Equating imaginary parts:

$2ab + b = 0$

$\Leftrightarrow b(2a+1) = 0$

$\Leftrightarrow b = 0 \text{ or } a = -\frac{1}{2}$

i.e. $\text{Im}(z) = 0$ (so that z is real) or $\text{Re}(z) = -\frac{1}{2}$

(b) If z is not real (i.e. $b \neq 0$), then $a = -\frac{1}{2}$.

Equating real parts in (*):

$a^2 - b^2 + a = k$

$\left(-\frac{1}{2}\right)^2 - b^2 - \frac{1}{2} = k$

$\therefore b^2 = \frac{1}{4} - k$

Since z is not real, there must be (non-zero real) solutions to this equation for b.

Therefore, $\frac{1}{4} - k > 0$, i.e. $k < \frac{1}{4}$.

5. $|i| = 1$ and $\arg i = \frac{\pi}{2}$, so $i = e^{i\frac{\pi}{2}}$.

Therefore $i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{-\frac{\pi}{2}}$, which is real.

9 DIFFERENTIATION

Mixed practice 9

1. $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

It is given that $\frac{dr}{dt} = 3$

$$\therefore \frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 2\pi r \times 3 = 6\pi r \text{ cm}^2\text{ s}^{-1}$$

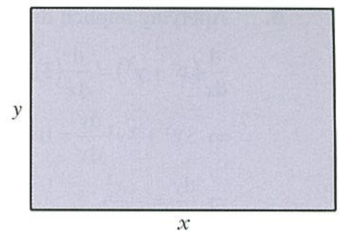
When $r = 20$: $\frac{dA}{dt} = 6\pi \times 20 = 120\pi \text{ cm}^2\text{ s}^{-1}$

2.
$$h'(x) = \lim_{t\to 0} \frac{h(x+t) - h(x)}{t}$$
$$= \lim_{t\to 0} \frac{[f(x+t) + g(x+t)] - [f(x) + g(x)]}{t}$$
$$= \lim_{t\to 0} \frac{[f(x+t) - f(x)] + [g(x+t) - g(x)]}{t}$$
$$= \lim_{t\to 0} \frac{f(x+t) - f(x)}{t} + \lim_{t\to 0} \frac{g(x+t) - g(x)}{t}$$
$$= f'(x) + g'(x)$$

3. (a) Perimeter $= 2x + 2y$

$40 = 2x + 2y$

$\therefore y = 20 - x$



Therefore,

Area $= x(20 - x)$

$= 20x - x^2$

(b) For maximum area, $\frac{dA}{dx} = 0$:

$A = 20x - x^2 \Rightarrow \frac{dA}{dx} = 20 - 2x$

$20 - 2x = 0 \Rightarrow x = 10$

$\frac{d^2A}{dx^2} = -2 < 0 \quad \therefore$ maximum

When $x = 10$, $y = 20 - x = 20 - 10 = 10$, i.e. a square.

4.
$$\frac{dy}{dx} = 5(\cos 3x)3 + 2x = 15\cos 3x + 2x$$

When $x = \pi$, $y = 5\sin 3\pi + \pi^2 = \pi^2$ and

$\frac{dy}{dx} = 15\cos 3\pi + 2\pi = -15 + 2\pi$

Gradient of normal is $m = \frac{-1}{-15 + 2\pi} = \frac{1}{15 - 2\pi}$

So equation of the normal is:

$y - y_1 = m(x - x_1)$

$y - \pi^2 = \frac{1}{15 - 2\pi}(x - \pi)$

$\therefore y = \frac{1}{15 - 2\pi}x + \pi^2 - \frac{\pi}{15 - 2\pi}$

5. (a) $s = at^2 + bt$

So $v = \frac{ds}{dt} = 2at + b \quad \cdots (*)$

and acceleration $= \frac{dv}{dt} = 2a$, a constant

(b) Substituting the given information into (*):

$t = 1, v = 1 \Rightarrow 1 = 2a + b \quad \cdots(1)$

$t = 2, v = 5 \Rightarrow 5 = 4a + b \quad \cdots(2)$

(2) – (1) gives $4 = 2a$

So $a = 2$

and then, from (1), $b = 1 - 2a = 1 - 4 = -3$

6. Applying implicit differentiation to $x^3 + y^3 = 3$:

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(3)$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2}$$

$$= -\left(\frac{x}{y}\right)^2 < 0 \text{ for all } x \text{ and } y$$

So the curve is always decreasing.

7. (a) Every non-constant polynomial has at least one (possibly complex) root; or equivalently, every polynomial of degree n has n roots (some of which may be repeated).

(b) Stationary points occur where the derivative is zero. Differentiating a polynomial of degree n gives a polynomial of degree $n - 1$, which has at most $n - 1$ distinct roots by the Fundamental Theorem of Algebra. Hence there are at most $n - 1$ distinct points where the derivative is zero, or at most $n - 1$ stationary points.

(c) $\frac{dy}{dx} = 3ax^2 + 2bx + c$

If $3ax^2 + 2bx + c = 0$ has only one root, then the discriminant is zero:

$$\Delta = (2b)^2 - 4(3a)c$$

$$0 = 4b^2 - 12ac$$

$$\therefore b^2 - 3ac = 0$$

8. (a) The zeros are where $f(x) = 0$:

$$x^4 - x = 0$$

$$\Leftrightarrow x(x^3 - 1) = 0$$

$$\Leftrightarrow x = 0 \text{ or } x^3 - 1 = 0$$

$$\Leftrightarrow x = 0,\ 1$$

(b) $f(x)$ is decreasing where $f'(x) < 0$:

$$f'(x) < 0$$

$$\Leftrightarrow 4x^3 - 1 < 0$$

$$\Leftrightarrow x < \frac{1}{\sqrt[3]{4}}$$

(c) $f''(x) = 12x^2$

$$f''(x) = 0$$

$$\Leftrightarrow 12x^2 = 0$$

$$\Leftrightarrow x = 0$$

(d) $f(x)$ is concave up where $f''(x) > 0$:

$$f''(x) > 0$$

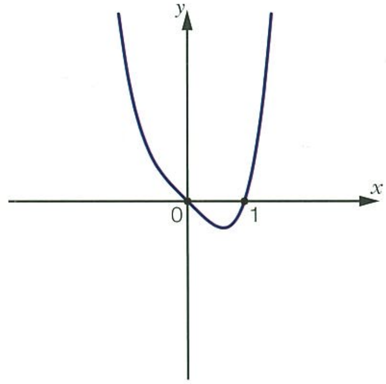
$$\Leftrightarrow 12x^2 > 0$$

$$\Leftrightarrow x^2 > 0$$

$$\Leftrightarrow x \in]-\infty, 0[\cup]0, \infty[$$

(e) At a point of inflexion, $f''(x) = 0$, so from (c) the only possibility is $x = 0$. However, there must also be a change in concavity from one side of the point to the other; and since, by (d), $f(x)$ is concave up on both sides of $x = 0$, this cannot be a point of inflexion.

