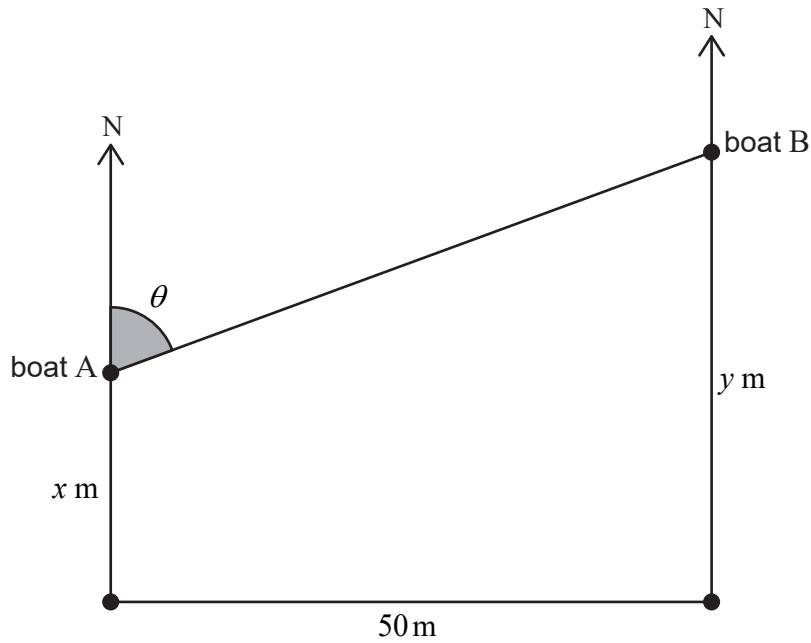Two boats A and B travel due north.

Initially, boat B is positioned 50 metres due east of boat A.

The distances travelled by boat A and boat B, after t seconds, are x metres and y metres respectively. The angle θ is the radian measure of the bearing of boat B from boat A. This information is shown on the following diagram.

diagram not to scale

(a)   Show that $y = x + 50 \cot \theta$.   [1]

At time T, the following conditions are true.

   Boat B has travelled 10 metres further than boat A.
   Boat B is travelling at double the speed of boat A.
   The rate of change of the angle θ is -0.1 radians per second.

(b)   Find the speed of boat A at time T.   [6]

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Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

Consider the function f defined by $f(x) = 90\,e^{-0.5x}$ for $x \in \mathbb{R}^+$.

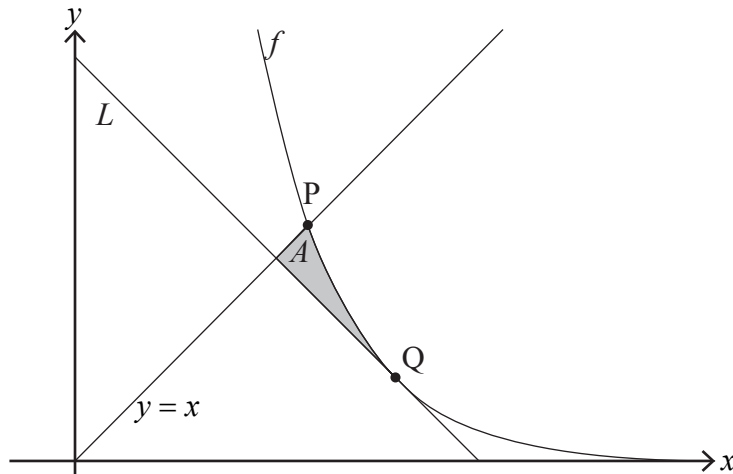The graph of f and the line $y = x$ intersect at point P.

(a) Find the x-coordinate of P. [2]

The line L has a gradient of -1 and is a tangent to the graph of f at the point Q.

(b) Find the exact coordinates of Q. [4]

(c) Show that the equation of L is $y = -x + 2\ln 45 + 2$. [2]

The shaded region A is enclosed by the graph of f and the lines $y = x$ and L.

(d) (i) Find the x-coordinate of the point where L intersects the line $y = x$.

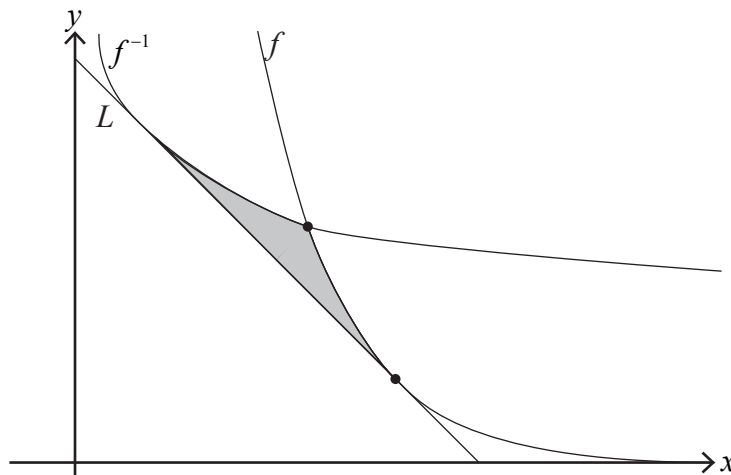(ii) Hence, find the area of A. [5]

(This question continues on the following page)

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(Question 10 continued)

The line L is tangent to the graphs of both f and the inverse function f^{-1}.

