9 DIFFERENTIATION

WHAT YOU NEED TO KNOW

- The derivative can be interpreted as the rate of change of one quantity as another changes, or as the gradient of a tangent to a graph.
- The notation for the derivative of $y = f(x)$ with respect to x is $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $f'(x)$.
  - Differentiating again gives the second derivative, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ or $f''(x)$, which can be interpreted as the rate of change (or gradient) of $f'(x)$.
- Differentiation from first principles involves looking at the limit of the gradient of a chord. The formula is:

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

- The basic rules of differentiation:
  - If $f(x) = x^n$ then $f'(x) = nx^{n-1}$.
  - $[kf(x)]' = kf'(x)$ for any constant k.
  - $[f(x) + g(x)]' = f'(x) + g'(x)$
- These derivatives are given in the Formula booklet:

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$\sin x$	$\cos x$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos x$	$-\sin x$	$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan x$	$\sec^2 x$	$\arctan x$	$\frac{1}{1+x^2}$
$\sec x$	$\sec x \tan x$	e^x	e^x
$\csc x$	$-\csc x \cot x$	$\ln x$	$\frac{1}{x}$
$\cot x$	$-\csc^2 x$	a^x	$a^x(\ln a)$
		$\log_a x$	$\frac{1}{x \ln a}$

- The following derivatives of common functions composed with linear expressions are very useful. (They are not in the Formula booklet but follow from those that are when the chain rule is applied.)

$f(x)$	$f'(x)$
$(ax+b)^n$	$an(ax+b)^{n-1}$
e^{ax+b}	ae^{ax+b}
$\ln(ax+b)$	$\dfrac{a}{ax+b}$
$\sin(ax+b)$	$a\cos(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$
$\tan(ax+b)$	$a\sec^2(ax+b)$

- Further rules of differentiation:
  - The chain rule is used to differentiate composite functions:

    $$y = g(u), \text{ where } u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

  - The product rule is used to differentiate two functions multiplied together:

    $$y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

  - The quotient rule is used to differentiate one function divided by another:

    $$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

- When a function is not written in the form $y = f(x)$, implicit differentiation is used. Differentiate each term separately, using the chain rule on all terms involving y:

  $$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}$$

- The equation of a tangent at the point (x_1, y_1) is given by $y - y_1 = m(x - x_1)$ where $m = f'(x_1)$. The equation of the normal at the same point is given by $y - y_1 = m(x - x_1)$ where $m = -\dfrac{1}{f'(x_1)}$.
- If a function is increasing, $f'(x) > 0$; if a function is decreasing, $f'(x) < 0$. If the graph is concave up, $f''(x) > 0$; if the graph is concave down, $f''(x) < 0$.
- Stationary points of a function are points where the gradient is zero, i.e. $f'(x) = 0$. The second derivative can be used to determine the nature of a stationary point.
  - At a local maximum, $f''(x) \le 0$.

- At a local minimum, $f''(x) \geq 0$.
- At a point of inflexion, $f''(x) = 0$ but $f'''(x) \neq 0$; there is a change in concavity of the curve.

- Optimisation problems involve setting up an expression and then finding the maximum or minimum value by differentiating or using a GDC.
  - If there is a constraint, it will be necessary to set up a second expression from this information and then substitute it into the expression to be optimised, thereby eliminating the second variable.
  - The optimal solution may occur at an end point of the domain as well as at a stationary point.
- The rate of change of a quantity often means the derivative with respect to time. Rates of change of more than two variables can be related using the chain rule:

  $$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

- Differentiate with respect to time to change an expression for displacement (s) into an expression for velocity (v) and then into one for acceleration (a):
  - $v = \dfrac{ds}{dt}$
  - $a = \dfrac{dv}{dt}$ $\left(\text{If velocity depends on displacement, } a = v\dfrac{dv}{ds}.\right)$

EXAM TIPS AND COMMON ERRORS

- Be careful when the variable is in the power. If you differentiate e^x with respect to x the answer is e^x **not** xe^{x-1}.
- Always use the product rule to differentiate a product. You cannot simply differentiate each element separately and multiply the answers together; the derivative of $x^2 \sin x$ is **not** $2x \cos x$.
- Make sure you are clear whether you have a product (such as $e^x \sin x$) or a composite function (such as $e^{\sin x}$) to differentiate. The latter is differentiated using the chain rule.
- It is sometimes easier to differentiate a quotient by turning it into a product (i.e. writing it as the numerator multiplied by the denominator raised to a negative power) and then differentiating using the product rule.
- Do not confuse the rules for differentiation and integration. Always check the sign when integrating or differentiating trigonometric functions, and carefully consider whether you should be multiplying or dividing by the coefficient of x.
- When differentiating trigonometric functions you **must** work in radians.

9.1 DIFFERENTIATION FROM FIRST PRINCIPLES

WORKED EXAMPLE 9.1

Use differentiation from first principles to prove that the derivative of x^3 is $3x^2$.

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

We start with the definition of the derivative at the point x (i.e. the formula for differentiation from first principles).

$$= \lim_{h\to 0}\left(\frac{(x+h)^3 - x^3}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{3x^2h+3xh^2+h^3}{h}\right)$$

We do not want to let the denominator tend to zero straight away, so first manipulate the numerator to get a factor of h that we can cancel with the h in the denominator.

$$= \lim_{h\to 0}(3x^2+3xh+h^2)$$

Divide top and bottom by h.

$$= 3x^2$$

Once there is no h in the denominator we can let $h \to 0$.

Practice questions 9.1

1. Prove from first principles that the derivative of x^4 is $4x^3$.

2. Prove from first principles that $\frac{\mathrm{d}}{\mathrm{d}x}(x^2-5x+2) = 2x-5$.

3. If $y = \frac{1}{x^2}$, use differentiation from first principles to show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-2}{x^3}$.

4. (a) Show that $\sqrt{a}-\sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$.

   (b) Show that $\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} = -\frac{h}{x\sqrt{x+h}+(x+h)\sqrt{x}}$.