(f)



9. (a) $y = x^{-1} \Rightarrow \frac{dy}{dx} = -x^{-2}$

So, at $x = p$, $\frac{dy}{dx} = -p^{-2} = -\frac{1}{p^2}$ and $y = \frac{1}{p}$

Therefore, the equation of the tangent is:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{p} = -\frac{1}{p^2}(x - p)$$

$$p^2y - p = -(x - p)$$

$$p^2y + x = 2p$$

(b) At Q, $x = 0$:

$$p^2y + 0 = 2p$$

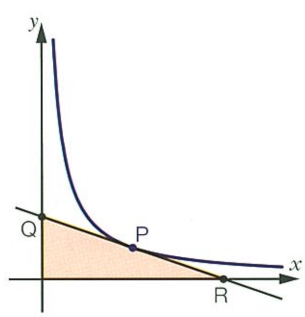
$$\Rightarrow y = \frac{2}{p}$$

At R, $y = 0$

$$p^2 \times 0 + x = 2p$$

$$\Rightarrow x = 2p$$

So the area of triangle OQR is



$$A = \frac{1}{2}bh = \frac{1}{2}(2p)\left(\frac{2}{p}\right) = 2$$, which is independent of p.

(c) Since $Q = \left(0, \frac{2}{p}\right)$ and $R = (2p, 0)$,

$$QR = \sqrt{(2p-0)^2 + \left(0 - \frac{2}{p}\right)^2}$$

$$= \sqrt{4p^2 + \frac{4}{p^2}}$$

$$= 2\sqrt{p^2 + p^{-2}}$$

(d) The value of p that minimises QR is the same as the value which minimises QR^2. For a minimum, $\frac{d}{dp}(QR^2) = 0$:

$$\frac{d}{dp}(4p^2 + 4p^{-2}) = 0$$
$$\Rightarrow 8p - 8p^{-3} = 0$$
$$\Rightarrow p^4 - 1 = 0$$
$$\Rightarrow p = \pm 1$$

But we know $p > 0$, so $p = 1$.

To verify that this is gives a minimum, check the second derivative at $p = 1$:

$$\frac{d^2}{dp^2}(QR^2) = 8 + 24p^{-4}$$
$$= 8 + 24(1)^{-4}$$
$$= 32 > 0 \quad \therefore \text{ minimum}$$

Going for the top 9

1. For stationary points, $\frac{dy}{dx} = 0$. Using implicit differentiation:

$$\frac{d}{dx}(y^2 + 4xy - x^2) = \frac{d}{dx}(20)$$
$$2y\frac{dy}{dx} + 4y + 4x\frac{dy}{dx} - 2x = 0$$
$$\frac{dy}{dx} = 0 \Rightarrow 4y - 2x = 0 \Rightarrow x = 2y$$

Substituting this into the original equation:

$$y^2 + 4(2y)y - (2y)^2 = 20$$
$$\Rightarrow y^2 + 8y^2 - 4y^2 = 20$$
$$\Rightarrow 5y^2 = 20$$
$$\Rightarrow y^2 = 4$$
$$\Rightarrow y = \pm 2$$

When $y = 2$, $x = 2 \times 2 = 4$.

When $y = -2$, $x = 2 \times (-2) = -4$.

So the stationary points are (4, 2) and (–4, –2).

2. Applying implicit differentiation to $x^2 + y^2 = 9$:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(9)$$
$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Then, differentiating again:

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2}$$
$$= -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$
$$= -\frac{y^2 + x^2}{y^3}$$
$$= -\frac{9}{y^3} \quad \text{(using the original equation)}$$

3. By Pythagoras' theorem, the distance d between a general point on the curve, (x, x^2), and the point (0, 9) satisfies:

$$d^2 = (x - 0)^2 + (x^2 - 9)^2$$
$$= x^2 + x^4 - 18x^2 + 81$$
$$= x^4 - 17x^2 + 81$$

The distance will be minimised when d^2 is minimised, and this will occur where $\frac{d}{dx}(d^2) = 0$.

$$\frac{d}{dx}(x^4 - 17x^2 + 81) = 0$$
$$\Leftrightarrow 4x^3 - 34x = 0$$
$$\Leftrightarrow 2x(2x^2 - 17) = 0$$
$$\Leftrightarrow x = 0 \text{ or } x = \pm\sqrt{\frac{17}{2}}$$

To check which point gives a minimum, consider the second derivative:

$$\frac{d^2}{dx^2}(d^2) = 12x^2 - 34$$

When $x = 0$, $\frac{d^2}{dx^2}(d^2) = -34 < 0 \quad \therefore$ maximum

When $x = \pm\sqrt{\frac{17}{2}}$,

$$\frac{d^2}{dx^2}(d^2) = 12\left(\pm\sqrt{\frac{17}{2}}\right)^2 - 34 = 68 > 0 \quad \therefore \text{ both minima}$$

Therefore, the closest points to (0, 9) on the curve $y = x^2$ are $\left(\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$ and $\left(-\sqrt{\frac{17}{2}}, \frac{17}{2}\right)$.

4. Consider the general cubic $y = ax^3 + bx^2 + cx + d$, where $a \neq 0$. Taking successive derivatives:

$$y' = 3ax^2 + 2bx + c$$
$$y'' = 6ax + 2b$$
$$y''' = 6a$$

At point(s) of inflexion, $y'' = 0$, i.e. $0 = 6ax + 2b$

$$\Leftrightarrow x = -\frac{b}{3a}$$

But there must also be a change in concavity for this to be a point of inflexion, which means that $y''' \neq 0$. This is the case here as $y''' = 6a \neq 0$ since $a \neq 0$.

Hence, every cubic has a (single) point of inflexion (and this occurs at $x = -\frac{b}{3a}$).

5. $a = \frac{dv}{dt}$

$= \frac{d}{dt}(s^2 + s)$

$= 2s\frac{ds}{dt} + \frac{ds}{dt}$

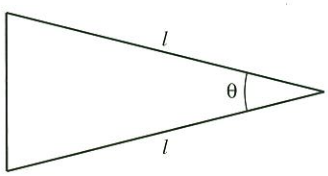
$= 2sv + v$

$= (2s+1)v$

$= (2s+1)(s^2+s)$

6. It is given that $\frac{dl}{dt} = 0.4$ and $\frac{d\theta}{dt} = -0.01$.

The area is $A = \frac{1}{2}ab\sin C = \frac{1}{2}l^2\sin\theta$



$\therefore \frac{dA}{dt} = \frac{1}{2}\left(2l\frac{dl}{dt}\sin\theta + l^2\cos\theta\frac{d\theta}{dt}\right)$ (by product rule and chain rule)

$= \frac{1}{2}\left(2l \times 0.4 \times \sin\theta + l^2\cos\theta \times (-0.01)\right)$

