5 SEQUENCES AND SERIES

WHAT YOU NEED TO KNOW

- The notation for sequences and series:
  - u_n represents the nth term of the sequence u.
  - S_n denotes the sum of the first n terms of the sequence.
  - $\sum_{r=k}^{n} u_r$ denotes the sum of the kth term to the nth term, so $S_n = \sum_{r=1}^{n} u_r$.
- Sequences can be described in two ways:
  - using recursive definitions to define how u_{n+1} depends on u_n
  - using deductive rules (the nth term formula) to define how u_n depends on n.
- An arithmetic sequence has a constant difference, d, between terms: $u_{n+1} = u_n + d$
  - $u_n = u_1 + (n-1)d$
  - $S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$
- A geometric sequence has a constant ratio, r, between terms: $u_{n+1} = ru_n$
  - $u_n = u_1 r^{n-1}$
  - $S_n = \frac{u_1(r^n - 1)}{r-1} = \frac{u_1(1-r^n)}{1-r}, r \neq 1$
  - When $|r| < 1$, S_n approaches a limit as n increases, called the sum to infinity: $S_\infty = \frac{u_1}{1-r}$

EXAM TIPS AND COMMON ERRORS

- Many questions on sequences and series involve forming and then solving simultaneous equations.
- For questions on geometric sequences, you may need to use logarithms or rules of exponents.
- You may need to use the list feature on your calculator to solve problems involving sequences.
- You only ever need to use the first sum formula for geometric sequences.
- Geometric sequences are often used to solve problems involving percentage increase or decrease, such as investments and mortgages.

5.1 ARITHMETIC SEQUENCES AND SERIES

WORKED EXAMPLE 5.1

The third term of an arithmetic sequence is 15 and the sixth term is 27.

(a) Find the tenth term.

(b) Find the sum of the first ten terms.

(c) The sum of the first n terms is 5250. Find the value of n.

> ⚠ You will often have to find a term of a sequence and a sum of a sequence in different parts of the same question. Make sure you are clear which you are being asked for.

(a) $u_3 = u_1 + 2d = 15 \quad \cdots (1)$

$u_6 = u_1 + 5d = 27 \quad \cdots (2)$

$(2)-(1)$: $3d = 12 \Rightarrow d = 4$ and so $u_1 = 7$

Hence, $u_{10} = u_1 + 9d = 7 + 9 \times 4 = 43$

> We can form two equations from the given information about the third and sixth terms.
> It is then clear that we need to solve these simultaneously for u_1 and d.

(b) $S_{10} = \frac{10}{2}(u_1 + u_{10})$

$= 5(7 + 43)$

$= 250$

> There are two possible formulae for the sum of an arithmetic sequence. Since we know the first and last terms (from part (a)), we use $S_n = \frac{n}{2}(u_1 + u_n)$.

(c) $5250 = \frac{n}{2}(2u_1 + (n-1)d)$

$= \frac{n}{2}(14 + 4(n-1))$

The solutions (from GDC) are $n = 50$ or -52.5; but n must be a positive integer, so $n = 50$.

> Again we need a formula for the sum, but this time the other formula is the one to use as n is the only unknown here. Form an equation and solve it using a GDC.

Practice questions 5.1

1. The fifth term of an arithmetic sequence is 64 and the eighth term is 46.

(a) Find the thirtieth term.

(b) Find the sum of the first twelve terms.

2. The first four terms of an arithmetic sequence are 16, 15.5, 14, 13.5.

(a) Find the twentieth term.

(b) Which term is equal to zero?

(c) The sum of the first n terms is 246. Find the possible values of n.

5.2 GEOMETRIC SEQUENCES AND SERIES

WORKED EXAMPLE 5.2

The second term of a geometric sequence is –5 and the sum to infinity is 12. Find the common ratio and the first term.

$u_2 = u_1 r = -5 \quad \cdots (1)$

$S_\infty = \dfrac{u_1}{1-r} = 12 \quad \cdots (2)$

From (2): $u_1 = 12(1-r)$

Substituting into (1): $12r(1-r) = -5$

$\therefore r = -0.316$ or $r = 1.32$ (from GDC)

Since the sum to infinity exists, $|r| < 1$ and so $r = -0.316$.

Hence $u_1 = 12(1-(-0.316)) = 15.8$

> We can form two equations from the given information about the second term and sum to infinity.
> It is then clear that we need to solve these simultaneously for u_1 and r.

> There is no need to rearrange the final equation as it can be solved using a GDC.

> The condition that $|r| < 1$ should be remembered as part of the formula for S_∞.

Practice questions 5.2

3. The fourth term of a geometric sequence is –16 and the sum to infinity is 32. Show that there is only one possible value of the common ratio and find this value.

4. The fifth term of a geometric series is 12 and the seventh term is 3. Find the two possible values of the sum to infinity of the series.

5. The sum of the first three terms of a geometric sequence is 38 and the sum of the first four terms is 65. Find the first term and the common ratio, $r > 1$.

6. The fifth term of a geometric sequence is 128 and the sixth term is 512.
   (a) Find the common ratio and the first term.
   (b) Which term has a value of 32 768?
   (c) How many terms are needed before the sum of all the terms in the sequence exceeds 100 000?

7. The first three terms of a geometric sequence are $2x + 4$, $x + 5$, $x + 1$, where x is a real number.
   (a) Find the two possible values of x.
   (b) Given that it exists, find the sum to infinity of the series.