   (c) Prove from first principles that the derivative of $\frac{1}{\sqrt{x}}$ is $-\frac{1}{2x\sqrt{x}}$.

9.2 THE PRODUCT, QUOTIENT AND CHAIN RULES

WORKED EXAMPLE 9.2

Differentiate $y = x\mathrm{e}^{\sin x}$.

Let $u = x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$

Let $v = e^{\sin x}$

This is a product so we need to use the product rule. It doesn't matter which function is $u(x)$ and which is $v(x)$.

For $\frac{\mathrm{d}v}{\mathrm{d}x}$, let $w = \sin x$. Then $v = e^w$,

$\frac{\mathrm{d}w}{\mathrm{d}x} = \cos x$ and $\frac{\mathrm{d}v}{\mathrm{d}w} = e^w$

Therefore

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}v}{\mathrm{d}w} \times \frac{\mathrm{d}w}{\mathrm{d}x}$$
$$= e^w \cos x$$
$$= e^{\sin x} \cos x$$

$v(x)$ is a composite function, so use the chain rule.

You do not have to set out your working in this much detail; from $v = \mathrm{e}^{\sin x}$ you can proceed straight to $\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{\sin x}\cos x$.

So $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} = x\cos x\, e^{\sin x} + e^{\sin x}$

Now apply the product rule.

There is no need to simplify (or factorise) your answer, unless you are asked to do so.

Practice questions 9.2

5. Differentiate $y = \mathrm{e}^{x^2} + \frac{\sin 3x}{2x}$.

6. Find the values of x for which the function $f(x) = \ln\left(\frac{2}{x^2 - 12}\right)$ has a gradient of 2.

7. Given that $f(x) = \frac{x^2 - 1}{x^2 + 2}$, find $f''(x)$ in the form $\frac{a - bx^2}{(x^2 + 2)^3}$.

8. (a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$.

(b) Given that $f(x) = \ln(\sec x + \tan x)$, show that $f'(x) = \sec x$.

(c) Hence evaluate $\int_0^{\frac{\pi}{6}} \sec x \,\mathrm{d}x$, giving your answer in the form $k\ln 3$.

9.3 TANGENTS AND NORMALS

WORKED EXAMPLE 9.3

Find the coordinates of the point where the normal to the curve $y = x^2$ at $x = a$ meets the curve again.

$\frac{dy}{dx} = 2x$

At $x = a$, $\frac{dy}{dx} = 2a$, i.e. gradient of tangent is $2a$

Therefore gradient of normal is $m = -\frac{1}{2a}$

> The normal is perpendicular to the tangent, so we need the gradient of the tangent first.

Equation of normal: $y - a^2 = -\frac{1}{2a}(x - a)$

> The normal is a straight line, so its equation is of the form $y - y_1 = m(x - x_1)$.

Intersection with $y = x^2$:

$$x^2 - a^2 = -\frac{1}{2a}(x - a)$$

> We need to find the point of intersection with $y = x^2$, so substitute $y = x^2$ into the equation of the normal.

$$\Rightarrow (x - a)(x + a) = -\frac{1}{2a}(x - a)$$

> Factorise the left-hand side (LHS) so that we have a common factor on both sides.

$$\Rightarrow (x - a)\left(x + a + \frac{1}{2a}\right) = 0$$

So $x = a$ or $x = -a - \frac{1}{2a}$ ($x = a$ was given)

> Move everything to the LHS and factorise. Do not divide by $(x - a)$ as this could result in the loss of a solution. Instead, we find all possible solutions and then reject any that are not relevant.

When $x = -a - \frac{1}{2a}$, $y = \left(-a - \frac{1}{2a}\right)^2$

> We can now find y by substituting into $y = x^2$.

So the coordinates of the point are

$$\left(-a - \frac{1}{2a}, \left(-a - \frac{1}{2a}\right)^2\right)$$

> ⚠ If asked to find the coordinates of a point, make sure you find both x- and y-coordinates.

Practice questions 9.3

9. Find the equation of the normal to the curve $y = e^{-3x^2}$ at the point where $x = 2$.

10. A tangent to the curve $y = \tan x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ is drawn at the point where $x = \frac{\pi}{4}$. Find the x-coordinate of the point where this tangent intersects the curve again.

9.4 IMPLICIT DIFFERENTIATION

WORKED EXAMPLE 9.4

Find the gradient of the curve $y^2 + \sin(xy) + x^3 = 4$ at the point (0, 2).

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(\sin(xy)) + \frac{d}{dx}(x^3) = \frac{d}{dx}(4)$$

We need to differentiate each term (on both sides) with respect to x.

$$\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$$

Use the chain rule on all terms containing y.

$$\frac{d}{dx}(\sin(xy)) = \cos(xy) \times \left(x\frac{dy}{dx} + 1 \times y\right)$$

For the $\sin(xy)$ term we will need to apply the product rule to xy (as well as the chain rule initially).

$$\therefore 2y\frac{dy}{dx} + \cos(xy)\left(x\frac{dy}{dx} + y\right) + 3x^2 = 0$$

Now put everything together.

At the point with $x = 0$ and $y = 2$:

$$4\frac{dy}{dx} + \cos 0 \times \left(0\frac{dy}{dx} + 2\right) + 3 \times 0 = 0$$

$$\Rightarrow 4\frac{dy}{dx} + 2 = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2}$$

If you are asked for the gradient at a particular point, you can substitute the given values into the differentiated equation without rearranging it.

Practice questions 9.4

11. Find the gradient of the normal to $y^2 + 3xy - 10x^2 = 0$ where $x = 1$, $y > 0$.

12. Find the coordinates of all the points on the curve $x^2 + y^2 - 3y = 10$ where $\frac{dy}{dx} = 0$.

13. Given that $\frac{x+1}{y-3} = 2xy^2$, find an expression for $\frac{dy}{dx}$.

14. The inverse function of $f(x) = \csc x$ is $f^{-1}(x) = \text{arccsc}\, x$.

(a) Given that $y = \text{arccsc}\, x$, find $\frac{dx}{dy}$ in terms of y.