$= \frac{l}{2}\left(0.8\sin\theta - 0.01l^2\cos\theta\right)$

So, when $l = 4$ and $\theta = \frac{\pi}{4}$:

$\frac{dA}{dt} = \frac{4}{2}\left(0.8\sin\frac{\pi}{4} - 0.01 \times 4^2\cos\frac{\pi}{4}\right)$

$= 2\left(0.8 \times \frac{\sqrt{2}}{2} - 0.01 \times 4^2 \times \frac{\sqrt{2}}{2}\right)$

$= 0.64\sqrt{2} = 0.905 \text{ m}^2\text{s}^{-1}$ (3 SF)

7. By the product rule:

$\frac{d}{dx}(x \times x^{-1}) = 1 \times x^{-1} + x\frac{d}{dx}(x^{-1})$

But also $\frac{d}{dx}(x \times x^{-1}) = \frac{d}{dx}(1) = 0$

$\therefore\ x^{-1} + x\frac{d}{dx}(x^{-1}) = 0$

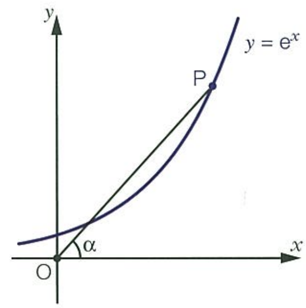
$\Rightarrow x\frac{d}{dx}(x^{-1}) = -x^{-1}$

$\Rightarrow \frac{d}{dx}(x^{-1}) = -x^{-2}$

8. (a) The point P has coordinates (p, p^2)

$\tan\alpha = \frac{e^p}{p}$

$\therefore \alpha = \arctan\left(\frac{e^p}{p}\right)$



(b) $y = e^x \Rightarrow \frac{dy}{dx} = e^x$

Therefore, at (p, e^p) the gradient of the tangent is $m = e^p$.

So the equation is

$y - y_1 = m(x - x_1)$

$y - e^p = e^p(x - p)$

$\Rightarrow y = e^p x + e^p(1-p)$

(c) At Q, $y = 0$. So

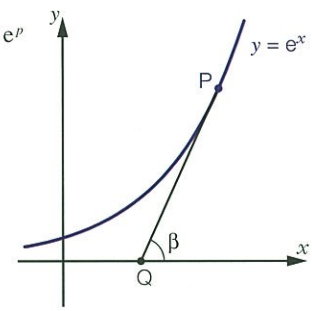
$0 = e^p x + e^p(1-p)$

$\Leftrightarrow x = p - 1$

Therefore, the coordinates of Q are $(p-1, 0)$.

(d) $\tan\beta = \frac{e^p}{p-(p-1)} = e^p$

$\therefore\ \beta = \arctan(e^p)$

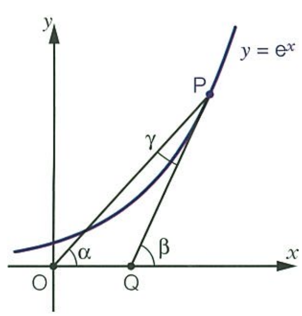


(e) Let $O\hat{P}Q = \gamma$

Then, in triangle OPQ,

$\alpha + (\pi - \beta) + \gamma = \pi$

$\Rightarrow \gamma = \beta - \alpha = \arctan(e^p) - \arctan\left(\frac{e^p}{p}\right)$



So

$$\tan\gamma = \tan\left[\arctan(e^p) - \arctan\left(\frac{e^p}{p}\right)\right]$$

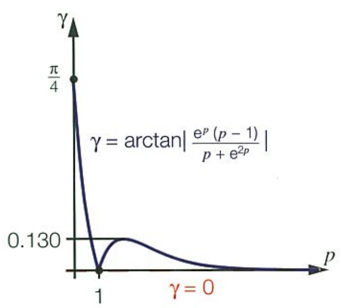
$$= \frac{e^p - \frac{e^p}{p}}{1 + e^p \times \frac{e^p}{p}} \quad \text{(using the } \tan(A-B) \text{ identity)}$$

$$= \frac{pe^p - e^p}{p + e^{2p}}$$

$$= \frac{e^p(p-1)}{p + e^{2p}}$$

Hence $\gamma = \arctan\left|\frac{e^p(p-1)}{p+e^{2p}}\right|$

From GDC, the graph is:



(f) From GDC, maximum value of γ for $p > 1$ is $\gamma = 0.130$ radians (3 SF).

(g) For the equation $e^x = kx$ to have exactly one solution, the line $y = kx$ must be a tangent to the curve $y = e^x$ at the solution point (p, e^p).

This will occur when $\gamma = 0$, which we see from the graph in part (e) is when $p = 1$, i.e. at the point $(1, e)$.

So, since $y = kx$,

$$k = \frac{y}{x} = \frac{e}{1} = e$$

10 INTEGRATION

Mixed practice 10

1. (a) $\int \sqrt{e^x}\,dx = \int (e^x)^{\frac{1}{2}}\,dx$

$$= \int e^{\frac{x}{2}}\,dx$$

$$= 2e^{\frac{x}{2}} + c$$

(b) $\int_0^{\ln 2} \frac{e^x}{\sqrt{e^x+1}}\,dx = \int_0^{\ln 2} e^x(e^x+1)^{-\frac{1}{2}}\,dx$

$$= \left[2(e^x+1)^{\frac{1}{2}}\right]_0^{\ln 2}$$

$$= 2(e^{\ln 2}+1)^{\frac{1}{2}} - 2(e^0+1)^{\frac{1}{2}}$$

$$= 2(\sqrt{3} - \sqrt{2})$$

(*Note*: Here the integration has been done by the reverse chain rule, but you may find it clearer to use the substitution $u = e^x + 1$.)

2. Use the cosine double angle identity to rewrite the integral:

$$\int_0^{\pi} \cos^2 5x\,dx = \frac{1}{2}\int_0^{\pi} (\cos 10x + 1)\,dx \quad (\text{from } \cos 2\theta = 2\cos^2\theta - 1)$$

$$= \frac{1}{2}\left[\frac{1}{10}\sin 10x + x\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\left(\frac{1}{10}\sin(10\pi) + \pi\right) - \left(\frac{1}{10}\sin 0 + 0\right)\right]$$

$$= \frac{\pi}{2}$$

3. Let $u = 4 - x$. Then $x = 4 - u$

and $\frac{du}{dx} = -1$ so that $dx = -du$. Hence

$$\int x\sqrt{4-x}\,dx = \int (4-u)\sqrt{u}(-du)$$

$$= \int u\sqrt{u} - 4\sqrt{u}\,du$$

$$= \int u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\,du$$

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}} + c$$

$$= \frac{2}{5}(4-x)^{\frac{5}{2}} - \frac{8}{3}(4-x)^{\frac{3}{2}} + c$$

4. (a) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Let $u = \cos x$

Then $\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$

$$\int \frac{\sin x}{\cos x}\,dx = \int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right)$$

$$= \int -\frac{1}{u}\,du$$

$$= -\ln|u| + c$$

$$= \ln|u^{-1}| + c$$

$$= \ln\left|(\cos x)^{-1}\right| + c$$

$$= \ln|\sec x| + c$$

(b) (i) Using the identity $\sec^2 x = 1 + \tan^2 x$:

$$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx$$

$$= \tan x - x + c$$

(ii) $\int \sec x \tan x\,dx = \sec x + c$

(iii) $\int \sec^2 x \tan x\,dx = \frac{1}{2}\tan^2 x + c$ (by the reverse chain rule, since $\frac{d}{dx}(\tan x) = \sec^2 x$, or by using the substitution $u = \tan x$).