5.3 APPLICATIONS

WORKED EXAMPLE 5.3

Daniel invests \$500 at the beginning of each year in a scheme that earns interest at a rate of 4% per annum, paid at the end of the year.

Show that the first year, n, in which the scheme is worth more than \$26 000 satisfies $n > \frac{\ln k}{\ln 1.04}$ where k is a constant to be found. Hence determine n.

Amount in the scheme at the end of the first year:
500×1.04

Generate the first and second terms of the sequence to establish a pattern.

Amount at the end of the second year:
$(500 + 500 \times 1.04) \times 1.04$
$= 500 \times 1.04 + 500 \times 1.04^2$

With more complicated questions of this type, it is always a good idea to write down the first few terms to see whether you have an arithmetic or geometric series, and to understand exactly how the series is being formed.

So, amount at the end of the nth year:
$500 \times 1.04 + 500 \times 1.04^2 + \ldots + 500 \times 1.04^n$

This is a geometric series with $u_1 = 500 \times 1.04$ and $r = 1.04$.

$$S_n = \frac{500 \times 1.04(1.04^n - 1)}{1.04 - 1}$$
$$= 13000(1.04^n - 1)$$

We can use the formula for the sum of the first n terms of a geometric sequence to form an inequality, which we solve to find n.

So, for the amount to exceed \$26 000:

$13000(1.04^n - 1) > 26000$
$\Rightarrow 1.04^n - 1 > 2$
$\Rightarrow 1.04^n > 3$

$\Rightarrow \ln(1.04^n) > \ln 3$
$\Rightarrow n \ln 1.04 > \ln 3$
$\Rightarrow n > \frac{\ln 3}{\ln 1.04} = 28.01$ (2 DP)

The unknown n is in the power, so use logarithms to solve the inequality.

Logarithms are covered in Chapter 2.

Therefore, $k = 3$ and $n = 29$.

Practice questions 5.3

8. A starting salary for a teacher is \$25 000 and there is an annual increase of 3%.

(a) How much will the teacher earn in their tenth year?

(b) How much will the teacher earn in total during a 35-year teaching career?

(c) Find the first year in which the teacher earns more than \$35 000.

(d) How many years would the teacher have to work in order to earn a total of \$1 million?

9. Beth repays a loan of \$10 500 over a period of n months. She repays \$50 in the first month, \$55 in the second, and so on, with the monthly repayments continuing to increase by \$5 each month.

(a) How much will Beth repay in the 28th month?

(b) Show that $n^2 + 19n - 4200 = 0$.

(c) Hence find the number of months taken to repay the loan in full.

10. A ball is dropped from a height of 3 m. Each time it hits the ground it bounces up to 90% of its previous height.

(a) How high does it bounce on the fifth bounce?

(b) On which bounce does the ball first reach a maximum height of less than 1 m?

(c) Assuming the ball keeps bouncing until it rests, find the total distance travelled by the ball.

11. Theo has a mortgage of \$127 000 which is to be repaid in annual instalments of \$7000. Once the annual payment has been made, 3.5% interest is added to the remaining balance at the beginning of the next year.

(a) Show that at the end of the third year the amount owing is given by

$127000 \times 1.035^2 - 7000(1 + 1.035 + 1.035^2)$

(b) By forming a similar expression for the amount owing after n years, show that

$$n > \frac{\ln k}{\ln 1.035} + 1$$

where k is a constant to be found.

(c) Hence find the number of years it will take Theo to pay off his mortgage.

Mixed practice 5

1. The fourth term of an arithmetic sequence is 17. The sum of the first twenty terms is 990. Find the first term, a, and the common difference, d, of the sequence.

2. The fourth, tenth and thirteenth terms of a geometric sequence form an arithmetic sequence. Given that the geometric sequence has a sum to infinity, find its common ratio correct to three significant figures.

3. Evaluate $\sum_{r=1}^{12} 4r + \left(\frac{1}{3}\right)^r$ correct to four significant figures.

4. Find an expression for the sum of the first 20 terms of the series

   $\ln x + \ln x^4 + \ln x^7 + \ln x^{10} + \ldots$

   giving your answer as a single logarithm.

5. A rope of length 300 m is cut into several pieces, whose lengths form an arithmetic sequence with common difference d. If the shortest piece is 1 m long and the longest piece is 19 m, find d.

6. Aaron and Blake each open a savings account. Aaron deposits \$100 in the first month and then increases his deposits by \$10 each month. Blake deposits \$50 in the first month and then increases his deposits by 5% each month. After how many months will Blake have more money in his account than Aaron?

Going for the top 5

1. (a) (i) Prove that the sum of the first n terms of a geometric sequence with first term a and common ratio r is given by:

   $$S_n = \frac{a(r^n - 1)}{r - 1}$$

   (ii) Hence establish the formula for the sum to infinity, clearly justifying any conditions imposed on the common ratio r.

   (b) Show that in a geometric sequence with common ratio r, the ratio of the sum of the first n terms to the sum of the next n terms is $1 : r^n$.

   (c) In a geometric sequence, the sum of the seventh term and four times the fifth term equals the eighth term.

   (i) Find the ratio of the sum of the first 10 terms to the sum of the next 10 terms.

   (ii) Does the sequence have a sum to infinity? Explain your answer.

2. Find the sum of all integers between 1 and 1000 which are not divisible by 7.