(b) Hence express the derivative of $\text{arccsc}\, x$ in terms of x.

9.5 STATIONARY POINTS

WORKED EXAMPLE 9.5

Find and classify the stationary points on the curve $y = 2x^3 - 9x^2 + 12x + 5$.

$\frac{dy}{dx} = 6x^2 - 18x + 12$

For stationary points, $\frac{dy}{dx} = 0$:

$6x^2 - 18x + 12 = 0$

$\Leftrightarrow x^2 - 3x + 2 = 0$

$\Leftrightarrow (x-1)(x-2) = 0$

$\Leftrightarrow x = 1$ or $x = 2$

> At stationary points the first derivative is zero, so we need to find $\frac{dy}{dx}$ and then solve the equation $\frac{dy}{dx} = 0$.

When $x = 1$, $y = 2 - 9 + 12 + 5 = 10$

When $x = 1$, $y = 2(2)^3 - 9(2)^2 + 12(2) + 5 = 9$

So stationary points are (1, 10) and (2, 9)

> Find the y-coordinates and give the full coordinates of the stationary points.

$\frac{d^2y}{dx^2} = 12x - 18$

> We can use the second derivative to determine the nature of the stationary points.

> Make sure you differentiate the original expression for $\frac{dy}{dx}$ and not a manipulated version (such as $x^2 - 3x + 2$ here).

At $x = 1$, $\frac{d^2y}{dx^2} = 12 - 18 = -6 < 0$

$\therefore$ (1, 10) is a local maximum.

At $x = 2$, $\frac{d^2y}{dx^2} = 12(2) - 18 = 6 > 0$

$\therefore$ (2, 9) is a local minimum.

> Apply the second derivative test by substituting the x values into $\frac{d^2y}{dx^2}$.

Practice questions 9.5

15. Find and classify the stationary points on the curve $y = x^3 - 3x + 8$.

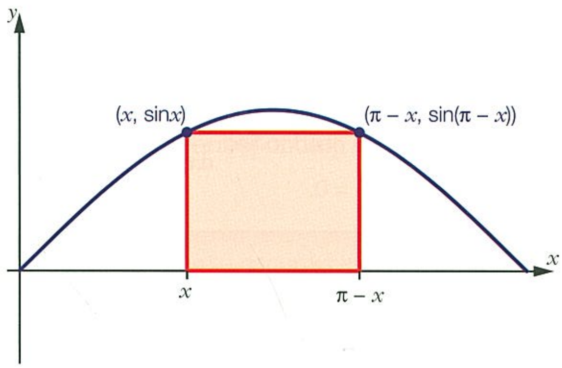
16. Find and classify the stationary points on the curve $y = x\sin x + \cos x$ for $0 < x < 2\pi$.

17. Find the maximum value of $y = \ln(x - \sin^2 x)$ for $0 < x \le 2\pi$.

9.6 OPTIMISATION WITH CONSTRAINTS

WORKED EXAMPLE 9.6

What is the area of the largest rectangle that can just fit under the curve $y = \sin x$, $0 \le x \le \pi$, if one side of the rectangle lies on the x-axis?



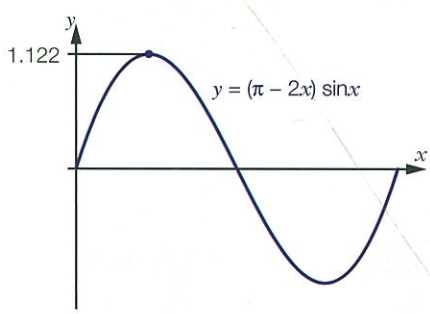
We start by defining the variables, taking the bottom left corner of the rectangle as the point (x, 0). Everything else then follows from the symmetry of the sine curve.

With harder optimisation problems a major difficulty can be defining the variables, as there may be more than one choice. In an unfamiliar situation, it is often a good idea to sketch a diagram to help.

Width of rectangle $= (\pi - x) - x = \pi - 2x$
So area $= (\pi - 2x)\sin x$

Find an expression for the quantity that needs to be optimised, in this case area.

Using a GDC for the graph of area against x:



Since this is a calculator question, we do not need to differentiate; but we do need to sketch the graph to justify that we have found a maximum (rather than any other stationary point) and that the maximum is not at an end point.

Therefore, the maximum area is 1.122 (4 SF).

Practice questions 9.6

18. An open cylindrical can has radius r and height h. The height and the radius can both change, but the volume remains fixed at $64\pi\,\text{cm}^2$. Find the minimum surface area of the can (including the base) and justify that the value you have found is a minimum.

19. Find the smallest surface area of a cone (including base) with volume $100\,\text{cm}^3$.

20. If the surface area is fixed, prove that the largest volume of a square-based cuboid is attained when it is a cube.

9.7 POINTS OF INFLEXION

WORKED EXAMPLE 9.7

The curve $y = x^3 - 6x^2 + 5x + 2$ has a point of inflexion. Find its coordinates.

$$\frac{dy}{dx} = 3x^2 - 12x + 5$$

$$\Rightarrow \frac{d^2y}{dx^2} = 6x - 12$$

At a point of inflexion, $\frac{d^2y}{dx^2} = 0$:

$6x - 12 = 0$

$\Leftrightarrow x = 2$

At a point of inflexion the second derivative is zero, so we need to find $\frac{d^2y}{dx^2}$ and solve the equation $\frac{d^2y}{dx^2} = 0$.

When $x = 2$:

$y = (2)^3 - 6(2)^2 + 5(2) + 2 = -4$

So the point of inflexion is at $(2, -4)$.

Find the y-coordinate.

If a question states that a curve has a point of inflexion and there is only one solution to $\frac{d^2y}{dx^2} = 0$, you can assume that you have found the point of inflexion; there is no need to check that $\frac{d^3y}{dx^3} \neq 0$.