5. (a) $x^2-4x+5=(x-2)^2-(-2)^2+5=(x-2)^2+1$

(b) $s=\int v\,\mathrm{d}t$

$=\int \frac{1}{t^2-4t+5}\,\mathrm{d}t$

$=\int \frac{1}{(t-2)^2+1}\,\mathrm{d}t$ by part (a)

$=\arctan(t-2)+c$

When $t=0$, $s=5$:

$5=\arctan(0-2)+c$

$\Rightarrow 5=-\arctan 2+c$

$\Rightarrow c=\arctan 2+5$

Therefore $s=\arctan(t-2)+\arctan 2+5$

(c) $a=\frac{\mathrm{d}v}{\mathrm{d}t}$

$=\frac{\mathrm{d}}{\mathrm{d}t}(t^2-4t+5)^{-1}$

$=-(t^2-4t+5)^{-2}(2t-4)$

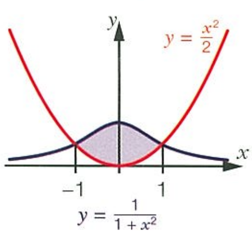
$=\frac{4-2t}{(t^2-4t+5)^2}$

(d) Velocity will reach a maximum when the denominator of $v=\frac{1}{t^2-4t+5}$ is minimum (the denominator is always positive).

By (a), $t^2-4t+5=(t-2)^2+1$, and this has a minimum of 1 (when $t=2$).

So $v_{\max}=\frac{1}{1}=1\ \mathrm{m\,s^{-1}}$

6. Solving simultaneously for the intersection points of the two curves:



$\frac{1}{1+x^2}=\frac{1}{2}x^2$

$\Leftrightarrow 2=x^2+x^4$

$\Leftrightarrow x^4+x^2-2=0$

$\Leftrightarrow (x^2+2)(x^2-1)=0$

$\Leftrightarrow x^2=-2$ or $x^2=1$

$x^2=-2$ does not give real solutions, $\therefore x=\pm 1$

Then

$\text{Area}=\int_{-1}^{1}\frac{1}{1+x^2}-\frac{1}{2}x^2\,\mathrm{d}x$

$=\left[\arctan x-\frac{1}{6}x^3\right]_{-1}^{1}$

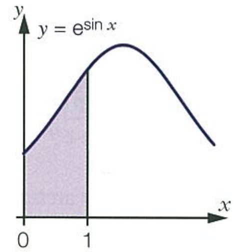
$=\left(\arctan 1-\frac{1}{6}(1)^3\right)-\left(\arctan(-1)-\frac{1}{6}(-1)^3\right)$

$=\left(\frac{\pi}{4}-\frac{1}{6}\right)-\left(-\frac{\pi}{4}+\frac{1}{6}\right)$

$=\frac{\pi}{2}-\frac{1}{3}$

7. From GDC:

$\int_0^1 \mathrm{e}^{\sin x}\,\mathrm{d}x=1.63187$



8. (a) $s=\int v\,\mathrm{d}t=\int t\sin t\,\mathrm{d}t$

By parts:

Let $u=t$ and $\frac{\mathrm{d}v}{\mathrm{d}t}=\sin t$

Then $\frac{\mathrm{d}u}{\mathrm{d}t}=1$ and $v=-\cos t$

$\int_0^{3\pi/2} t\sin t\,\mathrm{d}t=\left[t(-\cos t)\right]_0^{3\pi/2}-\int_0^{3\pi/2}1(-\cos t)\,\mathrm{d}t$

$=\left[-t\cos t\right]_0^{3\pi/2}+\int_0^{3\pi/2}\cos t\,\mathrm{d}t$

$=\left[-t\cos t+\sin t\right]_0^{3\pi/2}$

$=\left(-\frac{3\pi}{2}\cos\frac{3\pi}{2}+\sin\frac{3\pi}{2}\right)$

$-(-0\cos 0+\sin 0)$

$=-1\ \mathrm{m}$

(b) At maximum displacement, $\frac{\mathrm{d}s}{\mathrm{d}t}=0$, i.e. $v=0$:

$v=t\sin t=0$

$\Leftrightarrow t=0$ or $\sin t=0$

$\therefore t=0,\pi$ (for $0\le t\le\frac{3\pi}{2}$)

To check that $t=\pi$ is a local maximum (clearly $t=0$ isn't), consider $\frac{\mathrm{d}^2s}{\mathrm{d}t^2}$.

$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{\mathrm{d}v}{\mathrm{d}t}$

$=\frac{\mathrm{d}}{\mathrm{d}t}(t\sin t)$

$=\sin t+t\cos t$

When $t=\pi$:

$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\sin\pi+\pi\cos\pi=-\pi<0$ $\therefore$ local maximum.

At $t=\pi$, displacement from the initial position (using the integration in part (a)) is

$s=\left[-t\cos t+\sin t\right]_0^{\pi}$

$=(-\pi\cos\pi+\sin\pi)-(-0\cos 0+\sin 0)$

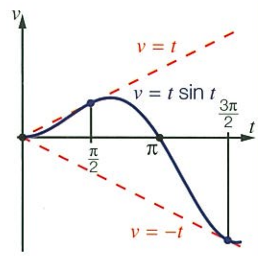
$=\pi\ \mathrm{m}$

This is greater than the displacement in part (a), so the maximum displacement is achieved after π s.

(c) Since $t = \pi$ is a local maximum for displacement, the ball changes direction here and subsequently passes back through the initial point before reaching a displacement of -1 m from the initial position at $t = \frac{3\pi}{2}$ (by part (a)). So:

- Distance travelled between $t = 0$ and $t = \pi$ is $d_1 = \pi$ m.
- Distance travelled between $t = \pi$ and $t = \frac{3\pi}{2}$ is $d_2 = \pi - (-1) = (\pi + 1)$ m

Therefore, total distance travelled is $d = d_1 + d_2 = (2\pi + 1)$ m.



(*Note*: The velocity graph is given here for information but is not strictly necessary to answer the question.)