(e)   Find the shaded area enclosed by the graphs of f and f^{-1} and the line L.   [2]

11.   [Maximum mark: 20]

The function f is defined by $f(x) = \dfrac{3x+2}{4x^2-1}$, for $x \in \mathbb{R}$, $x \neq p$, $x \neq q$.

(a)   Find the value of p and the value of q.   [2]

(b)   Find an expression for $f'(x)$.   [3]

The graph of $y = f(x)$ has exactly one point of inflexion.

(c)   Find the x-coordinate of the point of inflexion.   [2]

(d)   Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$, showing the values of any axes intercepts, the coordinates of any local maxima and local minima, and giving the equations of any asymptotes.   [5]

The function g is defined by $g(x) = \dfrac{4x^2-1}{3x+2}$, for $x \in \mathbb{R}$, $x \neq -\dfrac{2}{3}$.

(e)   Find the equations of all the asymptotes on the graph of $y = g(x)$.   [4]

(f)   By considering the graph of $y = g(x) - f(x)$, or otherwise, solve $f(x) < g(x)$ for $x \in \mathbb{R}$.   [4]

Do **not** write solutions on this page.

11. [Maximum mark: 18]

A function f is defined by $f(x) = \dfrac{ke^{\frac{x}{2}}}{1+e^x}$ where $x \in \mathbb{R}$, $x \geq 0$ and $k \in \mathbb{R}^+$.

The region enclosed by the graph of $y = f(x)$, the x-axis, the y-axis and the line $x = \ln 16$ is rotated $360°$ about the x-axis to form a solid of revolution.

(a) Show that the volume of the solid formed is $\dfrac{15k^2\pi}{34}$ cubic units. [6]

Pedro wants to make a small bowl with a volume of $300\,\text{cm}^3$ based on the result from part (a). Pedro's design is shown in the following diagrams.
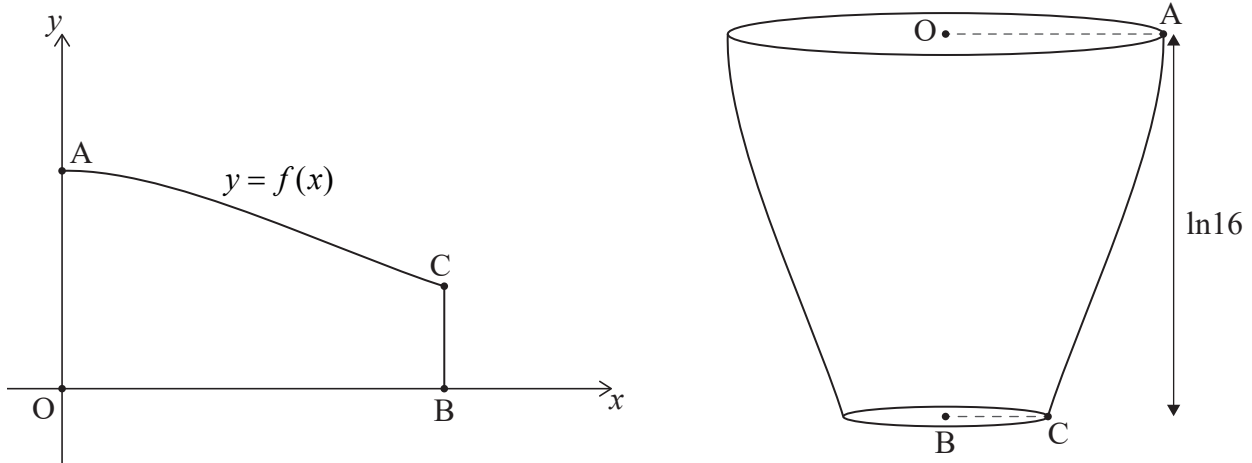
The vertical height of the bowl, BO, is measured along the x-axis. The radius of the bowl's top is OA and the radius of the bowl's base is BC. All lengths are measured in cm.

(b) Find the value of k that satisfies the requirements of Pedro's design. [2]

(c) Find

    (i) OA;

    (ii) BC. [4]

For design purposes, Pedro investigates how the cross-sectional radius of the bowl changes.

(d) (i) By sketching the graph of a suitable derivative of f, find where the cross-sectional radius of the bowl is decreasing most rapidly.

    (ii) State the cross-sectional radius of the bowl at this point. [6]

Do **not** write solutions on this page.

12. [Maximum mark: 21]

A function f is defined by $f(x) = \arcsin\left(\dfrac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$.

(a) Show that f is an even function. [1]

(b) By considering limits, show that the graph of $y = f(x)$ has a horizontal asymptote and state its equation. [2]

(c) (i) Show that $f'(x) = \dfrac{2x}{\sqrt{x^2}\left(x^2+1\right)}$ for $x \in \mathbb{R}$, $x \neq 0$.

   (ii) By using the expression for $f'(x)$ and the result $\sqrt{x^2} = |x|$, show that f is decreasing for $x < 0$. [9]

A function g is defined by $g(x) = \arcsin\left(\dfrac{x^2-1}{x^2+1}\right)$, $x \in \mathbb{R}$, $x \geq 0$.

(d) Find an expression for $g^{-1}(x)$, justifying your answer. [5]

(e) State the domain of g^{-1}. [1]

(f) Sketch the graph of $y = g^{-1}(x)$, clearly indicating any asymptotes with their equations and stating the values of any axes intercepts. [3]