Practice questions 9.7

21. Find the coordinates of the point of inflexion on the curve $y = x^3 - 12x^2 + 7$.

22. The graph of $y = 4x^3 - ax^2 + b$ has a point of inflexion at $(-1, 4)$. Find the values of a and b.

23. A curve has equation $y = (x^2 - a)e^x$.

(a) Find the range of values of a for which the curve has at least one point of inflexion.

(b) Given that one of the points of inflexion is a stationary point, find the value of a.

(c) For this value of a, sketch the graph of $y = (x^2 - a)e^x$.

9.8 INTERPRETING GRAPHS

WORKED EXAMPLE 9.8

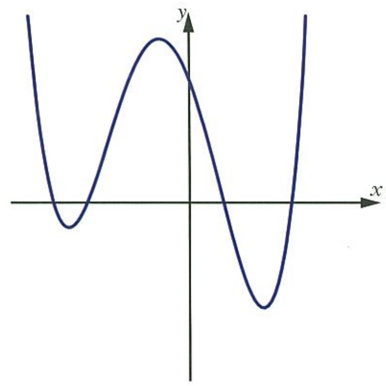
The graph shows $y = f'(x)$.

On the graph:

(a) Mark points corresponding to a local maximum of $f(x)$ with an A.

(b) Mark points corresponding to a local minimum of $f(x)$ with a B.

(c) Mark points corresponding to a point of inflexion of $f(x)$ with a C.



Local maximum points occur where the graph crosses the x-axis with a negative gradient.

A local maximum has $f'(x) = 0$ and $f''(x) < 0$, i.e. the gradient of the graph of $y = f'(x)$ is negative.

Be clear whether you are considering the graph of $y = f(x)$, $y = f'(x)$ or $y = f''(x)$.

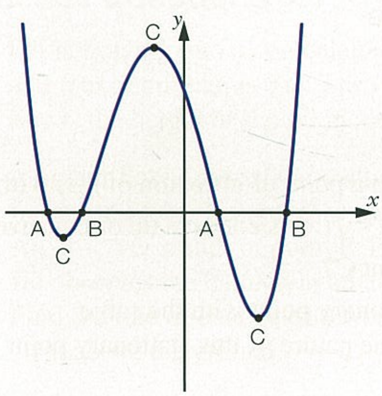
Local minimum points occur where the graph crosses the x-axis with a positive gradient.

A local minimum has $f'(x) = 0$ and $f''(x) > 0$, i.e. the gradient of the graph of $y = f'(x)$ is positive.

Points of inflexion occur at stationary points on the graph.

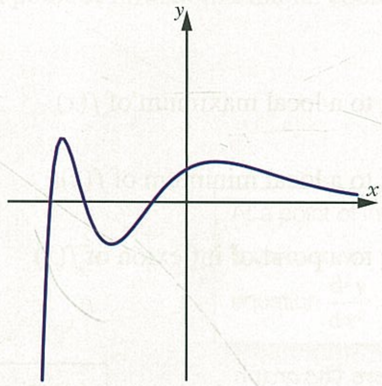
At a point of inflexion $f''(x) = 0$, i.e. the gradient of the graph of $y = f'(x)$ is zero (and the gradient is either positive on both sides of that point or negative on both sides).

Therefore:



Practice questions 9.8

24. The graph shows $y = f'(x)$.



On the graph:

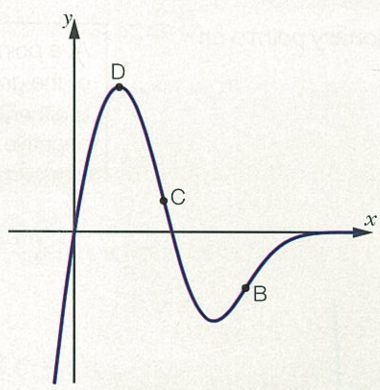
(a) Mark points corresponding to a local maximum of $f(x)$ with an A.

(b) Mark points corresponding to a local minimum of $f(x)$ with a B.

(c) Mark points corresponding to a point of inflexion of $f(x)$ with a C.

(d) Mark points corresponding to a zero of $f''(x)$ with a D.

25. The graph shows $y = f''(x)$.



(a) On the graph, mark points corresponding to a point of inflexion of $f(x)$ with an A.

(b) State whether at the point B, the graph of $y = f(x)$ is concave up or concave down.

(c) Is $f'(x)$ increasing or decreasing at the point C?

(d) Given that the graph of $y = f(x)$ has a stationary point with the same x-coordinate as the point marked D, state the nature of this stationary point and justify your answer.

9.9 RELATED RATES OF CHANGE

WORKED EXAMPLE 9.9

The volume of a spherical snowball is increasing at a constant rate of $12\,\text{cm}^3\,\text{s}^{-1}$. At what rate is the radius increasing when the radius is 4 cm?

Let V = volume in cm^3, r = radius in cm and t = time in seconds.

$\frac{dV}{dt} = 12$ and $\frac{dr}{dt}$ is needed.

> We start by defining the variables and writing down the given rate of change and the required rate of change.

By the chain rule:

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

> Set the required rate of change as the subject of the chain rule.
>
> $\frac{dV}{dt}$ is known but $\frac{dr}{dV}$ is not.

Since the snowball is spherical,

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$$

$$\therefore \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

> Use the geometric context to establish a link that enables us to find the unknown derivative.

$$\therefore \frac{dr}{dt} = \frac{1}{4\pi r^2} \times 12 = \frac{3}{\pi r^2}$$

> Substitute into the chain rule.

So, when $r = 4$:

$$\frac{dr}{dt} = \frac{3}{16\pi} = 0.0597 \text{ (3 SF)}$$

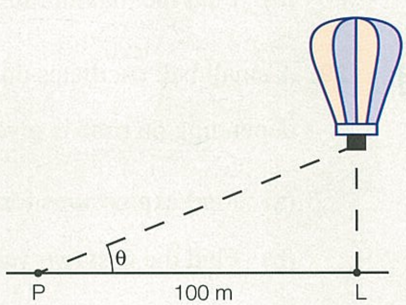
The radius is increasing at $0.0597\,\text{cm}\,\text{s}^{-1}$.