9. $\int_0^a \sin 2x \, dx = \frac{3}{4}$

$$\Rightarrow \left[-\frac{1}{2}\cos 2x\right]_0^a = \frac{3}{4}$$

$$\Rightarrow \left(-\frac{1}{2}\cos 2a\right) - \left(-\frac{1}{2}\cos 0\right) = \frac{3}{4}$$

$$\Rightarrow -\frac{1}{2}\cos 2a + \frac{1}{2} = \frac{3}{4}$$

$$\Rightarrow \cos 2a = -\frac{1}{2}$$

$$0 < a \le \pi \Rightarrow 0 < 2a \le 2\pi$$

$$\therefore 2a = \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$\Rightarrow a = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

10. (a) Let $u = 2^x$

Then $\frac{du}{dx} = (\ln 2)2^x$ so $dx = \frac{du}{(\ln 2)2^x} = \frac{du}{(\ln 2)u}$

$$\int 2^x \, dx = \int u \frac{du}{(\ln 2)u}$$

$$= \int \frac{u}{(\ln 2)u} \, du$$

$$= \int \frac{1}{\ln 2} \, du$$

$$= \frac{u}{\ln 2} + c$$

$$= \frac{2^x}{\ln 2} + c$$

(b) By the product rule:

$$\frac{d}{dx}(x\log_2 x) = \log_2 x + x\frac{1}{x\ln 2}$$

$$= \log_2 x + \frac{1}{\ln 2}$$

(c) $\int \log_2 x \, dx = \int 1 \times \log_2 x \, dx$

So, using integration by parts:

Let $u = \log_2 x$ and $\frac{dv}{dx} = 1$

Then $\frac{du}{dx} = \frac{1}{x\ln 2}$ and $v = x$

$$\int 1 \times \log_2 x \, dx = x\log_2 x - \int x\frac{1}{x\ln 2} \, dx$$

$$= x\log_2 x - \int \frac{1}{\ln 2} \, dx$$

$$= x\log_2 x - \frac{x}{\ln 2} + c$$

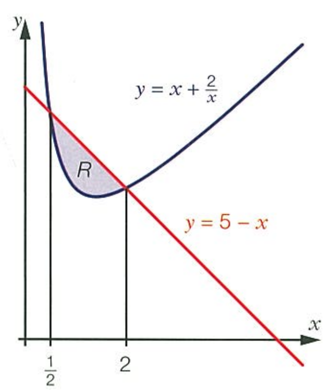
Alternatively, from part (b),

$$\frac{d}{dx}(x\log_2 x) - \frac{1}{\ln 2} = \log_2 x;$$

that is, $\frac{d}{dx}\left[x\log_2 x - \frac{1}{\ln 2}x\right] = \log_2 x$.

So $\int \log_2 x \, dx = x\log_2 x - \frac{x}{\ln 2} + c$

11. (a) From GDC:



Points of intersection are $x = \frac{1}{2}$ and $x = 2$.

$$\text{Area of } R = \int_{\frac{1}{2}}^{2} 5 - x - \left(x + \frac{2}{x}\right) dx$$
$$= \int_{\frac{1}{2}}^{2} 5 - 2x - \frac{2}{x}\, dx$$
$$= \left[5x - x^2 - 2\ln x\right]_{\frac{1}{2}}^{2}$$
$$= (6 - 2\ln 2) - \left(\frac{9}{4} - 2\ln\frac{1}{2}\right)$$
$$= \frac{15}{4} - \ln 4 + \ln\frac{1}{4}$$
$$= \frac{15}{4} - \ln 4 - \ln 4$$
$$= \frac{15}{4} - 2\ln 4$$

(b) $$V = \pi\int_{\frac{1}{2}}^{2} (5-x)^2 - \left(x + \frac{2}{x}\right)^2 dx$$
$$= \pi\int_{\frac{1}{2}}^{2} 25 - 10x + x^2 - \left(x^2 + 4 + \frac{4}{x^2}\right) dx$$
$$= \pi\int_{\frac{1}{2}}^{2} 21 - 10x - \frac{4}{x^2}\, dx$$
$$= \pi\left[21x - 5x^2 + \frac{4}{x}\right]_{\frac{1}{2}}^{2}$$
$$= \pi\left[(42 - 20 + 2) - \left(\frac{21}{2} - \frac{5}{4} + 8\right)\right]$$
$$= \frac{27\pi}{4}$$

12. By parts:

Let $u = x^2$ and $\frac{dv}{dx} = \cos 2x$

Then $\frac{du}{dx} = 2x$ and $v = \frac{1}{2}\sin 2x$

$$\int x^2 \cos 2x\, dx = x^2 \frac{1}{2}\sin 2x - \int 2x\left(\frac{1}{2}\sin 2x\right) dx$$
$$= \frac{1}{2}x^2 \sin 2x - \int x\sin 2x\, dx$$

Then, integrating $\int x\sin 2x\, dx$ by parts again:

Let $u = x$ and $\frac{dv}{dx} = \sin 2x$

Then $\frac{du}{dx} = 1$ and $v = -\frac{1}{2}\cos 2x$

$$\int x\sin 2x\, dx = x\left(-\frac{1}{2}\cos 2x\right) - \int 1\times\left(-\frac{1}{2}\cos 2x\right) dx$$
$$= -\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x\, dx$$
$$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$$

So, substituting this into the first expression gives

$$\int_0^{\pi/4} x^2\cos 2x\, dx = \left[\frac{1}{2}x^2\sin 2x\right]_0^{\frac{\pi}{4}} - \int_0^{\pi/4} x\sin 2x\, dx$$
$$= \left[\frac{1}{2}x^2\sin 2x\right]_0^{\frac{\pi}{4}} - \left[-\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{4}\left[2x^2\sin 2x + 2x\cos 2x - \sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{4}\left[\left(\frac{\pi^2}{8} + 0 - 1\right) - (0 + 0 - 0)\right]$$
$$= \frac{1}{4}\left(\frac{\pi^2}{8} - 1\right)$$

13. (a) By observation,

$$x^2 + 3 = (x+2)(x-2) + 7$$

So $\frac{x^2+3}{x+2} = x - 2 + \frac{7}{x+2}$

i.e. $A = 1$, $B = -2$, $C = 7$

(Alternatively, you can do this by polynomial long division or by formally comparing coefficients.)