> Evaluate when the radius is 4 cm.

Practice questions 9.9

26. Oil is flowing into a conical container with vertex on the bottom which has height 10 m and maximum radius 3 m. The oil is flowing into the cone at a rate of $2\,\text{m}^3\,\text{s}^{-1}$. At what rate is the height of the oil increasing when the petrol tank is filled to half of its capacity?

27. A hot air balloon rises vertically upwards from point L with a constant speed of $6\,\text{m}\,\text{s}^{-1}$. An observer stands at point P, 100 m from L. He observes the balloon at an angle of elevation θ. Find the rate of change of θ when the balloon is 50 m above the ground.



9.10 KINEMATICS

WORKED EXAMPLE 9.10

If the displacement, s, at time t is given by $s = 4t^2 - 3e^{-3t}$, find the time when the minimum velocity occurs in the form $\ln k$ where k is a rational number.

$v = \frac{ds}{dt} = 8t + 9e^{-3t}$

We first need to find an expression for the velocity.

$\frac{dv}{dt} = 8 - 27e^{-3t}$

At a local minimum, $\frac{dv}{dt} = 0$:

If there is a local minimum, it will occur when $\frac{dv}{dt} = 0$.

$8 - 27e^{-3t} = 0 \Rightarrow e^{3t} = \frac{27}{8}$

$\Rightarrow t = \frac{1}{3}\ln\left(\frac{27}{8}\right) = \ln\left(\frac{3}{2}\right)$

Use the laws of logarithms to put the result into the required form.

Laws of logarithms are covered in Chapter 2.

$\frac{d^2v}{dt^2} = 81e^{-3t} > 0$ for all t

Therefore $t = \ln\left(\frac{3}{2}\right)$ is a minimum.

We now need to check that this is a local minimum (rather than a maximum or point of inflexion). If it is not a minimum, the minimum velocity would occur at the end point.

Practice questions 9.10

28. A hiker has a displacement s km, at a time t hours, modelled by $s = t^3 - 4t$, $t \geq 0$.

(a) Find the time it takes for the hiker to return to his original position (where he stops).

(b) Find the maximum displacement from the starting point.

(c) Find the maximum speed of the hiker.

29. A small ball oscillates on a spring so that its displacement from the starting position depending on time is given by $s = \frac{2}{3}\sin\left(\frac{3\pi t}{2}\right)$.

(a) Find expressions for the velocity and acceleration of the ball at time t.

(b) Find the first two values of t for which the speed of the ball equals $\frac{\pi}{2}$.

Mixed practice 9

1. The radius of a circle is increasing at the constant rate of $3\,\text{cm}\,\text{s}^{-1}$. Find the rate of increase of the area when the radius is 20 cm.

2. If $h(x) = f(x) + g(x)$, prove from first principles that $h'(x) = f'(x) + g'(x)$.

3. A rectangle has perimeter 40 cm. One side of the rectangle has length x cm.

(a) Find an expression for the area of the rectangle in terms of x.

(b) Prove that the rectangle with the largest area is a square.

4. Find the equation of the normal to the curve $y = 5\sin 3x + x^2$ when $x = \pi$, giving your answer in the form $y = mx + c$.

5. A particle is moving with displacement s at time t.

(a) A model of the form $s = at^2 + bt$ is applied. Show that the particle moves with constant acceleration.

It is known that when $t = 1$, $v = 1$, and when $t = 2$, $v = 5$.

(b) Find a and b.

6. Show that the curve $x^3 + y^3 = 3$ is always decreasing.

7. (a) State the Fundamental Theorem of Algebra.

(b) Explain why a polynomial of degree n can have at most $n - 1$ stationary points.

(c) The cubic graph $y = ax^3 + bx^2 + cx + d$ has one stationary point. Show that $b^2 - 3ac = 0$.

8. Consider $f(x) = x^4 - x$.

(a) Find the zeros of $f(x)$.

(b) Find the region in which $y = f(x)$ is decreasing.

(c) Solve the equation $f''(x) = 0$.

(d) Find the region in which $y = f(x)$ is concave up.

(e) Hence explain why $y = f(x)$ has no points of inflexion.

(f) Sketch the curve $y = f(x)$.

9. The point P lies on the curve $y = \frac{1}{x}$ with $x = p$, $p > 0$.

(a) Show that the equation of the tangent to the curve at P is $p^2y + x = 2p$.

The tangent to the curve at P meets the y-axis at Q and meets the x-axis at R.

(b) Show that the area of the triangle OQR (where O is the origin) is independent of p.

(c) Show that the distance QR is given by $2\sqrt{p^2 + p^{-2}}$.

(d) Find the value of p that minimises the distance QR.

Going for the top 9

1. Find the stationary points on the curve $y^2 + 4xy - x^2 = 20$.

2. If $x^2 + y^2 = 9$, show that $\frac{d^2y}{dx^2} = -\frac{9}{y^3}$ (for $y \neq 0$).

3. Find the point(s) on the curve $y = x^2$ closest to (0, 9).

4. Prove that all cubic curves have a point of inflexion.

5. An object has speed v at a displacement s, linked by $v = s^2 + s$.
Find an expression for the acceleration in terms of the displacement.

6. An isosceles triangle has two equal sides of length l with angle θ between them. The length of the equal sides is increasing at a rate of $0.4\,\mathrm{m\,s^{-1}}$, and the angle is decreasing at a rate of 0.01 radians per second. Find the rate of change of the area of the triangle when $l = 4$ and $\theta = \frac{\pi}{4}$.

7. By differentiating $x \times \frac{1}{x}$ using the product rule, prove that $\frac{d}{dx}(x^{-1}) = -x^{-2}$.

8. The point P lies on the curve $y = e^x$ with $x = p$, $p > 0$. O is the origin.

(a) Find the angle that [OP] makes with the positive x-axis.

(b) Find the equation of the tangent to the curve at P.

(c) The tangent intersects the x-axis at Q. Find the coordinates of Q.

(d) Find the angle that [QP] makes with the positive x-axis.

(e) Sketch the graph of angle $O\hat{P}Q$ against p, stating the intersections with any axes and the equations of any asymptotes.

(f) Find the maximum value of the angle $O\hat{P}Q$ when $p > 1$.

(g) The equation $e^x = kx$ has one solution with $k > 0$. Find the exact value of k.

10 INTEGRATION

WHAT YOU NEED TO KNOW

- Integration is the reverse process of differentiation.
- The basic rules of integration:
  - For all rational $n \neq -1$, $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$, where C is the constant of integration.
  - $\int kf(x) \, dx = k\int f(x) \, dx$ for any constant k.
  - $\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx$
- Definite integration deals with integration between two points.
  - If $\int f(x)dx = F(x)$ then $\int_a^b f(x) \, dx = F(b) - F(a)$, where a and b are the limits of integration.
  - $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$
  - $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$
- These integrals are given in the Formula booklet:

Function	Integral
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
e^x	$e^x + C$
a^x	$\frac{1}{\ln a}a^x + C$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\arcsin\left(\frac{x}{a}\right) + C, \ \lvert x\rvert < a$

- The following integrals of common functions composed with linear expressions are very useful. (They are not in the Formula booklet but can be obtained from those that are by reversing the chain rule.)

Function	Integral
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert+C$
$\sin kx$	$-\frac{1}{k}\cos kx+C$
$\cos kx$	$\frac{1}{k}\sin kx+C$
$\sec^2 kx$	$\frac{1}{k}\tan kx+C$
e^{kx}	$\frac{1}{k}e^{kx}+C$

- Further integration methods:
  - Integration by substitution can be useful in a variety of circumstances. One particularly common case is where there is a composite function and (a multiple of) *the derivative of the inner part* of that function. (This can also be achieved by simply reversing the chain rule.)
    - Two useful substitutions are: (1) if a function involves $a^2 - x^2$, try $x = a\sin u$; (2) if a function involves $a^2 + x^2$, try $x = a\tan u$.
  - Integration by parts can be used on some integrals involving the product of functions: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
  - Some integrals may require the use of trigonometric identities. Common examples are:
    - $\int \sin^2 x \, dx = \int \frac{1}{2}(1-\cos 2x)dx$
    - $\int \cos^2 x \, dx = \int \frac{1}{2}(1+\cos 2x)dx$
    - $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx$
- Integration can be used to calculate areas:
  - The area between a curve and the x-axis from $x = a$ to $x = b$ is given by $\int_a^b y \, dx$.
  - The area between a curve and the y-axis from $y = a$ to $y = b$ is given by $\int_a^b x \, dy$.
  - The area between two curves is given by $\int_a^b f(x) - g(x)dx$, where $f(x) > g(x)$ and a and b are the intersection points.

- The volume of revolution is given by:
  - $V = \int_a^b \pi y^2 \, dx$ for rotation around the x-axis from $x = a$ to $x = b$
  - $V = \int_a^b \pi x^2 \, dy$ for rotation around the y-axis from $y = a$ to $y = b$
  - $V = \int_a^b \pi\left(f(x)^2 - g(x)^2\right) dx$, where $f(x) > g(x)$ and a and b are the intersection points, for rotation of the region between two curves around the x-axis.
- Integrate with respect to time to change an expression for acceleration (a) into an expression for velocity (v) and then into one for displacement (s):
  - $v = \int a \, dt$
  - $s = \int v \, dt$
    - The displacement between times a and b is $\int_a^b v \, dt$.
    - The distance travelled between times a and b is $\int_a^b |v| \, dt$.

EXAM TIPS AND COMMON ERRORS

- Don't forget the '+ C' for indefinite integration – it is part of the answer and you must write it every time. However, it can be ignored for definite integration.
- Make sure you know how to use your calculator to evaluate definite integrals. You can also check your answer on the calculator when you are asked to find the exact value of the integral.
- Always look out for integrals which are in the Formula booklet. Don't forget that $\frac{1}{\cos^2 x}$ is $\sec^2 x$.
- You cannot integrate products or quotients by integrating each part separately.
- When integrating fractions, always check whether the numerator is the derivative of the denominator. If this is the case, the answer is the natural logarithm of the denominator: $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$.
- You may have to simplify a fraction before integrating. This can be achieved by splitting into separate fractions or by polynomial division (or equivalent method).
- If you have a product of functions to integrate, try substitution unless you recognise the situation as a typical 'integration by parts' integral (where the two functions are often of different 'types', such as $\int x \sin x \, dx$, $\int x^2 e^x \, dx$). One important example of integration by parts that you should remember is $\int \ln x \, dx$, which is split into the product $\int 1 \times \ln x \, dx$.

10.1 INTEGRATING EXPRESSIONS

WORKED EXAMPLE 10.1

Find $\int x(e^x + e^{x^2})dx$.

$$\int x(e^x + e^{x^2})dx = \int xe^x dx + \int xe^{x^2} dx$$

We can split up the integral of a sum.

For $\int xe^x dx$:

$u = x$ and $\frac{dv}{dx} = e^x$

$\Rightarrow \frac{du}{dx} = 1$ and $v = e^x$

This is a product of two functions and is therefore a good candidate for integration by parts. Choose the function that is simpler to integrate as $\frac{dv}{dx}$ (here this should be e^x as its integral is the same as itself).

$$\therefore \int xe^x dx = xe^x - \int 1 \times e^x dx$$
$$= xe^x - e^x + c$$

Apply the integration by parts formula

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

For $\int xe^{x^2} dx$: Let $u = x^2$; then

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\int xe^{x^2} dx = \int xe^u \frac{du}{2x}$$
$$= \frac{1}{2}\int e^u du$$
$$= \frac{1}{2}e^u + c'$$
$$= \frac{1}{2}e^{x^2} + c'$$

This product has one part (x) that is a multiple of the derivative of the 'inner function' x^2 of the other part, which makes it ideal for substitution. Make the substitution u = 'inner function'.