(b) $$\int \frac{x^2+3}{x+2}\, dx = \int x - 2 + \frac{7}{x+2}\, dx$$
$$= \frac{1}{2}x^2 - 2x + 7\ln|x+2| + c$$

14. (a) $$V_A = \pi\int_0^9 x^2\, dy$$
$$= \pi\int_0^9 y\, dy$$
$$= \pi\left[\frac{1}{2}y^2\right]_0^9$$
$$= \frac{81\pi}{2}$$

(b) When the rectangle made up of regions A and B together is rotated 2π radians about the y-axis, it forms a cylinder of radius 3 and height 9. The volume of the cylinder is

$$V_{AB} = \pi r^2 h$$
$$= \pi \times 3^2 \times 9$$
$$= 81\pi$$

So

$$V_B = V_{AB} - V_A$$
$$= 81\pi - \frac{81\pi}{2}$$
$$= \frac{81\pi}{2}$$

15. (a) Using the double angle identity $\cos 2x = 1 - 2\sin^2 x$:

$$\int 2\sin^2 x \, dx = \int 1 - \cos 2x \, dx$$
$$= x - \frac{1}{2}\sin 2x + c$$
$$= x - \frac{1}{2}(2\sin x \cos x) + c$$
$$= x - \sin x \cos x + c$$

(b) At points of intersection:

$\sin x = 2\sin^2 x$

$\Leftrightarrow 2\sin^2 x - \sin x = 0$

$\Leftrightarrow \sin x(2\sin x - 1) = 0$

$\Leftrightarrow \sin x = 0$ or $\sin x = \frac{1}{2}$

$\therefore x = 0, \frac{\pi}{6}$ in the domain $\left[0, \frac{\pi}{2}\right]$

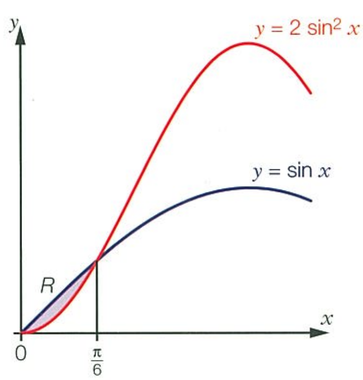
$x = 0: \ y = \sin 0 = 0$

$x = \frac{\pi}{6}: \ y = \sin\frac{\pi}{6} = \frac{1}{2}$

So the coordinates of the points of intersection are $(0, 0)$ and $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

(c) $A = \int_0^{\pi/6} \sin x - 2\sin^2 x \, dx$

$$= \left[-\cos x - (x - \sin x \cos x)\right]_0^{\frac{\pi}{6}} \quad \text{(using part (a))}$$
$$= \left[\sin x \cos x - \cos x - x\right]_0^{\frac{\pi}{6}}$$
$$= \left(\sin\frac{\pi}{6}\cos\frac{\pi}{6} - \cos\frac{\pi}{6} - \frac{\pi}{6}\right) - (\sin 0 \cos 0 - \cos 0 - 0)$$
$$= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2} - \frac{\pi}{6} - 1$$
$$= -\frac{\sqrt{3}}{4} - \frac{\pi}{6} - 1$$



(d)

$$V = \pi\int_0^{\pi/6} (\sin x)^2 - (2\sin^2 x)^2 \, dx$$
$$= \pi\int_0^{\pi/6} \sin^2 x - (1 - \cos 2x)^2 \, dx \qquad \text{(using } \cos 2\theta = 1 - 2\sin^2\theta\text{)}$$
$$= \pi\int_0^{\pi/6} \frac{1}{2}(1 - \cos 2x) - (1 - \cos 2x)^2 \, dx$$
$$= \pi\int_0^{\pi/6} \frac{1}{2}(1 - \cos 2x) - (1 - 2\cos 2x + \cos^2 2x) \, dx$$
$$= \pi\int_0^{\pi/6} -\frac{1}{2} + \frac{3}{2}\cos 2x - \cos^2 2x \, dx$$
$$= \pi\int_0^{\pi/6} -\frac{1}{2} + \frac{3}{2}\cos 2x - \frac{1}{2}(\cos 4x + 1) \, dx \qquad \text{(using } \cos 2\theta = 2\cos^2\theta - 1\text{)}$$
$$= \pi\int_0^{\pi/6} -1 + \frac{3}{2}\cos 2x - \frac{1}{2}\cos 4x \, dx$$
$$= \pi\left[-x + \frac{3}{4}\sin 2x - \frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{6}}$$
$$= \pi\left[\left(-\frac{\pi}{6} + \frac{3}{4}\sin\frac{\pi}{3} - \frac{1}{8}\sin\frac{2\pi}{3}\right) - \left(-0 + \frac{3}{4}\sin 0 - \frac{1}{8}\sin 0\right)\right]$$
$$= \pi\left(-\frac{\pi}{6} + \frac{3\sqrt{3}}{8} - \frac{\sqrt{3}}{16}\right)$$
$$= \pi\left(-\frac{\pi}{6} + \frac{5\sqrt{3}}{16}\right)$$

(e) $\int \arcsin x \, dx = \int 1 \times \arcsin x \, dx$, so by parts:

Let $u = \arcsin x$ and $\frac{dv}{dx} = 1$

Then $\frac{du}{dx} = \frac{1}{\sqrt{1 - x^2}}$ and $v = x$

$$\int 1 \times \arcsin x \, dx = x\arcsin x - \int x\frac{1}{\sqrt{1 - x^2}} \, dx$$
$$= x\arcsin x + \sqrt{1 - x^2} + c$$

(*Note*: The integration $\int x\frac{1}{\sqrt{1 - x^2}} \, dx$ has been done by the reverse chain rule here, but you could also do this with the substitution $u = 1 - x^2$.)

(f) When $y = 1$:

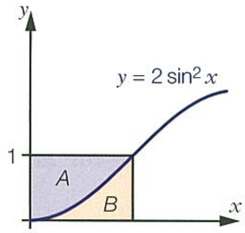
$1 = 2\sin^2 x$

$\Leftrightarrow \sin^2 x = \frac{1}{2}$

$\Leftrightarrow \sin x = \pm\frac{1}{\sqrt{2}}$

So the smallest positive value of x is $x = \frac{\pi}{4}$.

The area we want is the area of region A in the diagram.



Area of A = area of rectangle − area of B

$$= \left(\frac{\pi}{4}\times 1\right) - \int_0^{\pi/4} 2\sin^2 x \, dx$$

$$= \frac{\pi}{4} - \left[x - \sin x \cos x\right]_0^{\pi/4} \quad \text{(by part (a))}$$

$$= \frac{\pi}{4} - \left[\frac{\pi}{4} - \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{2}$$

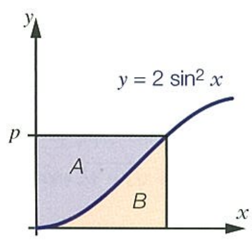
(g) Similarly to (f), when $y = p$:

$$p = 2\sin^2 x$$

$$\Leftrightarrow \sin^2 x = \frac{p}{2}$$

$$\Leftrightarrow \sin x = \pm\sqrt{\frac{p}{2}}$$

So the smallest positive value of x is $x = \arcsin\sqrt{\frac{p}{2}}$



Area of A = area of rectangle − area of B

$$= \left(p\arcsin\sqrt{\frac{p}{2}}\right) - \int_0^{\arcsin\sqrt{p/2}} 2\sin^2 x \, dx$$

$$= p\arcsin\sqrt{\frac{p}{2}} - \left[x - \sin x \cos x\right]_0^{\arcsin\sqrt{p/2}}$$

$$= p\arcsin\sqrt{\frac{p}{2}} - \left[x - \sin x\sqrt{1-\sin^2 x}\right]_0^{\arcsin\sqrt{p/2}}$$

$$= p\arcsin\sqrt{\frac{p}{2}} - \left[\left(\arcsin\sqrt{\frac{p}{2}} - \sqrt{\frac{p}{2}}\sqrt{1-\frac{p}{2}}\right) - (0)\right]$$

$$= p\arcsin\sqrt{\frac{p}{2}} - \arcsin\sqrt{\frac{p}{2}} + \frac{1}{2}\sqrt{2p - p^2}$$