When a product consists of a composite function and (a multiple of) the derivative of the 'inner part' of that function, always use substitution rather than integration by parts. So always check for this situation before trying integration by parts.

Therefore

$$\int x(e^x + e^{x^2})dx = xe^x - e^x + \frac{1}{2}e^{x^2} + C$$

Put the two integrals together.

Practice questions 10.1

1. Find $\int x\sin(x^2) + x^2 \sin x \, dx$.

2. Find $\int 1 + x\ln x \, dx$.

3. Find $\int \frac{e^x}{1 - 3e^x} dx$.

4. Find $\int \csc x \cot x \, dx$.

5. Find $\int \frac{7^x}{3^{2x}} dx$.

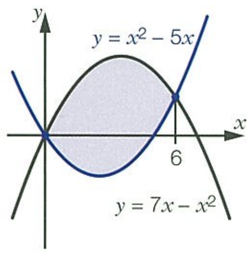
6. Find $\int \frac{\sqrt{x} + 1}{x} dx$.

10.2 FINDING AREAS

WORKED EXAMPLE 10.2

Find the area enclosed between the curves $y = x^2 - 5x$ and $y = 7x - x^2$.

Using GDC:



Sketch the graph on the GDC and use it to find the intersection points.

Intersections: $x = 0$ and $x = 6$

So area $= \int_0^6 (7x - x^2) - (x^2 - 5x)\,dx$

$= \int_0^6 12x - 2x^2\,dx$

$= \left[6x^2 - \frac{2}{3}x^3\right]_0^6 = 72$

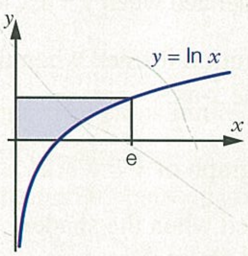
Write down the integral representing the area and carry out the definite integration.

The integral can be checked using your calculator.

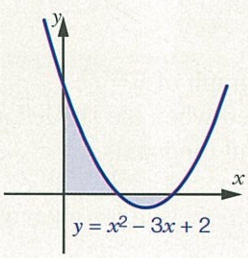
Practice questions 10.2

7. Find the area enclosed by the curves $y = \sin x$ and $y = \cos x$ between $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

8. The diagram shows the graph of $y = \ln x$. Find the shaded area.



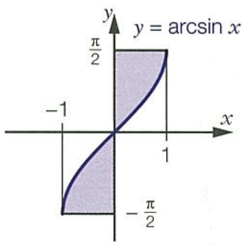
9. The diagram shows the graph of $y = x^2 - 3x + 2$. Find the shaded area.



10.3 VOLUMES OF REVOLUTION

WORKED EXAMPLE 10.3

Find the volume of revolution when the curve $y = \arcsin x$ is rotated 360° about the y-axis.



Sketch the graph to find the end points; since the volume of revolution is about the y-axis, y limits will be required.

When $x = -1, y = -\frac{\pi}{2}$. When $x = 1, y = \frac{\pi}{2}$.

$$\text{Volume} = \int_{-\pi/2}^{\pi/2} \pi x^2 \, dy$$

$$= \int_{-\pi/2}^{\pi/2} \pi (\sin y)^2 \, dy$$

$$= \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} 1 - \cos 2y \, dy$$

$$= \frac{\pi}{2} \left[y - \frac{1}{2}\sin 2y \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi^2}{2}$$

Use the formula for the volume of revolution for rotation around the y-axis. Rearrange the equation to get x in terms of y: $y = \arcsin x \Rightarrow x = \sin y$

Use the cosine double angle identity for the integration:

$\cos 2y = 1 - 2\sin^2 y \Rightarrow \sin^2 y = \frac{1}{2}(1 - \cos 2y)$

Trigonometric identities are covered in Chapter 6.

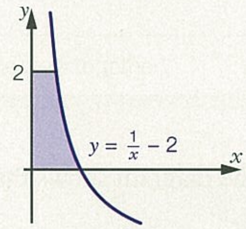
Practice questions 10.3

10. Find the exact volume generated when $y = e^{3x}$, for $1 < x < 3$, is rotated 360° around the x-axis.

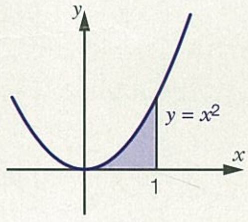
11. Find the volume of revolution formed when the curve $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated 180° around the x-axis.

12. The diagram shows the graph of $y = \frac{1}{x} - 2$.

Find the volume generated when the shaded area is rotated 2π radians about the y-axis.



13. The diagram shows the graph of $y = x^2$.
Find the volume generated when the shaded area is rotated 2π radians about the y-axis.



10.4 KINEMATICS

WORKED EXAMPLE 10.4

The velocity, $v\,\text{m}\,\text{s}^{-1}$, of a ball is given by $v = t^2 - 4t$, where t is measured in seconds. Initially the displacement is zero.

(a) Find the displacement when $t = 10$.

(b) Find the distance travelled in the first 10 seconds.

(a) $s = \int_0^{10} v\,\text{d}t$

$= \int_0^{10} t^2 - 4t\,\text{d}t$

$= \frac{400}{3}\text{m}$ (from GDC)

'Initially' means when $t = 0$, so we integrate between the limits $t = 0$ and $t = 10$.

(b) Distance $= \int_0^{10} |v|\,\text{d}t$

$= \int_0^{10} |t^2 - 4t|\,\text{d}t$

$= \frac{464}{3}\text{m}$ (from GDC)

To find the distance travelled, we need to take the modulus before integrating, as the velocity may be negative for part of the journey.

To answer a question of this type without a GDC, first sketch the velocity–time graph and then separate out the parts above the axis and below the axis.

Practice questions 10.4

14. The acceleration of a car for $0 \le t \le 5$ is modelled by $a = 5(1 - e^{-2t})$, where a is measured in $\text{m}\,\text{s}^{-2}$ and t is measured in seconds. The car is initially at rest.

(a) Find the velocity of the car after t seconds.