(h) Since the area of the shaded region A in (g) is also given by $\int_0^p x \, dy = \int_0^p \arcsin\left(\sqrt{\frac{y}{2}}\right) dy$, we have

$$\int_0^p \arcsin\left(\sqrt{\frac{y}{2}}\right) dy = p\arcsin\sqrt{\frac{p}{2}} - \arcsin\sqrt{\frac{p}{2}} + \frac{1}{2}\sqrt{2p - p^2}$$

So

$$\int \arcsin\left(\sqrt{\frac{y}{2}}\right) dy = y\arcsin\sqrt{\frac{y}{2}} - \arcsin\sqrt{\frac{y}{2}} + \frac{1}{2}\sqrt{2y - y^2} + c$$

Let $x = \frac{y}{2}$ so that $y = 2x$

Then $\frac{dx}{dy} = \frac{1}{2}$ and $dy = 2dx$

Substituting this into the formula for $\int \arcsin\left(\sqrt{\frac{y}{2}}\right) dy$ gives

$$\int \arcsin\left(\sqrt{x}\right)(2dx) = 2x\arcsin\sqrt{x} - \arcsin\sqrt{x} + \frac{1}{2}\sqrt{2(2x) - (2x)^2} + c$$

$$\Rightarrow 2\int \arcsin\left(\sqrt{x}\right) dx = 2x\arcsin\sqrt{x} - \arcsin\sqrt{x} + \frac{1}{2}\sqrt{4x - 4x^2} + c$$

$$\Rightarrow \int \arcsin\left(\sqrt{x}\right) dx = x\arcsin\sqrt{x} - \frac{1}{2}\arcsin\sqrt{x} + \frac{1}{4}\sqrt{4x - 4x^2} + c$$

$$= x\arcsin\sqrt{x} - \frac{1}{2}\arcsin\sqrt{x} + \frac{1}{2}\sqrt{x - x^2} + c$$

Going for the top 10

1. Let $x = \sin\theta$

Then $\frac{dx}{d\theta} = \cos\theta \Rightarrow dx = \cos\theta \, d\theta$

$$\int \sqrt{1-x^2}\, dx = \int \sqrt{1-\sin^2\theta}\cos\theta \, d\theta$$

$$= \int \sqrt{\cos^2\theta}\cos\theta \, d\theta$$

$$= \int \cos^2\theta \, d\theta$$

$$= \int \frac{1}{2}(\cos 2\theta + 1)\, d\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \theta\right) + c$$

$$= \frac{1}{2}(\sin\theta\cos\theta + \theta) + c$$

$$= \frac{1}{2}\left(\sin\theta\sqrt{1-\sin^2\theta} + \theta\right) + c$$

$$= \frac{1}{2}\left(x\sqrt{1-x^2} + \arcsin x\right) + c$$

2. (a) $C - S = \int \frac{\cos x}{\cos x + \sin x}\,\mathrm{d}x - \int \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x$

$$= \int \frac{\cos x - \sin x}{\cos x + \sin x}\,\mathrm{d}x$$

$$= \ln|\cos x + \sin x| + c$$

(*Note*: Here the integration has been done by the reverse chain rule, but you could also use the substitution $u = \cos x + \sin x$.)

(b) $C + S = \int \frac{\cos x}{\cos x + \sin x}\,\mathrm{d}x + \int \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x$

$$= \int \frac{\cos x + \sin x}{\cos x + \sin x}\,\mathrm{d}x$$

$$= \int 1\,\mathrm{d}x$$

$$= x + c$$

Then $S = \frac{1}{2}\left[(C+S) - (C-S)\right]$

$$= \frac{1}{2}\left(x - \ln|\cos x + \sin x|\right) + \tilde{c}$$

i.e. $\int \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x = \frac{1}{2}\left(x - \ln|\cos x + \sin x|\right) + c$

3. $\int \cos^3 x\,\mathrm{d}x = \int \cos x \cos^2 x\,\mathrm{d}x$

$$= \int \cos x(1 - \sin^2 x)\mathrm{d}x$$

$$= \int \cos x - \cos x \sin^2 x\,\mathrm{d}x$$

$$= \sin x - \frac{1}{3}\sin^3 x + c$$

4. (a) $V = \pi\int_{\pi/3}^{2\pi/3} \sin^2 x - \left(\frac{1}{2}\right)^2 \mathrm{d}x$

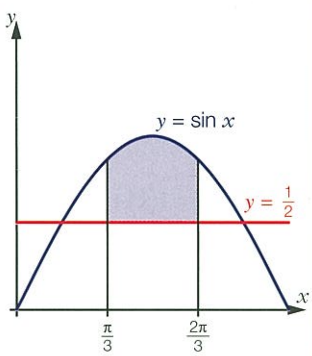
$$= \pi\int_{\pi/3}^{2\pi/3} \frac{1}{2}(1 - \cos 2x) - \frac{1}{4}\mathrm{d}x$$

$$= \frac{\pi}{4}\int_{\pi/3}^{2\pi/3} 1 - 2\cos 2x\,\mathrm{d}x$$

$$= \frac{\pi}{4}\left[x - \sin 2x\right]_{\pi/3}^{2\pi/3}$$

$$= \frac{\pi}{4}\left[\left(\frac{2\pi}{3} - \sin\frac{4\pi}{3}\right) - \left(\frac{\pi}{3} - \sin\frac{2\pi}{3}\right)\right]$$

$$= \frac{\pi}{4}\left(\frac{\pi}{3} + \sqrt{3}\right)$$



(b) The volume of revolution of the curve $y = \sin x$ around the line $y = \frac{1}{2}$ is the same as that of the curve $y = \sin x - \frac{1}{2}$ around the x-axis (i.e. translating everything down by $\frac{1}{2}$).