(b) Find the displacement from the initial position after t seconds.

(c) Find the maximum velocity of the car.

15. The velocity of a wave is modelled by $v = \cos 5t$. When $t = 0$, $s = 0$. Show that the acceleration a and displacement s are related by $a = ks$, where k is a constant to be determined.

16. A bird has acceleration modelled by $2e^{-t}\,\text{m}\,\text{s}^{-2}$ due north, where t is the time in seconds. The bird is initially travelling with speed $8\,\text{m}\,\text{s}^{-1}$ north and is $100\,\text{m}$ south of a tree.

(a) Find an expression for the velocity of the bird at time t.

(b) According to the model, when does the bird reach the tree?

Mixed practice 10

1. Find: (a) $\int \sqrt{e^x}\, dx$ (b) $\int_0^{\ln 2} \frac{e^x}{\sqrt{e^x+1}}\, dx$

2. Find the exact value of $\int_0^{\pi} \cos^2 5x\, dx$.

3. Use a substitution to find $\int x\sqrt{4-x}\, dx$.

4. (a) Show that $\int \tan x\, dx = \ln|\sec x| + c$.
   (b) Find the following integrals:
   (i) $\int \tan^2 x\, dx$ (ii) $\int \sec x \tan x\, dx$ (iii) $\int \sec^2 x \tan x\, dx$

5. (a) Write $x^2 - 4x + 5$ in the form $(x-p)^2 + q$.
   The velocity of a ball is given by $v = \frac{1}{t^2 - 4t + 5}$, where v is measured in m s^{-1} and t is measured in seconds. The ball is initially 5 m away from a flag.
   (b) Find the displacement of the ball from the flag after t seconds.
   (c) Find the acceleration of the ball after t seconds.
   (d) Find the maximum velocity of the ball.

6. Find the area enclosed by the curves $y = \frac{1}{1+x^2}$ and $y = \frac{1}{2}x^2$.

7. Evaluate $\int_0^1 e^{\sin x}\, dx$.

8. A ball's velocity, $v\ \text{m s}^{-1}$, after time t seconds is given by $v = t\sin t$.
   (a) Find the displacement of the ball from the initial position when $t = \frac{3\pi}{2}$ seconds.
   (b) After how long has the ball reached its maximum displacement in the first $\frac{3\pi}{2}$ seconds?
   (c) Find the distance travelled by the ball in the first $\frac{3\pi}{2}$ seconds.

9. Find a if $\int_0^a \sin 2x\, dx = \frac{3}{4}$, $0 < a \le \pi$.

10. (a) Show that $\int 2^x \, dx = \frac{1}{\ln 2} 2^x + c$.

(b) Find $\frac{d}{dx}(x \log_2 x)$.

(c) Find $\int \log_2 x \, dx$.

11. The region between the curves $y = x + \frac{2}{x}$ and $y = 5 - x$ is labelled R.

(a) Find the exact area of R.

(b) Find the exact volume generated when R is rotated a full turn around the x-axis.

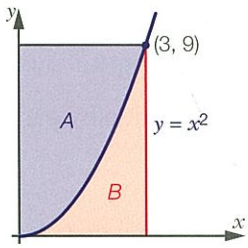
12. Find the exact value of $\int_0^{\pi/4} x^2 \cos 2x \, dx$.

13. (a) Find constants A, B and C such that $\frac{x^2+3}{x+2} = Ax + B + \frac{C}{x+2}$.

(b) Hence find $\int \frac{x^2+3}{x+2} \, dx$.

14. (a) Find the volume generated when the region marked A is rotated 2π radians around the y-axis.

(b) Find the volume generated when the region marked B is rotated 2π radians around the y-axis.



15. Consider the functions $f(x) = \sin x$ and $g(x) = 2\sin^2 x$ over the domain $0 \le x \le \frac{\pi}{2}$.

(a) Show that $\int 2\sin^2 x \, dx = x - \sin x \cos x + c$.

(b) Find the exact coordinates of the points of intersection of $f(x)$ and $g(x)$.

The region enclosed between the two curves is labelled R.

(c) Find the exact area of R.

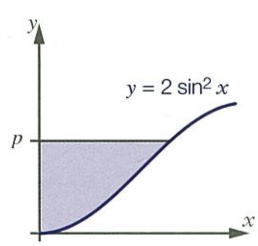
(d) Find the volume of the solid generated when R is rotated 360° around the x-axis.

(e) Use integration by parts to find $\int \arcsin x \, dx$.

(f) Find the area enclosed by $g(x)$, the y-axis and the line $y = 1$.

(g) Find the shaded area in terms of p.

(h) Hence find $\int \arcsin \sqrt{x} \, dx$ for $0 < x < 1$.



Going for the top 10

1. Find $\int \sqrt{1-x^2}\,dx$.

2. Let $S = \int \frac{\sin x}{\cos x + \sin x}\,dx$ and $C = \int \frac{\cos x}{\cos x + \sin x}\,dx$.

(a) Find an expression for $C - S$.

(b) Hence find $\int \frac{\sin x}{\cos x + \sin x}\,dx$.

3. Find $\int \cos^3 x\,dx$.

4. Consider the region bounded by the curve $y = \sin x$ and the line $y = \frac{1}{2}$ between $x = \frac{\pi}{3}$ and $x = \frac{2\pi}{3}$. Find the volume generated when the region is rotated a full turn around:

(a) the x-axis

(b) the line $y = \frac{1}{2}$.

5. Use integration by parts to find $\int e^x \sin x\,dx$.

6. (a) (i) Show that $\sqrt{\frac{1-3x}{1+3x}} = \frac{1-3x}{\sqrt{1-9x^2}}$.

(ii) Hence find $\int \sqrt{\frac{1-3x}{1+3x}}\,dx$.

(b) (i) Use integration by parts to find $\int \sec^3 x\,dx$.

(ii) Hence find $\int_0^1 \sqrt{x^2+3}\,dx$, giving your answer in the form $a + b\ln 3$ where a and b are rational numbers.