$$V = \pi\int_{\pi/3}^{2\pi/3} \left(\sin x - \frac{1}{2}\right)^2 \mathrm{d}x$$

$$= \pi\int_{\pi/3}^{2\pi/3} \sin^2 x - \sin x + \frac{1}{4}\,\mathrm{d}x$$

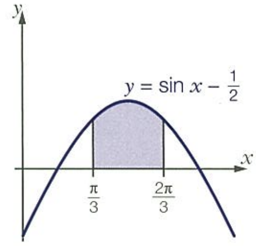
$$= \pi\int_{\pi/3}^{2\pi/3} \frac{1}{2}(1 - \cos 2x) - \sin x + \frac{1}{4}\,\mathrm{d}x$$

$$= \frac{\pi}{4}\int_{\pi/3}^{2\pi/3} 3 - 2\cos 2x - 4\sin x\,\mathrm{d}x$$

$$= \frac{\pi}{4}\left[3x - \sin 2x + 4\cos x\right]_{\pi/3}^{2\pi/3}$$

$$= \frac{\pi}{4}\left(2\pi - \sin\frac{4\pi}{3} + 4\cos\frac{4\pi}{3}\right) - \left(\pi - \sin\frac{2\pi}{3} + 4\cos\frac{2\pi}{3}\right)$$

$$= \frac{\pi}{4}\left(\pi + \sqrt{3} - 4\right)$$



5. By parts:

Let $u = \mathrm{e}^x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ and $v = -\cos x$

$$\int \mathrm{e}^x \sin x\,\mathrm{d}x = -\mathrm{e}^x\cos x - \int -\mathrm{e}^x \cos x\,\mathrm{d}x$$

$$= -\mathrm{e}^x\cos x + \int \mathrm{e}^x \cos x\,\mathrm{d}x$$

For $\int \mathrm{e}^x \cos x\,\mathrm{d}x$, use parts again:

Let $u = \mathrm{e}^x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

Then $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ and $v = \sin x$

$$\int \mathrm{e}^x \cos x\,\mathrm{d}x = \mathrm{e}^x \sin x - \int \mathrm{e}^x \sin x\,\mathrm{d}x$$

Substituting this into the expression for $\int \mathrm{e}^x \sin x\,\mathrm{d}x$ above:

$$\int \mathrm{e}^x \sin x\,\mathrm{d}x = -\mathrm{e}^x\cos x + \int \mathrm{e}^x \cos x\,\mathrm{d}x$$

$$= -\mathrm{e}^x\cos x + \left(\mathrm{e}^x \sin x - \int \mathrm{e}^x \sin x\,\mathrm{d}x\right)$$

$$\therefore 2\int \mathrm{e}^x \sin x\,\mathrm{d}x = -\mathrm{e}^x\cos x + \mathrm{e}^x \sin x$$

Hence $\int \mathrm{e}^x \sin x\,\mathrm{d}x = \frac{1}{2}\mathrm{e}^x(\sin x - \cos x) + c$

6. (a) (i) $\sqrt{\frac{1-3x}{1+3x}} = \sqrt{\frac{1-3x}{1+3x}} \times \sqrt{\frac{1-3x}{1-3x}}$

$$= \frac{\left(\sqrt{1-3x}\right)^2}{\sqrt{(1+3x)(1-3x)}}$$

$$= \frac{1-3x}{\sqrt{1-9x^2}}$$

(ii) $\int \sqrt{\frac{1-3x}{1+3x}}\,dx = \int \frac{1-3x}{\sqrt{1-9x^2}}\,dx$

$$= \int \frac{1}{\sqrt{1-9x^2}}\,dx - \int \frac{3x}{\sqrt{1-9x^2}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}}\,dx - \left(\frac{1}{-6}\right)\int \frac{-18x}{\sqrt{1-9x^2}}\,dx$$

$$= \frac{1}{3}\arcsin\left(\frac{x}{\frac{1}{3}}\right) + \frac{2}{6}\sqrt{1-9x^2} + C$$

$$= \frac{1}{3}\left(\arcsin(3x) + \sqrt{1-9x^2}\right) + C$$

(b) (i) $\int \sec^3 x\,dx = \int \sec x \sec^2 x\,dx$

Let $u = \sec x$ and $\frac{dv}{dx} = \sec^2 x$

Then $\frac{du}{dx} = \sec x \tan x$ and $v = \tan x$

$$\int \sec x \sec^2 x\,dx = \sec x \tan x - \int \sec x \tan x \tan x\,dx$$

$$= \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x\left(\sec^2 x - 1\right)\,dx$$

$$= \sec x \tan x - \int \sec^3 x - \sec x\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$

$$= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x| + C$$

$$\therefore 2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

Hence $\int \sec^3 x\,dx = \frac{1}{2}\left(\sec x \tan x + \ln|\sec x + \tan x|\right) + C$

(ii) Let $x = \sqrt{3}\tan\theta$

Then $\frac{dx}{d\theta} = \sqrt{3}\sec^2\theta \Rightarrow dx = \sqrt{3}\sec^2\theta\,d\theta$

To change the limits:

At $x = 0$, $0 = \sqrt{3}\tan\theta \Rightarrow \theta = 0$

At $x = 1$, $1 = \sqrt{3}\tan\theta \Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$

$$\Rightarrow \theta = \frac{\pi}{6}$$

So $\int_0^1 \sqrt{x^2+3}\,dx = \int_0^{\pi/6} \sqrt{(\sqrt{3}\tan\theta)^2 + 3}\left(\sqrt{3}\sec^2\theta\,d\theta\right)$

$$= \int_0^{\pi/6} \sqrt{3\tan^2\theta + 3}\left(\sqrt{3}\sec^2\theta\,d\theta\right)$$

$$= \int_0^{\pi/6} \sqrt{3}\sqrt{\tan^2\theta + 1}\left(\sqrt{3}\sec^2\theta\,d\theta\right)$$

$$= 3\int_0^{\pi/6} \sec^3\theta\,d\theta$$

$$= 3\left[\frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\right]_0^{\frac{\pi}{6}}$$

$$= \frac{3}{2}\left[\left(\sec\frac{\pi}{6}\tan\frac{\pi}{6} + \ln\left|\sec\frac{\pi}{6} + \tan\frac{\pi}{6}\right|\right) - \left(\sec 0 \tan 0 + \ln|\sec 0 + \tan 0|\right)\right]$$

$$= \frac{3}{2}\left(\frac{2}{3} + \ln\sqrt{3}\right)$$

$$= \frac{3}{2}\left(\frac{2}{3} + \frac{1}{2}\ln 3\right)$$

$$= 1 + \frac{3}{4}\ln 3$$

11 PROBABILITY AND STATISTICS

Mixed practice 11

1. (a) Let X = height of a tree. Then $X \sim N(26.2, 5.6^2)$.

$$P(X > 30) = 1 - P(X < 30)$$
$$= 1 - 0.75129\ldots \quad \text{(from GDC)}$$
$$= 0.249$$

(b) Let Y = the number of trees out of the 16 that are more than 30 m tall.

Then $Y \sim B(16, 0.249)$.

$$P(Y \geq 2) = 1 - P(Y \leq 1)$$
$$= 1 - 0.06487\ldots \quad \text{(from GDC)}$$
$$= 0.935$$

2. Probability distribution of X:

x	1	2	3	4
$P(X = x)$	k	$4k$	$9k$	$16k$

$$k + 4k + 9k + 16k = 1 \Leftrightarrow k = \frac{1}{30}$$

$$E(X) = 1(k) + 2(4k) + 3(9k) + 4(16k)$$
$$= 100k = \frac{100}{30} = 3.33$$